Memoirs on Differential Equations and Mathematical Physics
VOLUME 29, 2003, 1–30

S. Kharibegashvili

ON THE CORRECT FORMULATION OF SOME BOUNDARY VALUE PROBLEMS FOR SYMMETRIC HYPERBOLIC SYSTEMS OF FIRST ORDER IN A DIHEDRAL ANGLE

Abstract. For one class of symmetric hyperbolic systems of first order we study some boundary value problems in a dihedral angle none of whose faces meets an exterior cone of rays. To this class of systems belong Maxwell and Dirac equations, equations of crystal optics and also some other systems of equations of the mathematical physics.

2000 Mathematics Subject Classification. 35L50.
Key words and phrases: Symmetric hyperbolic systems, boundary value problems, dihedral angle, Maxwell and Dirac systems of crystal optics.

რეზიუმე. ნაშრომში პირველი რიგის სიმეტრიულ ჰიპერბოლურ სისტემათა ერთი კლასისათვის შესწავლილია ზოგიერთი სასაზღვრო ამოცანა ორწახნაგა კუთხეში, რომლის არცერთი წახნაგი არ ეხება სხივთა გარე კონუსს. ამ სისტემათა კლასს მიეკუთვნებიან მაქსველის, დირაკის, კრისტალთა ოპტიკის და მათემატიკური ფიზიკის სხვა განტოლებათა სისტემები.

1. Introduction

For hyperbolic systems of first order with two independent variables the boundary value problems, analogous to the Goursat problem, have been investigated in [1]–[4]. As is known, the Goursat problem or the characteristic problem for a hyperbolic equation of second order admits when passing from two-dimensional to multi-dimensional case different statements. For example, the characteristic problem for a multi-dimensional wave equation can be formulated both in a conic domain, whose boundary is a characteristic conoid, and in a dihedral angle, whose faces are characteristic planes [5]–[10]. The situation is similar for the Darboux problems [11]–[14]. Things become more complicated for a correct statement of the characteristic problem, when in a multi-dimensional case we pass from one equation to a system of equations of hyperbolic type. For example, despite the fact that for a hyperbolic system of second order, split in the principal part, the Goursat problem with the Dirichlet boundary condition on the characteristic conoid is posed correctly [15], in [16] we can find an example of a hyperbolic system of second order which is non-split in the principal part and for which the corresponding characteristic problem has an infinite set of linearly independent solutions. The complexity is that even for a non-split in the principal part strictly hyperbolic system whose cone of normals consists of infinitely smooth sheets, the cones of rays corresponding to these sheets may have strong singularities [17, p. 586]. Therefore difficulties arise already upon formulating the characteristic problem when we have to point out a carrier of boundary data. In this direction the works [18] and [19] are worth mentioning.

In our previous paper [20] we suggested an approach allowing one to formulate for one class of symmetric hyperbolic systems of first order the correct characteristic problems in dihedral angles. For these problems we proved the theorem on the uniqueness of a solution. As regards the question of the solvability of the problems, it was solved only in the case in which the faces, being the carriers of the data, meet the exterior cone of rays of the system of equations under consideration. This class of systems involves, for example, the well-known in the mathematical physics systems of differential equations of Maxwell, Dirac and crystal optics. At the end of the above-mentioned paper [20], for each of these systems we presented correct statements of characteristic problems in dihedral angles.

In the present work, for the same class of symmetric hyperbolic systems of first order as in [20], along with the uniqueness of the solution we have proved the existence of the solution of the boundary value problem in the case, in which the faces do not meet the exterior cone of rays of the system.

Note also that the Cauchy problem and mixed problems for symmetric hyperbolic systems of first order have been studied in [21]–[24].

2. Statement of the Boundary Value Problem. A Priori Estimate

In a space R^{n+1} of variables $x_1,\dots,x_n$ and t we consider a system of differential equations of first order of the type

$$Lu \equiv Eu_t + \sum_{i=1}^{n} A_i u_{x_i} + Bu = F, \tag{1}$$

where A_i and B are the given real $(m\times m)$-matrices, E is the unit $(m\times m)$-matrix, F is the given and u is the unknown m-dimensional real vectors, $n>1$, $m>1$.

Below, matrices A_i are assumed to be symmetric and constant. In this case system (1) is hyperbolic [17, p. 587].

Denote by $D=\left\{(x_1,\dots,x_n,t)\in R^{n+1}: \alpha_0^i t+\sum_{j=1}^{n}\alpha_j^i x_j<0, i=1,2\right\}$ a dihedral angle bounded by hyperplanes $\widetilde{S}_1: \alpha_0^1 t+\sum_{j=1}^{n}\alpha_j^1 x_j=0$ and $\widetilde{S}_2: \alpha_0^2 t+\sum_{j=1}^{n}\alpha_j^2 x_j=0$, where $\alpha^i=(\alpha_1^i,\dots,\alpha_n^i,\alpha_0^i)$ is the unit vector of the outer normal to ∂D at a point of the side $S_i=\widetilde{S}_i\cap\partial D$, $j=1,2$, $\alpha^1\neq\alpha^2$.

For the sake of simplicity we assume that $\alpha_0^i<0$, $i=1,2$.

Let us consider the boundary value problem formulated as follows: find in the domain D a solution u of system (1) by the boundary conditions

$$\Gamma^i u|_{S_i}=f^i, \quad i=1,2, \tag{2}$$

where Γ^i are the given real constants of the $(\varkappa_i\times m)$-matrix, and $f^i=(f_1^i,\dots,f_{\varkappa_i}^i)$ are the given $\varkappa_i$-dimensional real vectors, $i=1,2$.

Remark 1. Depending on the geometric orientation of the dihedral angle D, below we will indicate the method of constructing the matrices Γ^i, $i=1,2$, for which boundary value problem (1), (2) is posed correctly.

Since the matrix $Q(\xi')=-\sum_{i=1}^{n}A_i\xi_i$, $\xi'=(\xi_1,\dots,\xi_n)\in R^n$ is symmetric, its characteristic roots are real. We arrange them in decreasing order: $\widetilde{\lambda}_1(\xi')\geq\widetilde{\lambda}_2(\xi')\geq\cdots\widetilde{\lambda}_m(\xi')$. Multiplicities $k_1,\dots,k_s$ of these roots are assumed to be constant, i.e. independent of ξ', and we put

$$\lambda_1(\xi')=\widetilde{\lambda}_1(\xi')=\cdots=\widetilde{\lambda}_{k_1}(\xi')>\lambda_2(\xi')=\widetilde{\lambda}_{k_1+1}(\xi')=\cdots=\widetilde{\lambda}_{k_1+k_2}(\xi')>\cdots$$
$$\cdots>\lambda_s(\xi')=\widetilde{\lambda}_{m-k_s+1}(\xi')=\cdots=\widetilde{\lambda}_m(\xi'), \xi'\in R^n\backslash\{(0,\dots,0)\}. \tag{3}$$

Note that by (3) and due to continuous dependence of the polynomial on its coefficients, $\lambda_1(\xi'),\dots,\lambda_s(\xi')$ are continuous homogeneous functions of degree 1 [25].

Since the matrix $Q(\xi')$ is symmetric, there exists an orthogonal matrix $T=T(\xi')$ such that

$$(T^{-1}QT)(\xi')=\operatorname{diag}(\underbrace{\lambda_1(\xi'),\dots,\lambda_1(\xi')}_{k_1},\dots,\underbrace{\lambda_s(\xi'),\dots,\lambda_s(\xi')}_{k_s}). \tag{4}$$

According to (3) and (4), the cone of normals

$$K = \{\xi = (\xi_1, \dots, \xi_n, \xi_0) \in R^{n+1} : \det(E\xi_0 - Q(\xi')) = 0\}$$

of system (1) consists of separate sheets

$$K_i = \{\xi = (\xi', \xi_0) \in R^{n+1} : \xi_0 - \lambda_i(\xi') = 0\}, \quad i = 1, \dots, s.$$

Because of the fact that

$$\lambda_j(\xi') = -\lambda_{s+1-j}(-\xi'), \quad 0 \le j \le \left[\frac{s+1}{2}\right], \tag{5}$$

the cones K_j and K_{s+1-j} are centrally symmetric with respect to the point $(0, \dots, 0)$, where $[a]$ is an integral part of number a.

Remark 2. In case s is an odd number, we have $j = s+1-j$ for $j = \left[\frac{s+1}{2}\right]$. Therefore the cone K_j for $j = \left[\frac{s+1}{2}\right]$ is centrally symmetric with respect to the point $(0, \dots, 0)$. In this case, to simplify our exposition for $s = 2s_0 + 1$ we assume that

$$\lambda_{s_0+1}(\xi') \equiv 0, \quad \left[\frac{s+1}{2}\right] = s_0 + 1, \tag{6}$$

i.e. K_{s_0+1} is the hyperplane $\pi_0 : \xi_0 = 0$.

Note that condition (6) is fulfilled for some systems of equations of first order appearing in the mathematical physics, for example, for systems of equations of Maxwell and crystal optics.

Remark 3. Below it will be assumed that $\pi_0 \cap K_{s_0} = \{(0, \dots, 0)\}$ for even $s = 2s_0$. According to (3) and (5), this means that the cones $K_1, \dots, K_{s_0}$, are placed on one side from $\pi_0 : \xi_0 = 0$, and $K_{s_0+1}, \dots, K_{2s_0}$ on the other side, i.e.,

$$\lambda_1(\xi') > \dots > \lambda_{s_0}(\xi') > 0 > \lambda_{s_0+1}(\xi') > \dots > \lambda_{2s_0}(\xi'), \tag{7}$$
$$\xi' \in R^n \backslash \{(0, \dots, 0)\}.$$

If $s = 2s_0 + 1$ is odd, then by (3), (5) and (6) we automatically have $\pi_0 \cap K_{s_0} = \{(0, \dots, 0)\}$ and, consequently,

$$\lambda_1(\xi') > \dots > \lambda_{s_0}(\xi') > \lambda_{s_0+1}(\xi') \equiv 0 > \lambda_{s_0+2}(\xi') > \dots > \lambda_{2s_0+1}(\xi'), \tag{8}$$
$$\xi' \in R^n \backslash \{(0, \dots, 0)\}.$$

In this case $K_1, \dots, K_{s_0}$ are placed on one side from $\pi_0 = K_{s_0+1}$, and $K_{s_0+2}, \dots, K_{2s_0+1}$ on the other side. It follows from (5)-(8) that for multiplicities k_j of roots λ_j the equalities

$$k_j = k_{s+1-j}, \quad j = 1, \dots, \left[\frac{s+1}{2}\right]$$

are valid.

Taking into account (7) and (8), we consider in the half-space $\xi_0 < 0$ the following sets:

$$\begin{aligned} G_0 =&\Big\{\xi = (\xi', \xi_0) \in R^{n+1}\backslash\{(0,\dots,0)\} : \lambda_{i_0}(\xi') < \xi_0 < 0\Big\},\\ &i_0 = \Big[\frac{s+1}{2}\Big] + 1,\\ G_i =&\Big\{\xi = (\xi', \xi_0) \in R^{n+1}\backslash\{(0,\dots,0)\} : \lambda_{i_0+i}(\xi') < \xi_0 < \lambda_{i_0+i-1}(\xi')\Big\},\\ &i = 1,\dots,s_0-1,\\ G_{s_0} =&\Big\{\xi = (\xi', \xi_0) \in R^{n+1}\backslash\{(0,\dots,0)\} : \xi_0 < \lambda_s(\xi')\Big\}. \end{aligned}$$

It is easy to see that

$$\begin{gathered} \partial G_0 = \pi_0 \cup K_{i_0}, \quad \partial G_i = K_{i_0+i} \cup K_{i_0+i-1}, \quad i = 1,\dots,s_0-1, \quad \partial G_{s_0} = K_s,\\ \{(\xi', \xi_0) \in R^{n+1} : \xi_0 < 0\} \cup \{(0,\dots,0)\} = \Big(\bigcup_{i=0}^{s_0} G_i\Big) \cup \Big(\bigcup_{j=i_0}^{s} K_j\Big). \end{gathered} \tag{9}$$

In case, when problem (1),(2) is characteristic, i.e. both faces S_1 and S_2 are the characteristic planes of system (1), by virtue of (7), (8) and by our assumption above that $\alpha_0^i < 0$, $i = 1, 2$, there exist natural numbers s_1 and s_2 such that

$$s_i \ge i_0 = \Big[\frac{s+1}{2}\Big] + 1, \quad i = 1, 2; \quad \alpha^i \in K_{s_i}, \quad i = 1, 2. \tag{10}$$

If problem (1), (2) is non-characteristic, i.e. the faces S_1 and S_2 are non-characteristic planes of system (1), then since $\alpha_0^i < 0$, $i = 1, 2$ reasoning analogously and taking into account (7)-(9), we conclude that instead of (10) there exist non-negative integers p_1 and p_2 such that

$$0 \le p_i \le s_0, \quad \alpha^i \in G_{p_i}, \quad i = 1, 2. \tag{11}$$

The case, when one of the faces S_1 or S_2 in problem (1), (2) is characteristic and the other face is non-characteristic, is treated similarly,

Below we will restrict ourselves to the consideration of case (11), i.e. when both faces S_1 and S_2 are non-characteristic planes of system (1).

By $Q_0(\xi) \equiv E\xi_0 + \sum_{i=1}^n A_i\xi_i = E\xi_0 - Q(\xi')$ we denote the characteristic matrix of system (1) and consider the question on the reduction of the quadratic form $(Q_0(\xi)\eta, \eta)$ to the canonical form, when $\xi \in G_{p_i}$, where $\eta \in R^m$, and $(\cdot\,,\cdot)$ denotes the scalar product in the Euclidean space R^m.

By virtue of (4), for $\eta = T\zeta$ we have

$$\begin{aligned} (Q_0(\xi)\eta, \eta) &= ((T^{-1}Q_0T)(\xi)\zeta, \zeta) = ((E\xi_0 - (T^{-1}QT)(\xi'))\zeta, \zeta) =\\ &= (\xi_0 - \lambda_1(\xi'))\zeta_1^2 + \dots + (\xi_0 - \lambda_1(\xi'))\zeta_{k_1}^2 + (\xi_0 - \lambda_2(\xi'))\zeta_{k_1+1}^2 +\\ &\quad + \dots + (\xi_0 - \lambda_2(\xi'))\zeta_{k_1+k_2}^2 + \dots + (\xi_0 - \lambda_s(\xi'))\zeta_{m-k_s+1}^2 +\\ &\qquad + \dots + (\xi_0 - \lambda_s(\xi'))\zeta_m^2. \end{aligned} \tag{12}$$

By the definition of the sets G_i and by inequalities (3), (7) and (8), for $\xi=(\xi',\xi_0)\in G_{p_i}$ we have

$$\begin{aligned} &[\xi_0-\lambda_j(\xi')]|_{G_{p_i}}<0, \quad j=1,\dots,i_0+p_i-1;\\ &[\xi_0-\lambda_j(\xi')]|_{G_{p_i}}>0, \quad j=i_0+p_i,\dots s. \end{aligned} \tag{13}$$

If we denote by $\varkappa_i^+$ and $\varkappa_i^-$ the positive and negative indices of inertia of the quadratic form $(Q_0(\xi)\eta,\eta)|_{\xi\in G_{p_i}}$, then according to (12) and (13) the equalities

$$\varkappa_i^-=\sum_{j=1}^{i_0+p_i-1}k_j, \quad \varkappa_i^+=\sum_{j=i_0+p_i}^{s}k_j, \quad \varkappa_i^-+\varkappa_i^+=m \tag{14}$$

are valid.

If now $\zeta=C^i(\xi)\eta$ is an arbitrary non-degenerated linear transformation which reduces the quadratic form $(Q_0(\xi)\eta,\eta)|_{\xi\in G_{p_i}}$ to the canonical form, then by (14) and due to the invariance of the indices of inertia of the quadratic form with respect to the non-degenerated linear transformations, we have

$$(Q_0(\xi)\eta,\eta)|_{\xi\in G_{p_i}}=\sum_{j=1}^{\varkappa_i^+}[\Lambda_{ij}^+(\xi,\eta)]^2-\sum_{j=1}^{\varkappa_i^-}[\Lambda_{ij}^-(\xi,\eta)]^2. \tag{15}$$

Here

$$\begin{gathered} \Lambda_{ij}^-(\xi,\eta)=\sum_{p=1}^{m}c_{jp}^i(\xi)\eta_p, \quad \Lambda_{ij}^+(\xi,\eta)=\sum_{p=1}^{m}c_{(\varkappa_i^-+j)p}^i(\xi)\eta_p, \qquad (16)\\ C^i=C^i(\xi)=(c_{jp}^i(\xi))_{j,p=1}^m, \quad \xi\in G_{p_i}. \end{gathered}$$

In accordance with (11) and (16), in boundary conditions (2) we take Γ^i as the matrix of order $(\varkappa_i\times m)$, where $\varkappa_i=\varkappa_i^-$, $i=1,2$, whose Γ_{jp}^i elements are given by the equalities

$$\Gamma_{jp}^i=c_{jp}^i(\alpha^i), \quad i=1,2; \quad j=1,\dots\varkappa_i^-; \quad p=1,\dots,m. \tag{17}$$

Along with problem (1), (2), in the domain D we consider the boundary value problem

$$L^*v\equiv -Ev_t-\sum_{i=1}^{n}A_iv_{x_i}+B'v=G, \tag{18}$$

$$\Gamma_*^i v|_{S_i}=g^i, \quad i=1,2, \tag{19}$$

where Γ_*^i is the matrix of order $(\varkappa_i^+\times m)$ whose Γ_{*jp}^i elements are given by the equalities

$$\Gamma_{*jp}^i=c_{(\varkappa_i^-+j)p}^i(\alpha^i), \quad i=1,2; \quad j=1,\dots\varkappa_i^+; \quad p=1,\dots,m, \tag{20}$$

and B' denotes transposition of the matrix B.

Obviously,

$$C^i(\alpha^i)=\begin{pmatrix}\Gamma^i\\ \Gamma_*^i\end{pmatrix}, \quad i=1,2.$$

Remark 4. It can be easily verified that problem (1), (2) in assumption (17) and problem (18), (19) in assumption (20) are self-conjugate. For example, if $u,v\in C^1(\overline{D})$, $\operatorname{diamsupp} u<+\infty$, $\operatorname{diamsupp} v<+\infty$ and $\Gamma^i u|_{S_i}=\Gamma^i_* v|_{S_i}=0$, $i=1,2$, then $(Lu,v)_{L_2(D)}=(u,L^*v)_{L_2(D)}$.

Remark 5. If by virtue of (4) we take as the matrix C^i the orthogonal matrix $T^{-1}=T^1$, then taking into account (11) and (13), equality (12) for $\xi=\alpha^i=(\alpha^i_1,\dots,\alpha^i_n,\alpha^i_0)$ can be rewritten in the form

$$\begin{aligned}(Q_0(\alpha^i)\eta,\eta)&=-\sum_{j=1}^{\varkappa_i^-}\beta_j(\xi)\zeta_j^2+\sum_{j=\varkappa_i^-+1}^{m}\beta_j(\xi)\zeta_j^2=\\&=-\sum_{j=1}^{\varkappa_i^-}\beta_j(\xi)\zeta_j^2+\sum_{j=1}^{\varkappa_i^+}\beta_{\varkappa_i^-+j}(\xi)\zeta^2_{\varkappa_i^-+j}=\\&=-\sum_{j=1}^{\varkappa_i^-}\left[\beta_j^{\frac12}(\xi)\sum_{p=1}^m T_{pj}\eta_p\right]^2+\sum_{j=1}^{\varkappa_i^+}\left[\beta^{\frac12}_{\varkappa_i^++j}(\xi)\sum_{p=1}^m T_{p(\varkappa_i^++j)}\eta_p\right]^2,\end{aligned}\tag{21}$$

where $T=(T_{kl})^m_{k,l=1}$, and T' denotes transposition of the matrix T, $T_{kl}=T_{kl}(\xi')=T_{kl}(\alpha^i_*)$, $\alpha^i_*=(\alpha^i_1,\dots,\alpha^i_n)$. Here $\beta_j(\xi)|_{\xi=\alpha^i}$ are positive, of the kind $|\alpha^i_0-\lambda_{kj}(\alpha^i_*)|$. In this case, by virtue of (15)–(17) and (21), in the boundary conditions (2) as elements Γ^i_{jp} of the matrix Γ^i we have to take

$$\Gamma^i_{jp}=T_{pj}(\alpha^i_*),\quad i=1,2;\quad j=1,\dots,\varkappa_i^-;\quad p=1,\dots,m. \tag{22}$$

Below we assume that the elements of the matrix B in system (1) are the bounded measurable functions in $\overline{D}$, i.e. $B\in L_\infty(\overline{D})$. Let us introduce in our consideration the following weighted spaces:

$$\begin{aligned}W^1_{2,\lambda}(D)=&\{u\in L_{2,\mathrm{loc}}(D): u\exp(-\lambda t)\in W^1_2(D)\},\\&\|u\|_{W^1_{2,\lambda}(D)}=\|u\exp(-\lambda t)\|_{W^1_2(D)},\\L_{2,\lambda}(D)=&\{F\in L_{2,\mathrm{loc}}(D): F\exp(-\lambda t)\in L_2(D)\},\\&\|F\|_{L_{2,\lambda}(D)}=\|F\exp(-\lambda t)\|_{L_2(D)},\\L_{2,\lambda}(S_i)=&\{f\in L_{2,\mathrm{loc}}(S_i): f\exp(-\lambda t)\in L_2(S_i)\},\ i=1,2,\\&\|f\|_{L_{2,\lambda}(S_i)}=\|f\exp(-\lambda t)\|_{L_2(S_i)},\end{aligned}$$

where λ is the real parameter, and $L_{2,\mathrm{loc}}(D)$, $W^1_2(D)$, $L_{2,\mathrm{loc}}(S_i)$, $i=1,2$, are the well-known functional spaces (see [26, p. 384]).

Let $\lambda_{\max}(P)$ be the largest characteristic number of the non-negatively defined symmetric matrix BB' at the point $P\in\overline{D}$. Then because of the fact that $B\in L_\infty(\overline{D})$, we have

$$\lambda_0^2=\sup_{P\in\overline{D}}\lambda_{\max}(P)<+\infty. \tag{23}$$

Lemma 1. *In assumption* (17), *for any solution* $u \in W^1_{2,\lambda}(D)$ *of problem* (1), (2) *for* $\lambda > \lambda_0$ *the a priori estimate*

$$\|u\|_{L_{2,\lambda}(D)} \leq \frac{1}{\sqrt{\lambda - \lambda_0}} \sum_{i=1}^{2} \sum_{j=1}^{\varkappa_i} \|f^i_j\|_{L_{2,\lambda}(S_i)} + \frac{1}{\lambda - \lambda_0} \|F\|_{L_{2,\lambda}(D)}, \qquad (24)$$

where $\varkappa_i = \varkappa_i^-$, $i = 1, 2$ *is valid.*

Proof. Let us introduce into the consideration a new unknown function $w(x,t) = u(x,t)\exp(-\lambda t)$, $\lambda = \mathrm{const} > 0$. Then for $w(x,t)$ we obtain the following system of equations:

$$L_\lambda w \equiv E w_t + \sum_{i=1}^{n} A_i w_{x_i} + B_\lambda w = F_\lambda, \qquad (25)$$

where $B_\lambda = B + \lambda E$, $F_\lambda = F\exp(-\lambda t)$. Note that if $u \in W^1_{2,\lambda}(D)$, then $F \in L_{2,\lambda}(D)$ and $w \in W^1_2(D)$, $F_\lambda \in L_2(D)$ and boundary conditions (2) take the form

$$\Gamma^i w|_{S_i} = f^i_\lambda, \quad i = 1, 2, \qquad (26)$$

where $f^i_\lambda = f^i \exp(-\lambda t)$, $i = 1, 2$, and owing to the theory of a function trace, $f^i_\lambda \in L_2(S_i)$ holds (see [26, p. 253]).

Taking into account system (25), the integration by parts results in

$$2\int_D (L_\lambda w, w) dD = \int_{\partial D} (Q_0(\alpha) w, w) ds + \int_D (2B_\lambda w, w) dD =$$
$$= \sum_{i=1}^{2} \int_{S_i} (Q_0(\alpha^i) w, w) ds + \int_D (2B_\lambda w, w) dD, \qquad (27)$$

where $Q_0(\alpha) = E\alpha_0 + \sum_{j=1}^n A_j \alpha_j$, $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_0)$ is the unit vector of the outer normal to ∂D.

By (11), (14)-(17) and (26) we have

$$(Q_0(\alpha^i) w, w)|_{S_i} = \left(\sum_{j=1}^{\varkappa_i^+} [\Lambda^+_{ij}(\alpha^i, w)]^2 \right)\Bigg|_{S_i} - \left(\sum_{j=1}^{\varkappa_i^-} [\Lambda^-_{ij}(\alpha^i, w)]^2 \right)\Bigg|_{S_i} =$$
$$= \left(\sum_{j=1}^{\varkappa_i^+} [\Lambda^+_{ij}(\alpha^i, w)]^2 \right)\Bigg|_{S_i} - \left(\sum_{j=1}^{\varkappa_i^-} \left[\sum_{p=1}^{m} c^i_{jp}(\alpha^i) w_p \right]^2 \right)\Bigg|_{S_i} \geq$$
$$\geq -\left(\sum_{j=1}^{\varkappa_i^-} \left[\sum_{p=1}^{m} \Gamma^i_{jp} w_p \right]^2 \right)\Bigg|_{S_i} = -\sum_{j=1}^{\varkappa_i^-} (f^i_{\lambda j})^2 = -\sum_{j=1}^{\varkappa_i} (f^i_{\lambda j})^2, \quad i = 1, 2. \qquad (28)$$

In view of (23), we find that in the domain D

$$(2B_\lambda w, w)_{L_2(D)} = 2\lambda (w, w)_{L_2(D)} + 2(Bw, w) \geq$$
$$\geq 2\lambda (w, w)_{L_2(D)} - 2(Bw, Bw)^{\frac{1}{2}}_{L_2(D)} (w, w)^{\frac{1}{2}}_{L_2(D)} =$$
$$= 2\lambda (w, w)_{L_2(D)} - 2(B'Bw, w)^{\frac{1}{2}}_{L_2(D)} (w, w)^{\frac{1}{2}}_{L_2(D)} \geq 2\lambda (w, w)_{L_2(D)} -$$

$$-2\lambda_0(w,w)^{\frac{1}{2}}_{L_2(D)}(w,w)^{\frac{1}{2}}_{L_2(D)} = 2(\lambda-\lambda_0)(w,w)_{L_2(D)}. \tag{29}$$

Next, according to (25), for $\lambda - \lambda_0 > 0$ we have

$$2\int_D (L_\lambda w, w)dD = 2(F_\lambda, w)_{L_2(D)} \le 2\|F_\lambda\|_{L_2(D)}\|w\|_{L_2(D)} \le$$
$$\le \frac{1}{\lambda-\lambda_0}\|F_\lambda\|^2_{L_2(D)} + (\lambda-\lambda_0)\|w\|^2_{L_2(D)}. \tag{30}$$

It follows from (28)–(30) that

$$(\lambda-\lambda_0)\|w\|^2_{L_2(D)} \le \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|f^i_{\lambda j}\|^2_{L_2(S_i)} + \frac{1}{\lambda-\lambda_0}\|F_\lambda\|^2_{L_2(D)},$$

whence, with regard for the fact that

$$\|w\|_{L_2(D)}=\|u\|_{L_{2,\lambda}D},\ \ \|f^i_{\lambda j}\|_{L_2(S_i)}=\|f^i_j\|_{L_{2,\lambda}(S_i)},\ \ \|F_\lambda\|_{L_2(D)}=\|F\|_{L_{2,\lambda}(D)}$$

we immediately get the required inequality (24). □

Remark 6. From (11), (14) and (17) follows completely definite dependence of the structure and the number of boundary conditions in (2) on the geometric orientation of the dihedral angle D. Estimate (24) implies that the solution of problem (1), (2) of the class $W^1_{2,\lambda}(D)$ is unique for $\lambda > \lambda_0$.

Analogously, word for word we can prove that the following lemma is valid.

Lemma 2. *In assumption* (20)*, for any solution* $v \in W^1_{2,\lambda}(D)$ *of problem* (18)*,* (19) *for* $\lambda > \lambda_0^*$ *the a priori estimate*

$$\|v\|_{L_{2,\lambda}(D)} \le \frac{1}{\sqrt{\lambda-\lambda_0^*}}\sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|g^i_j\|_{L_{2,\lambda}(S_i)} + \frac{1}{\lambda-\lambda_0^*}\|G\|_{L_{2,\lambda}(D)} \tag{31}$$

is valid; here $\varkappa_i = \varkappa_i^+$*,* $i = 1, 2$*, and the number* λ_0^* *is defined analogously to* (23) *by means of the symmetric matrix* BB' (*note that in reality* $\lambda_0^* = \lambda_0$ [27, p. 291]*).*

It can be easily verified that for any $u, \omega \in W^1_{2,\lambda}(D)$ such that $\Gamma^i u|_{S_i} = \Gamma^i_* v|_{S_i} = 0$, $i = 1, 2$, the equality

$$(e^{-\lambda t}Lu, e^{-\lambda t}\omega)_{L_2(D)} = (e^{-\lambda t}u, L^*_\lambda e^{-\lambda t}\omega)_{L_2(D)}, \tag{32}$$

where $L^*_\lambda = -E\frac{\partial}{\partial t} - \sum_{i=1}^n A_i\frac{\partial}{\partial x_i} + B'_\lambda$, $B'_\lambda = B' + \lambda E$ is valid.

We can rewrite estimate (31) in the form

$$\|e^{-\lambda t}v\|_{L_2(D)} \le \frac{1}{\sqrt{\lambda-\lambda_0^*}}\sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|e^{-\lambda t}g^i_j\|_{L_2(S_i)} + \frac{1}{\lambda-\lambda_0^*}\|L^*_\lambda e^{-\lambda t}v\|_{L_2(D)},$$

from which it follows that for any vector function $\omega \in W^1_{2,\lambda}(D)$ satisfying the homogeneous, corresponding to (19), boundary conditions, i.e.

$$\Gamma^i_*\omega|_{S_i} = 0, \quad i = 1, 2, \tag{33}$$

the a priori estimate

$$\|e^{-\lambda t}\omega\|_{L_2(D)} \le \frac{1}{\lambda-\lambda_0^*}\|L_\lambda^* e^{-\lambda t}\omega\|_{L_2(D)} \tag{34}$$

is valid.

Remark 7. In accordance with equality (32), we can introduce the notion of a weak generalized solution u of problem (1), (2) of the class $L_{2,\lambda}$ with the homogeneous boundary conditions, i.e.

$$\Gamma^i u|_{S_i} = 0, \quad i = 1, 2, \tag{35}$$

as follows. Let $F \in L_{2,\lambda}(D)$. The vector function $u \in L_{2,\lambda}(D)$ is said to be a weak generalized solution of problem (1), (35) of the class $L_{2,\lambda}$, if for any $\omega \in W^1_{2,\lambda}(D)$ satisfying homogeneous boundary conditions (33) the equality

$$(e^{-\lambda t}u, L_\lambda^* e^{-\lambda t}\omega)_{L_2(D)} = (e^{-\lambda t}F, e^{-\lambda t}\omega)_{L_2(D)} \tag{36}$$

is valid.

It can be easily verified that if u is a solution of problem (1), (35) from the space $W^1_{2,\lambda}(D)$, then it will also be a weak generalized solution of that problem of the class $L_{2,\lambda}$.

The existence of a weak generalized solution of problem (1), (35) of the class $L_{2,\lambda}$ for $\lambda > \lambda_0^*$ follows from the following considerations. By virtue of inequality (34), for the right-hand side of equality (36) the estimate

$$|(e^{-\lambda t}F, e^{-\lambda t}\omega)_{L_2(D)}| \le \|e^{-\lambda t}F\|_{L_2(D)}\|e^{-\lambda t}\omega\|_{L_2(D)} =$$
$$= \|F\|_{L_{2,\lambda}(D)}\|e^{-\lambda t}\omega\|_{L_2(D)} \le \frac{1}{\lambda-\lambda_0^*}\|F\|_{L_{2,\lambda}(D)}\|L_\lambda^* e^{-\lambda t}\omega\|_{L_2(D)} \tag{37}$$

is valid.

Expressions (36) and (37) show that the functional $(e^{-\lambda t}u, L_\lambda^* e^{-\lambda t}\omega)_{L_2(D)}$ with respect to $L_\lambda^* e^{-\lambda t}\omega$ can be extended to the entire space $L_2(D)$ in continuity. Thus, according to the Riesz theorem on the representation of a functional over the space $L_2(D)$, there exists the vector function $w \in L_2(D)$ such that for any $\omega \in W^1_{2,\lambda}(D)$ satisfying the homogeneous boundary conditions (33), the equality

$$(w, L_\lambda^* e^{-\lambda t}\omega)_{L_2(D)} = (e^{-\lambda t}F, e^{-\lambda t}\omega)_{L_2(D)}$$

is valid. This equality by virtue of (36) implies that the vector function $u = e^{\lambda t}w \in L_{2,\lambda}(D)$ is the weak generalized solution of problem (1), (35) of the class $L_{2,\lambda}$.

Remark 8. Along with the weak generalized solution of problem (1), (35) of the class $L_{2,\lambda}$ we can introduce the notion of a strong generalized solution of that problem of the class $L_{2,\lambda}$. The vector function $u \in L_{2,\lambda}(D)$ is said to be a strong generalized solution of problem (1), (35) if there exists a sequence of vector functions $u_p \in W^1_{2,\lambda}(D)$ satisfying the homogeneous boundary conditions (35) and

$$\lim_{p\to\infty} \|u_p - u\|_{L_{2,\lambda}(D)} = 0, \quad \lim_{p\to\infty} \|Lu_p - F\|_{L_{2,\lambda}(D)} = 0.$$

It is easy to verify that the solution of problem (1), (35) from the space $W^1_{2,\lambda}(D)$ is the strong generalized solution of problem (1), (35) of the class $L_{2,\lambda}$, and the strong generalized solution is the weak generalized solution of that problem of the class $L_{2,\lambda}$.

The notion of a strong generalized solution of inhomogeneous problem (1), (2) is introduced analogously. Let $f^i \in L_{2,\lambda}(S_i)$, $i = 1,2$, in the boundary condition (2), and the right-hand side F of equation (1) belong to the space $L_{2,\lambda}(D)$. The vector function $u \in L_{2,\lambda}(D)$ is said to be a strong generalized solution of inhomogeneous problem (1), (2) of the class $L_{2,\lambda}$ if there exists a sequence of vector functions $u_p \in W^1_{2,\lambda}(D)$ such that

$$\lim_{p\to\infty} \|\Gamma^i u_p|_{S_i} - f^i\|_{L_{2,\lambda}(S_i)} = 0, \quad i = 1,2,$$
$$\lim_{p\to\infty} \|u_p - u\|_{L_{2,\lambda}(D)} = 0, \quad \lim_{p\to\infty} \|Lu_p - F\|_{L_{2,\lambda}(D)} = 0.$$

Note that the uniqueness of the strong generalized solution of inhomogeneous problem (1), (2) of the class $L_{2,\lambda}$ follows directly from the a priori estimate (24).

3. The Solvability of Inhomogeneous Boundary Value Problem (1), (2)

Below, for the sake of simplicity and without restriction of generality we assume that

$$S_i : t + (-1)^i \sigma_i x_1 = 0, \quad t \geq 0; \quad 0 < \sigma_i = \text{const} < +\infty, \quad i = 1,2. \tag{38}$$

In this case

$$\alpha^i = (\alpha^i_1, 0, \dots, 0, \alpha^i_0), \quad \alpha^i_1 = \frac{(-1)^{i-1}\sigma_i}{\sqrt{1+\sigma_i^2}}, \quad \alpha^i_0 = \frac{-1}{\sqrt{1+\sigma_i^2}}, \quad i = 1,2. \tag{39}$$

$$\begin{aligned} Q(\alpha^i) &= -\sum_{j=1}^n A_j \alpha^i_j = -\alpha^i_1 A_1, \\ Q_0(\alpha^i) &= E\alpha^i_0 + \sum_{j=1}^n A_j \alpha^i_j = \alpha^i_0 E + \alpha^i_1 A_1, \quad i = 1,2. \end{aligned} \tag{40}$$

If μ_j are the characteristic numbers of the symmetric matrix A_1 which are enumerated with regard for the multiplicity in increasing order, then by (3), (4) and by inequalities $-\alpha^1_1 < 0$, $-\alpha^2_1 > 0$ we have

$$(T_1^{-1} Q(\alpha^1) T_1) = -\alpha^1_1 \operatorname{diag}(\underbrace{\mu_1, \dots, \mu_1}_{k_1}, \dots, \underbrace{\mu_s, \dots, \mu_s}_{k_s}), \tag{41}$$

$$(T_2^{-1}Q(\alpha^2)T_2) = -\alpha_1^2 \operatorname{diag}(\underbrace{\mu_s,\dots,\mu_s}_{k_s},\dots,\underbrace{\mu_1,\dots,\mu_1}_{k_1}), \tag{42}$$

$$T_1^{-1}A_1T_1 = \operatorname{diag}(\underbrace{\mu_1,\dots,\mu_1}_{k_1},\dots,\underbrace{\mu_s,\dots,\mu_s}_{k_s}), \tag{43}$$

$$T_2^{-1}A_1T_2 = \operatorname{diag}(\underbrace{\mu_s,\dots,\mu_s}_{k_s},\dots,\underbrace{\mu_1,\dots,\mu_1}_{k_1}), \tag{44}$$

$$\lambda_j(\alpha_1^1,\dots,\alpha_n^1)=-\alpha_1^1\mu_j,\ \ \lambda_j(\alpha_1^2,\dots,\alpha_n^2)=-\alpha_1^2\mu_{s-j+1},\ \ j=1,\dots,s, \tag{45}$$

$$\mu_1<\mu_2<\dots<\mu_s, \tag{46}$$

where T_1 and T_2 are the constant orthogonal matrices, and the columns of the matrix T_2 are obtained by replacing the columns of the matrix T_1, i.e., $T_{2kl} = T_{1k(m-l+1)}, k,l = 1,\dots,m$.

Since $-\alpha_1^1 < 0$, $-\alpha_1^2 > 0$ by virtue of (7), (8) and (43)-(46), in case $s = 2s_0$ is even, we have the inequalities

$$\mu_1 < \dots < \mu_{s_0} < 0 < \mu_{s_0+1} < \dots < \mu_{2s_0}, \tag{47}$$

while if $s = 2s_0 + 1$ is odd, then

$$\mu_1 < \dots < \mu_{s_0} < \mu_{s_0+1} = 0 < \mu_{s_0+2} < \dots < \mu_{2s_0+1}. \tag{48}$$

By (11), (38), (45)-(48) and owing to the definition of the sets G_j, the condition $\alpha^1 \in G_{p_1}$ is equivalent to

$$\begin{cases}\sigma_1 > \mu_{s_0+1}^{-1}, & s=2s_0,\\ \sigma_1 > \mu_{s_0+2}^{-1}, & s=2s_0+1,\end{cases} \quad \text{for} \quad p_1 = 0;$$

$$\begin{cases}\mu_{s_0+1+p_1}^{-1} < \sigma_1 < \mu_{s_0+p_1}^{-1}, & s=2s_0,\\ \mu_{s_0+2+p_1}^{-1} < \sigma_1 < \mu_{s_0+1+p_1}^{-1}, & s=2s_0+1,\end{cases} \quad \text{for} \quad 1 \le p_1 \le s_0-1;$$

$$\sigma_1 < \mu_s^{-1} \quad \text{for} \quad p_1 = s_0.$$

For the sake of simplicity, we restrict ourselves to the case $s = 2s_0$, $1 \le p_1 \le s_0 - 1$. Then

$$\mu_{s_0+1+p_1}^{-1} < \sigma_1 < \mu_{s_0+p_1}^{-1}, \quad s = 2s_0, \quad 1 \le p_1 \le s_0 - 1. \tag{49}$$

Analogously, the condition $\alpha^2 \in G_{p_2}$ for $s = 2s_0$, $1 \le p_2 \le s_0 - 1$ is equivalent to

$$-\mu_{s_0-p_2}^{-1} < \sigma_2 < -\mu_{s_0-p_2+1}^{-1}, \quad s = 2s_0, \quad 1 \le p_2 \le s_0 - 1. \tag{50}$$

To prove the existence of the strong generalized solution of inhomogeneous problem (1), (2) of the class $L_{2,\lambda}$ we first introduce the functional space $\overset{\circ}{\Phi}{}_{\alpha}^{k}(\overline{D}_\tau)$ and then prove that the problem under consideration is solvable in that space.

Suppose $D_\tau = \{(x_1,\dots,x_n,t) \in D : t < \tau\}$, $\tau > 0$ and denote by $D_{0\tau}$ the set of points at which the domain D_τ and the two-dimensional plane of variables x_1 and t intersect. Let $S_{i\tau} = \partial D_\tau \cap S_i$, $i = 1, 2$. By (38) it is obvious that

$$D_\tau = \{(x_1, \dots, x_n, t) \in R^{n+1} : -\sigma_2^{-1} t < x_1 < \sigma_1^{-1} t, \quad 0 < t < \tau\},$$
$$D_{0\tau} = \{(x_1, t) \in R^2 : -\sigma_2^{-1} t < x_1 < \sigma_1^{-1} t, \quad 0 < t < \tau\}.$$

Denote by $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$, $k \geq 1$, $\alpha \geq 0$, the space of functions $u(x_1, \dots, x_n, t)$ of the class $C^k(\overline{D}_\tau)$ for which

$$\partial_{x_1}^{i_1} \partial_t^{i_2} u(0, x_2, \dots, x_n, 0) = 0, \quad -\infty < x_i < +\infty, \quad i = 2, \dots, n,$$
$$0 \leq i_1 + i_2 \leq k, \quad \partial_{x_1}^{i_1} = \frac{\partial^{i_1}}{\partial x_1^{i_1}}, \quad \partial_t^{i_2} = \frac{\partial^{i_2}}{\partial t^{i_2}},$$

and also partial Fourier transformations $\widehat{u}(x_1, \xi_2, \dots, \xi_n, t)$ of which with respect to the variables $x_2, \dots, x_n$ are continuous functions in $\overline{G}_\tau = \{(x_1, \xi_2, \dots, \xi_n, t) \in R^{n+1} : (x_1, t) \in \overline{D}_{0\tau},\ \widetilde{\xi} = (\xi_2, \dots, \xi_n) \in R^{n-1}\}$ together with their partial derivatives with respect to the variables x_1 and t up to the k-th order inclusive, and satisfy the following estimates: for any natural N there exist positive, independent of $\widetilde{\xi} = (\xi_2, \dots, \xi_n)$, numbers $\widetilde{C}_N = \widetilde{C}_N(x_1, t)$ and $\widetilde{K}_N = \widetilde{K}_N(x_1, t)$ such that for $(x_1, t) \in \overline{D}_{0\tau}$ and $|\widetilde{\xi}|^2 = |\xi_2|^2 + \cdots + |\xi_n|^2 > \widetilde{K}_N^2$ the inequalities

$$|\partial_{x_1}^{i_1} \partial_t^{i_2} \widetilde{u}(x_1, \widehat{\xi}, t)| \leq \widetilde{C}_N t^{k+\alpha-i_1-i_2} \exp(-N|\widetilde{\xi}|), \quad 0 \leq i_1 + i_2 \leq k, \tag{51}$$

hold, and for $(x_1, t) \in \overline{D}_{0\tau} \backslash \{(0, 0)\}$

$$\widetilde{C}_N^\circ(x_1, t) = \sup_{(\widetilde{x}_1, \widetilde{t}) \in \overline{D}_{0t}} \widetilde{C}_N(\widetilde{x}_1, \widetilde{t}) < +\infty,$$
$$\widetilde{K}_N^\circ(x_1, t) = \sup_{(\widetilde{x}_1, \widetilde{t}) \in \overline{D}_{0t}} \widetilde{K}_N(\widetilde{x}_1, \widetilde{t}) < +\infty.$$

Analogously we introduce spaces $\overset{\circ}{\Phi}{}^k_\alpha(S_{i\tau})$, $i = 1, 2$. Note that the trace $u|_{S_{i\tau}}$ of the function u from the space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$ belongs to the space $\overset{\circ}{\Phi}{}^k_\alpha(S_{i\tau})$. We denote the space of Fourier transformations $\widehat{u}(x_1, \xi_2, \dots, \xi_n, t)$ with respect to the variables $x_2, \dots, x_n$ of the functions $u(x_1, x_2, \dots, x_n, t)$ belonging to the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$ by $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_{0\tau})$ and take into account that the variable $\widetilde{\xi} = (\xi_2, \dots, \xi_n)$ is regarded as a parameter.

Remark 9. Below we will consider the boundary value problem for system (1) in the domain D_τ, i.e. instead of (2) we will consider the boundary conditions

$$\Gamma^j u|_{S_{j\tau}} = f^j, \quad j = 1, 2, \tag{52}$$

note that while studying problem (1), (52) in the space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$ it will be required of the coefficient B and the functions F, f^i that $F \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$, $f^j \in \overset{\circ}{\Phi}{}^k_\alpha(S_{i\tau})$, $j = 1, 2$; $B \in C^k(\overline{D}_\tau)$. Moreover, we will assume that the elements of the matrix B depend only on the variables x_1 and t and are the bounded functions.

If u is the solution of problem (1), (52) from the space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$, then after the Fourier transformation with respect to the variables $x_2, \dots, x_n$ the system of equations (1) and the boundary conditions (52) take the form

$$E\widehat{u}_t + A_1\widehat{u}_{x_1} + i\Big(\sum_{j=2}^{n} A_j\xi_j\Big)\widehat{u} + B\widehat{u} = \widehat{F}, \tag{53}$$

$$\Gamma^j\widehat{u}|_{\gamma_{j\tau}} = \widehat{f^j}, \quad j = 1, 2, \tag{54}$$

where $\widehat{u}$, $\widehat{F}$, $\widehat{f^j}$ are the Fourier transformations respectively of the functions u, F, f^j with respect to the variables $x_2, \dots, x_n$, and $\gamma_{j\tau} : t+(-1)^j\sigma_j x_1 = 0$, $0 \le t \le \tau$, $j = 1, 2$, are the sides of the triangular domain $D_{0\tau}$ in the plane of variables x_1, t introduced by us above. In these equalities here $i = \sqrt{-1}$.

Remark 10. Thus, after the Fourier transformation with respect to the variables $x_2, \dots, x_n$ the spatial problem (1), (52) is reduced to the plane problem (53), (54) with the parameters $\xi_2, \dots, \xi_n$ in the domain $D_{0\tau} = \{(x_1, t) \in R^2 : -\sigma_2^{-1}t < x_1 < \sigma_1^{-1}t, \quad 0 < t < \tau\}$ of the plane of variables x_1, t. It is easy to see that in a class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$ of vector functions defined by inequalities (51) the above-mentioned reduction is equivalent.

As a result of substitution $\widehat{u} = T_1 v$, by virtue of (40), (41), instead of system (53) and boundary conditions (54), with respect to the new unknown vector function v we have

$$Ev_t + Av_{x_1} + i\Big(\sum_{j=2}^{n} \widetilde{A}_j\xi_j\Big)v + \widetilde{B}v = \widetilde{F}, \tag{55}$$

$$\Gamma^j T_1 v|_{\gamma_{j\tau}} = \widehat{f^j}, \quad j = 1, 2. \tag{56}$$

Here

$$\begin{gathered} A = \operatorname{diag}(\underbrace{\mu_1, \dots, \mu_1}_{k_1}, \dots, \underbrace{\mu_s, \dots, \mu_s}_{k_s}), \\ \widetilde{A}_j = T_1^{-1}A_jT_1, \quad j = 2, \dots, n, \quad \widetilde{B} = T_1^{-1}BT_1, \end{gathered} \tag{57}$$

and $\widetilde{F} = T_1\widehat{F}$.

Let $L_j(x_1^\circ, t^\circ) : x_1 = z_j(x_1^\circ, t^\circ; \widetilde{t}) = x_1^\circ - \mu_j t^\circ + \mu_j\widetilde{t}$, $t = \widetilde{t}$ be the parametric writing of the characteristic of the j-th family of system (55), coming out of the point $(x_1^\circ, t^\circ) \in \overline{D}_{o\tau}$, $1 \le j \le s$, towards decreasing values of the variable t, i.e. $\widetilde{t} \le t^\circ$. Denote by $\omega_j(x_1, t)$ the ordinate of the point at which the characteristic $L_j(x_1, t)$, $P(x_1, t) \in \overline{D}_{0\tau}$ and the curve $\gamma_{1\tau}$ or $\gamma_{2\tau}$ intersect, depending on the index j of the characteristic L_j and on the location of the point $P(x, t)$ in $\overline{D}_{0\tau}$; the latter curve we denote by $\gamma_{i(P)\tau}$. Owing to the above-said, it becomes obvious that

$$0 \le \omega_j(x_1, t) \le t, \quad (x_1, t) \in \overline{D}_{0\tau}, \quad j = 1, \dots, s. \tag{58}$$

It is not difficult to verify that

$$\omega_j|_{\gamma_{1\tau}} = \begin{cases} t, & j=1,\dots,s_0+p_1, \\ \tau_j t, & j=s_0+p_1+1,\dots,s, \end{cases}$$
$$\omega_j|_{\gamma_{2\tau}} = \begin{cases} \tau_j t, & j=1,\dots,s_0-p_2, \\ t, & j=s_0-p_2+1,\dots,s. \end{cases} \tag{59}$$

Here

$$\tau_j = \begin{cases} (\sigma_2\mu_j+1)\sigma_1(\sigma_1\mu_j-1)^{-1}\sigma_2^{-1}, & j=1,\dots,s_0-p_2, \\ (\sigma_1\mu_j-1)\sigma_2(\sigma_2\mu_j+1)^{-1}\sigma_1^{-1}, & j=s_0+p_1+1,\dots,s, \end{cases}$$

and by (47), (49), (50) and the fact that $\gamma_{1\tau}$ and $\gamma_{2\tau}$ are not characteristics of system (55), we have

$$0<\tau_j<1, \quad j=1,\dots,s_0-p_2, \quad s_0+p_1+1,\dots,s. \tag{60}$$

Remark 11. The functions v, $\widetilde{F}$, $\widehat{f}^j$, $j=1,2$, with the exclusion of independent variables x_1 and t, depend also on the parameters $\xi_2,\dots,\xi_n$. To simplify our writing, these parameters will be omitted below. For example, instead of $v(x_1,\xi_2,\dots,\xi_n,t)$ we will write $v(x_1,t)$.

Integrating the (q_j+l)-th equation of system (55), where $q_1=0$, $q_j=k_1+\dots,+k_{j-1}$, $l=1,\dots,k_j$ along the j-th characteristic $L_j(x_1,t)$, coming out of the point $P(x_1,t)\in\overline{D}_{0\tau}$ towards decreasing values of the variable t from the point $P(x_1,t)$ to the point where $L_j(x_1,t)$ and the curve $\gamma_{1\tau}$ or $\gamma_{1\tau}$ intersect depending on the index j of the characteristic L_j and on the location of the point P in $\overline{D}_{0\tau}$, we get

$$v_{q_j+l}(x_1,t) = v_{q_j+l}(\gamma_{j(P)\tau}(\omega_j(x_1,t)),\omega_j(x_1,t)) + \\ + \int_{\omega_j(x_1,t)}^{t} \Big[\sum_{p=1}^{m} \Lambda_{jlp} v_p\Big](z_j(x_1,t;\widetilde{t}),\widetilde{t})d\widetilde{t} + F^1_{jl}, \tag{61}$$
$$1\le j\le s; \quad l=1,\dots,k_j,$$

where Λ_{jlp} are the completely definite linear scalar functions with respect to the parameters $\xi_2,\dots,\xi_n$, and F^1_{jl} are the completely definite scalar functions. Here $\gamma_{j(P)\tau}(t)$ is the function which describes equation of the curve $\gamma_{j(P)\tau}$, i.e. $\gamma_{j(P)\tau}: x_1=\gamma_{j(P)\tau}(t)$, $0\le t\le\tau$.

Suppose

$$\varphi_{q_j+l}(t) = v_{q_j+l}|_{\gamma_{1\tau}} = v_{q_j+l}(\sigma_1^{-1}t,t), \ \ j=1,\dots,s_0+p_1; \ l=1,\dots,k_j,$$
$$\psi_{q_j+l}(t) = v_{q_j+l}|_{\gamma_{2\tau}} = v_{q_j+l}(-\sigma_2^{-1}t,t), \tag{62}$$
$$j=s_0-p_2+1,\dots,s; \ \ l=1,\dots,k_j.$$

Taking into account (62), we can rewrite the system of equations (61) in the form of one equation

$$v(x_1,t) = \chi(x_1,t) + \sum_{j=1}^{s}\int_{\omega_j(x_1,t)}^{t} \Lambda^1_j v(z_j(x_1,t;\widetilde{t})d\widetilde{t} + F^1, \tag{63}$$

where Λ_j^1 is the matrix of order $m \times m$. Its elements are linear functions with respect to the parameters $\xi_2, \dots, \xi_n$, $j = 1, \dots, s$, and $\chi(x_1, t) = \{(\varphi_{q_j+l}(\omega_j(x_1,t)),\ j = 1, \dots, s_0 + p_1;\ l = 1, \dots, k_j),\ (\psi_{q_j+l}(\omega_j(x_1,t)),\ j = s_0 - p_2 + 1, \dots, s;\ l = 1, \dots, k_j)\}$. It can be easily verified that a number of components φ_{q_j+l} of the vector χ is equal to $\varkappa_1$, and a number of components ψ_{q_j+l} is equal to $\varkappa_2$.

Substituting expression (63) for the vector function v into boundary conditions (56) and taking into account equalities (59), we obtain

$$\begin{aligned} G_0^1\varphi(t) + \sum_{j=s_0+p_1+1}^{s} G_j^1\psi(\tau_j t) + [T_1(v)](t) = \widehat{f}^1, \quad 0 \le t \le \tau, \\ G_0^2\psi(t) + \sum_{j=1}^{s_0-p_2} G_j^2\varphi(\tau_j t) + [T_2(v)](t) = \widehat{f}^2, \quad 0 \le t \le \tau, \end{aligned} \tag{64}$$

where

$$\varphi(t) = (\varphi_{q_j+l}(t),\ j = 1, \dots, s_0 + p_1;\ l = 1, \dots, k_j) = (\varphi_1(t), \dots, \varphi_{\varkappa_1}(t)),$$
$$\psi(t) = (\psi_{q_j+l}(t), j = s_0 - p_2 + 1, \dots, s; l = 1, \dots, k_j) = (\psi_{m-\varkappa_2+1}(t), \dots, \psi_m(t)).$$

Here G_j^1 and G_j^2 are the completely definite constant matrices, and T_1 and T_2 by virtue of (59) and (60) are linear integral operators of Volterra type whose kernels depend on the parameters $\xi_2, \dots, \xi_n$.

Denote by T_1^1 the matrix of order $m \times \varkappa_1$ which consists of the first $\varkappa_1$ columns of the orthogonal matrix T_1 appearing in (56) and by T_1^2 the matrix of order $m \times \varkappa_2$ consisting of the last $\varkappa_2$ columns of the matrix T_1. Then by virtue of (56) and (59) we can easily check that G_0^j, $j = 1, 2$, appearing in system (64) are square matrices of order $\varkappa_j \times \varkappa_j$, and

$$G_0^j = \Gamma^j \times T_1^j, \quad j = 1, 2, \tag{65}$$

where the elements of the matrix Γ^j are defined from equalities (17).

In the assumption that

$$\det(\Gamma^j \times T_1^j) \ne 0, \quad j = 1, 2, \tag{66}$$

we solve equations (64) with respect to φ and ψ by means of (65) and find that

$$\varphi(t) - \sum_{j=1}^{s_0-p_2} \sum_{p=s_0+p_1+1}^{s} G_{1jp}\varphi(\tau_j\tau_p t) = [T_3(v)](t) + f^3(t), \quad 0 \le t \le \tau, \tag{67}$$

$$\psi(t) - \sum_{j=1}^{s_0-p_2} \sum_{p=s_0+p_1+1}^{s} G_{2jp}\psi(\tau_j\tau_p t) = [T_4(v)](t) + f^4(t), \quad 0 \le t \le \tau, \tag{68}$$

where G_{ljp}, $l = 1, 2$, are the completely definite constant square matrices of order $\varkappa_l \times \varkappa_l$, and T_3 and T_4 are linear integral operators of Volterra type whose kernels depend linearly on the parameters $\xi_2, \dots, \xi_n$.

Remark 12. As is seen from our reasoning, when conditions (66) are fulfilled, problem (1), (52) in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$ is equivalent to the problem of finding a system of functions v, φ and ψ from the system of integral functional equations (63), (67) and (68), where $v \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_{0\tau})$; φ, $\psi \in \overset{\circ}{\Phi}{}^k_\alpha([0,\tau])$, $F^1 \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_{0\tau})$; f^3, $f^4 \in \overset{\circ}{\Phi}{}^k_\alpha([0,\tau])$. The system of type (63), (67), (68) in the spaces under consideration has been studied in [28]. According to the results obtained in this paper, there exists the real number ρ_0 depending only on the elements of the constant matrices G_{1jp} and G_{2jp} appearing in (67) and (68) such that when $k+\alpha>\rho_0$, for any $F^1 \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_{0\tau})$; f^3, $f^4 \in \overset{\circ}{\Phi}{}^k_\alpha([0,\tau])$ there exists the unique solution v, φ, ψ of the system of equations (63), (67), (68) respectively from the spaces $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_{0\tau})$, $\overset{\circ}{\Phi}{}^k_\alpha([0,\tau])$, $\overset{\circ}{\Phi}{}^k_\alpha([0,\tau])$ for which the following estimates for $|\widetilde{\xi}|>K$ are valid:

$$
\begin{gathered}
|\partial^{i_1}_{x_1}\partial^{i_2}_t v(x_1,\widetilde{\xi},t)| \le M^* t^{k+\alpha-i_1-i_2}\exp[M_*(1+|\widetilde{\xi}|)]\exp(-N|\widetilde{\xi}|), \qquad (69)\\
|\partial^{i_1+i_2}_t \varphi(t,\widetilde{\xi})| \le M^* t^{k+\alpha-i_1-i_2}\exp[M_*(1+|\widetilde{\xi}|)]\exp(-N|\widetilde{\xi}|),\\
|\partial^{i_1+i_2}_t \psi(t,\widetilde{\xi})| \le M^* t^{k+\alpha-i_1-i_2}\exp[M_*(1+|\widetilde{\xi}|)]\exp(-N|\widetilde{\xi}|),\\
(x_1,t)\in \overline{D}_{0\tau}, \quad 0\le i_1+i_2\le k, \quad |\widetilde{\xi}|^2=\xi_2^2+\dots+\xi_n^2,
\end{gathered}
$$

where the values $K=K(x_1,t,N)$, $M^*=M^*(x_1,t,N,f^3,f^4,F^1)$ and $M_*=M_*(\Lambda^1_j,G_{1jp},G_{2jp})$ are independent of $\widetilde{\xi}\in R^{n-2}$. Moreover, if for some $\tau_0\in(0,\tau)$

$$
F^1|_{\overline{D}_{0\tau_0}}=0, \quad f^j|_{[0,\tau_0]}=0, \quad j=3,4,
$$

then

$$
v|_{\overline{D}_{0\tau_0}}=0, \quad \varphi|_{[0,\tau_0]}=0, \quad \psi|_{[0,\tau_0]}=0
$$

as well.

Remark 13. According to Plancherel's equality, in considering the question on the solvability of problem (1), (52) in the spaces $L_{2,\lambda}(D_\tau)$ and $W^1_{2,\lambda}(D_\tau)$ the use will be made of the following equivalent norms:

$$
\begin{gathered}
\|u\|^2_{L_{2,\lambda}(D_\tau)}=\int\limits_{D_\tau}|u(x_1,\dots,x_n,t)|^2e^{-2\lambda t}dxdt=\\
=\int\limits_{D_{0\tau}}e^{-2\lambda t}dx_1dt\int\limits_{R^{n-1}}|u(x_1,\dots,x_n,t)|^2dx_2\dots dx_n=\\
=\int\limits_{\widehat{D}_\tau}|\widehat{u}(x_1,\xi_2,\dots,\xi_n,t)|^2e^{-2\lambda t}dx_1d\xi_2\dots d\xi_ndt=\|\widehat{u}\|^2_{L_{2,\lambda}(\widehat{D}_\tau)}, \qquad (70)\\
\|u\|^2_{W^1_{2,\lambda}(D_\tau)}=\int\limits_{D_\tau}\left[|u_{x_1}|^2+\dots+|u_{x_n}|^2+|u_t|^2+|u|^2\right]e^{-2\lambda t}dxdt=\\
=\int\limits_{\widehat{D}_\tau}[(1+|\widehat{\xi}|^2)|\widehat{u}|^2+|\widehat{u}_{x_1}|^2+|\widehat{u}_t|^2]e^{-2\lambda t}dx_1d\xi_2\dots d\xi_ndt, \qquad (71)
\end{gathered}
$$

where $\widehat{D}_\tau = D_{0\tau} \times R^{n-1}_{\widetilde{\xi}}$, $\widetilde{\xi} = (\xi_2, \dots, \xi_n)$, $\widehat{u} = \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}} u$ are partial Fourier transformation of the function u with respect to the variables $(x_2, \dots, x_n) = \widetilde{x}$. It is evident that for a finite τ the weighted spaces $L_{2,\lambda}(D_\tau)$ and $W^1_{2,\lambda}(D_\tau)$ with the parameter λ coincide with ordinary spaces $L_2(D_\tau)$ and $W^1_2(D_\tau)$.

Analogously, by means of the Fourier transformation and by Plancherel's equality we introduce an equivalent norm in the space $L_{2,\lambda}(S_{j\tau})$:

$$\|u\|^2_{L_{2,\lambda}(S_{j\tau})} = \int_{S_{j\tau}} |u(x_1, \dots, x_n, t)|^2 e^{-2\lambda t} dS_{j\tau} =$$
$$= \int_{\gamma_{j\tau}} e^{-2\lambda t} d\gamma_{j\tau} \int_{R^{n-1}} |\widehat{u}(x_1, \xi_2, \dots, \xi_n, t)|^2 d\xi_2 \dots d\xi_n =$$
$$= \int_{\widehat{S}_{j\tau}} |\widehat{u}(x_1, \xi_2, \dots, \xi_n, t)|^2 e^{-2\lambda t} d\widehat{S}_{j\tau} = \|\widehat{u}\|_{L_{2,\lambda}(\widehat{S}_{j\tau})}, \quad j = 1, 2, \tag{72}$$

where $\widehat{S}_{j\tau} = \gamma_{j\tau} \times R^{n-1}_{\widetilde{\xi}}$.

Let $F \in L_{2,\lambda}(D)$ and $f^j \in L_{2,\lambda}(S_j)$, $j = 1, 2$, in problem (1), (2). Obviously, $F_\tau = F|_{D_\tau} \in L_{2,\lambda}(D_\tau)$, $f^j_\tau = f^j|_{S_{j\tau}} \in L_{2,\lambda}(S_{j\tau})$, $j = 1, 2$. Therefore by (70) and (72) we have $\widehat{F}_\tau \in L_{2,\lambda}(\widehat{D}_\tau)$ and $\widehat{f}^j_\tau \in L_{2,\lambda}(\widehat{S}_{j\tau})$, $j = 1, 2$. Since the spaces $C^\infty_0(\widehat{D}_\tau)$ and $C^\infty_0(\widehat{S}_{j\tau})$, $j = 1, 2$, of finite finitely differentiable functions in $\widehat{D}_\tau$ and $\widehat{S}_{j\tau}$ are dense respectively in $L_{2,\lambda}(\widehat{D}_\tau)$ and $L_{2,\lambda}(\widehat{S}_{j\tau})$, $j = 1, 2$, there exist sequences of functions $\widehat{F}_{\tau p} \in C^\infty_0(\widehat{D}_\tau)$ and $\widehat{f}^j_{\tau p} \in C^\infty_0(\widehat{S}_{j\tau})$, $j = 1, 2$; $p = 1, 2, \dots$, such that

$$\lim_{p\to\infty} \|\widehat{F}_{\tau p} - \widehat{F}_\tau\|_{L_{2,\lambda}(\widehat{D}_\tau)} = 0, \quad \lim_{p\to\infty} \|\widehat{f}^j_{\tau p} - \widehat{f}^j_\tau\|_{L_{2,\lambda}(\widehat{S}_{j\tau})} = 0, \quad j = 1, 2. \tag{73}$$

Because of the fact that $\widehat{F}_{\tau p} \in C^\infty_0(\widehat{D}_\tau)$ and $\widehat{F}_{\tau p} \in C^\infty_0(\widehat{S}_{j\tau})$, $j = 1, 2$, we have $\widehat{F}_{\tau p}|_{\widehat{D}_{\tau_0}} = 0$, $\widehat{f}^j_{\tau p}|_{\widehat{S}_{j\tau_0}} = 0$ for sufficiently small τ_0, $0 < \tau_0 < \tau$, and by (51) we find that

$$F_{\tau p} = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{F}_{\tau p}) \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau), \quad f^j_{\tau p} = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{f}^j_{\tau p}) \in \overset{\circ}{\Phi}{}^k_\alpha(S_{i\tau})$$

for any k and α, $p = 1, 2, \dots$, where $\mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}$ are ordinary Fourier transformation with respect to the variables $\xi_2, \dots, \xi_n$. Therefore, according to Remark 12, problem (1), (52) has the solution $u_{\tau p} \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau)$ for $F = F_{\tau p}$, and $f^j = f^j_{\tau p} = (f^j_{1\tau p}, \dots, f^j_{\varkappa_j \tau p})$, $k \geq 1$, $\alpha \geq 0$.

By (71) it is evident that $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau) \subset W^1_{2,\lambda}(D_\tau)$. Consequently, $u_{\tau p} \in W^1_{2,\lambda}(D_\tau)$. Noticing that $Q_0(\alpha)|_{\partial D_\tau \cap \{t=\tau\}} = E$ as $\alpha|_{\partial D_\tau \cap \{t=\tau\}} = (0, \dots, 0, 1)$, where $Q_0(\alpha) = E\alpha_0 + \sum_{j=1}^n A_j \alpha_j$, and $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_0)$ is the unit vector of the outer normal to ∂D_τ, we have

$$\int_{\partial D_\tau \cap \{t=\tau\}} (Q_0(\alpha) w, w) ds = \int_{\partial D_\tau \cap \{t=\tau\}} |w|^2 ds \geq .$$

Therefore for the solution $u \in W^1_{2,\lambda}(D_\tau)$ of problem (1), (52) in the domain D_τ we repeat word for word the same reasoning as in proving Lemma 1 and see that for $\lambda > \lambda_0$ the estimate

$$\|u\|_{L_{2,\lambda}(D_\tau)} \leq \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2} \sum_{j=1}^{\varkappa_i} \|f^i_j\|_{L_{2,\lambda}(S_{i\tau})} + \frac{1}{\lambda-\lambda_0} \|F\|_{L_{2,\lambda}(D_\tau)} \qquad (74)$$

is valid.

Inequality (74) implies that

$$\|u_{\tau l} - u_{\tau p}\|_{L_{2,\lambda}(D_\tau)} \leq$$
$$\leq \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2} \sum_{j=1}^{\varkappa_i} \|f^i_{j\tau l} - f^i_{j\tau p}\|_{L_{2,\lambda}(S_{i\tau})} + \frac{1}{\lambda-\lambda_0} \|F_{\tau l} - F_{\tau p}\|_{L_{2,\lambda}(D_\tau)},$$

whence, owing to (70), (72) and (73), it follows that the sequence $\{u_{\tau p}\}$ is fundamental in $L_{2,\lambda}(D_\tau)$. Therefore taking into account the fact that the space $L_{2,\lambda}(D_\tau)$ is complete, there exists the vector function $u_\tau \in L_{2,\lambda}(D_\tau)$ such that $u_{\tau p} \to u_\tau$, $Lu_{\tau p} = F_{\tau p} \to F_\tau$ in $L_{2,\lambda}(D)$ and $\Gamma^i u_{\tau p}|_{S_{i\tau}} = f^i_{\tau p} \to f^i_\tau$ in $L_{2,\lambda}(S_{i\tau})$ as $p \to \infty$. The latter with regard for (74) means that problem (1), (52) has the unique strong generalized solution u_τ of the class $L_{2,\lambda}$, $\lambda > \lambda_0$, for any $F_\tau \in L_{2,\lambda}(D_\tau)$ and $f^j_\tau \in L_{2,\lambda}(S_{j\tau})$, $j = 1, 2$, for which the estimate

$$\|u_\tau\|_{L_{2,\lambda}(D_\tau)} \leq \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2} \sum_{j=1}^{\varkappa_i} \|f^i_{j\tau}\|_{L_{2,\lambda}(S_{i\tau})} + \frac{1}{\lambda-\lambda_0} \|F_\tau\|_{L_{2,\lambda}(D_\tau)} \qquad (75)$$

is valid.

Note that if we take into account that $F_\tau = F|_{D_\tau}$ and $f^i_\tau = f^i|_{S_{i\tau}}$, $i = 1, 2$, and hence $F_{\tau_1}|_{D_{\tau_1}} = F_{\tau_2}|_{D_{\tau_1}}$ and $f^i_{\tau_1}|_{S_{i\tau_1}} = f^i_{\tau_2}|_{S_{i\tau_1}}$ for $\tau_2 \geq \tau_1$, then by virtue of (75) we can conclude that $u_{\tau_1}|_{D_{\tau_1}} = u_{\tau_2}|_{D_{\tau_1}}$ for $\tau_2 \geq \tau_1$.

Thus the vector function is defined correctly in the whole domain D. Its narrowing on D_τ is the strong generalized solution $u_\tau = u|_{D_\tau}$ of the class $L_{2,\lambda}$ of problem (1), (52) for $F_\tau = F|_{D_\tau} \in L_{2,\lambda}(D_\tau)$ and $f^j_\tau = f^j|_{S_{j\tau}} \in L_{2,\lambda}(S_{j\tau})$, $j = 1, 2$.

In fact, we can show that u is the strong generalized solution of problem (1), (2) of the class $L_{2,\lambda}$ in D for sufficiently large λ. Indeed, we can construct the above-considered vector function u in somewhat different way. Since $F \in L_{2,\lambda}(D)$ and $f^j \in L_{2,\lambda}(S_j)$, $j = 1, 2$, there exist sequences of the functions $\widehat{F}_p \in C^\infty_0(\widehat{D})$ and $\widehat{f}^j_p \in C^\infty_0(\widehat{S}_j)$, $j = 1, 2$; $p = 1, 2, \dots$ such that

$$\lim_{p\to\infty} \|\widehat{F}_p - \widehat{F}\|_{L_{2,\lambda}(\widehat{D})} = 0, \quad \lim_{p\to\infty} \|\widehat{f}^j_p - \widehat{f}^j\|_{L_{2,\lambda}(\widehat{S}_j)} = 0, \quad j = 1, 2. \qquad (76)$$

Analogously, for $F = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{F}_p)$ and $f^j = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{f}^j_p)$ problem (1), (52) can be reduced to the same system of integral functional equations (63), (67) and (68) for v with the domain of definition D and for φ and ψ with the domain of definition $[0, +\infty)$, respectively, instead of D_τ and $[0, \tau]$ in the case of problem (1), (52). Since under that reduction the right-hand sides of system (63), (67), (68) satisfy by virtue of $\widehat{F}^1_p \in C^\infty_0(\widehat{D})$ and $\widehat{f}^j_p \in C^\infty_0(\widehat{S}_j)$ the conditions $F^1 = F^1_p \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_{0\tau})$, $f^j = f^j_p \in \overset{\circ}{\Phi}{}^k_\alpha[0,\tau]$, $j = 3,4$, for arbitrary $k \geq 1$, $\alpha \geq 0$ and $\tau > 0$, according Remark 12 there exists the unique solution v_p, φ_p, ψ_p of the system of equations (63), (67), (68) with the domain of definition D for v_p and $[0,+\infty)$ for φ_p and ψ_p, where $v_p \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_{0\tau})$; φ_p, $\psi_p \in \overset{\circ}{\Phi}{}^k_\alpha([0,\tau])$ for any $\tau > 0$. Moreover, estimate (69) in which the value M_* does not depend on τ, is valid. Taking into account (70)–(72), this implies that for the solution $u = u_p = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(v_p)$ of problem (1), (2) for $F = F_p = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{F}_p)$ and $f^j = f^j_p = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{f}^j_p)$ the inclusion

$$u_p \in W^1_{2,\lambda}(D), \quad \lambda > \widetilde{\lambda}_1,$$

holds, where $\widetilde{\lambda}_1$ is a real number depending only on coefficients of system (1) and on elements of matrices Γ^i, $i = 1,2$, in the boundary conditions (2). Therefore, according to (24), for $\lambda > \lambda^1_0 = \max(\lambda_0, \widetilde{\lambda}_1)$ we have the inequality

$$\|u_l - u_p\|_{L_{2,\lambda}(D)} \leq \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2} \sum_{j=1}^{\varkappa_i} \|f^i_{jl} - f^i_{jp}\|_{L_{2,\lambda}(S_i)} + \frac{1}{\lambda-\lambda_0} \|F_l - F_p\|_{L_{2,\lambda}(D)}$$

from which, owing to (70), (72) and (76), it follows that the sequence $\{u_p\}$ is fundamental in $L_{2,\lambda}(D)$ and hence there exists the vector function $u \in L_{2,\lambda}(D)$ such that $u_p \to u$, $Lu_p = F_p \to F$ in $L_{2,\lambda}(D)$ and $\Gamma^i u_p|_{S_i} = f^i_p \to f^i$ in $L_{2,\lambda}(S_i)$ as $p \to \infty$. This means that the strong generalized solution of inhomogeneous problem (1), (2) of the class $L_{2,\lambda}$ exists and its uniqueness follows directly from the a priori estimate (24).

Theorem 1. *Let conditions* (66) *be fulfilled. Then for any* $F \in L_2(D_\tau)$ *and* $f^i \in L_2(S_{i\tau})$, $i = 1,2$, $0 < \tau = \text{const} < +\infty$, *there exists the unique strong solution* u *of problem* (1), (52) *of the class* $L_{2,\lambda}$ *for which the estimate*

$$\|u\|_{L_{2,\lambda}(D_\tau)} \leq \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2} \sum_{j=1}^{\varkappa_i} \|f^i_j\|_{L_{2,\lambda}(S_{i\tau})} + \frac{1}{\lambda-\lambda_0} \|F\|_{L_{2,\lambda}(D_\tau)}$$

is valid for $\lambda > \lambda_0$.

Note that for a finite positive τ the spaces $L_{2,\lambda}(D_\tau)$ and $L_{2,\lambda}(S_{i\tau})$ coincide respectively with the spaces $L_2(D_\tau)$ and $L_2(S_{i\tau})$ for any $\lambda \in (-\infty,+\infty)$.

Theorem 2. *Let conditions* (66) *be fulfilled. Then for* $\lambda > \lambda^1_0$ *for any* $F \in L_{2,\lambda}(D)$ *and* $f^i \in L_{2,\lambda}(S_i)$, $i = 1,2$, *there exists the unique strong solution* u *of problem* (1), (2) *of the class* $L_{2,\lambda}$ *for which estimate* (24) *is valid.*

Remark 14. Taking now into account (21) and (22), we give some examples of matrices Γ^i under the boundary conditions (2) for which conditions (66), appearing in Theorems 1 and 2, are automatically fulfilled. By (17), as matrices Γ^i we take the matrix of order $(\varkappa_i \times m)$ consisting of the first $\varkappa_i$ rows of the matrix $C^i(\alpha^i)$ which, according to equality (15), reduces the quadratic form $(Q_0(\alpha^i)\eta, \eta)$ to the canonical one, i.e. for $\zeta = C^i(\alpha^i)\eta$ we have

$$(Q_0(\alpha^i)\eta, \eta) = -\sum_{j=1}^{\varkappa_i^-} \zeta_j^2 + \sum_{j=1}^{\varkappa_i^+} \zeta_{\varkappa_i^- + j}^2.$$

As is known such a matrix $C^i(\alpha^i)$, and hence Γ^i, are defined non-uniquely. By equalities (4), (41)-(44), as matrices $C^i(\alpha^i)$ we can take the orthogonal matrix T_i^{-1}, $i = 1, 2$, where T_1 and T_2 are defined in (43) and (44). In this case, by the definition of matrices T_1^i, $i = 1, 2$, from (65), (43) and (44) we can easily get that $\Gamma^i \times T_1^i = \widetilde{E}_{\varkappa_i}$, $i = 1, 2$, where $\widetilde{E}_{\varkappa_1}$ is the unit matrix of order $\varkappa_1$, and $\widetilde{E}_{\varkappa_2} = (e_{pq})$ is the square matrix of order $\varkappa_2$, where

$$e_{pq} = \begin{cases} 1 & \text{for} \quad p + q = \varkappa_2 + 1, \\ 0 & \text{for} \quad p + q \neq \varkappa_2 + 1. \end{cases}$$

Thus under such a choice of matrices Γ^i in (2) conditions (66) will be fulfilled.

Remark 15. For the sake of simplicity, we restricted ourselves above to the case in which $s = 2s_0$, $1 \leq p_i \leq s_0 - 1$, $i = 1, 2$, and thus inequalities (49) and (50) have been fulfilled. Minor modifications on our reasoning allow one to prove the validity of Theorems 1 and 2 in case $s = 2s_0$ and $p_i = 0$ or $p_i = s_0$, $i = 1, 2$, and also in case $s = 2s_0 + 1$, $0 \leq p_i \leq s_0$, $i = 1, 2$.

4. Smoothness of a Solution of Boundary Problem (1), (2)

By Theorem 2, if $F \in L_{2,\lambda}(D)$ and $f^i \in L_{2,\lambda}(S_i)$, $i = 1, 2$, $\lambda > \lambda_0^1$, then the strong solution u of problem (1), (2) belongs to the space $L_{2,\lambda}(D)$. Below it will be shown that under additional smoothness of data of problem $F|_{D_{\tau_0}} \in W_{2,\lambda}^1(D_{\tau_0})$ and $f^i|_{S_{i\tau_0}} \in W_{2,\lambda}^1(S_{i\tau_0})$, $i = 1, 2$, $\tau_0 > 0$, the above-indicated solution u of problem (1), (2) in the domain $D_{\tau_0,\tau}$ would belong to the space $W_{2,\lambda}^1(D_{\tau_0,\tau})$, where $D_{\tau_0,\tau} = D_{\tau_0} \setminus \overline{D}_\tau$, $0 < \tau < \tau_0$. Moreover, in case $-\sigma_2^{-1} \leq \sigma \leq \sigma_1^{-1}$, the planes $S_\sigma^o : \sigma t - x_1 = 0$ will be non-characteristic planes of system (1). To simplify our writing, we retain the same notation for the narrowing of the vector function to a subset of domain of its definition. It is also assumed that the norm $\|F\|_{W_{2,\lambda}^1(D_\tau)}$, being the function of the variable τ, decreases as $\tau \to +0$ not as slowly as the power function, i.e. $\|F\|_{W_{2,\lambda}^1(D_\tau)} = O(\tau^l)$, where l is a positive constant. The similar conditions are imposed on vector functions f^i, $i = 1, 2$. For simplicity we assume that $F|_{D_\varepsilon} = 0$ and $f^i|_{S_{i\varepsilon}} = 0$, $i = 1, 2$, where ε is a fixed sufficiently small positive number. Then according to our reasoning in the proof of Theorem 2, we have

$$u|_{D_\varepsilon} = 0. \tag{77}$$

Under transformation of variables $y_1 = \frac{x_1}{t}$, $y_i = x_i$, $i = 2, \dots, n$, $y_{n+1} = t$ the domain $D_{\tau_0,\tau}$ turns by virtue of (38) into the domain $\Omega_{\tau_0,\tau} = \{y = (y_1, \dots, y_{n+1}) \in R^{n+1} : -\sigma_2^{-1} < y_1 < \sigma_1^{-1}, -\infty < y_i < +\infty, i = 2, \dots, n, \tau < y_{n+1} < \tau_0\}$ and system (1) with new variables $y_1, \dots, y_{n+1}$ takes the form

$$\widetilde{L}\widetilde{u} \equiv E\widetilde{u}_{y_{n+1}} + \frac{1}{y_{n+1}}(A_1 - y_1 E)\widetilde{u}_{y_1} + \sum_{i=2}^{n} A_i \widetilde{u}_{y_i} + \widetilde{B}\widetilde{u} = \widetilde{F}, \tag{78}$$

where $\widetilde{u}(y) = u(x,t)$, $\widetilde{F}(y) = F(x,t)$, $\widetilde{B}(y) = B(x,t)$.

By (2), for the vector function $\widetilde{u}$ we have

$$\Gamma^i \widetilde{u}|_{\widetilde{S}_{i\tau_0,\tau}} = \widetilde{f}^i, \quad i = 1,2, \tag{79}$$

where $\widetilde{S}_{i\tau_0,\tau} = \{y \in R^{n+1} : y_1 = (-1)^{i-1}\sigma_i^{-1}, -\infty < y_j < +\infty, j = 2, \dots, n, \tau < y_{n+1} < \tau_0\}$, $\widetilde{f}^i(y) = f^i(x,t)$, $i = 1,2$. It is clear that $\widetilde{S}_{i\tau_0,\tau} = \partial\Omega_{\tau_0,\tau} \cap \{y_1 = (-1)^{i-1}\sigma_i^{-1}\}$, $i = 1,2$.

Obviously, the condition $u \in W^1_{2,\lambda}(D_{\tau_0,\tau})$ is equivalent to $\widetilde{u} \in W^1_{2,\lambda}(\Omega_{\tau_0,\tau})$. In the same manner as in getting (25), for the new unknown function $(\widetilde{w})(y) = \widetilde{u}(y)\exp(-\lambda y_{n+1})$ we obtain by means of (78) the following system of equations:

$$\widetilde{L}_\lambda \widetilde{w} \equiv E\widetilde{w}_{y_{n+1}} + \frac{1}{y_{n+1}}(A_1 - y_1 E)\widetilde{w}_{y_1} + \sum_{i=2}^{n} A_i \widetilde{w}_{y_i} + \widetilde{B}_\lambda \widetilde{w} = \widetilde{F}_\lambda, \tag{80}$$

where $\widetilde{B}_\lambda = \widetilde{B} + \lambda E$, $\widetilde{F}_\lambda = \widetilde{F}\exp(-\lambda y_{n+1})$. Note that if $\widetilde{u} \in W^1_{2,\lambda}(\Omega_{\tau_0,\tau})$, then $\widetilde{F} \in L_{2,\lambda}(\Omega_{\tau_0,\tau})$ and $\widetilde{w} \in W^1_2(\Omega_{\tau_0,\tau})$, $\widetilde{F}_\lambda \in L_2(\Omega_{\tau_0,\tau})$ and conditions (79) take the form

$$\Gamma^i \widetilde{w}|_{\widetilde{S}_{i\tau_0,\tau}} = \widetilde{f}^i_\lambda, \quad i = 1,2, \tag{81}$$

where $\widetilde{f}^i_\lambda = \widetilde{f}^i \exp(-\lambda y_{n+1}) \in L_2(\widetilde{S}_{i\tau_0,\tau})$, $i = 1,2$.

Taking into account (80), the integration by parts yields

$$\begin{aligned} 2\int_{\Omega_{\tau_0,\tau}} (\widetilde{L}_\lambda \widetilde{w}, \widetilde{w})dy &= \int_{\partial\Omega_{\tau_0,\tau}} (\widetilde{Q}_0(\alpha)\widetilde{w}, \widetilde{w})ds + \\ &+ \int_{\Omega_{\tau_0,\tau}} \frac{1}{y_{n+1}} (\widetilde{w}, \widetilde{w})dy + \int_{\Omega_{\tau_0,\tau}} (2B_\lambda \widetilde{w}, \widetilde{w})dy, \end{aligned} \tag{82}$$

where $\widetilde{Q}_0(\alpha) = E\alpha_0 + \sum_{j=1}^{n} \widetilde{A}_j \alpha_j$, $\widetilde{A}_j = A_j$ for $j \neq 1$, $\widetilde{A}_1 = \frac{1}{y_{n+1}}(A_1 - y_1 E)$, $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_0)$ is the unit vector of the outer normal to $\partial\Omega_{\tau_0,\tau}$.

Since

$$\alpha|_{\widetilde{S}_{i\tau_0,\tau}} = ((-1)^{i-1}, 0, \dots, 0),\ i = 1,2;\ \alpha|_{\partial\Omega_{\tau_0,\tau} \cap \{y_{n+1} = \tau\}} = (0, \dots, 0, -1),$$
$$\alpha|_{\partial\Omega_{\tau_0,\tau} \cap \{y_{n+1} = \tau_0\}} = (0, \dots, 0, 1), \quad y_1|_{\widetilde{S}_{i\tau_0,\tau}} = (-1)^{i-1}\sigma_i,\ i = 1,2,$$

from (82) it follows that

$$2\int_{\Omega_{\tau_0,\tau}}(\widetilde{L}_\lambda,\widetilde{w},\widetilde{w})dy=\int_{\widetilde{S}_{1\tau_0,\tau}}\frac{1}{y_{n+1}}((A_1-\sigma_1^{-1}E)\widetilde{w},\widetilde{w})ds-$$
$$-\int_{\widetilde{S}_{2\tau_0,\tau}}\frac{1}{y_{n+1}}((A_1+\sigma_2^{-1}E)\widetilde{w},\widetilde{w})ds+\int_{\partial\Omega_{\tau_0,\tau}\cap\{y_{n+1}=\tau_0\}}(\widetilde{w},\widetilde{w})ds-$$
$$-\int_{\partial\Omega_{\tau_0,\tau}\cap\{y_{n+1}=\tau\}}(\widetilde{w},\widetilde{w})ds+\int_{\Omega_{\tau_0,\tau}}((2B_\lambda+\frac{1}{y_{n+1}}E)\widetilde{w},\widetilde{w})dy. \qquad (83)$$

By virtue of (39) and (40), for the characteristic matrix $Q_0(\alpha)=E\alpha_0+\sum_{j=1}^n A_j\alpha_j$ of system (1) we have

$$Q_0(\alpha^i)=\alpha_0^iE+\alpha_1^iA_1=-\frac{1}{\sqrt{1+\sigma_i^2}}E+\frac{(-1)^{i-1}}{\sqrt{1+\sigma_i^2}}\sigma_iA_1=$$
$$=(-1)^{i-1}\frac{\sigma_i}{\sqrt{1+\sigma_i^2}}(A_1+(-1)^i\sigma_i^{-1}E),\quad i=1,2. \qquad (84)$$

From (84),(28) and (79),(81) it immediately follows that

$$\int_{\widetilde{S}_{1\tau_0,\tau}}\frac{1}{y_{n+1}}((A_1-\sigma_1^{-1}E)\widetilde{w},\widetilde{w})ds-\int_{\widetilde{S}_{2\tau_0,\tau}}\frac{1}{y_{n+1}}((A_1+\sigma_2^{-1}E)\widetilde{w},\widetilde{w})ds=$$
$$=\sum_{i=1}^2\int_{\widetilde{S}_{i\tau_0,\tau}}\frac{\sqrt{1+\sigma_i^2}}{\sigma_iy_{n+1}}(Q_0(\alpha^i)\widetilde{w},\widetilde{w})ds\geq-\sum_{i=1}^2\int_{\widetilde{S}_{i\tau_0,\tau}}\frac{\sqrt{1+\sigma_i^2}}{\sigma_iy_{n+1}}\sum_{j=1}^{\varkappa_i}(\widetilde{f}_{\lambda j}^i)^2ds\geq$$
$$\geq-\frac{c_1}{\tau}\sum_{i=1}^2\sum_{j=1}^{\varkappa_i}\int_{\widetilde{S}_{i\tau_0,\tau}}(\widetilde{f}_{\lambda j}^i)^2ds=-\frac{c_1}{\tau}\sum_{i=1}^2\sum_{j=1}^{\varkappa_i}\|\widetilde{f}_{\lambda j}^i\|^2_{L_2(\widetilde{S}_{i\tau_0,\tau})}, \qquad (85)$$

where $c_1=\max_{i=1,2}\frac{\sqrt{1+\sigma_i^2}}{\sigma_i}$, while $\frac{1}{y_{n+1}}|_{\widetilde{S}_{i\tau_0,\tau}}\leq\frac{1}{\tau}$.

Owing to (23) and the fact that $\frac{1}{y_{n+1}}|_{\Omega_{\tau_0,\tau}}\geq\frac{1}{\tau_0}$, similarly to (29) we have

$$\Big(\Big(2\widetilde{B}_\lambda+\frac{1}{y_{n+1}}E\Big)\widetilde{w},\widetilde{w}\Big)_{L_2(\Omega_{\tau_0,\tau})}\geq2\Big(\lambda-\lambda_0+\frac{1}{2\tau_0}\Big)(\widetilde{w},\widetilde{w})_{L_2(\Omega_{\tau_0,\tau})}=$$
$$=2(\lambda-\lambda_0+\frac{1}{2\tau_0})\|\widetilde{w}\|_{L_2(\Omega_{\tau_0,\tau})}. \qquad (86)$$

For $\tau<\varepsilon$, by (77) we find that

$$\int_{\partial\Omega_{\tau_0,\tau}\cap\{y_{n+1}=\tau_0\}}(\widetilde{w},\widetilde{w})ds-\int_{\partial\Omega_{\tau_0,\tau}\cap\{y_{n+1}=\tau\}}(\widetilde{w},\widetilde{w})ds=$$
$$=\int_{\partial\Omega_{\tau_0,\tau}\cap\{y_{n+1}=\tau_0\}}(\widetilde{w},\widetilde{w})ds\geq0. \qquad (87)$$

By means of (80) and analogously to (30), for $\lambda>\lambda_0$ we get

$$2\int_{\Omega_{\tau_0,\tau}}(\widetilde{L}_\lambda,\widetilde{w},\widetilde{w})dy=2(\widetilde{F}_\lambda,\widetilde{w})_{L_2(\Omega_{\tau_0,\tau})}\leq$$

$$\le (\lambda-\lambda_0)\|\widetilde{w}\|^2_{L_2(\Omega_{\tau_0,\tau})}+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}^2_\lambda\|_{L_2(\Omega_{\tau_0,\tau})}. \tag{88}$$

From (83), (85)-(88) follows

$$(\lambda-\lambda_0)\|\widetilde{w}\,\|_{L_2(\Omega_{\tau_0,\tau})}+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}_\lambda\|^2_{L_2(\Omega_{\tau_0,\tau})}\ge$$
$$\ge -\frac{c_1}{\tau}\sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|\widetilde{f}^i_{\lambda j}\|_{L_2(\widetilde{S}_{i\tau_0,\tau})}+2\Big(\lambda-\lambda_0+\frac{1}{2\tau_0}\Big)\|\widetilde{w}\|_{L_2(\Omega_{\tau_0,\tau})}. \tag{89}$$

According to (89), we have

$$(\lambda-\lambda_0)\|\widetilde{w}\|^2_{L_2(\Omega_{\tau_0,\tau})}\le \frac{c_1}{\tau}\sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|\widetilde{f}^i_{\lambda j}\|^2_{L_2(\widetilde{S}_{i\tau_0,\tau})}+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}_\lambda\|^2_{L_2(\Omega_{\tau_0,\tau})},$$

whence, taking into account the fact that

$$\|\widetilde{w}\|_{L_2(\Omega_{\tau_0,\tau})}=\|\widetilde{u}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})},\quad \|\widetilde{f}^i_{\lambda j}\|_{L_2(\widetilde{S}_{i\tau_0,\tau})}=\|\widetilde{f}^i_j\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})},\quad i=1,2,$$
$$\|\widetilde{F}_\lambda\|_{L_2(\Omega_{\tau_0,\tau})}=\|\widetilde{F}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}$$

we arrive at

$$\|u\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}\le$$
$$\le\Big(\frac{c_1}{\tau}\Big)^{\frac12}\frac{1}{\sqrt{\lambda-\lambda_0}}\sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|\widetilde{f}^i_j\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}. \tag{90}$$

Let now $u\in W^2_{2,\lambda}(D_{\tau_0,\tau})$ or, what comes to the same thing, $\widetilde{u}\in W^2_{2,\lambda}(\Omega_{\tau_0,\tau})$. Below it will be assumed that elements of the matrix B are continuous and bounded together with their partial derivatives of first order with respect to variables $x_1,\dots,x_n,t$ in the closed domain $\overline{D}_{\tau_0,\tau}$, and in this case $\widetilde{B}_{y_i}\in L_\infty(\overline{\Omega}_{\tau_0,\tau})$, $i=1,\dots,n+1$. Therefore, if $\mu_i(P)$ is the largest characteristic number of the nonnegatively defined symmetric matrix $\widetilde{B}'_{y_i}\widetilde{B}_{y_i}$ at the point $P\in\overline{\Omega}_{\tau_0,\tau}$, then we will have

$$\mu_0^2=\sup_{P\in\overline{\Omega}_{\tau_0,\tau}}\max_{1\le i\le n+1}\mu_i(P)<+\infty. \tag{91}$$

Differentiating system (78) and the boundary conditions (79) with respect both to y_p, $2\le p\le n$, and to the vector function $v^p=\widetilde{u}_{y_p}$ in the domain $\Omega_{\tau_0,\tau}$, we obtain the following problem:

$$Ev^p_{y_{n+1}}+\frac{1}{y_{n+1}}(A_1-y_1E)v^p_{y_1}+\sum_{i=2}^{n}A_iv^p_{y_i}+\widetilde{B}v^p=\widetilde{F}_{y_p}-\widetilde{B}_{y_p}\widetilde{u}, \tag{92}$$

$$\Gamma^iv^p|_{\widetilde{S}_{i\tau_0,\tau}}=\widetilde{f}^i_{y_p},\quad i=1,2. \tag{93}$$

Since by our assumption $\widetilde{u}\in W^2_{2,\lambda}(\Omega_{\tau_0,\tau})$, we get $v^p\in W^1_{2,\lambda}(\Omega_{\tau_0,\tau})$ and, applying estimate (90) to the solution v^p of problem (92), (93) and to $\widetilde{u}$, with regard for (91) and (77) for $\tau < \varepsilon$ we obtain

$$\|v^p\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})} \leq \left(\frac{c_1}{\tau}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i} \|\widetilde{f}^i_{y_pj}\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+$$

$$+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}_{y_p}-\widetilde{B}_{y_p}\widetilde{u}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})} \leq \left(\frac{c_1}{\tau}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i} \|\widetilde{f}^i_{y_pj}\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+$$

$$+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}_{y_p}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}+\frac{\mu_0}{\lambda-\lambda_0}\|\widetilde{u}\|_{L_{2,\lambda}(\Omega\tau_0,\tau)} \leq$$

$$\leq \left(\frac{c_1}{\tau}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i} \|\widetilde{f}^i_{y_pj}\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}_{y_p}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}+$$

$$+\frac{\mu_0}{\lambda-\lambda_0}\left[\left(\frac{c_1}{\tau}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i} \|\widetilde{f}^i_j\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}\right], \tag{94}$$

$$p=2,\dots,n.$$

By virtue of (78), from the equality $\frac{1}{y_{n+1}}(y_{n+1}\widetilde{L}\widetilde{u})_{y_{n+1}}=\frac{1}{y_{n+1}}(y_{n+1}\widetilde{F})_{y_{n+1}}$ for v^p, where $p=n+1$, we arrive with regard for (79) at the following problem:

$$Ev^{n+1}_{y_{n+1}}+\frac{1}{y_{n+1}}(A_1-y_1E)v^{n+1}_{y_1}+\sum_{i=2}^{n}A_iv^{n+1}_{y_i}+\widetilde{B}v^{n+1}=$$

$$=\frac{1}{y_{n+1}}(y_{n+1}\widetilde{F})_{y_{n+1}}-Ev^{n+1}-\sum_{i=2}^{n}A_iv^i-(\widetilde{B}+y_{n+1}\widetilde{B}_{y_{n+1}})\widetilde{u}, \tag{95}$$

$$\Gamma^iv^{n+1}|_{\widetilde{S}_{i\tau_0,\tau}}=\widetilde{f}^i_{y_{n+1}}, \quad i=1,2. \tag{96}$$

For the solution v^{n+1} of problem (95), (96), by (90)-(94) we have

$$\|v^{n+1}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})} \leq \left(\frac{c_1}{\tau}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i} \|\widetilde{f}^i_{y_{n+1}j}\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+$$

$$+\frac{1}{\lambda-\lambda_0}\Big\|\frac{1}{y_{n+1}}(y_{n+1}\widetilde{F})_{y_{n+1}}-Ev^{n+1}-$$

$$-\sum_{p=2}^{n}A_pv^p-(\widetilde{B}+y_{n+1}\widetilde{B}_{y_{n+1}})\widetilde{u}\Big\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})} \leq$$

$$\leq \left(\frac{c_1}{\tau}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i} \|\widetilde{f}^i_{y_{n+1}j}\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+$$

$$+\frac{1}{\lambda-\lambda_0}\Big\|\frac{1}{y_{n+1}}(y_{n+1}\widetilde{F})_{y_{n+1}}\Big\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}+$$

$$+\sum_{p=2}^{n}\|A_p\|\left\{\left(\frac{c_1}{\tau}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i} \|\widetilde{f}_{y_pj}\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+\right.$$

$$+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}_{y_p}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}+\frac{\mu_0}{\lambda-\lambda_0}\times$$

$$\times\Big[\Big(\frac{c_1}{\tau}\Big)^{\frac{1}{2}}\frac{1}{\sqrt{\lambda-\lambda_0}}\sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|\widetilde{f}^i_j\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+\frac{1}{\lambda-\lambda_0}\|\widetilde{F}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}\Big]\Big\}+$$

$$+(1+\tau_0)\mu_0\times$$

$$+(1+\tau_0)\mu_0\Big[\Big(\frac{c_1}{\tau}\Big)^{\frac{1}{2}}\frac{1}{\sqrt{\lambda-\lambda_0}}\sum_{i=1}^{2}\sum_{j=1}^{\varkappa_i}\|\widetilde{f}^i_j\|_{L_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+$$

$$\frac{1}{\lambda-\lambda_0}\|\widetilde{F}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}\Big]+\frac{1}{\lambda-\lambda_0}\|v^{n+1}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}. \qquad (97)$$

Solving inequality (97) with respect to $\|v^{n+1}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}$ for $\lambda>\lambda_0+1$, i.e. for $\frac{1}{\lambda-\lambda_0}<1$, we get

$$\|v^{n+1}\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}\le C_1\Big[\sum_{i=1}^{2}\|\widetilde{f}^i\|_{W^1_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+\|\widetilde{F}^i\|_{W^1_{2,\lambda}(\Omega_{\tau_0,\tau})}\Big], \qquad (98)$$

where C_1 is the positive constant depending on the coefficients of system (1), values σ_1, σ_2, τ, τ_0, λ_0, μ_0 and parameter λ, but independent of $\widetilde{u}$, and hence of v^{n+1}, $\widetilde{f}^i$, $\widetilde{F}$.

According to our assumption above, the planes $S^0_\sigma:\sigma t-x_1=0$ for $-\sigma_2^{-1}\le\sigma\le\sigma_1^{-1}$ are non-characteristic for system (1), and this is equivalent to the condition

$$\det(A_1-y_1E)\neq 0,\quad -\sigma_2^{-1}\le y_1\le\sigma_1^{-1}. \qquad (99)$$

By (99), system (78) in $\overline{\Omega}_{\tau_0,\tau}$ can be rewritten as

$$\widetilde{u}_{y_1}=y_{n+1}(A_1-y_1E)^{-1}\Big[\widetilde{F}-E\widetilde{u}_{y_n+1}-\sum_{i=2}^{n}A_i\widetilde{u}_{y_i}-\widetilde{B}\widetilde{u}\Big],$$

whence owing to (90), (94) and (98), for $v^1=\widetilde{u}_{y_1}$ we obtain

$$\|v^1\|_{L_{2,\lambda}(\Omega_{\tau_0,\tau})}\le C_2\Big[\sum_{i=1}^{2}\|\widetilde{f}^i\|_{W^1_{2,\lambda}(\widetilde{S}_{i\tau_0,\tau})}+\|\widetilde{F}\|_{W^1_{2,\lambda}(\Omega_{\tau_0,\tau})}\Big], \qquad (100)$$

where the positive constant C_2 does not depend on $\widetilde{u}$, $\widetilde{f}^i$, $\widetilde{F}$.

Combining now inequalities (90), (94), (98) and (100) and getting back to the initial independent variables $x_1,\dots,x_n,t$, for the solution $u\in L_{2,\lambda}(D)\cap W^2_{2,\lambda}(D_{\tau_0,\tau})$ of problem (1), (2) we find the estimate

$$\|u\|_{W^1_{2,\lambda}(D_{\tau_0,\tau})}\le C_3\Big[\sum_{i=1}^{2}\|f^i\|_{W^1_{2,\lambda}(S_{i\tau_0,\tau})}+\|F\|_{W^1_{2,\lambda}(D_{\tau_0,\tau})}\Big] \qquad (101)$$

with the positive constant C_3, not depending on u and f^i, F, $S_{i\tau_0,\tau}=S_i\cap D_{\tau_0,\tau}$.

In a similar way we can prove that for the solution $u \in L_{2,\lambda}(D) \cap W^{k+1}_{2,\lambda}(D_{\tau_0,\tau})$, $k>1$, of problem (1), (2) the estimate

$$\|u\|_{W^k_{2,\lambda}(D_{\tau_0,\tau})} \le C_4\bigg[\sum_{i=1}^{2}\|f^i\|_{W^k_{2,\lambda}(S_{i\tau_0,\tau})}+\|F\|_{W^k_{2,\lambda}(D_{\tau_0,\tau})}\bigg] \tag{102}$$

is valid, where the positive constant C_4 does not depend on u, f^i, F.

When constructing the solution $u \in L_{2,\lambda}(D)$ of problem (1), (2) whose existence was stated in Theorem 2, we approximated the vector functions $\widehat{F}_\tau = \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}(F)|_{\widehat{D}_\tau}$, $\widehat{f}^j_\tau = \mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}(f^j)|_{\widehat{S}_{j\tau}}$, $j=1,2$; $\tau=\tau_0$, by the vector functions $\widehat{F}_{\tau p} \in C^\infty_0(\widehat{D}_\tau)$, $\widehat{f}^j_{\tau p} \in C^\infty_0(\widehat{S}_{j\tau}))$, $j=1,2$; $p=1,2,\dots$ respectively in the spaces $L_{2,\lambda}(\widehat{D}_\tau)$, $L_{2,\lambda}(\widehat{S}_{j\tau})$, i.e. there took place equalities (73). Now approximating vector functions $\widehat{F}_{\tau p}$ and $\widehat{f}^j_{\tau p}$ can be constructed as follows. We take the function $\chi_p(\widetilde{\xi}) \in C^\infty(R^{n-1})$ such that $0 \le \chi(\widetilde{\xi}) \le 1$, $\widetilde{\xi} \in R^{n-1}$, $\chi(\widetilde{\xi})=1$ for $|\widetilde{\xi}| \le p$ and $\chi_p(\widetilde{\xi})=0$ for $|\widetilde{\xi}| \ge p+1$. If $F_\tau \in W^1_{2,\lambda}(D_\tau)$, $f^j_\tau \in W^1_{2,\lambda}(S_{j\tau})$, $j=1,2$, then as is known [26, p. 205], there exist vector functions $F^*_{\tau p} \in C^\infty(\overline{D}_\tau) \cap W^1_{2,\lambda}(D_\tau)$, $f^{*j}_{\tau p} \in C^\infty(\overline{S}_{j\tau}) \cap W^1_{2,\lambda}(S_{j\tau})$ such that

$$\|F_{\tau p}-F^*_{\tau p}\|_{W^1_{2,\lambda}(D_\tau)} < \frac{1}{p}, \quad \|f^j_{\tau p}-f^{*j}_{\tau p}\|_{W^1_{2,\lambda}(S_{j\tau})} < \frac{1}{p}. \tag{103}$$

Assume

$$\begin{gathered}\widehat{F}_{\tau p} = \chi_p(\widetilde{\xi})\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}(F^*_{\tau p}), \quad \widehat{f}^j_{\tau p} = \chi_p(\widetilde{\xi})\mathcal{F}_{\widetilde{x}\to\widetilde{\xi}}(f^{*j}_{\tau p}), \quad j=1,2,\\ F_{\tau p} = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{F}_{\tau p}), \quad f^j_{\tau p} = \mathcal{F}^{-1}_{\widetilde{\xi}\to\widetilde{x}}(\widehat{f}^j_{\tau p}), \quad j=1,2.\end{gathered} \tag{104}$$

Note that if $F|_{D_\varepsilon}=0$, $f^j|_{S_{j\varepsilon}}=0$, $j=1,2$, the vector functions $F^*_{\tau p}$, $\widehat{F}_{\tau p}$, $f^{*j}_{\tau p}$, $\widehat{f}^j_{\tau p}$, $j=1,2$, $0<\varepsilon<\tau=\tau_0$, possess the same property, i.e.

$$F^*_{\tau p}|_{D_\varepsilon} = \widehat{F}_{\tau p}|_{\widehat{D}_\varepsilon}=0, \quad f^{*j}_{\tau p}|_{S_{j\varepsilon}} = \widehat{f}^j_{\tau p}|_{\widehat{S}_{j\varepsilon}}=0, \quad j=1,2. \tag{105}$$

By equality (71), for the norm of the space $W^1_{2,\lambda}(D_\tau)$, for properties of the function $\chi_p(\widetilde{\xi})$ and for relations (103) and (104) we, obviously, have

$$\lim_{p\to\infty}\|F_\tau - F_{\tau p}\|_{W^1_{2,\lambda}(D_\tau)}=0. \tag{106}$$

Analogously,

$$\lim_{p\to\infty}\|f^j_\tau - f^j_{\tau p}\|_{W^1_{2,\lambda}(S_{j\tau})}=0. \tag{107}$$

holds.

Owing to (105) and to our construction above, it is evident that

$$F_{\tau p} \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau), \quad f^j_{\tau p} \in \overset{\circ}{\Phi}{}^k_\alpha(S_{j\tau}), \quad j=1,2,$$

for any k and α, $p=1,2,\dots$. Therefore according to Remark 12, problem (1), (52) has the solution $u_{\tau p} \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_\tau) \subset W^2_{2,p}(D_\tau)$, $\tau=\tau_0$. Now estimate

(101) implies

$$\|u_{\tau l}-u_{\tau p}\|_{W^1_{2,\lambda}(D_{\tau_0,\tau})}\leq$$
$$\leq C_3\Bigg[\sum_{i=1}^{2}\|f^i_{\tau l}-f^i_{\tau p}\|_{W^1_{2,\lambda}(S_{i\tau_0,\tau})}+\|F_{\tau l}-F_{\tau p}\|_{W^1_{2,\lambda}(D_{\tau_0,\tau})}\Bigg],$$

from which by (106) and (107) it follows that the sequence $\{u_{\tau p}\}$ is fundamental in $W^1_{2,\lambda}(D_{\tau_0,\tau})$. Consequently, since the space $W^1_{2,\lambda}(D_{\tau_0,\tau})$ is complete, there exists the vector function $u^0_\tau\in W^1_{2,\lambda}(D_{\tau_0,\tau})$ such that $u_{\tau p}\to u^0_\tau$ as $p\to\infty$ in the space $W^1_{2,\lambda}(D_{\tau_0,\tau})$. On the other hand, by virtue of (74), as is shown while proving Theorem 2, the same sequence $\{u_{\tau p}\}$ converges to the solution $u_\tau\in L_{2,\lambda}(D_\tau)$ of problem (1), (52) in the space $L_{2,\lambda}(D_\tau)$. But u_τ is the narrowing of the solution u of problem (1), (2) of the class $L_{2,\lambda}$ to the domain D_τ. Therefore by the uniqueness of the solution of problems (1), (2) and (1), (52) of the class $L_{2,\lambda}$ we obtain

$$u|_{D_{\tau_0,\tau}}=u^\circ_\tau\in W^1_{2,\lambda}(D_{\tau_0,\tau}),$$

which was to be demonstrated.

Just in the similar manner, on the basis of estimate (102) we can prove that under additional smoothness of data of the problem $F\in W^k_{2,\lambda}(D_{\tau_0,\tau})$, $f^i\in W^k_{2,\lambda}(S_{i\tau_0})$, $i=1,2$, $0<\tau<\tau_0$, the solution u of problem (1), (2) belongs to the space $W^k_{2,\lambda}(D_{\tau_0,\tau})$.

Acknowledgement

The present work was supported by the INTAS grant No 00136.

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(Received 20.11.2002)

Author's address:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0193, Georgia
E-mail: khar@rmi.acnet.ge