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Mathematics

Circular Configurations of Polygonal Linkages

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ABSTRACT. We deal with cyclic and tangential configurations of planar polygonal linkages. For an arbitrary open polygonal chain, it is shown that the set of all cyclic configurations is a one-dimensional smooth submanifold of the moduli space of linkage. A similar result is true for tangential configurations. More detailed results are obtained for open chains with three links and regular chains. We also present a number of other observations about cyclic, tangential and bicentric configurations of open polygonal chains. *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *polygonal linkage, moduli space, cyclic configuration, tangential configuration.*

1. This paper is concerned with the investigation of certain special configurations of planar polygonal linkages in the framework of an approach described in [1]. Specifically, we deal with the ***cyclic*** and ***tangential*** [2] configurations of a planar polygonal linkage [3]. The important role of cyclic configurations in the theory of polygonal linkages was revealed in [1], [4]. The results of [1, 4] were further developed in [5-7]. This paper follows the same paradigm but there are two essential novelties: 1) we deal with open polygonal chains (linkages) and 2) tangential configurations are included in the consideration. For brevity, cyclic and tangential configurations will be referred to as ***circular*** configurations.

As usual, under a ***cyclic polygon*** we understand a polygon which can be inscribed in a circle, i.e., there exists a point *(circumcenter)* equidistant from all vertices of the polygon (see, e.g., [2]). By way of duality, a ***tangential polygon*** is defined as a polygon which has an inscribed circle, whose center is called the *incenter* of polygon [2]. A polygon is called *bicentric* if it is cyclic and tangential. Study of cyclic and tangential polygons has a long history starting with elementary classical results such as Ptolemy theorem and Brahmagupta formula (see, e.g., [2]). Fundamental results on the geometry of cyclic, tangential and bicentric polygons were obtained by J.Steiner, J.-V.Poncelet and N.Fuss (see, e.g., [2]). We take an essentially different view of these classical results, based on the concept *of moduli space of polygonal linkage* [3].

Polygonal linkages (or, equivalently, polygons with fixed lengths of the sides [2]) were actively studied from various points of view for at least 150 years (cf., e.g., [3]). They are also called ***polygonal chains*** [3] and we use the last term because one can then conveniently speak of *open (polygonal) chains*, where the last vertex need not coincide with the first one, or *closed (polygonal) chains*. We only consider planar polygonal chains, i.e., it is assumed that all vertices belong to the same Euclidean plane. Classical results about planar polygonal chains and the approach developed by the second-named author in [1] provided a natural background and main guidelines for our research.

Our main results state that circular configurations form one-dimensional submanifolds of the moduli space of an open polygonal chain (Theorems 1, 2). We also present a number of results on circular configurations of open chains with three and four sides. Theorems 1 and 2 have been proved by the second-named author. The detailed results about circular configurations of polygonal chains with three sides and regular k-chains have been obtained by the first-named author. Our approach is based on a few paradigms of singularity theory, for which we refer to [8]. We believe that our results may be of interest on their own and also because they confirm some general conjectures formulated in [1, 7].

2. Let us present first the necessary definitions and constructions. Recall that a polygonal k-chain L is defined by a k-tuple of nonnegative numbers l_i (called *sidelengths* of L). In the case of a closed polygonal chain it is always assumed that each of the sidelengths is not greater than the sum of all other ones [3]. A polygonal chain is called *regular* if all sidelengths are equal. Such chains naturally arise from symmetric random walks and play a certain role in theoretical physics and biology [3]. The *planar configuration space* C(L) of a polygonal k-chain is defined as the collection of all k-tuples of points v_i in Euclidean plane such that the distance between v_i and v_{i+1} is equal to l_i (for closed chains it is assumed that $v_{k+1} = v_1$). Each such collection of points is called a *configuration* of L and one can speak of *cyclic, tangential* and *bicentric configurations*. For a configuration of open chain, one can consider a (virtual) *closing side* defined as the segment joining the last and the first vertex. A configuration is called *convex* if the corresponding polygon is convex. Factoring C(L) over the natural diagonal action of SO(2) one obtains the *(planar)* ***moduli space*** M(L) [4]. Moduli spaces, as well as configuration spaces, are endowed with natural topologies induced by Euclidean metric.

It is easy to see that the moduli space can be identified with a subset of configurations such that $v_1 = (0,0)$, $v_2=(l_1,0)$. It is well-known that the moduli space of an open k-chain is homeomorphic to T^{k-1}, while for a closed k-chain the moduli space has a natural structure of compact orientable real-algebraic set of dimension k – 3. Let us say that a closed polygonal k-chain is *degenerate* if it has an *aligned configuration*, i.e., a configuration where all vertices lie on the same straight line. It is well known that this happens if and only if there exists a k-tuple of "signs" $s_i = \pm 1$ such that $\Sigma s_i l_i = 0$. The moduli space M(L) of a closed k-chain is smooth (does not have singular points) if and only if L is nondegenerate (see, e.g., [3]).

Next, for any configuration V of L with vertices $v_i=(x_i,y_i)$, its *signed area* A(V) is defined by the formula

$$2A(V) = (x_1y_2 - x_2y_1) + \ldots + (x_ky_1 - x_1y_k).$$

If moduli space is smooth then this formula defines a smooth function A on compact manifold M(L) and one may consider its critical points. For generic closed chains and open k-chains, the critical points of A are given by the cyclic configurations [4] and the so-called *diacyclic* configurations [7], respectively. Recall that a cyclic configuration of an open chain is called diacylic if it is cyclic and its (virtual) closing side is a diameter of the circumscribed circle.

A natural general approach to the study of cyclic configurations is based on the fact that they are real solutions to a system of quadratic equations. A similar interpretation is possible for tangential configurations. For example, for a 3-chain Q the moduli space can be naturally identified with the set of all real solutions to the following system of two equations in four unknowns: $\{(x - l_1)^2 + y^2 = (l_2)^2, (x-s)^2 + (y-t)^2 = (l_3)^2\}$, where $v_3 = (x,y)$, $v_4 = (s,t)$ are the "movable" vertices of configuration. A configuration is cyclic if and only if there exist real numbers p, q (coordinates of circumcenter) satisfying the system (chain) of (three) equations:

$$p^2 + q^2 = (p - l_1)^2 + q^2 = (p - x)^2 + (q - y)^2 = (p - s)^2 + (q - t)^2.$$

In total we obtain a system of 5 quadratic equations in 6 unknowns x, y, s, t, p, q such that the two pairs (x,y) and (s,t) of its real solutions give the cyclic configurations of Q. A direct computation shows that the Jacobian of this system is generically of rank 5, which implies that the set of its solutions is generically one-dimensional (cf. [7]). One can further investigate this system using methods of real algebraic geometry and singularity theory.

In particular, one can compute the bifurcation diagram [8] in the space of parameters (l_1,l_2) of this system and then use Ehresmann theorem to describe the structure of cyclic configurations for arbitrary sidelengths. The same can be done for tangential configurations. This approach is easy to realize when the number of links is small but in general it encounters considerable technical difficulties. For this reason in the present note we treat a number of situations in which the structure of circular configurations can be described by more elementary geometric methods.

3. We start by considering cyclic configurations. A configuration is called *quasicyclic* (*quasitangential*) if it is cyclic (respectively, tangential) or aligned. Recall that for a diacyclic configuration V of an open k-chain L one can consider its "*double*" which is defined as the configuration of closed 2k-chain obtained as the union of V with its reflection in the line through the first and last vertices [7]. Obviously, the double of a diacritical configuration of regular k-chain is a cyclic configuration of a closed 2k-chain D(L) with the sidelengths obtained by joining the sequence of sidelengths of L with the same sequence taken in the reverse order (cf. [7]). Notice that the doubles are never non-degenerate in the sense of Section 2 since they always have aligned configurations. Hence the moduli spaces of doubles are not smooth and one cannot automatically speak of the critical points of A on M(D(L)) but of course one can still consider cyclic configurations of D(L). These remarks are helpful since many interesting results have been established for cyclic configurations of closed chains (see, e.g., [4, 9]) and the construction of "double" can be used to obtain similar results for open chains. Finally, let R_k denote an open regular k-chain with the sidelength vector $\mathbf{1}_k = (1, \dots ,1)$.

Theorem 1. *Each open polygonal k-chain L has a one-dimensional set of cyclic configurations in M(L). For an open regular polygonal k-chain R_k, the set of cyclic configurations is a connected smooth one-dimensional submanifold (arc) in $M(R_k)$. The regular k-gon is a nondegenerate maximum of A on $M(R_k)$.*

Proof. Fix a "nearly-aligned" configuration of the first three links, with the angles close to but smaller than π. Let C denote the circle passing through the first three links. Without loss of generality we can assume that the radius of C is bigger than any sidelength of L. Hence, for each given point p in C, we can find a point q in C such that the distance between p and q is equal to any of the sidelengths. This implies that we can place consecutive vertices of L on C by properly rotating the links, which yields a configuration inscribed in C. Since the center C is determined by the position of the first three links, this construction has two degrees of freedom and passing to the moduli space gives a one-dimensional family of cyclic configurations.

For regular chain R_k of perimeter k, it's obvious that this construction works if and only if the radius of C is not less than one. Moreover, from the construction it follows that we get an arc in the moduli space with boundary points given by two aligned configurations, one of diameter k and the other of diameter 1. The last statement of the theorem can be proved by finding the index of Hessian of A at the regular k-gon and showing that it is non-degenerate and its index is k – 3. This is a matter of a routine calculation, which completes the proof.

Using Lagrange multipliers it is easy to see that critical points of A satisfy a certain square system of algebraic equations in Cartesian coordinates. In fact, there will be 2k – 4 equations for 2k – 4 variables. The first k – 1 of these equations (namely, the defining equations of the moduli space) are quadratic and the rest k – 3 equations (Lagrange equations) are of (algebraic) degree k. Since the regular k-gon is a non-degenerate maximum it follows that it is a simple real solution of the aforementioned system, which, as is well known, implies that there exists a nearby real solution for each sidelength vector sufficiently close to $\mathbf{1}_k$.

Corollary 1. *There is a neighborhood U of the point* $\mathbf{1}_k = (1, \dots, 1)$ *in the space of sidelengths such that, for all sidelength vectors belonging to U, the corresponding k-chain has cyclic configurations close to the regular k-gon.*

By results of D.Robbins [9] the area of a cyclic polygon is explicitly computable as a root of the so-called *generalized Heron polynomial* with the coefficients expressible through the sidelengths. Since the critical values of A are given by the areas of diacyclic configurations [7] we arrive at the following conclusion.

Corollary 2. *Critical values of A on $M(R_k)$ can be found as real roots of an explicitly computable real polynomial in one indeterminate*.

For an arbitrary k-chain L, an obvious application of Lusternik-Schnirelmann category theory gives a lower estimate for the number of critical points of A on M(L).

Proposition 1. For any open polygonal k-chain L, the number c(A) of critical points of A on M(L) is not smaller than k.

Since the moduli space of an open k-chain is a (k-1)-torus T^{k-1}, the estimate follows from Lusternik-Schnirelmann theorem and the well-known fact that LS-cat T^n = n+1. As follows from the next proposition, this estimate is not exact. Indeed, as was shown in [7], A is a Morse function on the moduli space of a generic 3-chain. Hence by Morse theory the lower estimate for the number of its critical points is given by the sum of Betti numbers of moduli space. Since the latter is diffeomorphic to torus T^2, this gives four as a lower estimate for c(A). The following observations were experimentally worked out by the first-named author by playing with a "carpenter ruler". An outline of rigorous proof is given below.

Proposition 2. The signed area A is a Morse function on the moduli space of a generic open 3-chain and generically has four non-degenerate critical points (maximum, minimum and two saddles). The extremal values of A can be explicitly calculated in terms of sidelengths. For a regular 3-chain R_3, A has four critical points on the moduli space $M(R_3)$, its extremal values are $\pm(1+\sqrt{3})/2$.

Playing with a "carpenter ruler" with 3 sides it is easy to visualize those critical points and see that there are no other ones. Maximum and minimum are attained at a convex diacyclic configuration taken with positive or negative orientation, respectively. By reflecting a convex diacyclic configuration in the diameter of circumcircle one obtains a convex cyclic hexagon with known sidelengths. By [9] the area of such a cyclic hexagon can be explicitly calculated in terms of sidelengths. Since the area of this hexagon is equal to two times the maximum of A on M(L), the second statement follows. The case of regular 3-chain R_3 can be analyzed by direct calculations.

As was conjectured in [7], the signed area is a Morse function for generic k-chain with arbitrary k. If this is really so, then by the same reasoning $c(A) \geq 2k-1$ for a generic open k-chain. It would be interesting to precisely characterize the chains for which A is a Morse function. The first named author constructed examples of 4-chains for which A is a Morse function and the latter estimate is exact. Exactness of the estimate in general case remains unsolved. Not surprisingly, quite detailed information appeared available for regular 4-chains.

Proposition 3. The number of critical points of A on $M(R_4)$ is equal to 6 (maximum, minimum and four saddles). The maximal value of A can be calculated as the maximal root of an explicitly given polynomial of degree 76.

It is easy to figure out these configurations by using a "carpenter ruler" with 4 links and then proving that there are no other critical points. The second statement follows by using the aforementioned construction of "double" and description of generalized Heron polynomial given in [9]. An important conclusion is that in this case A is not a Morse function on the moduli space because otherwise by above said it should have not less than 8 critical points.

In view of the above said, an upper estimate for the number of diacyclic configurations is given by the Bezout number of the polynomial system for the critical points of A. By the remark about algebraic degrees of the equations, this number is $2^{k-1}k^{k-3}$, which is obviously a very rough estimate. An intriguing problem is to obtain better estimates and, in particular, to obtain exact estimates for small k.

One way to obtain better upper estimates is suggested by the results of [9]. For a positive integer m, define a positive integer d(m) by formula $2d(m) = [(2m+1)(2m!)]/(m!)^2 - 2^{2m}$. For example, d(1) = 1, d(2) = 5, d(3) = 38. D.Robbins proved that, for even n = 2m+2, the maximal number of different critical values of A on the moduli space of n-gons is not smaller than 2d(m) [9]. In fact, for n=4 and n=6 this estimate is exact. More precisely, this upper bound is realized on certain "extremal" n-gons which are sufficiently close to but different from the regular n-gon [9].

Note that this refers to the *number of critical values* but there are no obvious reasons why values of A at different critical points cannot coincide. In other words, the *number of cyclic configurations* can be a priori bigger than 2d(m). However, for the aforementioned "c-extremal" linkages, the number of different cyclic configurations is exactly 2d(m). This seems remarkable and suggests the following

Conjecture 1. The number of different cyclic configurations of a (2m+2)-gon linkage does not exceed 2d(m).

We verified this for m=1 and m=2 but were unable to solve the case of 8-gons. Anyway, this seems to be a reasonable conjecture for closed chains and using our construction of "double" we can formulate a similar conjecture for open k-chains. Namely, let D_k denote the maximal number of different diacyclic configurations of an open k-chain. Supposing that the doubles of two different diacyclic configurations are never equal *as points of moduli space* of the doubled closed 2k-chain linkage and that Conjecture 1 is valid we come to the second conjecture.

Conjecture 2. The number D_k cannot exceed 2d(k-1).

If true, this would give a much better upper estimate than Bezout number of Lagrange system. We have proven Conjecture 2 for k=3 using computer calculations but already for k=4 the problem appears too difficult.

4. Let us now switch to tangential configurations. Such configurations did not seem to have been studied earlier but it turned out that they can be treated analogously to the cyclic ones.

Theorem 2. *Each open polygonal k-chain has a one-dimensional set of tangential configurations. For an open regular polygonal k-chain R_k, the set of quasitangential configurations is a smooth one-dimensional submanifold of $M(R_k)$.*

Proof. We can proceed as in the proof of Theorem 1. Let us fix a configuration of R_k with the angles close to but smaller than π. Next construct a circle C tangent to the first three links. By properly choosing the first two angles the radius of C can be made arbitrarily big. This in turn guarantees that we can draw a tangent to C from each consecutive vertex and place the next vertex at the prescribed distance from the preceding one. As above, this implies that in such way we obtain a one-dimensional subset of tangential configurations in the moduli space. For a regular chain, this construction works for arbitrary positions of the first three links and we obtain an arc of tangential configurations with the endpoints given by the same aligned configurations as in the proof of Theorem 1. It is again routine to make this argument completely rigorous.

For 3-chains, one can get rather detailed information about tangential configurations. We say that a tangential configuration is *strictly tangential* if the segment joining the first and last vertices is tangent to the incircle.

Proposition 4. A strictly tangential configuration of a 3-chain L(a,b,c) exists if and only if $a + c > b$. If this is the case, then L(a,b,c) also has a bicentric configuration. For the bicentric configuration, its inradius r, circumradius R, and the distance Δ between the incenter and circumcenter can be calculated by the following formulas, where we put $d = a + c - b$:

$$r^2 = abcd/(a+c)^2,\ 4R^2 = (ab + cd)\ (ac + bd)\ (ad + bc)\ /\ abcd,\ (R - \Delta)^{-2} + (R + \Delta)^{-2} = r^{-2}.$$

The first statement follows from the well-known criterion for tangential quadrilaterals [2]. By the same criterion, if $a+c > b$ each convex configuration of quadrilateral Q = Q(a,b,c,a+c-b) is tangential. Hence the convex cyclic configuration of Q will be a bicentric configuration of L. It remains to refer to the well-known formulas for r, R, Δ of a bicentric polygon [2].

Reflecting on the results presented above and in [1], the second author realized that they are particular cases of the following general statement which may serve as a paradigm for developing Morse theory for polynomial functions on configuration spaces of linkages.

Theorem 3. (Morse theory in algebraic context). *Let* $f, g_1, \ldots, g_k,\ k < n$ *be algebraically independent real polynomials in* n *variables such that* $g_1, \ldots, g_k$ *define a proper polynomial mapping. Suppose that the level surface* $X_a = \{ g_1=a_1, \ldots, g_k= a_k\}$ *is smooth. Then the critical values of restriction* f_a *of polynomial* f *to* X_a *are the real roots of a real polynomial in one variable* H(f, g, a) *whose coefficients can be algebraically expressed through the coefficients of polynomials* $f, g_1, \ldots, g_k$ *and numbers* $a_1, \ldots, a_k$.

For generic $a = (a_1, \ldots, a_k)$, *all critical points of restriction* f_a *are nondegenerate in the sense of Morse,* f_a *is a perfect Morse function, and its Morse index at any critical point can be expressed through the signature of a certain quadratic form in* $n + k$ *variables* Q(f, g) *whose coefficients are real polynomials in n variables which can be algebraically expressed through the coefficients of polynomials* $f, g_1, \ldots, g_k$.

The proof and algorithms for computing the coefficients of H(f, g, a) and Q(f, g) will be given elsewhere.

მათემატიკა

სახსრული მრავალკუთხედების წრიული კონფიგურაციები

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ნაშრომში განხილულია ბრტყელი სახსრული მრავალკუთხედების წრეწირში ჩახაზვადი და ტანგენციური კონფიგურაციები. ნაჩვენებია, რომ გახსნილი მრავალკუთხა ჯაჭვის წრეწირში ჩახაზვადი კონფიგურაციები ქმნიან ერთგანზომილებიან ქვესიმრავლეს მოდულების სივრცეში. ანალოგიური შედეგი მიღებულია ტანგენციური კონფიგურაციებისათვის. უფრო დეტალური შედეგები მიღებულია სამგვერდიანი ჯაჭვებისათვის და რეგულარი ჯაჭვებისათვის. მოყვანილია აგრეთვე სხვა შედეგები წრეწირში ჩახაზვადი, ტანგენციური და ბიცენტრული კონფიგურაციების შესახებ.

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