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ON THE SOLVABILITY OF SOME BOUNDARY VALUE PROBLEMS FOR SYMMETRIC FIRST ORDER HYPERBOLIC SYSTEMS IN A DIHEDRAL ANGLE

Abstract. The paper suggests an approach, which makes it possible to state well-posed characteristic problems in dihedral angles for a class of symmetric first order hyperbolic systems. That class involves the systems of differential equations of Maxwell, Dirac, and crystal optics for which well-posed characteristic problems in dihedral angles are presented.

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რეზიუმე. ნაშრომში შემოთავაზებულია მიდგომა, რომელიც ორწახნაგა კუთხეში პირველი რიგის სიმეტრიულ ჰიპერბოლურ სისტემათა ერთი კლასისთვის იძლევა მახასიათებელი ამოცანების კორექტულად დასმის საშუალებას. ამ კლასს მიეკუთვნებიან მაქსველის, დირაკის და კრისტალთა ოპტიკის განტოლებათა სისტემები, რომელთათვისაც მოყვანილია კორექტულად დასმული მახასიათებელი ამოცანები ორწახნაგა კუთხეში.

1. Introduction

It is known that the Goursat problem or the so-called characteristic problem for the second order hyperbolic equation admits different statements when passing from two-dimensional multi-dimensional case. For example, the characteristic problem for the multi-dimensional wave equation can be formulated both in a conic domain, whose boundary is a characteristic conoid, and in a dihedral angle, whose faces are characteristic surfaces. In addition, on the boundary of the domain one can consider the boundary Dirichlet condition, and these problems are assumed to be posed correctly [1–7]. As to the correct statement of the characteristic problem, the situation becomes more complicated when in the multi-dimensional case we pass from one equation to a system of hyperbolic equations. For instance, despite the fact that for a split in the principal part second order hyperbolic system the Goursat problem with the Dirichlet data on a characteristic conoid is well-posed [8], we can find in [9] an example of a non-split in the principal part second order hyperbolic system for which the corresponding homogeneous characteristic problem has an infinite set of linearly independent solutions. The fact that even for a non-split in the principal part strictly hyperbolic system, whose cone of normals consists of infinitely smooth sheets, the cones of rays corresponding to these sheets may have strong singularities [10, p. 586] already is a complexity. Therefore difficulties arise already in the course of the statement of the characteristic problem in which a carrier of the boundary data should be pointed out. In this direction the works [11, 12] are worth noticing.

In the present work we suggest an approach allowing us to formulate correct boundary value problems for a class of symmetric first order hyperbolic systems, among them are the characteristic problems in dihedral angles. To that class of systems belong, for instance, the well-known from the mathematical physics systems of differential equations of Maxwell, Dirac and crystal optics.

At the end of the paper for each of these systems we present well-posed statements of characteristic problems in dihedral angles.

As for the Cauchy and mixed type problems for symmetric first order hyperbolic systems, they are well studied and treated in [13–16].

2. Statement of the Boundary Value Problem. A Priori Estimate

In the space R^{n+1} of variables $x_1, \dots, x_n$ and t we consider a system of first order differential equations of the type

$$Lu \equiv Eu_t + \sum_{i=1}^{n} A_i u_{x_i} + Bu = F, \tag{1}$$

where A_i, B are given real $(m \times m)$-matrices, E is the unit $(m \times m)$-matrix, F is a given and u is an unknown m-dimensional real vector, $n > 1$, $m > 1$.

Below, the matrices A_i will be assumed to be symmetric and constant. In this case the system (1) is hyperbolic [10, p.587].

Denote by $D = \Big\{(x_1,\dots,x_n,t) \in R^{n+1} : \alpha_0^i t + \sum_{j=1}^n \alpha_j^i x_j < 0,\ i=1,2\Big\}$ a dihedral angle bounded by the hypersurfaces $\widetilde{S_1} : \alpha_0^1 t + \sum_{j=1}^n \alpha_j^1 x_j = 0$ and $\widetilde{S_2} : \alpha_0^2 t + \sum_{j=1}^n \alpha_j^2 x_j = 0$, where $\alpha^i = (\alpha_1^i,\dots,\alpha_n^i,\alpha_0^i)$ is the unit vector of the outer to ∂D normal at the point of the face $S_i = \widetilde{S}_i \cap \partial D$, $i=1,2$, $\alpha^1 \neq \alpha^2$. For the sake of simplicity of our exposition, below it will be assumed that $\alpha_0^i < 0$, $i=1,2$.

Let us consider the boundary value problem formulated as follows: find in the domain D a solution u of the system (1) by the boundary conditions

$$\Gamma^i u\Big|_{S_i} = f^i, \quad i=1,2, \tag{2}$$

where Γ^i are given real constants $(\varkappa_i \times m)$-matrices, and $f^i = (f_1^i,\dots,f_{\varkappa_i}^i)$ are given $\varkappa_i$-dimensional real vectors, $i=1,2$.

Remark 1. Below, depending on the geometric orientation of the dihedral angle D, we will indicate a method of constructing the matrices Γ^i, $i=1,2$, for which the boundary value problem (1), (2) will be well-posed.

Get now back to the system (1). Since the matrix $Q(\xi') = -\sum_{i=1}^n A_i\xi_i$, $\xi' = (\xi_1,\dots,\xi_n) \in R^n$, is symmetric, its characteristic roots are real. We enumerate them in decreasing order: $\widetilde{\lambda}_1(\xi') \geq \widetilde{\lambda}_2(\xi') \geq \dots \geq \widetilde{\lambda}_m(\xi')$. Below, multiplicities $k_1,\dots,k_s$ of these roots will be assumed to be constant, i.e., independent of ξ', and we assume

$$\lambda_1(\xi') = (\widetilde{\lambda}_1)(\xi') = \dots = \widetilde{\lambda}_{k_1}(\xi') > \lambda_2(\xi') = \widetilde{\lambda}_{k_1+1}(\xi') = \dots = \widetilde{\lambda}_{k_1+k_2}(\xi') >$$
$$> \lambda_s(\xi') = \widetilde{\lambda}_{m-k_s+1}(\xi') = \dots = \widetilde{\lambda}_m(\xi'), \xi' \in R^n \backslash \{(0,\dots,0)\}. \tag{3}$$

Note that by virtue of (3) and continuous dependence of the roots of the polynomial on its coefficients, $\lambda_1(\xi'),\dots,\lambda_s(\xi')$ are continuous homogeneous functions of degree 1 [17].

Since the matrix $Q(\xi')$ is symmetric, there exists an orthogonal matrix $T = T(\xi')$ such that

$$(T^{-1}QT)(\xi') = \operatorname{diag}(\underbrace{\lambda_1(\xi'),\dots,\lambda_1(\xi')}_{k_1},\dots,\underbrace{\lambda_s(\xi'),\dots,\lambda_s(\xi')}_{k_s}). \tag{4}$$

By (3) and (4), the normal cone

$$K = \{\xi = (\xi_1,\dots,\xi_n,\xi_0) \in R^{n+1} : \det(E\xi_0 - Q(\xi')) = 0\}$$

of the system (1) consists of separate sheets

$$K_i = \{\xi = (\xi',\xi_0) \in R^{n+1} : \xi_0 - \lambda_i(\xi') = 0\}, \quad i=1,\dots,s.$$

Since

$$\lambda_j(\xi') = -\lambda_{s+1-j}(-\xi'), \quad 0 \leq j \leq \Big[\frac{s+1}{2}\Big], \tag{5}$$

the cones K_j and K_{s+1-j} are centrally symmetric with respect to the point $(0,\dots,0)$, where $[a]$ denotes the integer part of the number a.

Remark 2. In case s is an odd number, we have $j = s+1-j$ for $j = \left[\frac{s+1}{2}\right]$. Therefore the cone K_j for $j = \left[\frac{s+1}{2}\right]$ is centrally symmetric with respect to the point $(0,\dots,0)$. In this case, if $s = 2s_0 + 1$, to simplify our exposition we assume that

$$\lambda_{s_0+1}(\xi') \equiv 0, \quad \left[\frac{s+1}{2}\right] = s_0 + 1, \tag{6}$$

i.e., K_{s_0+1} is the hypersurface $\pi_0 : \xi_0 = 0$.

Remark 3. Below it will be assumed that $\pi_0 \cap K_{s_0} = \{(0,\dots,0)\}$ for the even $s = 2s_0$. By (3) and (5), this means that the cones $K_1,\dots,K_{s_0}$ are located on one face of $\pi_0 : \xi_0 = 0$, while $K_{s_0+1},\dots,K_{2s_0}$ on another face, i.e.,

$$\lambda_1(\xi') > \dots > \lambda_{s_0}(\xi') > 0 > \lambda_{s_0+1}(\xi') > \dots > \lambda_{2s_0}(\xi'), \tag{7}$$
$$\xi' \in R^n \backslash \{(0,\dots,0)\}.$$

For the odd $s = 2s_0 + 1$, by virtue of (3), (5) and (6) there automatically takes place $\pi_0 \cap K_{s_0} = \{(0,\dots,0)\}$ and, consequently,

$$\lambda_1(\xi') > \dots > \lambda_{s_0}(\xi') > \lambda_{s_0+1}(\xi') \equiv 0 > \lambda_{s_0+2}(\xi') > \dots > \lambda_{2s_0+1}(\xi'), \tag{8}$$
$$\xi' \in R^n \backslash \{(0,\dots,0)\}.$$

In this case $K_1,\dots,K_{s_0}$ are located on one face of $\pi_0 = K_{s_0+1}$, while $K_{s_0+2},\dots,K_{2s_0+1}$ on another face. From (5)–(8) it easily follows that for the multiplicities k_j of the roots λ_j the equalities

$$k_j = k_{s+1-j}, \quad j = 1,\dots,\left[\frac{s+1}{2}\right],$$

are valid.

Consider the case where the problem (1), (2) is characteristic, i.e., both faces S_1 and S_2 are characteristic surfaces of the system (1). The latter implies that

$$\alpha^i = (\alpha^i_1,\dots,\alpha^i_n,\alpha^i_0) \in K = \left\{\xi \in R^{n+1} : \det(E\xi_0 + \sum_{j=1}^{n} A_j\xi_j) = 0\right\}, \; i = 1,2.$$

In this case, since $K = \cup_{j=1}^{s} K_j$ and by our assumption $\alpha^i_0 < 0$, $i = 1,2$, by virtue of (7) and (8) there exist natural numbers s_1 and s_2 such that

$$s_i > \left[\frac{s+1}{2}\right], \quad i = 1,2; \quad \alpha^i \in K_{s_i}, \quad i = 1,2. \tag{9}$$

Denote by $Q_0(\xi) \equiv E\xi_0 + \sum_{i=1}^{n} A_i\xi_i = E\xi_0 - Q(\xi')$ the characteristic matrix of the system (1) and consider the problem on reduction of the quadratic form $(Q_0(\xi)\eta,\eta)$ to the canonical form when $\xi \in K'_i = K_i \backslash \{(0,\dots,0)\}$, where $\eta \in R^m$ and $(\cdot,\cdot)$ denotes the scalar product in the Euclidean space R^m.

By (4), for $\eta = T\zeta$ we have

$$(Q_0(\xi)\eta,\eta) = ((T^{-1}Q_0T)(\xi)\zeta,\zeta) =$$
$$= ((E\xi_0 - (T^{-1}QT)(\xi'))\zeta,\zeta) = (\xi_0 - \lambda_1(\xi'))\zeta_1^2 + \cdots + (\xi_0 - \lambda_1(\xi'))\zeta_{k_1}^2 +$$
$$+ (\xi_0 - \lambda_2(\xi'))\zeta_{k_1+1}^2 + \cdots + (\xi_0 - \lambda_2(\xi'))\zeta_{k_1+k_2}^2 + \cdots +$$
$$+ \cdots + (\xi_0 - \lambda_s(\xi'))\zeta_{m-k_s+1}^2 + \cdots + (\xi_0 - \lambda_s(\xi'))\zeta_m^2. \tag{10}$$

Since for $\xi = (\xi',\xi_0) \in K_i'$ the equality $\xi_0 = \lambda_i(\xi')$ holds, taking into account (3), we will have

$$[\xi_0 - \lambda_j(\xi')]\Big|_{K_i'} < 0, \quad j = 1,\dots,i-1; \ [\xi_0 - \lambda_i(\xi')]\Big|_{K_i'} = 0,$$
$$[\xi_0 - \lambda_j(\xi')]\Big|_{K_j'} > 0, \quad j = i+1,\dots,s. \tag{11}$$

If we denote by $\varkappa_i^+$ and $\varkappa_i^-$ the positive and the negative indices of inertia of the quadratic form $(Q_0(\xi)\eta,\eta)\Big|_{\xi\in K_i'}$, then owing to (10) and (11) we obtain

$$\varkappa_i^- = k_1 + \cdots + k_{i-1}, \ \ \varkappa_i^+ = k_{i+1} + \cdots + k_s, \ (\mathrm{def})_i = k_i, \tag{12}$$

where $(\mathrm{def})_i$ is the defect of that form, and $\varkappa_i^- = 0$ for $i = 1$.

If now $\zeta = C^i\eta$ is any non-degenerate linear transformation reducing the quadratic form $(Q_0(\xi)\eta,\eta)\Big|_{\xi\in K_i'}$ to the canonical one, then by (12) and invariance of the indices of inertia of the quadratic form with respect to non-degenerate linear transformations we have

$$(Q_0(\xi)\eta,\eta)\Big|_{\xi\in K_i'} = \sum_{j=1}^{\varkappa_i^+}[\Lambda_{ij}^+(\xi,\eta)]^2 - \sum_{j=1}^{\varkappa_i^-}[\Lambda_{ij}^-(\xi,\eta)]^2. \tag{13}$$

Here

$$\Lambda_{ij}^-(\xi,\eta) = \sum_{p=1}^m c_{jp}^i(\xi)\eta_p, \quad \Lambda_{ij}^+(\xi,\eta) = \sum_{p=1}^m c_{\varkappa_i^-+j,p}^i(\xi)\eta_p, \tag{14}$$
$$C^i = C^i(\xi) = (c_{jp}^i(\xi)), \quad \xi \in K_i'.$$

According to (14) and (9), in the boundary conditions (2) in the capacity of the matrix Γ^i we take the matrix of order $(\varkappa_i \times m)$, where $\varkappa_i = \varkappa_{s_i}^-$, $i = 1,2$, and its elements Γ_{jp}^i are given by the equalities

$$\Gamma_{jp}^i = c_{jp}^{s_i}(\alpha^i), \quad i = 1,2; \quad j = 1,\dots,\varkappa_{s_i}^-; \quad p = 1,\dots,m. \tag{15}$$

Below, the elements of the matrix B in the system (1) are assumed to be bounded measurable functions in $\overline{D}$, i.e., $B \in L_\infty(\overline{D})$. Introduce the following weighted spaces:

$$W^1_{2,\lambda}(D) = \{u \in L_{2,\mathrm{loc}}(D) : u \exp(-\lambda t) \in W^1_2(D)\},$$
$$\|u\|_{W^1_{2,\lambda}(D)} = \|u \exp(-\lambda t)\|_{W^1_2(D)},$$
$$L_{2,\lambda}(D) = \{F \in L_{2,\mathrm{loc}}(D) : F \exp(-\lambda t) \in L_2(D)\},$$
$$\|F\|_{L_{2,\lambda}(D)} = \|F \exp(-\lambda t)\|_{L_2(D)},$$
$$L_{2,\lambda}(S_i) = \{f \in L_{2,\mathrm{loc}}(S_i) : f \exp(-\lambda t) \in L_2(S_i)\}, \ i = 1, 2,$$
$$\|f\|_{L_{2,\lambda}(S_i)} = \|f \exp(-\lambda t)\|_{L_2(S_i)},$$

where λ is a real parameter and $L_{2,\mathrm{loc}}(D)$, $W^1_2(D)$, $L_{2,\mathrm{loc}}(S_i)$, $i = 1, 2$, are the well-known functional spaces [18, p. 384].

Let $\lambda_{\max}(P)$ be the largest characteristic number of the non-negative definite symmetric matrix $B'B$ at the point $P \in \overline{D}$ (the prime denotes matrix transposition). Then, because of the fact that $B \in L_\infty(\overline{D})$, we have

$$\lambda_0^2 = \sup_{P \in \overline{D}} \lambda_{\max}(P) < +\infty. \tag{16}$$

Lemma. *Under the assumption* (15), *for any solution* $u \in W^1_{2,\lambda}(D)$ *of the problem* (1), (2) *with* $\lambda > \lambda_0$ *the following a priori estimate is valid:*

$$\|u\|_{L_{2,\lambda}(D)} \leq \frac{1}{\sqrt{\lambda - \lambda_0}} \sum_{i=1}^{2} \sum_{j=1}^{\varkappa_i} \|f^i_j\|_{L_{2,\lambda}(S_i)} + \frac{1}{\lambda - \lambda_0} \|F\|_{L_{2,\lambda}(D)}, \tag{17}$$

where $\varkappa_i = \varkappa^-_{s_i}$, $i = 1, 2$.

Proof. Introduce a new unknown function $v(x,t) = u(x,t) \exp(-\lambda t)$, $\lambda = \mathrm{const} > 0$. Then for $v(x,t)$ we obtain the system of equations

$$L_\lambda v \equiv E v_t + \sum_{i=1}^{n} A_i v_{x_i} + B_\lambda v = F_\lambda, \tag{18}$$

where $B_\lambda = B + \lambda E$, $F_\lambda = F \exp(-\lambda t)$. Note that if $u \in W^1_{2,\lambda}(D)$, then $F \in L_{2,\lambda}(D)$ and $v \in W^1_2(D)$, $F_\lambda \in L_2(D)$ and the boundary conditions (2) will take the form

$$\Gamma^i v \Big|_{S_i} = f^i_\lambda, \quad i = 1, 2, \tag{19}$$

where $f^i_\lambda = f^i \exp(-\lambda t) \in L_2(S_i)$, $i = 1, 2$.

By virtue of (18), integration by parts yields

$$2 \int_D (L_\lambda v, v) dD = \int_{\partial D} (Q_0(\alpha) v, v) ds + \int_D (2 B_\lambda v, v) dD =$$
$$= \sum_{i=1}^{2} \int_{S_i} (Q_0(\alpha^i), v, v) ds + \int_D (2 B_\lambda v, v) dD, \tag{20}$$

where $Q_0(\alpha) = E\alpha_0 + \sum_{j=1}^n A_j \alpha_j$, and $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_0)$ is the unit vector of the outer normal to ∂D.

By (9), (12)–(15) and (19) we have

$$(Q_0(\alpha^i)v,v)\Big|_{S_i} = \Big(\sum_{j=1}^{\varkappa^+_{s_i}}\Big[\Lambda^+_{s_i,j}(\alpha^i,v)\Big]^2\Big)\Big|_{S_i} - \Big(\sum_{j=1}^{\varkappa^-_{s_i}}\Big[\Lambda^-_{s_i,j}(\alpha^i,v)\Big]^2\Big)\Big|_{S_i} =$$

$$= \Big(\sum_{j=1}^{\varkappa^+_{s_i}}\Big[\Lambda^+_{s_i,j}(\alpha^i,v)\Big]^2\Big)\Big|_{S_i} - \Big(\sum_{j=1}^{\varkappa^-_{s_i}}\Big[\sum_{p=1}^{m} c^{s_i}_{jp}(\alpha^i)v_p\Big]^2\Big|_{S_i} \geq$$

$$\geq -\Big(\sum_{j=1}^{\varkappa^-_{s_i}}\Big[\sum_{p=1}^{m}\Gamma^i_{jp}v_p\Big]^2\Big|_{S_i} = -\sum_{j=1}^{\varkappa^-_{s_i}}(f^i_{\lambda j})^2 = -\sum_{j=1}^{\varkappa_i}(f^i_{\lambda j})^2, \quad i=1,2. \qquad (21)$$

Taking into account (16), we find that in the domain D

$$(2B_\lambda v,v) = 2\lambda(v,v) + 2(Bv,v) \geq 2\lambda(v,v)-$$
$$-2(Bv,Bv)^{1/2}(v,v)^{1/2} = 2\lambda(v,v) - 2(B'Bv,v)^{1/2}(v,v)^{1/2} \geq$$
$$\geq 2\lambda(v,v) - 2\lambda_0(v,v)^{1/2}(v,v)^{1/2} = 2(\lambda-\lambda_0)(v,v). \qquad (22)$$

Next, by (18), for $\varepsilon = \lambda - \lambda_0 > 0$ we have

$$2\int\limits_D (L_\lambda v,v)dD = 2(F_\lambda,v)_{L_2(D)} \leq 2\|F_\lambda\|_{L_2(D)}\|v\|_{L_2(D)} \leq$$
$$\leq \frac{1}{\varepsilon}\|F_\lambda\|_{L_2(D)} + \varepsilon\|v\|^2_{L_2(D)} = \frac{1}{\lambda-\lambda_0}\|F_\lambda\|^2_{L_2(D)} + (\lambda-\lambda_0)\|v\|^2_{L_2(D)}. \qquad (23)$$

It follows from (20)–(23) that

$$(\lambda-\lambda_0)\|v\|^2_{L_2(D)} \leq \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|f^i_{\lambda j}\|^2_{L_2(S_i)} + \frac{1}{\lambda-\lambda_0}\|F_\lambda\|^2_{L_2(D)},$$

whence with regard for

$$\|v\|_{L_2(D)} = \|u\|_{L_{2,\lambda}(D)},\ \|f^i_{\lambda j}\|_{L_2(S_i)} = \|f^i_j\|_{L_{2,\lambda}(S_i)},\ \|F_\lambda\|_{L_2(D)} = \|F\|_{L_{2,\lambda}(D)}$$

we immediately obtain the desired inequality (17).

Remark 4. From (9), (12) and (15) it follows a completely definite dependence of the structure and number of boundary conditions in (2) on the geometric orientation of the dihedral angle D. The estimate (17) results in the unique solvability of the problem (1), (2) in the class $W^1_{2,\lambda}(D)$ for $\lambda > \lambda_0$.

3. The Existence of a Solution of the Boundary Value Problem with Homogeneous Boundary Conditions

Consider now the question on the solvability of the boundary value problem (1), (2) with homogeneous boundary conditions. Note that if

$v \in W_2^2(D) \cap \overset{\circ}{W}{}_2^1(D)$, then on the boundary ∂D we will have $v = 0$ and, consequently,

$$\partial D : v_t = \alpha_0 v_\alpha, \quad v_{x_i} = \alpha_i v_\alpha, \quad i = 1, \dots, n, \tag{24}$$

where $v_\alpha = \alpha_0 v_t + \sum_{i=1}^{n} \alpha_i v_{x_i}$, and $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_0)$ is the unit vector of the outer normal to ∂D.

For the sake of simplicity of our exposition, we introduce the following notation: $t = x_{n+1}$, $\alpha_0 = \alpha_{n+1}$, $A_{n+1} = E$. Then the principal part of the system (18) can be written in the form $L_\lambda^0 v \equiv \sum_{i=1}^{n+1} A_i v_{x_i}$. For $v \in W_2^2(D) \cap \overset{\circ}{W}{}_2^1(D)$, a simple integration by parts yields

$$\int_D (A_i v_{x_i}, v_{x_j x_j}) dD = \frac{1}{2} \int_{\partial D} (A_j \alpha_j v_{x_j}, v_{x_j}) ds, \quad i = j,$$

$$\int_D (A_i v_{x_i}, v_{x_j x_j}) dD = \int_{\partial D} (A_j \alpha_j v_{x_i}, v_{x_j}) ds - \frac{1}{2} \int_{\partial D} (A_i \alpha_i v_{x_j}, v_{x_j}) ds, \quad i \neq j,$$

whence it immediately follows that

$$\int_D (L_\lambda^0 v, v_{x_j x_j}) dG = \int_{\partial D} \Bigg(\sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_j v_{x_j}, v_{x_j} \Bigg) ds -$$

$$-\frac{1}{2} \int_{\partial D} \Bigg(\sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_i v_{x_j}, v_{x_j} \Bigg) ds + \frac{1}{2} \int_{\partial D} (A_j \alpha_j v_{x_j}, v_{x_j}) ds. \tag{25}$$

By (24), (25) and the equality $\alpha^2 = \sum_{i=1}^{n+1} \alpha_i^2 = 1$, we have

$$\int_D (L_\lambda^0 v, \Delta v) dD = \int_D \Bigg(L_\lambda^0 v, \sum_{j=1}^{n+1} v_{x_j x_j} \Bigg) dD =$$

$$= \sum_{j=1}^{n+1} \int_{\partial D} \Bigg(\sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_j v_{x_i}, v_{x_j} \Bigg) ds +$$

$$+ \frac{1}{2} \sum_{j=1}^{n+1} \int_{\partial D} \Bigg(\Bigg[A_j \alpha_j - \sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_i \Bigg] v_{x_j}, v_{x_j} \Bigg) ds =$$

$$= \sum_{j=1}^{n+1} \int_{\partial D} \Bigg(\sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_i \alpha_j^2 v_\alpha, v_\alpha \Bigg) ds +$$

$$+ \frac{1}{2} \sum_{j=1}^{n+1} \int_{\partial D} \Bigg(\Bigg[A_j \alpha_j - \sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_i \Bigg] \alpha_j^2 v_\alpha, v_\alpha \Bigg) ds =$$

$$= \frac{1}{2}\sum_{j=1}^{n+1}\int_{\partial D}\left(\left[A_j\alpha_j + \sum_{\substack{i=1\\ i\neq j}}^{n+1} A_i\alpha_i\right]\alpha_j^2 v_{\alpha,v_\alpha}\right)ds =$$

$$= \frac{1}{2}\sum_{j=1}^{n+1}\int_{\partial D}\left(\left(\sum_{i=1}^{n+1} A_i\alpha_i\right)\alpha_j^2 v_\alpha, v_\alpha\right)ds =$$

$$= \left[\frac{1}{2}\int_{\partial D}\left(\sum_{i=1}^{n+1} A_i\alpha_i v_\alpha, v_\alpha\right)ds\right]\left(\sum_{j=1}^{n+1}\alpha_j^2\right) = \frac{1}{2}\int_{\partial D}(Q_0(\alpha)v_\alpha, v_\alpha)ds, \qquad (26)$$

where $Q_0(\alpha) = \sum_{i=1}^{n+1} A_i\alpha_i = E\alpha_0 + \sum_{i=1}^{n} A_i\alpha_i$ is the characteristic matrix of the system (18).

Below, the elements of the matrix $\overline{D}$ will be assumed to be bounded in a closed domain $\overline{D}$ together with their partial derivatives of the first order, and in the condition (9) it will be assumed that

$$s_1 = s_2 = s, \quad \alpha^i \in K_s, \quad i = 1, 2. \qquad (27)$$

According to (18) and (26) and an anologous argument, we get

$$-\int_D (L_\lambda v, \Delta v - v)dD = -\int_D \left(L_\lambda v, \sum_{j=1}^{n+1} v_{x_j x_j} - v\right)dD =$$

$$= \sum_{j=1}^{n+1}\int_D (B_\lambda v_{x_j}, v_{x_j})dD + \sum_{j=1}^{n+1}\int_D (B_{\lambda x_j} v, v_{x_j})dD + \int_D (B_\lambda v, v)dD +$$

$$+\frac{1}{2}\int_{\partial D}(Q_0(\alpha)v, v)ds - \frac{1}{2}\int_{\partial D}(Q_0(\alpha)v_\alpha, v_\alpha)ds. \qquad (28)$$

Taking into account (10), (11) and (27), it is not difficult to see that the matrix $Q_0(\alpha)$, where α is the unit vector of the outer normal to ∂D, is non-positive. Therefore

$$-\frac{1}{2}\int_{\partial D}(Q_0(\alpha)v_\alpha, v_\alpha)ds \geq 0. \qquad (29)$$

Since $v \in W_2^2(D) \cap \overset{\circ}{W}{}_2^1(D)$, we have $v\Big|_{\partial D} = 0$ and

$$\frac{1}{2}\int_{\partial D}(Q_0(\alpha)v, v)ds = 0. \qquad (30)$$

Let $\lambda_{\max}^i(P)$ be the largest characteristic number of the non-negative definite symmetric matrix $B'_{x_i}B_{x_i}$ at the point $P \in \overline{D}$, $i = 1, \dots, n+1$. Then, because of the fact that by our assumption the elements of the matrix B_{x_i}, $i = 1, \dots, n+1$, are bounded in $\overline{D}$, we have

$$\lambda_*^2 = \max_{1\leq i\leq n+1}\sup_{P\in\overline{D}}\lambda_{\max}^i(P) < +\infty. \qquad (31)$$

By (16) and (31), analogously to how we have obtained the inequality (22), we have

$$\sum_{j=1}^{n+1}\int_D (B_\lambda v_{x_j}, v_{x_j})dD + \sum_{j=1}^{n+1}\int_D (B_{\lambda x_j} v, v_{x_j})dD + \int_D (B_\lambda v, v)dD \geq$$
$$\geq (\lambda-\lambda_0)\int_D \left(\sum_{j=1}^{n+1} v_{x_j}^2\right)dD - \lambda_* \sum_{j=1}^{n+1} \|v\|_{L_2(D)} \|v_{x_j}\|_{L_2(D)} +$$
$$+(\lambda-\lambda_0)\int_D v^2 dD \geq (\lambda-\lambda_0)\int_D \left(\sum_{j=1}^{n+1} v_{x_j}^2\right)dD -$$
$$-\lambda_* \|v\|_{L_2(D)} \left[(n+1)\sum_{j=1}^{n+1} \|v_{x_j}\|_{L_2(D)}^{\frac{1}{2}}\right]^{\frac{1}{2}} + (\lambda-\lambda_0)\int_D v^2 dD \geq$$
$$\geq (\lambda-\lambda_0)\int_D \left(\sum_{j=1}^{n+1} v_{x_j}^2\right)dD - \frac{\lambda_*}{2}\|v\|_{L_2(D)}^2 -$$
$$-\frac{\lambda_*}{2}(n+1)\sum_{j=1}^{n+1} \|v_{x_j}\|_{L_2(D)}^2 + (\lambda-\lambda_0)\int_D v^2 dD =$$
$$= (\lambda-\lambda_0-\frac{1}{2}\lambda_*(n+1))\int_D \left(\sum_{j=1}^{n+1} v_{x_j}^2\right)dD + (\lambda-\lambda_0-\frac{1}{2}\lambda_*)\int_D v^2 dD \geq$$
$$\geq (\lambda-\lambda_0-\frac{1}{2}(n+1)\lambda_*)\int_D \left(v^2+\sum_{j=1}^{n+1} v_{x_j}^2\right)dD =$$
$$= (\lambda-\lambda_0-\frac{1}{2}(n+1)\lambda_*)\|v\|_{W_2^1(D)}^2. \tag{32}$$

Next, by (29)–(32), it follows from (28) that for any $v \in W_2^2(D) \cap \overset{\circ}{W}{}_2^1(D)$ the inequality

$$\left|\int_D (L_\lambda v, \Delta v - v)dD\right| \geq (\lambda-\lambda_0-\frac{1}{2}(n+1)\lambda_*)\|v\|_{W_2^1(D)}^2 \tag{33}$$

holds.

From (33) for

$$\lambda > \lambda_0 + \frac{1}{2}(n+1)\lambda_*, \tag{34}$$

in the well-known manner we obtain the following inequality [19, p.51]:

$$\|L_\lambda^* w\|_{-1} \geq c\|w\|_{-1} \quad \forall w \in W_2^1(D), \tag{35}$$

where $L_\lambda^* w \equiv -Ew_t - \sum_{i=1}^n A_i w_{x_i} + B_\lambda' w$, the positive constant $c = c(\lambda, \lambda_0, \lambda_*)$ does not depend on w, and

$$\|w\|_{-1} = \sup_{v\in \overset{\circ}{W}{}_2^1(D)} \frac{(w,v)_{L_2(D)}}{\|v\|_{\overset{\circ}{W}{}_2^1(D)}}$$

is the norm in the negative Lax space $\overset{\circ}{W}{}_2^{-1}(D)$.

Consider now, under the assumption (27), the characteristic problem (18), (19) with homogeneous boundary conditions, i.e.,

$$L_\lambda v = F_\lambda, \tag{36}$$

$$\Gamma^i v\Big|_{S_i} = 0, \quad i = 1, 2. \tag{37}$$

Definition. The vector function $v \in L_2(D)$ will be called a weak solution of the problem (36), (37), where $F_\lambda \in L_2(D)$, if the identity

$$(v, L_\lambda^* w)_{L_2(D)} = (F_\lambda, w)_{L_2(D)} \tag{38}$$

is valid for any $w \in W_2^1(D)$.

It is easily seen that if a weak solution v of the problem (36), (37) belongs to the space $W_2^1(D)$, then it is a solution of that problem in the ordinary sense.

Let us show that when the inequality (34) is fulfilled, for any vector function $F_\lambda \in \overset{\circ}{W}{}_2^1(D)$ there exists a unique weak solution of the problem (36), (37) from the space $\overset{\circ}{W}{}_2^1(D)$. Indeed, considering the linear functional $(v, L_\lambda^* w)_{L_2(D)}$ on the space $\overset{\circ}{W}{}_2^{-1}(D)$, by virtue of (35) for any $w \in W_2^1(D)$ we have

$$|(F_\lambda, w)_{L_2(D)}| \le \|F_\lambda\|_{\overset{\circ}{W}{}_2^1(D)} \|w\|_{-1} \le c^{-1} \|L_\lambda^* w\|_{-1} \|F\|_{\overset{\circ}{W}{}_2^1(D)}. \tag{39}$$

The inequality (39) allows one to extend this functional to the entire space $\overset{\circ}{W}{}_2^{-1}(D)$ by continuity. Furthermore, using Riesz theorem on the functional representation on the space $\overset{\circ}{W}{}_2^{-1}(D)$, we obtain that there exists the vector function $v \in \overset{\circ}{W}{}_2^1(D)$, satisfying (38). To prove the uniqueness of the solution, it should be noted that for a weak solution v of the problem (36), (37) from the space $\overset{\circ}{W}{}_2^1(D)$ we have $v|_{\partial D} = 0$. Therefore, integrating (38) by parts, we get $L_\lambda v = F_\lambda$, $(x,t) \in D$. It remains only to note that in proving the a priori estimate (17), it has incidentally been shown that for any $v \in \overset{\circ}{W}{}_2^1(D)$ the inequality

$$\|v\|_{L_2(D)} \le \frac{1}{\lambda - \lambda_0} \|F_\lambda\|_{L_2(D)}$$

is valid.

Thus the following theorem is valid.

Theorem 1. *Let the conditions* (27) *and* (34) *be fulfilled. Then for any* $F_\lambda \in \overset{\circ}{W}{}_2^1(D)$ *there exists a unique solution of the problem* (36), (37) *from the space* $W_2^1(D)$.

Since the problem (1), (2) with the homogeneous boundary conditions

$$\Gamma^i u\Big|_{S_i} = 0, \quad i = 1, 2, \tag{40}$$

in the space $W_{2,\lambda}^1(D)$ is equivalent to the problem (36), (37) in the space $W_2^1(D)$, from Theorem 1 follows

Theorem 2. *Let the conditions* (27), (34) *be fulfilled. Then for any* $F \in \overset{\circ}{W}{}_2^1(D)$ *there exists a unique solution of the problem* (1), (40) *from the space* $W_{2,\lambda}^1(D)$.

Definition. Let $F \in L_{2,\lambda}(D)$. The function $u \in L_{2,\lambda}(D)$ will be called a strong solution of the problem (1), (40) of the class $L_{2,\lambda}$, if there exists a sequence of the functions $u_k \in W_{2,\lambda}^1(D)$ satisfying the homogeneous boundary conditions (40) such that

$$\lim_{k\to\infty} \|u - u_k\|_{L_{2,\lambda}(D)} = \lim_{k\to\infty} \|F - Lu_k\|_{L_{2,\lambda}(D)} = 0.$$

If $F \in L_{2,\lambda}(D)$, then because of the fact that the space of infinitely differentiable finite functions $C_0^\infty(D)$ is dense in $L_{2,\lambda}(D)$, there exists a sequence $F_k \in C_0^\infty(D)$ such that $F_k \to F$ in $L_{2,\lambda}(D)$. Since $F_k \in C_0^\infty(D)$, hence $F_k \in \overset{\circ}{W}{}_2^1(D)$. Therefore, according to Theorem 2, for $F = F_k$ there exists a unique solution $u_k \in W_{2,\lambda}^1(D)$ of the problem (1), (40). From the inequality (17) we have

$$\|u_k - u_p\|_{L_{2,\lambda}(D)} \le \frac{1}{\lambda - \lambda_0} \|F_k - F_p\|_{L_{2,\lambda}(D)},$$

from which it follows that the sequence $\{u_k\}$ is fundamental in $L_{2,\lambda}(D)$, since $F_k \to F$ in $L_{2,\lambda}(D)$. Because of the completeness of the space $L_{2,\lambda}(D)$, there exists a function $u \in L_{2,\lambda}(D)$ such that $u_k \to u$ and $Lu_k = F_k \to F$ in $L_{2,\lambda}(D)$. Consequently, u is a strong solution of the problem (1), (40) of the class $L_{2,\lambda}$. The uniqueness of the strong solution of the problem (1), (40) of the class $L_{2,\lambda}$ follows from the inequality (17).

Thus we have proved the following

Theorem 3. *Let the conditions* (27) *and* (34) *be fulfilled. Then for any* $F \in L_{2,\lambda}(D)$ *there exists a unique solution* u *of the problem* (1), (40) *of the class* $L_{2,\lambda}$ *for which the estimate*

$$\|u\|_{L_{2,\lambda}(D)} \le \frac{1}{\lambda - \lambda_0} \|F\|_{L_{2,\lambda}(D)}$$

is valid.

4. The Existence of a Solution of the Non-Homogeneous Boundary Value Problem

For $F \in L_{2,\lambda}(D)$, $f^i \in L_{2,\lambda}(S_i)$, $i = 1,2$, analogously to the homogeneous case we call the funetion $u \in L_{2,\lambda}(D)$ a strong solution of the non-homogeneous problem (1), (2) of the class $L_{2,\lambda}$, if there exists a sequence of functions $u_n \in W^1_{2,\lambda}(D)$ such that

$$\lim_{k\to\infty} \|u - u_k\|_{L_{2,\lambda}(D)} = \lim_{k\to\infty} \|f^i - \Gamma^i u_k|_{S_i}\|_{L_{2,\lambda}(S_i)} = \\ = \lim_{k\to\infty} \|F - Lu_k\|_{L_{2,\lambda}(D)} = 0. \qquad (41)$$

To simplify the question of the solvability of the non-homogeneous problem (1), (2) in the space $L_{2,\lambda}(D)$, the matrix B in the system (1) will be assumed below to be constant.

Since the spaces $C_0^\infty(D)$ and $C_0^\infty(S_i)$ of finite, continuously differentiable functions are dense, respectively, in the spaces $L_{2,\lambda}(D)$ and $L_{2,\lambda}(S_i)$, $i = 1,2$, and $F \in L_{2,\lambda}(D)$, $f^i \in L_{2,\lambda}(S_i)$, $i = 1,2$, there exist sequences of functions $F_k \in C_0^\infty(D)$ and $f_k^i \in C_0^\infty(S_i)$, $i = 1,2$, such that

$$\lim_{k\to\infty} \|f^i - f_k^i\|_{L_{2,\lambda}(S_i)} = 0, \quad i = 1,2, \lim_{k\to\infty} \|F - F_k\|_{L_{2,\lambda}(D)} = 0. \qquad (42)$$

First we will show that for sufficiently large $\lambda > 0$ and for any fixed k there exists a function $v_k \in C^\infty(\overline{D}) \cap W^2_{2,\lambda}(D)$ such that the equalities

$$(Lv_k - F_k)\Big| S_i = 0, \quad i = 1,2, \qquad (43)$$

$$\Gamma^i v_k\Big|_{S_i} = f_k^i, \quad i = 1,2, \qquad (44)$$

hold. Since $F_k \in C_0^\infty(D)$, we have that $F_k\Big|_{S_i} = 0$, $i = 1,2$. Therefore (43) is equivalent to the equalities

$$Lv_k\Big|_{S_i} = 0, \quad i = 1,2. \qquad (45)$$

It can be easily seen that in order to construct the function v it is sufficient to construct two functions v_k^1 and v_k^2 which possess the properties

$$v_k^i \in C^\infty(\overline{D}) \cap W^2_{2,\lambda}(D), \quad \operatorname{diam}\operatorname{supp} v_k^i < +\infty, \quad i = 1,2, \\ \operatorname{supp} v_k^1 \cap S_2 = \phi, \quad \operatorname{supp} v_k^2 \cap S_1 = \varnothing, \quad \operatorname{supp} v_k^1 \cap \operatorname{supp} v_k^2 = \varnothing, \\ Lv_k^i\Big|_{S_i} = 0, \quad \Gamma^i v_k^i\Big|_{S_i} = f_k^i, \quad i = 1,2, \qquad (46)$$

and then to take their sum, i.e., $v_k = v_k^1 + v_k^2$.

Introduce a new Cartesian rectangular system of coordinates connected with the independent variables $x' = (x_1', x_2', \dots, x_n', x_{n+1}' = t')$. The equation of the surface S_1 in this system has the form $x_{n+1}' = t' = 0$, i.e., $x' = \widetilde{A}x$, where $x = (x_1, \dots, x_n, x_{n+1} = t)$, and $\widetilde{A}$ is an orthogonal $(n+1) \times (n+1)$-matrix. We denote by D' the domain D in variables x'.

Taking into account that $\nabla_x = \widetilde{A}'\nabla_{x'}$, where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n+1}}\right)$, $\nabla_{x'} = \left(\frac{\partial}{\partial x'_1}, \dots, \frac{\partial}{\partial x'_{n+1}}\right)$ and the prime in the matrix $\widetilde{A}'$ denotes transposition, the system of equations (1) in variables x' will take the form

$$L'u \equiv A_0 u_{t'} + \sum_{i=1}^{n} A_{*i} u_{x'_i} + Bu = F, \tag{47}$$

where A_0, A_{*i}, $i = 1, \dots, n$, are also symmetric matrices, provided

$$A_0 = -Q_0(\alpha^1) = -\left(E\alpha_0^1 + \sum_{i=1}^{n} A_i\alpha_i^1\right) = -(E\alpha_0^1 - Q(\nu)). \tag{48}$$

Here $Q(\nu) = -\sum_{i=1}^{n} A_i\alpha_i^1$, $\nu = (\alpha_1^1, \dots, \alpha_n^1)$.

If $\widetilde{C}^s$ is matrix of the order $m \times m$ which reduces the quadratic form $(Q_0(\alpha^1)\eta, \eta)$ to the canonical type, then by virtue of (12)–(15) and (27) we have

$$((\widetilde{C}^s)^{-1})' Q_0(\alpha^1)(\widetilde{C}^s)^{-1} = \operatorname{diag}(\underbrace{-1, \dots, -1}_{\varkappa_s^-}, \underbrace{0, \dots, 0}_{k_s}), \tag{49}$$

$$\Gamma^1(\widetilde{C}^s)^{-1} = \left.\begin{pmatrix} 1\,0\,0 \cdots\cdots\cdots 0 \\ 0\,1\,0 \cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \underbrace{0\,0 \cdots 0\,1}_{\varkappa_1 = \varkappa_s^-} 0 \cdots 0 \end{pmatrix}\right\} \varkappa_1 = \varkappa_s^-. \tag{50}$$

In (49), it is taken into account that by virtue of (27) and according to (11) the expansion (13) does not involve the first sum $\sum_{j=1}^{\varkappa_i^+}[\Lambda_{ij}^+(\xi, \eta)]^2$ for $i = s$.

After the substitution $u = (\widetilde{C}^s)^{-1}w$, the equation (47) can equivalently be rewritten as

$$\widetilde{L}w \equiv \widetilde{A}_0 w_{t'} + \sum_{i=1}^{n} \widetilde{A}_i w_{x'_i} + \widetilde{B}w = \widetilde{F}, \tag{51}$$

where

$$\widetilde{A}_0 = -((\widetilde{C}^s)^{-1})' Q_0(\alpha^1)(\widetilde{C}^s)^{-1}, \quad \widetilde{A}_i = ((\widetilde{C}^s)^{-1})' A_{*i}(\widetilde{C}^s)^{-1}, \tag{52}$$
$$\widetilde{B} = ((\widetilde{C}^s)^{-1})' B(\widetilde{C}^s)^{-1}, \quad \widetilde{F} = ((\widetilde{C}^s)^{-1})' F.$$

As a result of that substitution $u = (\widetilde{C}^s)^{-1}w$, the boundary condition (2) for $i = 1$ by virtue of (50) will take the form

$$\overline{w}\big|_{S_1} = f^1, \tag{53}$$

where $\overline{w} = (w_1, \dots, w_{\varkappa_1})$, $w = (w_1, \dots, w_{\varkappa_1}, w_{\varkappa_1+1}, \dots, w_m)$.

Let us now construct the function $w_k^1 \in C^\infty(\overline{D}')$ satisfying the equalities

$$\widetilde{L}w_k^1\big|_{S_1} = 0, \tag{54}$$

$$\overline{w}_k^1\big|_{S_1} = f_k^1. \tag{55}$$

Suppose $k_0 = \operatorname{tg}\frac{\alpha}{3}$, where α is the angle lying between the faces S_1 and S_2, and introduce the function $\sigma_1(x)$ possessing the following properties [20, p. 19]:

$$\sigma_1(x) \in C^\infty[0,+\infty) : \sigma_1(x) =$$
$$= \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ 0, & x \ge 1, \end{cases} \quad \text{and} \quad 0 < \sigma_1(x) < 1 \quad \text{for} \quad \frac{1}{2} < x < 1. \tag{56}$$

Obviously, without restriction of generality we can assume that $x'_n = 0$ is the equation of the boundary of the face S_1 or, what comes to the same, of the edge of the dihedral angle D'.

Since the equation of the face S_1 has in the new variables $x'_1,\dots,x'_n,t'$ the form $t' = 0$, $x'_n > 0$, a solution $w_k^1 = (w_{k1}^1,\dots,w_{km}^1)$ of the system of equations (54) will be sought in the form

$$w_{ki}^1(x'_1,\dots,x'_n,t') = \sigma_1\Big(\frac{t'}{k_0 x'_n}\Big)\sigma_1(t')t'\omega_{ki}^+(x'_1,\dots,x'_n)+$$
$$+\sigma_1\Big(\frac{t'}{k_0 x'_n}\Big)\sigma_1(t')\omega_{ki}^-(x'_1,\dots,x'_n),\ i = 1,\dots,\varkappa_1\ (\varkappa_1 = \varkappa_s^-), \tag{57}$$
$$w_{ki}^1(x'_1,\dots,x'_n,t') =$$
$$= \sigma_1\Big(\frac{t'}{k_0 x'_n}\Big)\sigma_1(t')\omega_{ki}^-(x'_1,\dots,x'_n),\ i = \varkappa_1 + \varkappa_s^+ + 1,\dots,m, \tag{58}$$

where $\omega_{ki}^\pm$ are the functions to be defined.

Substituting (57) in the boundary condition (55) and taking into consideration (56) and the fact that $S_1 : t' = 0$, $x'_n > 0$, we obtain

$$\overline{\omega}_{ki}(x'_1,\dots,x'_n) = f_{ki}^1, \quad i = 1,\dots,\varkappa_1. \tag{59}$$

Owing to (49) and (57)–(59), the last $m - \varkappa_s^- = k_s$ equations of the system (54) with respect to an unknown vector

$$\psi = (\omega_{k\varkappa_1+\varkappa_s^++1}^-,\dots,\omega_{km}^-)$$

can be rewritten as follows:

$$L_1\psi = A_n^1\psi_{x'_n} + \sum_{i=1}^{n-1} A_i^1\psi_{x'_i} = g_k, \tag{60}$$

where g_k is the known vector function which is defined uniquely with respect to f_k^1, and the matrix A_i^1, $1 \le i \le n$, consists of the elements located at the intersection of the last k_s rows and columns of the matrix $\widetilde{A}_i$ from (52). Since the matrix $\widetilde{A}_i$ is symmetric, the matrix A_i^1 is likewise symmetric, $1 \le i \le n$.

Below it will be assumed that the matrix A_{*n} from (47), and hence the matrix $\widetilde{A}_n$ from (52), are positive definite. But then, according to the known criterion of the positive definiteness of a symmetric matrix, the matrix A_n^1 is positive definite, as well. That is,

$$(A_n^1\zeta,\zeta)>0 \quad \forall\zeta\in R^{k_s}, \quad |\zeta|\neq 0. \tag{61}$$

This implies that the system (60) is a symmetric hyperbolic system [10, p.587], and if we consider for (60) the Cauchy problem with homogeneous initial data on $x'_n=0$, i.e.,

$$\psi\Big|_{x'_n=0}=0, \tag{62}$$

then owing to the fact that $g_k\in C_0^\infty(\Omega)$, where $\Omega=\{x'\in R^n : x'_n>0\}$, the problem (60), (62) has a unique solution $\psi(x'_1,\dots,x'_n)$ of the class $C^\infty(\overline{\Omega})\cap W^2_{2,\lambda}(\Omega)$ for sufficiently large $\lambda>0$; note that $\psi\Big|_{\Omega_\rho}\equiv 0$ for sufficiently small ρ, where $\Omega_\rho=\Omega\cap\{0<x'_n<\rho\}$ [10, 21].

Knowing already ω^-_{ki} for $1\le i\le\varkappa_1$ and $\varkappa_1+\varkappa^+_S+1\le i\le m$, the rest unknowns ω^+_{ki}, $1\le i\le\varkappa_1+\varkappa^+_s$ in the representations (57), (58) can be defined immediately from the first $\varkappa_1+\varkappa^+_s$ equations of the system (54).

From the above reasoning it easily follows that the vector function w^1_k defined by the formulas (57), (58) satisfies the equalities (54), (55) and $w^1_k\in C^\infty(\overline{D}')\cap W^2_{2,\lambda}(D')$.

It is clear now that $v^1_k=(\widetilde{C}^s)^{-1}w^1_k$ in the original variables $x_1,\dots,x_n,t$ satisfies the equalities (46).

Analogously, we introduce the orthogonal coordinate system connected with the variables $x''_1,\dots,x''_n,x''_{n+1}=t''$, in which the equation of the face S_2 has the form $x''_{n+1}=t''=0$. We introduce the matrix A^2_n which is assumed to be positive definite , i.e.,

$$(A_n^2\zeta,\zeta)>0, \quad |\zeta|\neq 0, \tag{63}$$

and construct appropriately the function v^2_k, satisfying the equalities (46). From these constructions it follows that $v^i_k\in C^\infty(\overline{D})$, $i=1,2$, $\operatorname{supp} v^1_k\cap S_2=\varnothing$, $\operatorname{supp} v^2_k\cap S_1=\varnothing$, $\operatorname{supp} v^1_k\cap\operatorname{supp} v^2_k=\varnothing$. Therefore for sufficiently large $\lambda>0$ their sum $v_k=v^1_k+v^2_k\in C^\infty(\overline{D})\cap W^2_{2,\lambda}(D)$ satisfies the systems of equalities (43) and (44).

Since $v_k\in C^\infty(\overline{D})\cap W^2_{2,\lambda}(D)$ and $F_k\in C_0^\infty(D)$, by virtue of (43) we have $\widetilde{F}_k=F_k-Lv_k\in \overset{\circ}{W}{}^1_{2,\lambda}(D)$, and according to Theorem 2 the problem (1), (40) for $F=\widetilde{F}_k$ has a unique solution $\widetilde{u}_k\in W^1_{2,\lambda}(D)$. But with regard for (44), the vector function $u_k=\widetilde{u}_k+v_k$ will be a solution of the non-homogeneous problem (1), (2) for $F=F_k$, $f^i=f^i_k$, $i=1,2$, of the class $W^1_{2,\lambda}(D)$.

It now follows from (42) and the a priori estimate (17) that there exists a unique function $u\in L_{2,\lambda}(D)$ such that for the above-constructed sequence $u_k\in W^1_{2,\lambda}(D)$ the limiting equalities (41) take place.

Thus we have proved the following

Theorem 4. *Let the conditions* (27), (61) *and* (63) *be fulfilled. Then for sufficiently large positive* λ *and for any* $F \in L_{2,\lambda}(D)$, $f^i \in L_{2,\lambda}(S_i)$, $i = 1, 2$, *there exists a strong solution* u *of the problem* (1), (2) *of the class* $L_{2,\lambda}$ *for which the estimate* (17) *holds.*

Remark 5. Note that the cases where the faces S_1 or S_2 or the both are not characteristic can be considered analogously in connection with the expansion (10).

5. Some Examples of Systems of Differential Equations from Mathematical Physics

1^0. In the space of the variables x_1, x_2, x_3 and t we consider the nonhomogeneous system of Maxwell differential equations for electromagnetic field in the vacuum [10,p.182]

$$\widetilde{E}_t - \operatorname{rot} H = F_1, \quad H_t + \operatorname{rot} \widetilde{E} = F_2, \tag{64}$$

where $\widetilde{E} = (E_1, E_2, E_3)$ is the vector of the electric field, and $H = (H_1, H_2, H_3)$ is the vector of the magnetic field. Here the light velocity is adopted to be unity.

Assuming $U = (\widetilde{E}, H)$, $F = (F_1, F_2)$, the system (64) can be rewritten as

$$L_1 U \equiv U_t + \sum_{i=1}^{3} A_i U_{x_i} = F, \tag{65}$$

where A_i, $i = 1, 2, 3$, are quite definite real symmetric (6×6)-matrices. The characteristic determinant of the system (65) is equal to

$$p(\xi) = \det Q_0(\xi) = \xi_0^2 (\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2)^2, \tag{66}$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_0) \in R^4$, $Q_0(\xi) = \xi_0 E + \sum_{i=1}^{3} \xi_i A_i$ is the characteristic matrix of that system. E is the unit 6×6-matrix.

In connection with (4), (6) and (8), for the system (65) we have $s_0 = 1$, $s = 2s_0 + 1 = 3$, $k_1 = k_2 = k_3 = 2$, $\lambda_1(\xi') = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{1}{2}}$, $\lambda_2(\xi') \equiv 0$, $\lambda_3(\xi') = -(\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{1}{2}}$, $K_i : \xi_0 - \lambda_i(\xi') = 0$, $i = 1, 2, 3$. We take as D the dihedral angle $D : t > |x_3|$ whose faces are the characteristic surfaces $S_1 : t - x_3 = 0$, $t \geq 0$ and $S_2 : t + x_3 = 0$, $t \geq 0$.

If $\alpha^i = (\alpha_1^i, \alpha_2^i, \alpha_3^i, \alpha_0^i) = (\widetilde{\alpha}^i, \alpha_0^i)$ is the unit vector of the outer normal on S_i, then, as is easily verified,

$$(Q_0(\alpha^1)U, U) = -\frac{1}{\sqrt{2}}[(H_1 - E_1)^2 + (H_1 + E_2)^2 + E_3^2 + H_3^2], \tag{67}$$

$$(Q_0(\alpha^2)U, U) = -\frac{1}{\sqrt{2}}[(H_2 + E_1)^2 + (H_1 - E_2)^2 + E_3^2 + H_3^2]. \tag{68}$$

In connection with (13)–(15), now from (67) and (68) we obtain the following boundary conditions for the system (65):

$$(H_2 - E_1)\Big|_{S_1} = f_1^1, \ (H_1 + E_2)\Big|_{S_1} = f_2^1, \ E_3\Big|_{S_1} = f_3^1, \ H_3\Big|_{S_1} = f_4^1, \tag{69}$$

$$(H_2+E_1)\Big|_{S_2}=f_1^2,\ \ (H_1-E_2)\Big|_{S_2}=f_2^2,\ \ E_3\Big|_{S_2}=f_3^2,\ \ H_3\Big|_{S_2}=f_4^2. \quad (70)$$

Corresponding to the face S_1, the matrix $\widetilde{A}$ of passage from the variables x, t to x', t', the matrix $\widetilde{C}^s$ $(s=3)$ from (49) and the matrix A_{*3} from (47) have the form

$$\widetilde{A}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix},\quad \widetilde{C}^s=\frac{1}{\sqrt[4]{2}}\begin{pmatrix}1&0&0&0&1&0\\0&1&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&-1&0&1&0&0\\1&0&0&0&1&0\end{pmatrix},$$

$$A_{*3}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1&0&0&0&-1&0\\0&1&0&1&0&0\\0&0&1&0&0&0\\0&1&0&1&0&0\\-1&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

and, as is easily verified,

$$A_3^1=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

i.e., the condition (61) is fulfilled.

Analogously, to the face S_2 there correspond the matrices

$$\widetilde{A}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix},\quad \widetilde{C}^s=\frac{1}{\sqrt[4]{2}}\begin{pmatrix}1&0&0&0&1&0\\0&-1&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&1&0&1&0&0\\-1&0&0&0&1&0\end{pmatrix},$$

$$A_{*3}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0&-1&0\\0&1&0&1&0&0\\0&0&1&0&0&0\\0&1&0&1&0&0\\-1&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\quad A_3^2=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

i.e., the condition (63) is likewise fulfilled.

Therefore by Theorems 2 and 4, for sufficiently large $\lambda>0$ and for any $F\in L_{2,\lambda}(D)$, $f_k^i\in L_{2,\lambda}(S_i)$, $i=1,2$; $k=1,\dots,4$, there exists a unique strong solution U of the problem (65), (69), (70) of the class $L_{2,\lambda}$ for which the corresponding to that problem estimate (17) is valid. If $F\in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ and $f_k^i=0$, $i=1,2$; $k=1,\dots,4$, then this solution U will belong to the space $W^1_{2,\lambda}(D)$.

Note that other statements of characteristic problems for systems of Maxwell differential equations have been investigated in [22–25].

2^0. Consider the non-homogeneous system of Dirac differential equations in the complex form [10, p. 183]

$$\sum_{k=1}^{4} \mu_k \Big(\frac{\partial}{\partial x_k} - a_k \Big) u - \beta b u = F, \tag{71}$$

where the vector (a_1, a_2, a_3) is proportional to the magnetic potential, a_4 to the electric potential and b to the rest-mass; $F = (F_1, F_2, F_3, F_4)$ is the given and $u = (u_1, u_2, u_3, u_4)$ is an unknown 4-dimensional complex-valued vector function of the variables $x_1, x_2, x_3, x_4 = t$. Coefficients in the system (71) are the following matrices:

$$\mu_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad i = \sqrt{-1}.$$

If $u = w + iv$, then the system (71) with respect to the unknown 8-dimensional real vector $U = (w, v)$ will be written in the form

$$L_2 U \equiv \sum_{k=1}^{4} \sigma_k U_{x_k} + \sigma U = \widetilde{F}, \tag{72}$$

where $\widetilde{F} = (\operatorname{Re} F, \operatorname{Im} F)$, σ is a real (8×8)-matrix, and

$$\sigma_1 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i\mu_2 \\ -i\mu_2 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} \mu_3 & 0 \\ 0 & \mu_3 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} \mu_4 & 0 \\ 0 & \mu_4 \end{pmatrix}.$$

The system (71) is a symmetric first order hyperbolic system whose characteristic polynomial is equal to

$$p(\xi) = \det Q_0(\xi) = (\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_0^2)^4,$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_0) \in R^4$, $Q_0(\xi) = \xi_1\sigma_1 + \xi_2\sigma_2 + \xi_3\sigma_3 + \xi_0\sigma_4$ is the characteristic matrix of the system.

According to (4) and (7), for the system (72) we have

$$s_0 = 1, \quad s = 2s_0 = 2, \quad k_1 = k_2 = 4, \quad \lambda_1(\xi') = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{1}{2}},$$
$$\lambda_2(\xi') = -(\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{1}{2}}, \quad K_i : \xi_0 - \lambda_i(\xi') = 0, \quad i = 1, 2.$$

We rewrite the system (72) in the form of scalar equations and multiply the equations the numbers 1, 2, 5 and 8 by -1. Then we replace these equations according to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 8 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

In the new notation $\Lambda = (W, V, \omega_1, \omega_2)$, where $W = (v_1, w_1, -v_2)$, $V = (w_3, -v_3, -w_4)$, $\omega_1 = v_4$, $\omega_2 = -w_2$, the obtained system of equations in the matrix form can be rewritten as the symmetric system

$$L_3\Lambda \equiv \Lambda_t + \sum_{i=1}^{3} \widetilde{\sigma}_i \Lambda_{x_i} + \widetilde{\sigma}\Lambda = \widetilde{F}_1, \tag{73}$$

where

$$\widetilde{\sigma}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\sigma}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\sigma}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$\widetilde{F}_1$ is the given vector function, and σ_2 is a real (8×8)-dimensional matrix. Obviously, the systems (71) and (73) are equivalent.

It can be easily verified that if $L_3^\circ \equiv E\frac{\partial}{\partial t} + \sum_{i=1}^{3} \widetilde{\sigma}_i \frac{\partial}{\partial x_i}$ is the principal part of the operator (73), then

$$2(L_3^\circ\Lambda)\Lambda = 2\frac{\partial W}{\partial t}V - 2W \operatorname{rot} V - 2W \operatorname{grad} \omega_1 + 2V\frac{\partial V}{\partial t} +$$

$$+2V\operatorname{rot}W+2V\operatorname{grad}\omega_2+2\omega_1\frac{\partial\omega_1}{\partial t}-2\omega_1\operatorname{div}W+2\omega_2\frac{\partial\omega_2}{\partial t}+2\omega_2\operatorname{div}V,$$

whence

$$2(L_3^\circ\Lambda)\Lambda=(W^2+V^2)_t+ \\ +2\operatorname{div}[W\times V]+(\omega_1^2+\omega_2^2)_t+2\operatorname{div}[\omega_2V-\omega_1W], \tag{74}$$

where $[W\times V]$ is the vector product of the vectors W and V.

On the other hand, similarly to (29), for any $\Lambda\in W_2^1(D)$ we have

$$2\int\limits_D(L_3^\circ\Lambda)\Lambda dD=\int\limits_{\partial D}(Q_0(\alpha)\Lambda,\Lambda)ds=\sum_{i=1}^2\int\limits_{S_i}(Q_0(\alpha^i)\Lambda,\Lambda)ds, \tag{75}$$

where $Q_0(\alpha)=E\alpha_0+\sum_{j=1}^3\widetilde{\sigma}_j\alpha_j$, $\alpha=(\widetilde{\alpha},\alpha_0)=(\alpha_1,\alpha_2,\alpha_3,\alpha_0)$ is the unit vector of the outer normal to ∂D, and

$$D=\{(x_1,x_2,x_3,t)\in R^4:\alpha_0^it+\sum_{j=1}^3\alpha_j^ix_j<0,\quad i=1,2\}$$

is the dihedral angle whose faces are the characteristic surfaces $S_i:\alpha_0^it+\sum_{j=1}^3\alpha_j^ix_j=0$, $t\geq 0$, $i=1,2$, $\partial D=S_1\cup S_2$, where $\alpha^i=(\alpha_1^i,\alpha_2^i,\alpha_3^i,\alpha_0^i)\in K_2:\xi_0-\lambda_2(\xi')=\xi_0+(\xi_1^2+\xi_2^2+\xi_3^2)^{\frac12}=0$, $i=1,2$. Obviously,

$$\alpha_0^i<0,\quad(\alpha_0^i)^2=|\widetilde{\alpha}^i|^2,\quad\widetilde{\alpha}^i=(\alpha_1^i,\alpha_2^i,\alpha_3^i),\quad i=1,2. \tag{76}$$

By virtue of the known vector relations [10, p. 642]

$$W^2\widetilde{\alpha}^2=[W\times\widetilde{\alpha}]^2+[W\cdot\widetilde{\alpha}]^2,\quad V^2\widetilde{\alpha}^2=[V\times\widetilde{\alpha}]^2+[V\cdot\widetilde{\alpha}]^2,$$

(74)–(76) immediately imply

$$(Q_0(\alpha)\Lambda,\Lambda)=\frac{1}{\alpha_0}\Big[(W^2+V^2)\alpha_0^2+2[W\times V](\widetilde{\alpha})\alpha_0+ \\ +(\omega_1^2+\omega_2^2)\alpha_0^2+2[\omega_2V-\omega_1W]\widetilde{\alpha}\cdot\alpha_0= \\ =\frac{1}{\alpha_0}[(\omega_1\alpha_0-W\widetilde{\alpha})^2]+(\omega_2\alpha_0+V\widetilde{\alpha})^2+ \\ +(W^2+V^2)\alpha_0^2+2[W\times V]\cdot\widetilde{\alpha}\alpha_0-(W\cdot\widetilde{\alpha})^2-(V\cdot\widetilde{\alpha})^2\Big]= \\ =\frac{1}{\alpha_0}\Big[(\omega_1\alpha_0-W\widetilde{\alpha})^2+(\omega_2\alpha_0+V\widetilde{\alpha})^2]+[W\times\widetilde{\alpha}]^2+ \\ +2[W\times V]\cdot\widetilde{\alpha}\alpha_0+[V\times\widetilde{\alpha}]^2\Big]. \tag{77}$$

Assume that

$$I=[W\times\widetilde{\alpha}]^2+2[W\times V]\cdot\widetilde{\alpha}\alpha_0+[V\times\widetilde{\alpha}]^2. \tag{78}$$

Let first

$$\widetilde{\alpha}=\widetilde{\alpha}_0=(0,0,|\alpha_0|)=|\alpha_0|(0,0,1). \tag{79}$$

Now it is easy to verify that

$$[W\times\widetilde{\alpha}]^2=|\alpha_0|^2(W_1^2+W_2^2),\ 2[W\times V]\widetilde{\alpha}\alpha_0=2\alpha_0|\alpha_0|(W_1V_2-W_2V_1),$$

$$[V \times \widetilde{\alpha}]^2 = |\alpha_0|^2 (V_1^2 + V_2^2). \tag{80}$$

Therefore in the case of (79), since $|\alpha_0| = |\widetilde{\alpha}|$, $\alpha_0 < 0$, $\alpha_0^2 + |\widetilde{\alpha}|^2 = 1$ and thus $|\alpha_0|^2 = \frac{1}{2}$, we find from (78) and (80) that

$$I = \frac{1}{2}(W_1 - V_2)^2 + \frac{1}{2}(W_2 + V_1)^2. \tag{81}$$

Let T be the matrix of orthogonal transformation which is the rotation transforming the vector $\widetilde{\alpha}$ to the vector $\widetilde{\alpha}_0 = (0, 0, |\alpha_0|)$ and which does not change orientation of the space. As is known, the action of this transformation on the vector $x = (x_1, x_2, x_3)$ is given by the following equality [26, p. 68]:

$$Tx = x - \frac{(\widetilde{\alpha} + \widetilde{\alpha}_0) \cdot x}{\alpha_0^2 + \widetilde{\alpha} \cdot \widetilde{\alpha}_0} (\widetilde{\alpha} + \alpha_0) + \frac{2}{\alpha_0^2} (\widetilde{\alpha} \cdot x) \widetilde{\alpha}_0, \quad \widetilde{\alpha} \neq -\widetilde{\alpha}_0.$$

Using the properties of the vector and mixed products, we easily obtain

$$\begin{aligned} I &= [W \times \widetilde{\alpha}]^2 + 2[W \times V] \cdot \widetilde{\alpha}\alpha_0 + [V \times \widetilde{\alpha}]^2 = \\ &= [TW \times T\widetilde{\alpha}]^2 + 2[TW \times TV] \cdot T\widetilde{\alpha}\alpha_0 + [TV \times T\widetilde{\alpha}]^2 = \\ &= [TW \times \widetilde{\alpha}_0]^2 + 2[TW \times TV] \cdot \widetilde{\alpha}_0\alpha_0 + [TV \times \widetilde{\alpha}_0]. \end{aligned} \tag{82}$$

Let ν_1, ν_2 and ν_3 be the rows of the matrix T, i.e.,

$$T = \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ \nu_{31} & \nu_{32} & \nu_{33} \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}.$$

Using (79)–(81), from (82) we obtain

$$I = \frac{1}{2}(\nu_1 W - \nu_2 V)^2 + \frac{1}{2}(\nu_2 W + \nu_1 V)^2. \tag{83}$$

Now (77), (78) and (83) imply that

$$\begin{aligned} (Q_0(\alpha)\Lambda, \Lambda) = \frac{1}{\alpha_0} \Big[&(\omega_1\alpha_0 - W\widetilde{\alpha})^2 + (\omega_2\alpha_0 + V\widetilde{\alpha})^2 + \\ &+ \frac{1}{2}(\nu_1 W - \nu_2 V)^2 + \frac{1}{2}(\nu W + \nu_1 V)^2 \Big]. \end{aligned} \tag{84}$$

When $\widetilde{\alpha} = -\widetilde{\alpha}_0 = -|\alpha_0|(0, 0, 1)$, instead of (81) we have

$$I = \frac{1}{2}(W_1 + V_2)^2 + \frac{1}{2}(W_2 - V_1)^2. \tag{85}$$

Now, by virtue of (13)–(15), from (84) we obtain the following boundary conditions for the system (73):

$$\begin{aligned} &(\omega_1\alpha_0^i - W\widetilde{\alpha}^i)\Big|_{S_i} = f_1^i, \quad (\omega_2\alpha_0^i + V\widetilde{\alpha}^i)\Big|_{S_i} = f_2^i, \\ &(\nu_1^i W - \nu_2^i V)\Big|_{S_i} = f_3^i, \quad (\nu_2^i W + \nu_1^i V)\Big|_{S_i} = f_4^i, \quad i = 1, 2, \end{aligned} \tag{86}$$

where $\alpha^i = \alpha\Big|_{S_i}$, $\nu_j^i = \nu_j\Big|_{S_i}$, $i = 1, 2$. If $\widetilde{\alpha} = \widetilde{\alpha}_0$ or $\widetilde{\alpha} = -\widetilde{\alpha}_0$, say, on S_i, then by (77), (81) and (85) and in view of the fact that $\alpha_0 = -|\alpha_0| < 0$, the boundary conditions take the form

$$\begin{aligned} (\omega_1 + W_3)\Big|_{S_i} &= f_1^i, \quad (\omega_2 - V_3)\Big|_{S_i} = f_2^i, \\ (W_1 - V_2)\Big|_{S_i} &= f_3^i, \quad (W_2 + V_1)\Big|_{S_i} = f_4^i \end{aligned} \tag{87}$$

for $\widetilde{\alpha} = \alpha_0$, and

$$\begin{aligned} (\omega_1 - W_3)\Big|_{S_i} &= f_1^i, \quad (\omega_2 + V_3)\Big|_{S_i} = f_2^i, \\ (W_1 + V_2)\Big|_{S_i} &= f_3^i, \quad (W_2 - V_1)\Big|_{S_i} = f_4^i \end{aligned} \tag{88}$$

for $\widetilde{\alpha} = -\widetilde{\alpha}_0$.

By Theorem 2, for any $\widetilde{F}_1 \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ and $f_k^i = 0$, $i = 1, 2$; $k = 1, \dots, 4$, there exists, in the space $W^1_{2,\lambda}(D)$, a unique solution of the problem (73), (86) for which the estimate (17) corresponding to this problem holds.

In considering the non-homogeneous boundary value problem for the system (73), for simplicity we consider the case where the equality $\widetilde{\alpha} = \widetilde{\alpha}_0$ holds on S_1, and then we have the boundary conditions (87), and the equality $\widetilde{\alpha} = -\widetilde{\alpha}_0$ holds on S_2 and then we have the boundary conditions (88), i.e. $S_1 : t - x_3 = 0$, $t \geq 0$, $S_2 : t + x_3 = 0$, $t \geq 0$, and

$$\begin{aligned} (\omega_1 + (-1)^{i+1} W_3)\Big|_{S_i} &= f_1^i, \quad (\omega_2 + (-1)^i V_3)\Big|_{S_i} = f_2^i, \\ (W_1 + (-1)^i V_2)\Big|_{S_i} &= f_3^i, \quad (W_2 + (-1)^{i+1} V_1)\Big|_{S_i} = f_4^i, \quad i = 1, 2. \end{aligned} \tag{89}$$

The matrix $\widetilde{A}$ of transformation from the variables x, t to the variables x', t', which correspond to the face $S_1 : t - x_3 = 0$, $t > 0$, the matrix $\widetilde{C}^s (s = 2)$ from (49) and the matrix A_{*3} from (47) have the form

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad A_{*3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\widetilde{C}^s = \frac{1}{\sqrt[4]{2}} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$(\widetilde{C}^s)^{-1} = \sqrt[4]{2} \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

and, as one can easily verify,

$$A_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

i.e., the condition (61) is fulfilled.

Similarly, the following matrices correspond to the face $S_2 : t + x_3 = 0$, $t > 0$

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \widetilde{C}^s = \frac{1}{\sqrt[4]{2}} \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_{*3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$(\widetilde{C}^s)^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$A_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

i.e., the condition (63) is also fulfilled.

Therefore, by Theorem 4, for sufficiently large $\lambda > 0$ and any $\widetilde{F}_1 \in L_{2,\lambda}(D)$, $f_k^i \in L_{2,\lambda}(S_i)$, $i = 1, 2$; $k = 1, \dots, 4$, there exists a unique strong solution Λ of the problem (73), (89) from the class $L_{2,\lambda}$, for which the estimate (17) corresponding to this problem holds.

Remark 6. Since $\Lambda = (W, V, \omega_1, \omega_2)$, where $W = (v_1, w_1, -v_2)$, $V = (w_3, -v_3, -w_4)$, $\omega_1 = v_4$, $\omega_2 = -w_2$, the boundary conditions (89) can be rewritten in terms of the unknown functions w_j and v_j, $j = 1, \dots, 4$, as

$$(v_4 - v_2)\Big|_{S_1} = f_1^1,\ (w_4 - w_2)\Big|_{S_1} = f_2^1,\ (v_1 + v_3)\Big|_{S_1} = f_3^1,\ (w_1 + w_3)\Big|_{S_1} = f_4^1,$$
$$(v_4 + v_2)\Big|_{S_2} = f_1^2,\ (w_4 + w_2)\Big|_{S_2} = f_2^2,\ (v_1 - v_3)\Big|_{S_2} = f_3^2,\ (w_1 - w_3)\Big|_{S_2} = f_4^2.$$

3^0. The system of equations of crystal optics [10, p. 597] has the form

$$\frac{1}{c} E_\varepsilon \widetilde{E}_t - \operatorname{rot} H = F_1, \quad \frac{1}{c} \mu H_t + \operatorname{rot} \widetilde{E} = F_2, \tag{90}$$

where $\widetilde{E}$ and H are the same notation as in the Maxwell equation (64), c is the light velocity, μ is the magnetic penetrability constant, E_ε is the (3×3) diagonal matrix with the elements ε_1, ε_2 and ε_3 on the diagonal, and ε_i are the dielectric constants along three coordinate axes.

Using the notation $U = (\widetilde{E}, H)$, $F = (F_1, F_2)$, we rewrite the system (90) as follows:

$$\widetilde{E}_\varepsilon U_t + \sum_{i=1}^{3} A_i U_{x_i} = F, \tag{91}$$

where A_i, $i = 1, 2, 3$, are the same matrices as in (65), and

$$\widetilde{E}_\varepsilon = \operatorname{diag}\Big(\frac{1}{c}\varepsilon_1, \frac{1}{c}\varepsilon_3, \frac{1}{c}\varepsilon_3, \frac{1}{c}\mu, \frac{1}{c}\mu, \frac{1}{c}\mu\Big).$$

Since all the coefficients in (91) are real symmetric matrices and $\widetilde{E}_\varepsilon$ is positive definite, the system (91) is hyperbolic [10, p.587].

We put $\sigma_i = (\mu/c^2)\varepsilon_i$, $i = 1,2,3$, and suppose that $\sigma_1 > \sigma_2 > \sigma_3 > 0$. If K and S are, respectively, the cone of normals and that of rays for system (90), then, as is known, they are algebraic surfaces which are specified by the equations [10, p.598]

$$K : \prod_{i=1}^{3}(\rho^2 - \sigma_i\xi_0^2)\left[1-\sum_{i=1}^{3}\frac{\xi_i^2}{\rho^2 - \sigma_i\xi_0^2}\right]=0, \quad S : 1 - \sum_{i=1}^{3}\frac{x_i^2}{r^2 - \sigma_i^{-1}t^2}=0, \quad (92)$$

where $\rho^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$.

According to (4), (6) and (8), for the system (91) we have $s_0 = 2$, $s = 2s_0 + 1 = 5$, $k_1 = k_2 = k_4 = k_5 = 1$ $k_3 = 2$, $\varkappa_5^- = 5$; $\lambda_3(\xi') \equiv 0$, the rest $\lambda_i(\xi')$ are the roots of the first equation of (92) with respect to ξ_0 which define sheets $K_i : \xi_0 - \lambda_i(\xi') = 0$ of the cone of normals K.

The system (91) with respect to the new vector function $V = (\widetilde{E}_\varepsilon)^{\frac{1}{2}}U$ can be rewritten in the equivalent form

$$V_t + \sum_{i=1}^{3}\widetilde{A}_iV_{x_i} = \widetilde{F}, \quad (93)$$

where $\widetilde{A}_i = (\widetilde{E}_\varepsilon)^{-\frac{1}{2}}A_i(\widetilde{E}_\varepsilon)^{-\frac{1}{2}}$, $i = 1,2,3$, are real symmetric matrices, $\widetilde{F} = (\widetilde{E}_\varepsilon)^{-\frac{1}{2}}F$. Let D be a dihedral angle whose faces $S_i : \alpha_0^i t + \sum_{j=1}^{3}\alpha_j^i x_j = 0$, $t > 0$; $\alpha^i = (\alpha_1^i, \alpha_2^i, \alpha_3^i, \alpha_0^i) \in K_5$, $i = 1,2$, are characteristic surfaces of the system (93); $\widetilde{Q}_0(\xi)$ be the characteristic matrix of system (93) and $\widetilde{T}$ be the orthogonal matrix from the corresponding equation (3). Then by virtue of (10) and (11) applied to the system (93), we have

$$(\widetilde{Q}_0(\alpha^i)\eta,\eta) = -\sum_{j=1}^{\varkappa_5^-}[(\widetilde{\lambda}_j(\alpha^i) - \lambda_5(\alpha^i))^{\frac{1}{2}}\zeta_j]^2 =$$

$$= -\sum_{j=1}^{5}[(\widetilde{\lambda}_j(\alpha^i) - \lambda_5(\alpha^i))^{\frac{1}{2}}(\widetilde{T}'\eta)_j]^2 = -\sum_{j=1}^{5}[(\widetilde{\lambda}_j(\alpha^i) - \lambda_5(\alpha^i))^{\frac{1}{2}}\sum_{k=1}^{6}\widetilde{T}_{kj}\eta_k]^2 =$$

$$= -\sum_{j=1}^{5}\left[\sum_{k=1}^{6}\widetilde{c}_{jk}(\alpha^i)\eta_k\right]^2, \quad \alpha^i \in K_5, \quad i = 1,2, \quad (94)$$

where according to (3) we obtain $\widetilde{\lambda}_1 = \lambda_1$, $\widetilde{\lambda}_2 = \lambda_2$, $\widetilde{\lambda}_3 = \widetilde{\lambda}_4 = \lambda_3$, $\widetilde{\lambda}_5 = \lambda_4$, $\eta \in R^6$ and $(\widetilde{T}'\eta)_j$ is the j-th component of the vector $\widetilde{T}'\eta$, $\widetilde{c}_{jk}(\alpha^i) = (\widetilde{\lambda}_j(\alpha^i) - \lambda_5(\alpha^i))^{\frac{1}{2}}\widetilde{T}_{kj}$. By (13), (15) and (94), the boundary conditions (2) will take the form

$$\left(\sum_{k=1}^{6}\widetilde{c}_{jk}(\alpha^i)V_k\right)\Big|_{S_i} = f_j^i, \quad j = 1,\dots,5; \quad i = 1,2, \quad (95)$$

which with respect to the unknown $U = (\widetilde{E}_\varepsilon)^{-\frac{1}{2}} V$ of the initial system (91) will be written as

$$\left(\sum_{k=1}^{6} \widetilde{c}_{jk}(\alpha^i) d_k V_k \right)\Big|_{S_i} = f_j^i, \quad j = 1, \dots, 5; \quad i = 1, 2, \tag{96}$$

where d_k, $k = 1, \dots, 6$, are the diagonal elements of the matrix $(\widetilde{E}_\varepsilon)^{\frac{1}{2}}$, and α^i is the unit vector of the outer normal to S_i, $i = 1, 2$. Therefore by Theorems 2 and 3, for any $F \in L_{2,\lambda}(D)$ there exists a unique strong solution of the problem (91), (96) of the class $L_{2,\lambda}$ with the homogeneous boundary conditions, i.e., for $f_j^i = 0$, $j = 1, \dots, 5$; $i = 1, 2$. If $F \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$, then this solution will belong to the space $W^1_{2,\lambda}(D)$.

When considering the non-homogeneous boundary value problem (93), (95), to simplify our presentation we consider the dihedral angle D whose faces are the following characteristic surfaces:

$$S_1 : t - \sqrt{\sigma_2} x_3 = 0, \quad t \geq 0; \quad S_2 : t + \sqrt{\sigma_2} x_3 = 0, \quad t \geq 0.$$

The corresponding to the face S_1 matrix $\widetilde{A}$ of passage from variables x, t to x', t', the matrix $\widetilde{C}^s$ $(s = 5)$ from (49) and the matrix A_{*3} from (47) are of the form

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1+\sigma_2}} & \sqrt{\frac{\sigma_2}{1+\sigma_2}} \\ 0 & 0 & -\sqrt{\frac{\sigma_2}{1+\sigma_2}} & \frac{1}{\sqrt{1+\sigma_2}} \end{pmatrix}, \quad \widetilde{C} = \left(\widetilde{c}_{jk} = (\widetilde{\lambda}_j(\alpha^1) - \lambda_5(\alpha^1))^{\frac{1}{2}} \widetilde{T}_{kj} \right),$$

$$\widetilde{C}^{-1} = \left(\widetilde{c}_{jk}^{-1} = (\widetilde{\lambda}_k(\alpha^1) - \lambda_5(\alpha^1))^{-\frac{1}{2}} \widetilde{T}_{jk} \right), \quad A_{*3} = \frac{1}{\sqrt{1+\sigma_2}} (\sqrt{\sigma_2} E + \widetilde{A}_3),$$

where it is formally assumed that $\widetilde{\lambda}_6(\alpha^1) - \lambda_5(\alpha^1) = 1$; $\alpha^1 = (\alpha_1^1, \alpha_2^1, \alpha_3^1, \alpha_0^1) = \left(0, 0, \sqrt{\frac{\sigma_2}{1+\sigma_2}}, -\frac{1}{\sqrt{1+\sigma_2}}\right)$. As is easily seen, $\widetilde{Q}_0(\alpha^1) = \alpha_0^1 E + \alpha_3^1 \widetilde{A}_3$, and by the definition of the orthogonal matrix $\widetilde{T}$ from (3) and (94), the matrix $\widetilde{T}' \widetilde{A}_3 \widetilde{T}$ is diagonal, i.e.,

$$\widetilde{T}' \widetilde{A}_3 \widetilde{T} = \operatorname{diag}(-\mu_1, \dots, -\mu_6). \tag{97}$$

It is not difficult to see that in the given case

$$\begin{gathered} \widetilde{\lambda}_1 = \lambda_1 = \frac{1}{\sqrt{\sigma_2}} \alpha_3^1, \quad \widetilde{\lambda}_2 = \lambda_2 = \frac{1}{\sqrt{\sigma_1}} \alpha_3^1, \quad \widetilde{\lambda}_3 = \widetilde{\lambda}_4 = \lambda_3 = 0, \\ \widetilde{\lambda}_5 = \lambda_4 = -\frac{1}{\sqrt{\sigma_1}} \alpha_3^1, \quad \widetilde{\lambda}_6 = \lambda_5 = -\frac{1}{\sqrt{\sigma_2}} \alpha_3^1; \end{gathered} \tag{98}$$

and, at the same time, by virtue of (97)

$$(\widetilde{Q}_0(\alpha^1)\eta, \eta) = (\widetilde{Q}_0(\alpha^1) \widetilde{T}\zeta, \widetilde{T}\zeta) =$$

$$([\alpha_0^1 E - \operatorname{diag}(\alpha_3^1\mu_1,\dots,\alpha_3^1\mu_6)]\zeta,\zeta) = -\sum_{j=1}^{6}(\alpha_3^1\mu_j - \alpha_0^1)\zeta_j^2. \tag{99}$$

Comparing (94) and (99) and taking into account (98), we get

$$\mu_1 = \frac{1}{\sqrt{\sigma_2}},\ \ \mu_1 = \frac{1}{\sqrt{\sigma_1}},\ \ \mu_3 = \mu_4 = 0,\ \ \mu_5 = -\frac{1}{\sqrt{\sigma_1}},\ \ \mu_6 = -\frac{1}{\sqrt{\sigma_2}}. \tag{100}$$

In the case under consideration $k_s = 1$ and, consequently, the matrix A_3^1 from (61) is the scalar value which is equal to the element of (6×6)-matrix $(\widetilde{C}^{-1})' A_{*3} \widetilde{C}^{-1}$ lying in the right lower angle. If we take the lower row of the matrix $(\widetilde{C}^{-1})'$ as the vector ζ_0, then, as is easily seen,

$$A_3^1 = (A_{*3}\zeta_0,\zeta_0) = \frac{1}{\sqrt{1+\sigma_2}}((\sqrt{\sigma_2}E + \widetilde{A}_3)\zeta_0,\zeta_0) =$$

$$= \frac{1}{\sqrt{1+\sigma_2}}\left(\sqrt{\sigma_2}|\zeta_0|^2 - \sum_{j=1}^{6}\mu_j\zeta_{0j}^2\right) = \frac{1}{\sqrt{1+\sigma_2}}\sum_{j=1}^{6}(\sqrt{\sigma_2} - \mu_j)\zeta_{0j}^2. \tag{101}$$

It follows from (100) and (101) that the matrix A_{*3} for $\sigma_2 > 1$ is positive definite and $A_3^1 > 0$, since $\zeta_0 \neq 0$. Analogously, word by word we obtain that the scalar value A_3^2 from (63) is likewise positive for $\sigma_2 > 1$. Therefore by Theorem 4, in this case for sufficiently large $\lambda > 0$ and any $\widetilde{F} \in L_{2,\lambda}(D)$, $f_j^i \in L_{2,\lambda}(S_i)$, $i = 1,2$; $j = 1,\dots,5$, there exists a unique strong solution V of the problem (93), (95) of the class $L_{2,\lambda}$ for which the corresponding to that problem estimate (17) is valid.

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References

1. J. Hadamard, Lectures on Cauchy's problem in partial differential equations. *Yale University Press, New Haven,* 1923.
2. M. Riesz, L'intégrale de Riemann-Liouville et le problème de Cauchy. *Acta Math.* **81**(1949), 107–125.
3. L. Lundberg, The Klein-Gordon equation with light-cone data. *Comm. Math. Phys.* **62**(1978), No. 2, 107–118.
4. J. Tolen, Problème de Cauchy sur les deux hypersurfaces caractéristiques sécantes. *C.R. Acad. Sci. Paris Sér. A-B* **291**(1980), No. 1, A49–A52.
5. A. A. Borghardt and D. A. Karpenko, A characteristic problem for the wave equation with mass. (Russian) *Differentsial'nye Uravneniya* **20**(1984), No. 2, 302–308.
6. S. S. Kharibegashvili, The Goursat problems for some class of hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **17**(1981), No. 1, 157–164.
7. S. S. Kharibegashvili, Goursat and Darboux type problems for linear hyperbolic partial differential equations and systems. *Mem. Differential Equations Math. Phys.* **4**(1995), 1–127.
8. F. Cagnac, Problème de Cauchy sur un conoide caractéristique. *Ann. Mat. Pura Appl.* (4) **104**(1975), 355–393.

9. A. V. BITSADZE, To the analysis of the formulation of characteristic problems for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* **223**(1975), No. 6, 1289–1292.
10. R. COURANT, Partial differential equations. (Russian) *Mir, Moscow,* 1964; English original: *New York, London,* 1962.
11. A. V. BITSADZE, Some classes of partial differential equations. (Russian) *Nauka, Moscow,* 1981.
12. A. A. DEZIN, Invariant hyperbolic systems and Goursat problem. (Russian) *Dokl. Akad. Nauk SSSR* **135**(1960), No. 5, 1042–1045.
13. K. O. FRIEDRICHS, Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.* **7**(1954), 345–392.
14. K. O. FRIEDRICHS, Symmetric positive linear differential equations. *Comm. Pure Appl. Math.* **11**(1958), 333–418.
15. P. D. LAX AND R. S. PHILLIPS, Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure Appl. Math.* **13**(1960), No. 3, 427–456.
16. G. F. DUFF, Mixed problems for linear systems of first order equations. *Canad. J. Math.* **10**(1958), 127–160.
17. B. V. VERBITSKIĬ, One global property of matrix-functions depending on several variables. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **1**(1978), 8–17.
18. H. TRIEBEL, Interpolation theory, Function spaces, Differential operators. *VEB Deutscher Verlag der Wissenschaften, Berlin,* 1978.
19. V. N. VRAGOV, Boundary value problems for nonclassical equations of mathematical physics. (Russian) *NSU, Novosibirsk,* 1983.
20. R. NARASIMHAN, Analysis on real and complex manifolds. *North-Holland Publishing Company, Amsterdam,* 1968.
21. S. MIZOHATA, Theory of partial differential equations. (Russian) *Mir, Moscow,* 1977.
22. K. KUBOTA AND T. OHKUBO, On well posedness of mixed problems for Maxwell's equations, I. *Hokkaido Math. J.* **7**(1978), 142–168.
23. K. KUBOTA AND T. OHKUBO, On well posedness of mixed problems for Maxwell's equations, II. *Hokkaido Math. J.* **7**(1978), 308–327.
24. T. OHKUDO, Regularity of solutions to hyperbolic mixed problems with uniformly characteristic boundary. *Hokkaido Math. J.* **10**(1981), 93–123.
25. S. S. KHARIBEGASHVILI, On the solvability of some multi-dimensional versions of the Goursat problem for a first order hyperbolic system. (Russian) *Differentsial'nye Uravneniya* **35**(1999), No. 8, 1112–1121.
26. J. MILNOR AND J. STASHEFF, Characteristic classes. *Princeton, New Jersey,* 1974.

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