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ON SOME NONLOCAL PROBLEMS FOR A HYPERBOLIC EQUATION OF SECOND ORDER ON A PLANE

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Abstract. Some nonlocal problems for a hyperbolic equation of second order with two unknown variables are formulated and studied. The conditions for the data of the problem are found, which in one cases guarantee the correctness of the problem and in another cases the existence of infinitely many linearly independent solutions of the corresponding homogeneous problem. We consider both the generalized and the classical solutions. The question when the smoothness of a solution raises together with the corresponding raise of the data of the nonlocal problem, is also considered.

რეზიუმე. მეორე რიგის ჰიპერბოლური განტოლებისათვის ორი დამოუკიდებელი ცვლადის შემთხვევაში დასმული და გამოკვლეულია ზოგიერთი არალოკალური ამოცანა. ნაპოვნია პირობები ამოცანის მონაცემებზე, რომლებიც ერთ შემთხვევაში უზრუნველყოფენ მის კორექტულობას, ხოლო სხვა შემთხვევაში–შესაბამისი ერთგვაროვანი ამოცანის წრფივად დამოუკიდებელ უსასრულო რაოდენობის ამონახსნთა არსებობას. ამასთან, განიხილება როგორც განზოგადებული, ისე კლასიკური ამონახსნები. აგრეთვე, განხილულია ამონახსნის სიგლუვის აწევის საკითხი არალოკალური ამოცანის მონაცემთა შესაბამისი სიგლუვის აწევისას.

As is known, the nonlocal problems for partial differential equations arise in mathematical modelling of some physical and biological processes. A great number of works (see, e.g., [1]–[16]) are devoted to the investigation of equations of elliptic and parabolic type. In this direction, the works [8], [17]–[22] are worth mentioning in which the equations of hyperbolic type have been studied.

Below we will formulate and investigate some nonlocal problems for a hyperbolic equation of second order with two unknown variables.

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1. Statement of the Problem and Its Reduction to the Integral Differential Equation

Consider a hyperbolic equation of second order

$$Lu := u_{tt} - u_{xx} + a_1 u_t + a_2 u_x + a_3 u = a_4 \tag{1}$$

in a characteristic quadrangle D with vertices at the points $O(0,0)$, $A(1,1)$, $B(-1,1)$ and $C(0,2)$. Here, a_i, $i = \overline{1,4}$, are the given continuous functions in D. Let $J : OA \to OC$ be a continuous mapping transforming the point $P \in OA$ into the point $J(P) \in OC$, i.e., if $P = (x,x) \in OA$, then $J(P) = (0, 2\lambda(x)) \in OC$, where $\lambda : [0,1] \to [0,1]$ is the given continuous function.

For equation (1) in the domain D we consider the nonlocal problem which is formulated as follows: find a regular in the domain D solution $u(x,t)$ of equation (1), continuous in $\overline{D}$ and satisfying the conditions

$$u(P) = \varphi(P), \quad P \in OB, \tag{2}$$

$$u(P) = \alpha(P) u\big(J(P)\big) + \beta(P), \quad P \in OA, \tag{3}$$

where φ and α, β are the given continuous functions on the segments OB and OA, respectively, satisfying the concordance condition $\varphi(0) = \alpha(0)\varphi(0) + \beta(0)$ in case $J(O) = O$.

In new variables $\xi = 2^{-1}(t+x)$, $\eta = 2^{-1}(t-x)$ the problem (1), (2), (3) in the domain $\Omega : 0 < \xi < 1$, $0 < \eta < 1$ of the plane of variables ξ, η is rewritten in the form

$$v_{\xi\eta} + a v_\xi + b v_\eta + c v = g, \tag{4}$$

$$v(0,\eta) = \varphi(\eta), \quad 0 \le \eta \le 1, \tag{5}$$

$$v(\xi,0) = \alpha(\xi) v\big(\lambda(\xi), \lambda(\xi)\big) + \beta(\xi), \quad 0 \le \xi \le 1, \tag{6}$$

where $v(\xi,\eta) := u(\xi-\eta, \xi+\eta)$, $g(\xi,\eta) := a_4(\xi-\eta,\xi+\eta)$, $a = \frac{1}{2}(a_1 + a_2)$, $b = \frac{1}{2}(a_1 - a_2)$, $c = a_3$. As is mentioned above, for $J(O) = O$, i.e., for $\lambda(0) = 0$, the concordance condition $\varphi(0) = \alpha(0)\varphi(0) + \beta(0)$ is assumed to be fulfilled.

As is known, under the assumption that a_ξ, $b_\eta \in C(\overline{\Omega})$ any solution $v(\xi,\eta)$ of equation (4) of the class $C(\overline{\Omega}) \cap C^2(\Omega)$ can be represented in the form ([23, p. 172])

$$v(\xi,\eta) = R(\xi,0;\xi,\eta)v(\xi,0) + R(0,\eta;\xi,\eta)v(0,\eta) - R(0,0;\xi,\eta)v(0,0) +$$
$$+ \int_0^\xi \Big[b(\sigma,0)R(\sigma,0;\xi,\xi) - \frac{\partial R(\sigma,0;\xi,\eta)}{\partial\sigma}\Big] v(\sigma,0)\, d\sigma +$$
$$+ \int_0^\eta \Big[a(0,\tau)R(0,\tau;\xi,\eta) - \frac{\partial R(0,\tau;\xi,\eta)}{\partial\tau}\Big] v(0,\tau)\, d\tau +$$

$$+\int_0^{\xi} d\sigma \int_0^{\eta} R(\sigma,\tau;\xi,\eta)g(\sigma,\tau)\,d\tau, \tag{7}$$

where $R(\xi_1,\eta_1;\xi,\eta)$ is the Riemann function for equation (4). Below, in case of need, the coefficients a_1, a_2 of equation (1) will be assumed to belong to the class $C^1(\overline{D})$.

Substituting (7) in (6) and taking into account (5), we obtain with respect to the unknown function $\psi(\xi) = v(\xi,0)$ the following integro-functional equation:

$$\psi(\xi) - \alpha_0(\xi)\psi\big(\lambda(\xi)\big) = \int_0^{\lambda(\xi)} K(\xi,\sigma)\psi(\sigma)\,d\sigma + f(\xi), \quad 0 \le \xi \le 1. \tag{8}$$

Here,

$$\alpha_0(\xi) = \alpha(\xi)R\big(\lambda(\xi),0;\lambda(\xi),\lambda(\xi)\big), \tag{9}$$

$$K(\xi,\sigma) = \alpha(\xi)\Big[b(\sigma,0)R\big(\sigma,0;\lambda(\xi),\lambda(\xi)\big) - \frac{\partial R(\sigma,0;\lambda(\xi),\lambda(\xi))}{\partial\sigma}\Big],$$

$$f(\xi) = \alpha(\xi)\Big[R\big(0,\lambda(\xi);\lambda(\xi),\lambda(\xi)\big)\varphi\big(\lambda(\xi)\big) - R\big(0,0;\lambda(\xi),\lambda(\xi)\big)\varphi(0)+$$

$$+\int_0^{\lambda(\xi)} \Big[a(0,\tau)R\big(0,\tau;\lambda(\xi),\lambda(\xi)\big) - \frac{\partial R(0,\tau;\lambda(\xi),\lambda(\xi))}{\partial\tau}\Big]\varphi(\tau)d\tau+$$

$$+\int_0^{\lambda(\xi)} d\sigma \int_0^{\lambda(\xi)} R\big(\sigma,\tau;\lambda(\xi),\lambda(\xi)\big)g(\sigma,\tau)\,d\tau + \beta(\xi). \tag{10}$$

Remark 1. Below, we will first restrict ourselves to the investigation of the problem (1), (2), (3) in a class of generalized solutions $u(x,t)$ of the class $C(\overline{D})$, i.e., when $u \in C(\overline{D})$ and there exists a sequence of functions $u_n \in C^2(\overline{D})$ such that $u_n \to u$ and $Lu_n \to a_4$ as $n \to \infty$ in the norm of the space $C(\overline{D})$. In this case, it is obvious that $(u, L^*\omega)_{L_2(D)} = (a_4,\omega)_{L_2(D)} \forall\, \omega \in C_0^{\infty}(D)$, $L^*\omega := \omega_{tt} - \omega_{xx} - (a_1\omega)_t - (a_2\omega)_x + a_3\omega$. Note that representation (7) holds likewise for generalized solutions of equation (1) of the class $C(\overline{D})$ ([22]). In this case, the problem (1), (2), (3) is equivalent to the integro-functional equation (8) in the class $C([0,1])$.

2. Investigation of the Integro-Functional Equation (8) and the Theorems on the Solvability of the Problem (1), (2), (3)

2.1. In this section we will consider the functional part of equation (8), i.e., of the equation

$$T\psi(\xi) := \psi(\xi) - \alpha_0(\xi)\psi\big(\lambda(\xi)\big) = f(\xi), \quad 0 \le \xi \le 1, \tag{11}$$

where $\alpha_0(\xi)$ and $f(\xi)$ are the given continuous functions, and $\alpha_0(\xi)$ is given by equality (9).

Many works are devoted to the investigation of equations of type (11) in different spaces of functions. The detailed bibliography concerning this problem the reader can find in the monograph [24]. Below, we will cite some results dealing with the solvability of equation (11) in the space of continuous functions.

Lemma 1. *If the condition*

$$\big|\alpha(\xi)\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau, \quad 0 \le \xi \le 1, \tag{12}$$

is fulfilled, then equation (11) *is uniquely solvable in the class* $C([0,1])$, *i.e., for any function* $f \in C([0,1])$ *equation* (11) *has a unique solution* $\psi \in C([0,1])$, *and the estimate*

$$\big\|\psi\big\|_{C([0,1])} \le \frac{1}{1-q}\big\|f\big\|_{C([0,1])}, \tag{13}$$

where $q = \max\limits_{0\le\xi\le1} |\alpha(\xi)| \exp[-\int\limits_0^{\lambda(\xi)} a(\lambda(\xi),\tau)\,d\tau] < 1$, *holds.*

Proof. Since ([23], p. 170)

$$R\big(\lambda(\xi),0;\lambda(\xi),\lambda(\xi)\big) = \exp\bigg[-\int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau\bigg],$$

by virtue of (9), the condition (12) is equivalent to the condition

$$\big|\alpha_0(\xi)\big| < 1, \quad 0 \le \xi \le 1. \tag{14}$$

When fulfilling inequality (14), equation (11) is uniquely solvable in the class $C([0,1])$ by the contraction mapping principle and this solution is representable in the form of the Neumann's series ([25], p. 211)

$$\psi = \Big[\sum_{i=0}^{\infty} T_0^i\Big] f. \tag{15}$$

Here, the operator $T_0 : C([0,1]) \to C([0,1])$ acts by the formula

$$T_0 f(\xi) = \alpha_0(\xi) f\big(\lambda(\xi)\big), \quad 0 \le \xi \le 1, \tag{16}$$

where $T_0^0 = Id$ is the unit operator.

It follows from (14) and (16) that

$$\big\|T_0\big\|_{C([0,1])\to C([0,1])} \le \max_{0\le\xi\le1} \big|\alpha_0(\xi)\big| = q < 1. \tag{17}$$

Now, by (15) and (17), for the norm of the operator $T^{-1}: C([0,1]) \to C([0,1])$, inverse to T, from (11) we have

$$\big\|T^{-1}\big\|_{C([0,1])\to C([0,1])} \le \sum_{i=0}^{\infty} q^i = \frac{1}{1-q}. \tag{18}$$

Estimate (13) follows directly from (18). □

Remark 2. The condition (12), ensuring the unique solvability of equation (11), can be appreciably weakened depending on specific properties of the continuous mapping $\lambda : [0,1] \to [0,1]$.

Lemma 2. *Let the continuous mapping $\lambda : [0,1] \to [0,1]$ be strictly monotonically increasing. Denote by $I_1 := \{\xi \in [0,1] : \lambda(\xi) = \xi\}$ a set of fixed points of the mapping λ. If the condition*

$$\big|\alpha(\xi)\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau \quad \forall\, \xi \in I_1 \tag{19}$$

is fulfilled, then equation (11) for any function $f \in C([0,1])$ has the unique solution $\psi \in C([0,1])$ for which the estimate

$$\big\|\psi\big\|_{C([0,1])} \le c\big\|f\big\|_{C([0,1])} \tag{20}$$

is valid. Here the positive constant c does not depend on f, i.e.,

$$\big\|T^{-1}\big\|_{C([0,1])\to C([0,1])} \le c.$$

Proof. Since the mapping λ is continuous, the set I_1 is closed and, moreover, nonempty by the Bauer's theorem on a fixed point ([25], p.613). If $I_1 = [0,1]$, then the condition (19) coincides with the condition (12), and hence Lemma 2 is reduced to the already proven Lemma 1. Let now $I_1 \ne [0,1]$. Then the open set $(0,1)\setminus I_1$ consists of a finite or countable union of non-intersecting connected open intervals $J_1, J_2, \dots, J_n, \dots$ ([26], p.48). Note that since the function $\lambda = \lambda(\xi)$ is strictly monotonically increasing, every of those intervals $J_k = (\tau_k, \sigma_k)$ is invariant with respect to the mapping λ, i.e., $\lambda : J_k \to J_k$, $k = 1, \dots, n, \dots$. Moreover, their ends are the fixed points of the mapping λ, i.e., $\lambda(\tau_k) = \tau_k$, $\lambda(\sigma_k) = \sigma_k$, except may be the cases when $\tau_k = 0$, or $\sigma_k = 1$, and either

$$\lambda(\xi) < \xi \quad \forall\, \xi \in J_k \tag{21}$$

or

$$\lambda(\xi) > \xi \quad \forall\, \xi \in J_k. \tag{22}$$

We will restrict ourselves to the consideration of case (21), when $\lambda(\tau_k) = \tau_k$, $\lambda(\sigma_k) = \sigma_k$. Let us show that the Neumann series (15) provides the unique continuous solution of equation (11) on a closed invariant segment $\overline{J}_k = [\tau_k, \sigma_k]$. As far as the series (15) is formally a solution of equation

(11), for its solvability on $\overline{J}_k$ it is sufficient to prove that the series (15) converges uniformly on the segment $\overline{J}_k$. Since τ_k, $\sigma_k \in I_1$ the function $\lambda = \lambda(\xi)$ is continuous on $\overline{J}_k$, and the condition (19) is, by virtue of (9), equivalent to the condition

$$\big|\alpha_0(\xi)\big| < 1 \quad \forall\, \xi \in I_1, \tag{23}$$

therefore for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that $\delta(\varepsilon) < \frac{1}{2}(\sigma_k - \tau_k)$ and

$$\big|\alpha_0(\xi)\big| < q \quad \forall\, \xi \in \big[\tau_k, \tau_k + \delta\big] \cup \big[\sigma_k - \delta, \delta_k\big], \quad 0 < q = q(\varepsilon, \delta) < 1. \tag{24}$$

By (21), we can easily verify that

$$\lim_{n\to\infty} \lambda_n(\xi) = \tau_k \quad \forall\, \xi \in J_k, \tag{25}$$

where $\lambda_n(\xi) = \lambda(\lambda_{n-1}(\xi))$, $\lambda_0(\xi) = \xi$, $n = 1, 2, \dots$. Indeed, since $\lambda : J_k \to J_k$ and there takes place inequality (21), the sequence $\{\lambda_n(\xi)\}_{n=1}^{\infty}$ is monotonically decreasing, bounded below by the number τ_k. Therefore this sequence has the limit $\lim\limits_{n\to\infty} \lambda_n(\xi) = \widetilde{\lambda} \geq \tau_k$. Should $\widetilde{\lambda} > \tau_k$, by the continuity of the function $\lambda = \lambda(\xi)$ we would obtain $\lambda(\widetilde{\lambda}) = \lim\limits_{n\to\infty} \lambda(\lambda_n(\xi)) = \lim\limits_{n\to\infty} \lambda_{n+1}(\xi) = \widetilde{\lambda}$, which contradicts inequality (21) for $\xi = \widetilde{\lambda} \in J_k$.

Next, by virtue of (21) and (25), there exists a natural number m such that

$$\tau_k < \lambda_m(\sigma_k - \delta) < \tau_k + \delta. \tag{26}$$

Taking into account equality (16), it can be easily seen that

$$T_0^i f(\xi) = \alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big) \cdots \alpha_0\big(\lambda_{i-1}(\xi)\big) f\big(\lambda_i(\xi)\big), \quad i \geq 1. \tag{27}$$

Furthermore, by (21), (25) and (26) we note that for any $\xi \in \overline{J}_k$ a number of points from the set $\{\lambda_n(\xi)\}_{n=0}^{\infty}$, belonging to the interval $(\lambda_m(\sigma_k - \delta), \sigma_k - \delta)$, does not exceed m. In its turn, for $i > m$, by (24) and (27), we have

$$\big|T_0^i f(\xi)\big| \leq M^m q^{i-m} \big|f(\lambda_i(\xi))\big| \quad \forall\, \xi \in \overline{J}_k, \quad i > m, \tag{28}$$

where $M = \max\limits_{0\leq\xi\leq 1} |\alpha_0(\xi)|$.

Inequality (28) implies that

$$\big\|T_0^i\big\|_{C([0,1])\to C([0,1])} \leq M^m q^{i-m}, \quad i > m. \tag{29}$$

Owing to (29), for $n > m$ we have

$$\Big\|\sum_{i\geq n} T_0^i\Big\|_{C([0,1])\to C([0,1])} \leq \sum_{i\geq n} M^m q^{i-m} = \frac{M^m}{1-q} q^{n-m}. \tag{30}$$

From (30) follows uniform convergence of the Neumann series (15) on the segment $\overline{J}_k$. This, in its turn, means that the unique continuous solution of equation (11) on $\overline{J}_k$ is representable by formula (15). As for the unique solvability of equation (11) in a class of continuous functions on a closed set

I_1 of fixed points of the mapping $\lambda : [0,1] \to [0,1]$, the latter indeed takes place because by virtue of (23) and the equality $\lambda(\xi) = \xi$ for $\xi \in I_1$, the solution of equation (11) on I_1 is representable by the formula

$$\psi(\xi) = \frac{f(\xi)}{1 - \alpha_0(\xi)}, \quad \xi \in I_1. \tag{31}$$

The fact that formula (15) on $\overline{J}_k$, $k = 1, 2, \dots$ and formula (31) on I_1 provide one and the same value at common points of the segment $[0,1]$, follows from the uniqueness of values of the solution ψ of the same equation (11) at those points, calculated by formulas (15) and (31).

Since the operator $T : C([0,1]) \to C([0,1])$ from (11) is continuous and, as above, has its inverse T^{-1}, by the Banach theorem on the open mapping ([25], p. 453), the operator $T^{-1} : C([0,1]) \to C([0,1])$ is likewise continuous and, hence, the estimate (20) is valid for the solution of equation (11).

The case (22) is considered analogously. □

Lemma 3. *Let the continuous mapping* $\lambda : [0,1] \to [0,1]$ *be strictly monotonically increasing and*

$$\lambda(0) = 0, \quad \lambda(\xi) < \xi \quad \textit{for} \quad 0 < \xi \le 1. \tag{32}$$

If

$$\big|\alpha(0)\big| < 1, \tag{33}$$

then for any function $f \in C([0,1])$ *equation* (11) *has the unique solution* $\psi \in C([0,1])$ *for which the estimate* (20) *is valid. If, however,* $|\alpha(0)| > 1$, *then equation* (11) *for any function* $f \in C([0,1])$ *has the solution* $\psi \in C([0,1])$, *although the homogeneous equation, corresponding to* (11), *has an infinite set of linearly independent solution in a class of continuous functions* $C([0,1])$.

Proof. Taking into account that in fulfilling the condition (32) the set $I_1 = \{0\}$ and inequality (33) is equivalent to (19), the first part of the statement of Lemma 3, i.e., the unique solvability of equation (11) and the estimate (20) in the class $C([0,1])$, follows from Lemma 2.

Let now the inequality

$$\big|\alpha_0(0)\big| = \big|\alpha(0)\big| > 1 \tag{34}$$

be fulfilled.

Without restriction of generality, we may assume

$$f(0) = 0, \tag{35}$$

since, otherwise, instead of ψ we would consider the function $\widetilde{\psi} = \psi - \frac{f(0)}{1-\alpha_0(0)}$.

Fix a sufficiently small number $\varepsilon > 0$. By the condition (32), for any $\xi \in (0, \lambda(\varepsilon))$ there exists the unique natural number $n_1 = n_1(\xi)$ such that

$$\lambda(\varepsilon) < \lambda_{-n_1}(\xi) \le \varepsilon, \tag{36}$$

where $\lambda_{-1}(\xi)$ is the function, inverse to the strictly monotone function $\lambda(\xi)$, $\lambda_{-n}(\xi) = \lambda_{-1}(\lambda_{-(n-1)}(\xi))$, $\lambda_0(\xi) = \xi$, $n \ge 1$. Similarly, for any $\xi \in (\varepsilon, 1]$ there exists the unique natural number $n_2 = n_2(\xi)$ such that

$$\lambda(\varepsilon) \le \lambda_{n_2}(\xi) < \varepsilon, \tag{37}$$

where $\lambda_n(\xi) = \lambda(\lambda_{n-1}(\xi))$, $\lambda_0(\xi) = \xi$, $n \ge 1$.

We can easily verify that every continuous on the set $(0, 1]$ solution ψ of equation (11) is, with regard for (36) and (37), representable by the formula ([27], p. 18)

$$\psi(\xi) = \begin{cases} \psi_0(\xi), & \lambda(\varepsilon) \le \xi \le \varepsilon, \\ \big(T_0^{-n_1(\xi)}\psi_0\big)(\xi) - \sum\limits_{i=1}^{n_1(\xi)} (T_0^{-i} f)(\xi), & 0 < \xi < \lambda(\varepsilon), \\ \big(T_0^{n_2(\xi)}\psi_0\big)(\xi) + \sum\limits_{i=0}^{n_2(\xi)-1} (T_0^{i} f)(\xi), & \xi > \varepsilon, \end{cases} \tag{38}$$

where $\psi_0(\xi)$ is an arbitrary function of the class $C([\lambda(\varepsilon), \varepsilon])$, satisfying the condition $\psi_0(\varepsilon) - \alpha_0(\varepsilon)\psi_0(\lambda(\varepsilon)) = f(\varepsilon)$; $(T_0\psi)(\xi) = \alpha_0(\xi)\psi(\lambda(\xi))$ and, respectively, $(T_0^{-1}\psi)(\xi) = \alpha_0^{-1}(\lambda_{-1}(\xi))\psi(\lambda_{-1}(\xi))$.

Let us prove that the function ψ, given by formula (38) on the set $(0, 1]$ and predetermined by zero at the point $\xi = 0$, belongs to the class $C([0, 1])$. Thus it is sufficient to show that

$$\lim_{\xi \to 0+} \psi(\xi) = 0. \tag{39}$$

Towards this end, we take in formula (38) the number $\varepsilon > 0$ so small that the inequality

$$\big|\alpha_0^{-1}(\xi)\big| < q = \text{const} < 1, \quad 0 \le \xi \le \varepsilon, \tag{40}$$

by virtue of (34) to be valid.

Inequality (40) implies

$$\big|(T_0^{-i}\psi)(\xi)\big| \le q^i \big\|\psi\big\|_{C([0,1])} \quad \text{for} \quad i \le n_1(\xi). \tag{41}$$

By the continuity of the function f and equality (35), for any $\delta > 0$ there exists the number $\delta_1 = \delta_1(\delta) > 0$, $\delta_1 < \lambda(\varepsilon)$, such that

$$\big|f(\xi)\big| < \delta, \quad 0 \le \xi \le \delta_1. \tag{42}$$

Take now a natural number k, which will be chosen below, and introduce into consideration a number

$$\delta_2 = \lambda_k(\delta_1) < \lambda(\varepsilon). \tag{43}$$

Obviously, $0 < \delta_2 < \delta_1 < \varepsilon$.

From (36) and (43), we have

$$n_1(\xi) > k \quad \text{for} \quad 0 < \xi < \delta_2. \tag{44}$$

From (38) for $0 < \xi < \delta_2$ and by (40)–(44), we now obtain

$$\begin{aligned}
\big|\psi(\xi)\big| &= \Big|\big(T_0^{-n_1(\xi)}\psi_0\big)(\xi) - \sum_{i=1}^{n_1(\xi)} \big(T_0^{-i}f\big)(\xi)\Big| \le \\
&\le \big|\big(T_0^{-n_1(\xi)}\psi_0\big)(\xi)\big| + \sum_{i=1}^{n_1(\xi)} \big|(T_0^{-i}f)(\xi)\big| \le \\
&\le q^{n_1(\xi)}\big\|\psi_0\big\|_{C([\lambda(\varepsilon),\varepsilon])} + \sum_{i=1}^{k} \big|(T_0^{-i}f)(\xi)\big| + \sum_{i=k+1}^{n_1} \big|(T_0^{-i}f)(\xi)\big| \le \\
&\le q^k\big\|\psi_0\big\|_{C([\lambda(\varepsilon),\varepsilon])} + \frac{1-q^{k+1}}{1-q}\big\|f\big\|_{C([0,\delta_1])} + q^{k+1}\frac{1-q^{n_1-k}}{1-q}\big\|f\big\|_{C([0,\varepsilon])} \le \\
&\le q^k\big\|\psi_0\big\|_{C([\lambda(\varepsilon),\varepsilon])} + \frac{1}{1-q}\delta + \frac{q^{k+1}}{1-q}\big\|f\big\|_{C(0,\varepsilon])} \le \\
&\le \frac{\delta}{1-q} + q^k\Big(\big\|\psi_0\big\|_{C([\lambda(\varepsilon),\varepsilon])} + \frac{q}{1-q}\|f\|_{C([0,\varepsilon])}\Big).
\end{aligned} \tag{45}$$

By virtue of (40), it follows from (45) that for an arbitrarily small number $\delta_0 > 0$ there exist positive numbers $\delta = \delta(\delta_0)$ and $k = k(\delta_0)$ such that

$$\big|\psi(\xi)\big| < \delta_0, \quad 0 < \xi < \delta_2 = \lambda_k(\delta_1),$$

from which we also obtain (39). All this proves the second part of lemma 3 because the function $\psi_0 \in C([\lambda(\varepsilon),\varepsilon])$ in formula (38) can be arbitrary and satisfying the condition $\psi_0(\varepsilon) - \alpha_0(\varepsilon)\psi_0(\lambda(\varepsilon)) = f(\varepsilon)$. □

Remark 3. It should be noted that the particular case of Lemma 3, when the coefficient α_0 in equation (11) is constant, has been considered in [28]. The analogue of that lemma in a class of continuous functions with power growth in zero can be found in [27].

In the case in which the mapping λ realizes continuous homeomorphism of the segment $[0,1]$ into itself, the condition (12) in Lemma 1 can be weakened. Indeed, we have

Lemma 4. *If* $\lambda : [0,1] \to [0,1]$ *is a continuous homeomorphism of the segment* $[0,1]$ *into itself, and the condition*

$$\big|\alpha(\xi)\big| \ne \exp \int_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau, \quad 0 \le \xi \le 1 \tag{46}$$

is fulfilled, then equation (11) *is uniquely solvable in the class* $C([0,1])$.

Proof. By the continuity of the functions α, λ and a, it follows from (46) that either the condition (12) is fulfilled and then equation (11) is, by Lemma 1, uniquely solvable in the class $C([0,1])$, or there takes place the inequality

$$\big|\alpha(\xi)\big| > \exp\int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau, \quad 0\le\xi\le 1. \tag{47}$$

Rewrite equation (11) as follows:

$$\psi(\xi)-\alpha_0^{-1}\big(\lambda_{-1}(\xi)\big)\psi\big(\lambda_{-1}(\xi)\big) = -\alpha_0^{-1}\big(\lambda_{-1}(\xi)\big)f\big(\lambda_{-1}(\xi)\big), \quad 0\le\xi\le 1, \tag{48}$$

where $\lambda_{-1}:[0,1]\to[0,1]$ is the continuous, inverse to λ, mapping, which exists by the condition of Lemma 4. With regard for inequality (47), we have

$$\big|\alpha_0^{-1}\big(\lambda_{-1}(\xi)\big)\big| < 1, \quad 0\le\xi\le 1.$$

Therefore, just as in proving Lemma 1, owing to the principle of contracted mappings, equation (48) and, hence, equation (11) is uniquely solvable in the class $C([0,1])$. □

Lemma 5. *Let $\lambda:[0,1]\to[0,1]$ be the continuous homeomorphism of the segment $[0,1]$ into itself, leaving the ends of that segment fixed. If the condition*

$$\big|\alpha(\xi)\big| > \exp\int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau \quad \forall\,\xi\in I_1$$

is fulfilled, then for any function $f\in C([0,1])$ equation (11) has the unique solution $\psi\in C([0,1])$ for which estimate (20) is valid.

The same reasoning as in proving Lemmas 3 and 4 allows us to prove the following

Lemma 6. *Let the continuous mapping $\lambda:[0,1]\to[0,1]$ be strictly monotonically increasing, $\lambda(0)=0$ and $\lambda(\xi_0)=\xi_0$ for some $\xi_0\in(0,1)$, and $\lambda(\xi)<\xi$ for $\xi_0<\xi\le 1$. Let the condition*

$$\big|\alpha(\xi)\big| \neq \exp\int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau, \quad 0\le\xi\le\xi_0$$

be fulfilled. If

$$\big|\alpha(\xi_0)\big| < \exp\int\limits_0^{\lambda(\xi_0)} a\big(\lambda(\xi_0),\tau\big)\,d\tau,$$

then for any function $f \in C([0,1])$ *equation* (11) *has the unique solution* $\psi \in C([0,1])$ *for which estimate* (20) *is valid. If, however,*

$$\big|\alpha(\xi_0)\big| > \exp \int\limits_0^{\lambda(\xi_0)} a\big(\lambda(\xi_0), \tau\big)\, d\tau,$$

then equation (11) *for any function* $f \in C([0,1])$ *has the solution* $\psi \in C([0,1])$, *although the homogeneous equation, corresponding to* (11), *has an infinite set of linearly independent solutions in the class of continuous functions* $C([0,1])$.

Remark 4. Below, for the solvability of equation (11) we will give the conditions of somewhat different nature than those appearing in Lemmas 1–6.

Introduce into consideration the set $\Lambda_n = \{\xi \in [0,1] : \lambda_k(\xi) \neq \xi, k = 1, \dots, n-1, \lambda_n(\xi) = \xi\}$ for $n = 2, 3, \dots$, where, as above, $\lambda_m(\xi) = \lambda(\lambda_{m-1}(\xi))$, $\lambda_0(\xi) = \xi$, $m > 1$. Note that the set Λ_n may be empty, e.g., for $\lambda(\xi) \equiv \xi$, $n \geq 2$. In case $n = 1$, we put $\Lambda_1 = I_1$. For $\lambda(\xi) = 1 - \xi$, it is obvious that $\Lambda_1 = I_1 = \{\frac{1}{2}\}$, $\Lambda_2 = [0,1] \setminus \{\frac{1}{2}\}$, $\Lambda_n = \varnothing$ for $n > 2$.

Lemma 7. *The necessary conditions for the solvability of equation* (11) *in the class* $C([0,1])$ *for any function* $f \in C([0,1])$ *is the fulfilment of the following inequalities:*

$$\alpha_0(\xi) \neq 1 \quad \forall\, \xi \in \Lambda_1, \tag{49}$$
$$\alpha_0(\xi)\alpha\big(\lambda(\xi)\big) \cdots \alpha_0\big(\lambda_{n-1}(\xi)\big) \neq 1 \;\; \forall\, \xi \in \Lambda_n, \;\; n = 2, 3, \dots.$$

Proof. Take an arbitrary function $f \in C([0,1])$. By the assumption on the solvability of equation (11), for an arbitrary solution $\psi \in C([0,1])$ of that equation the equality

$$\psi(\xi) - \alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big) \cdots \alpha_0\big(\lambda_{n-1}(\xi)\big)\psi\big(\lambda_n(\xi)\big) = \\ = \sum_{i=0}^{n-1} T_0^i f(\xi), \quad 0 \leq \xi \leq 1, \tag{50}$$

is valid, where the operator $T_0 : C([0,1]) \to C([0,1])$ acts by formula (16). For $\xi \in \Lambda_n$, from (50) we obtain

$$\big[1 - \alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big) \cdots \alpha_0\big(\lambda_{n-1}(\xi)\big)\big]\psi(\xi) = \sum_{i=0}^{n-1} T_0^i f(\xi) \quad \forall\, \xi \in \Lambda_n. \tag{51}$$

Assume that for some $\xi \in \Lambda_n$ the condition (49) is violated, i.e.,

$$1 - \alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big) \cdots \alpha_0\big(\lambda_{n-1}(\xi)\big) = 0. \tag{52}$$

Since for $\xi \in \Lambda_n$ the points ξ, $\lambda(\xi), \dots, \lambda_{n-1}(\xi)$ are different, there exists the continuous function $f \in C([0,1])$ such that $f(\xi) = 1$, $f(\lambda_k(\xi)) = 0$, $k = 1, \dots, n-1$, and hence

$$\sum_{i=0}^{n-1} T_0^i f(\xi) = f(\xi) + \alpha_0(\xi) f\big(\lambda(\xi)\big) + \cdots +$$
$$+ \alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big)\cdots\alpha_0\big(\lambda_{n-2}(\xi)\big) f\big(\lambda_{n-1}(\xi)\big) = 1. \tag{53}$$

But then from (51)–(53) we would get that $0 \cdot \psi(\xi) = 1$, which is impossible. □

Lemma 8. *Let $\Lambda_n = \varnothing$ for $n > n_0$. Assume $k_n = n$ for $\Lambda_n \neq \varnothing$, and $k_n = 1$ for $\Lambda_n = \varnothing$. Let m be the least common multiple of numbers $k_1, k_2, \dots, k_{n_0}$. Put $E_{n_0} = [0,1] \setminus \bigcup\limits_{k=1}^{n_0} \Lambda_k$. Assume that*

$$\alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big)\cdots\alpha_0\big(\lambda_{m-1}(\xi)\big) \neq 1 \quad \forall\, \xi \in \bigcup_{k=1}^{n_0} \Lambda_k, \tag{54}$$

$$\big|\alpha(\xi)\big| < \exp \int_0^{\lambda(\xi)} a\big(\lambda(\xi), \tau\big)\, d\tau \quad \forall\, \xi \in \overline{E}_{n_0}, \tag{55}$$

where $\overline{E}_{n_0}$ is closure of set E_{n_0}. Then for any function $f \in C([0,1])$ equation (11) *has the unique solution $\psi \in C([0,1])$ for which estimate* (20) *is valid.*

Proof. By the assumption, $[0,1] = \bigcup\limits_{k=1}^{n_0} \Lambda_k \cup E_{n_0}$, and the sets $\bigcup\limits_{k=1}^{n_0} \Lambda_k$ and E_{n_0}, and hence the set $\overline{E}_{n_0}$, are invariant with respect to the mapping $\lambda : [0,1] \to [0,1]$. Indeed, if, for example, the number $\xi \in \Lambda_k$, then by means of iteration we obtain the following sequence of numbers

$$\lambda(\xi), \lambda_2(\xi), \dots, \lambda_{k-1}(\xi), \lambda_k(\xi) = \xi, \lambda_{k+1}(\xi) = \lambda(\xi),$$

i.e., $\lambda(\xi) \in \Lambda_k$, and hence $\lambda(\Lambda_k) = \Lambda_k$, $k = 1, \dots, n_0$. Let now $\xi \in E_{n_0}$. Since $E_{n_0} = [0,1] \setminus \bigcup\limits_{k=1}^{n_0} \Lambda_k$ and $\Lambda_n = \varnothing$ for $n > n_0$, it is obvious that $\lambda(\xi) \in E_{n_0}$, as well. Therefore $\lambda(E_{n_0}) \subset E_{n_0}$, and hence by the continuity of the mapping λ, $\lambda(\overline{E}_{n_0}) \subset \overline{E}_{n_0}$ likewise.

Since the number m is divided by k_i, therefore $\Lambda_{k_i} \subset \Lambda_m^* = \{\xi \in [0,1] : \lambda_m(\xi) = \xi\}$, $1 \leq i \leq n_0$, and hence $\bigcup\limits_{k=1}^{n_0} \Lambda_k \subset \Lambda_m^*$. Thus by equality (50) for $n = m$ and by (54), equation (11) is uniquely solvable on the invariant (with respect to the mapping λ) set $\bigcup\limits_{k=1}^{n_0} \Lambda_k$, and this solution is given by the formula

$$\psi(\xi) = \big[1 - \alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big)\cdots\alpha_0\big(\lambda_{m-1}(\xi)\big)\big]^{-1} \sum_{i=0}^{m-1} T_0^i f(\xi), \qquad \forall\, \xi \in \bigcup_{k=1}^{n_0} \Lambda_k. \tag{56}$$

Inequality (55), which is valid on the compact $\overline{E}_{n_0}$, results in the inequality

$$\max_{\xi\in\overline{E}_{n_0}} \big|\alpha_0(\xi)\big| = q_0 < 1.$$

Therefore, by the contraction mapping principle equation (11) is uniquely solvable on the invariant (with respect to the mapping λ) set $\overline{E}_{n_0}$, and just as in proving Lemma 1, this solution is representable by the Neumann series (15). It can be easily seen that for $f \in C([0,1])$ this solution, defined by formula (56) for $\xi \in \bigcup\limits_{k=1}^{n_0} \Lambda_k$ and by the Neumann series (15) for $\xi \in E_{n_0}$, belongs to the class $C([0,1])$. The validity of estimate (20) follows, just as in proving Lemma 2, from the Banach theorem on the inverse mapping. □

2.2. In this section we will consider the question on the solvability of equation (8) and hence of the problem (1), (2), (3), due to their equivalence.

Remark 5. We rewrite equation (8) in terms of the operator

$$T\psi - K_0\psi = f, \quad f \in C([0,1]), \tag{57}$$

where the operator $T : C([0,1]) \to C([0,1])$ acts by formula (11), while the operator $K_0 : C([0,1]) \to C([0,1])$ acts by the formula

$$K_0\psi(\xi) = \int\limits_0^{\lambda(\xi)} K(\xi,\sigma)\psi(\sigma)\, d\sigma.$$

Since the functions $K(\xi,\sigma)$ and $\lambda(\xi)$ are continuous, the operator $K_0 : C([0,1]) \to C([0,1])$ is entirely continuous ([29], p. 225). Therefore it directly follows from the reversibility of the operator T that equation (57), and hence the problem (1), (2), (3) in the class of continuous functions, possess the Fredholm property. Consequently, due to the above-proven lemmas on the reversibility of the operator T, the following theorem is valid.

Theorem 1. *The problem* (1), (2), (3) *is fredholmian in the class of generalized solutions $u(x,t)$ of equation* (1) *of the class $C(\overline{D})$, if at least one of the following conditions is fulfilled:*

1. $\big|\alpha(\xi)\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\, d\tau$, $0 \le \xi \le 1$*;*

2. *The continuous mapping* $\lambda : [0,1] \to [0,1]$ *is strictly monotonically increasing, and inequality*

$$\big|\alpha(\xi)\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau \ \ \forall\, \xi \in I_1 := \big\{\xi \in [0,1] : \lambda(\xi) = \xi\big\}$$

is fulfilled;

3. *The continuous mapping* $\lambda : [0,1] \to [0,1]$ *is strictly monotonically increasing, and the conditions*

$$\lambda(0) = 0, \quad \lambda(\xi) < \xi \ \ \textit{for} \ \ 0 < \xi \leq 1, \ \ \big|\alpha(0)\big| < 1$$

are fulfilled;

4. *The mapping* $\lambda : [0,1] \to [0,1]$ *is a continuous homeomorphism of the segment* $[0,1]$ *into itself, and*

$$\big|\alpha(\xi)\big| \neq \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau, \ \ 0 \leq \xi \leq 1$$

holds.

5. *The mapping* $\lambda : [0,1] \to [0,1]$ *is a continuous homeomorphism of the segment* $[0,1]$ *into itself, leaving the ends of that segment fixed, and the inequality*

$$\big|\alpha(\xi)\big| > \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau \ \ \forall\, \xi \in I_1$$

is fulfilled.

6. *The continuous mapping* $\lambda : [0,1] \to [0,1]$ *is strictly monotonically increasing,* $\lambda(0) = 0$ *and* $\lambda(\xi_0) = \xi_0$ *for some number* $\xi_0 \in (0,1)$*, where* $\lambda(\xi) < \xi$ *if* $\xi_0 < \xi \leq 1$ *and*

$$\big|\alpha(\xi)\big| \neq \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau, \ \ 0 \leq \xi \leq \xi_0,$$

$$\big|\alpha(\xi_0)\big| < \exp \int\limits_0^{\lambda(\xi_0)} a\big(\lambda(\xi_0),\tau\big)\,d\tau;$$

7. *Let* $\Lambda_n = \{\xi \in [0,1] : \lambda_k(\xi) \neq \xi,\ k = 1,\dots,n-1, \lambda_n(\xi) = \xi\} = \varnothing$ *for* $n > n_0$*. Denote by* m *the least common multiple of numbers* $k_1, k_2, \dots, k_{n_0}$*, where* $k_i = i$ *and* $k_i = 1$ *for* $\Lambda_i \neq \varnothing$ *and* $\Lambda_i = \varnothing$*, respectively. Let* $E_{n_0} =$

$[0,1] \setminus \bigcup\limits_{k=1}^{n_0} \Lambda_k$, *and let the conditions*

$$\alpha_0(\xi)\alpha_0\big(\lambda(\xi)\big)\cdots\alpha_0\big(\lambda_{m-1}(\xi)\big) \neq 1 \ \ \forall\, \xi \in \bigcup_{k=1}^{n_0} \Lambda_k,$$

$$\big|\alpha(\xi)\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\, d\tau \quad \forall\, \xi \in \overline{E}_{n_0}$$

be fulfilled.

Now we distinguish the cases in which the problem (1), (2), (3) is uniquely solvable in the class $C(\overline{D})$.

Theorem 2. *If at least one of the conditions of Theorem* 1 *is fulfilled, then the problem* (1), (2), (3) *is uniquely solvable in the class* $C(\overline{D})$ *for any* $a_4 \in C(\overline{D})$, $\varphi \in C(OB)$, $\beta \in C(OA)$ *in the absence, or for sufficient smallness of lower coefficients* a_1, a_2, a_3 *of equation* (1) *in the norm of the space* $C^1(\overline{D})$ *under the corresponding assumption that these coefficient are smooth.*

Proof of Theorem 2 follows immediately from the above-proven Lemmas 1–8 and from the fact that the reversibility of the operator remains valid for sufficiently small perturbations of that operator with respect to the norm in the corresponding space ([25], p. 212).

Theorem 3. *Let the continuous mapping* $\lambda : [0,1] \to [0,1]$ *satisfy the condition* $\lambda(\xi) \leq \xi \ \forall\, \xi \in [0,1]$ *and the functional equation* (11) *for any* $f \in C([0,1])$ *have a unique solution* $\psi \in C([0,1])$ *for which the estimate*

$$\big\|\psi\big\|_{C([0,\xi])} \leq c\big\|f\big\|_{C([0,\xi])} \quad \forall \in [0,1] \tag{58}$$

with the positive constant c *independent of* f *and* $\xi \in [0,1]$, *is valid. Then the problem* (1), (2), (3) *for any* $a_4 \in C(\overline{D})$, $\varphi \in C(OB)$, $\beta \in C(OA)$ *has a unique solution* $u \in C(\overline{D})$ *for which the estimate*

$$\big\|u\big\|_{C(\overline{D}_{x,y})} \leq$$
$$\leq c_0\Big(\big\|\varphi\big\|_{C(OB_{x,y})} + \big\|\beta\big\|_{C(OA_{x,y})} + \big\|a_4\big\|_{C(\overline{D}_{x,y})}\Big) \ \ \forall\, (x,y) \in \overline{D}, \tag{59}$$

with the positive constant c_0 *independent both of* φ, β, a_4 *and of the point* $(x,y) \in (\overline{D})$, *is valid, where* $D_{x,y}$ *is the characteristic rectangle bounded by the characteristics of equation* (1), *coming out of the points* $O(0,0)$ *and* $Q(x,y)$; $OA_{x,y} = OA \cap \overline{D}_{x,y}$, $OB_{x,y} = OB \cap \overline{D}_{x,y}$.

Proof. As is mentioned above, the problem (1), (2), (3) is equivalent to equation (8), rewritten in the form of the operator (57). We will solve equation

(57) in the class $C([0,1])$ by using the method of successive approximations:

$$\psi_0 = 0, \ \ T\psi_n = K_0\psi_{n-1} + f, \quad n = 1, 2, \dots. \tag{60}$$

By the method of mathematical induction we show that

$$\big|\psi_n(\xi) - \psi_{n-1}(\xi)\big| \le M M_1^{n-1} \frac{\xi^{n-1}}{(n-1)!} \big\|f\big\|_{C([0,\xi])}, \quad n \ge 1, \tag{61}$$

where M and M_1 are the positive numbers, independent of f.

Indeed, let (61) be fulfilled for $n = m$, and we prove this inequality for $n = m + 1$. By the condition of Theorem 3, equation (11) for any $f \in C([0,1])$ has a unique solution $\psi \in C([0,1])$ for which estimate (58) is valid, i.e., for the norm of the inverse operator $T^{-1} : C([0,1]) \to C([0,1])$ the estimate

$$\big\|T^{-1} f\big\|_{C([0,\xi])} = \big\|\psi\big\|_{C([0,\xi])} \le c\big\|f\big\|_{C([0,\xi])}, \quad 0 \le \xi \le 1, \tag{62}$$

holds.

By virtue of (60), we have

$$T\psi_{m+1} - T\psi_m = K_0\big(\psi_m - \psi_{m-1}\big),$$

from which we obtain

$$\psi_{m+1} - \psi_m = T^{-1}K_0\big(\psi_m - \psi_{m-1}\big) \tag{63}$$

owing to the fact that the operator is invertible.

Since by our assumption $\lambda(\xi) \le \xi$, $0 \le \xi \le 1$, introducing into consideration the integral operator $\widetilde{K}_0$, which acts by the formula

$$\widetilde{K}_0\psi(\xi) = \int\limits_0^\xi \big|K(\xi,\sigma)\big|\psi(\sigma)\,d\sigma,$$

we easily see that

$$\big|K_0\psi(\xi)\big| \le \int\limits_0^{\lambda(\xi)} \big|K(\xi,\sigma)\big|\psi(\sigma)\,d\sigma \le \int\limits_0^\xi \big|K(\xi,\sigma)\big|\psi(\sigma)\,d\sigma =$$

$$= \big|\widetilde{K}_0|\psi(\xi)|\big| \le d_0 \int\limits_0^\xi \big|\psi(\sigma)\big|\,d\sigma, \quad d_0 = \max_{0\le\xi,\sigma\le 1} \big|K(\xi,\sigma)\big|.$$

Therefore, taking into account estimate (61) for $n = m$, it immediately follows from (62) and (63) that

$$\big|\psi_{m+1}(\xi) - \psi_m(\xi)\big| \le c\big\|K_0(\psi_m - \psi_{m-1})\big\|_{C([0,\xi])} =$$

$$= c \max_{0\le\eta\le\xi} \big|K_0(\psi_m - \psi_{m-1})(\eta)\big| \le c\,d_0 \max_{0\le\eta\le\xi} \int\limits_0^\eta \big|\psi_m(\sigma) - \psi_{m-1})(\sigma)\big|\,d\sigma \le$$

$$\leq c\, d_0 \int\limits_0^\xi M M_1^{m-1} \frac{\sigma^{m-1}}{(m-1)!} \|f\|_{C([0,\sigma])} d\sigma =$$

$$= c\, d_0 M M_1^{m-1} \frac{\xi^m}{m!} \|f\|_{C([0,\sigma])}. \tag{64}$$

From (64) we arrive at (61) for $n = m + 1$ if in the capacity of M and M_1 we take $M = c$ and $M_1 = cd_0$. The validity of inequality (61) for $n = 1$ follows by virtue of (60) from the equality $\psi_1(\xi) - \psi_0(\xi) = \psi_1(\xi) = T^{-1}f$ and from estimate (62).

From (61), taking into account the equality

$$\psi_n = \psi_0 + \big(\psi_1 - \psi_0\big) + \big(\psi_2 - \psi_1\big) + \cdots + \big(\psi_n - \psi_{n-1}\big),$$

in a standard way we obtain the convergence of the sequence $\{\psi_n\}_{n=1}^\infty$ in the class $C([0,1])$ to some function $\psi \in C([0,1])$ which is, in fact, a solution of equation (8) for which the estimate

$$\big|\psi(\xi)\big| \leq M e^{M_1 \xi} \|f\|_{C([0,\xi])}, \quad 0 \leq \xi \leq 1, \tag{65}$$

is valid.

The uniqueness of the solution $\psi \in C([0,1])$ of equation (8) is proved analogously.

Finally, from the existence of the unique solution $\psi \in C([0,1])$ of equation (8), as well as from estimate (65) and representation (7) follows the existence of the solution $u \in C(\overline{D})$ of the problem (1), (2), (3) for which estimate (59) is valid. □

Theorem 4. *Let the continuous mapping $\lambda : [0,1] \to [0,1]$ be strictly monotonically increasing, satisfy the condition $\lambda(\xi) \leq \xi \ \forall \xi \in [0,1]$, and the inequality*

$$\big|\alpha(\xi)\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi), \tau\big)\, d\tau \ \ \forall \xi \in I_1 := \big\{\xi \in [0,1] : \lambda(\xi) = \xi\big\}$$

be fulfilled. Then the problem (1), (2), (3) *for any $a_4 \in C(\overline{D})$, $\varphi \in C(OB)$, $\beta \in C(OA)$ has the unique solution $u \in C(\overline{D})$ for which estimate* (59) *is valid.*

Proof. From the proof of Lemma 2 and the condition $\lambda(\xi) \leq \xi \ \forall \xi \in [0,1]$ follow the unique solvability of equation (11) in the class $C([0,1])$ and estimate (58) for its solution. Therefore Theorem 4 is the direct corollary of Theorem 3. □

Analogously, the direct corollary of Theorem 4 in the case $I_1 = \{0\}$ is then following

Theorem 5. *Let the continuous mapping* $\lambda : [0,1] \to [0,1]$ *be strictly monotonically increasing, and* $\lambda(0) = 0$*,* $\lambda(\xi) < \xi$ *for* $0 < \xi \leq 1$*. Then, if* $|\alpha(0)| < 1$*, the problem* (1), (2), (3) *for any* $a_4 \in C(\overline{D})$*,* $\varphi \in C(OB)$*,* $\beta \in C(OA)$ *has the unique solution* $u \in C(\overline{D})$ *for which estimate* (59) *is valid.*

Theorem 6. *Let the continuous mapping* $\lambda : [0,1] \to [0,1]$ *be strictly monotonically increasing, and* $\lambda(0) = 0$*,* $\lambda(\xi) < \xi$ *for* $0 < \xi \leq 1$*. Then, if* $|\alpha(0)| > 1$*, the problem* (1), (2), (3) *in the class* $C(\overline{D})$ *is normally Hausdorf solvable* ([30]*, p.* 108)*, and its index* $\varkappa = +\infty$*. In particular, the homogeneous problem, corresponding to the problem* (1), (2), (3)*, has in the class* $C(\overline{D})$ *an infinite set of linearly independent solutions.*

Proof. As is said above, the problem (1), (2), (3) in the class $C(\overline{D})$ is equivalent to equation (57) in the class $C([0,1])$. In addition, equation (11), being the functional part of equation (57), is, according to Lemma 2, solvable for every right-hand side $f \in C([0,1])$, and the homogeneous equation, corresponding to (11), has an infinite set of linearly independent solutions in $C([0,1])$. Consequently, equation (11) under the conditions of Theorem 6 is normally Hausdorf solvable, and its index is equal to $+\infty$. Therefore equation (57), different from equation (11) only by the compact operator $-K_0 : C([0,1]) \to C([0,1])$, likewise possesses this property, because the property for being normally solvable in the Banach space and to have an index, equal to $+\infty$, is stable for compact perturbations ([31]). Thus Theorem 6 is complete. □

Relying on Lemma 6, the following theorem is proved analogously.

Theorem 7. *Let the continuous mapping* $\lambda : [0,1] \to [0,1]$ *be strictly monotonically increasing, and* $\lambda(0) = 0$*,* $\lambda(\xi_0) = \xi_0$ *for some* $\xi_0 \in (0,1)$*, where* $\lambda(\xi) < \xi$ *for* $\xi_0 < \xi \leq 1$*. Moreover, let the condition*

$$\big|\alpha(\xi)\big| \neq \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi), \tau\big)\, d\tau, \quad 0 \leq \xi \leq \xi_0,$$

be fulfilled. Then if

$$\big|\alpha(\xi_0)\big| > \exp \int\limits_0^{\lambda(\xi_0)} a\big(\lambda(\xi_0), \tau\big)\, d\tau,$$

then the problem (1), (2), (3) *in the class* $C(\overline{D})$ *is normally Hausdorf solvable, and its index* $\varkappa = +\infty$*. In particular, the homogeneous problem, corresponding to the problem* (1), (2), (3)*, has in the class* $C(\overline{D})$ *an infinite set of linearly independent solutions.*

3. Smoothness of the Solution of Problem (1), (2), (3)

3.1. Below, we will present the conditions imposed on the data of the problem (1), (2), (3) which allow one to prove its solvability in the class $C^k(\overline{D})$, $k \geq 1$, in case $\lambda(0) = 0$. For $k \geq 2$ we will, evidently, deal with the classical solution of equation (1).

Regarding the data of the problem (1), (2), (3), it will be assumed that they possess the following conditions of smoothness:

$$\begin{gathered} a_1, a_2 \in C^k(\overline{D}), \ \ a_3, a_4 \in C^{k-1}(\overline{D}), \ \ \varphi \in C^k(OB), \\ \beta \in C^k(OA), \ \ \lambda \in C^k([0,1]), \ \ k \geq 1. \end{gathered} \tag{66}$$

In this case, in equation (8) we have

$$\alpha_0(\xi), \ \ \lambda(\xi), \ \ f(\xi) \in C^k([0,1]), \ \ K(\xi,\sigma) \in C^k([0,1]\times[0,1]) \tag{67}$$

and the problem (1), (2), (3) in the class $C^k(\overline{D})$ is equivalent to the integro-functional equation (8) in the class $C^k([0,1])$.

As is known, the function $\psi \in C^k([0,1])$ can be uniquely defined by its derivative $\psi^{(k)} \in C([0,1])$ and numbers $\psi(0), \psi^{(1)}, \dots \psi^{(n-1)}(0)$ by the formula

$$\psi(\xi) = \sum_{i=0}^{k-1} \frac{\psi^{(i)}(0)}{i!}\xi^i + \frac{1}{(k-1)!}\int\limits_0^\xi (\xi-\tau)^{k-1}\psi^{(k)}(\tau)\,d\tau. \tag{68}$$

Analogously, the equalities

$$\psi^{(j)}(\xi) = \sum_{i=0}^{k-j-1} \frac{\psi^{(j+i)}(0)}{i!}\xi^i + $$

$$+\frac{1}{(k-j-1)!}\int\limits_0^\xi (\xi-\tau)^{k-j-1}\psi^{(k)}(\tau)\,d\tau, \quad j = 0,\dots,k-1 \tag{69}$$

hold.

Under the assumption that equation (8) has the solution ψ of the class $C^k([0,1])$, taking into account (66) and (67), differentiating i–times equation $0 \leq i \leq k$, and supposing that in the obtained equalities $\xi = 0$ with respect to unknowns $\psi^{(i)}(0)$, $i = 0, 1\dots, k-1$, we get the linear system

$$A\Psi = F, \tag{70}$$

where $\Psi = (\psi(0), \psi^{(1)}(0), \dots, f^{(k-1)}(0))$, $F = (f(0), f^{(1)}(0), \dots, f^{(k-1)}(0))$. Here, the matrix A of order $k \times k$ is the lower triangular matrix whose diagonal elements are the numbers $a_{ii} = 1 - \alpha_0(0)(\lambda^{(1)}(0))^i = 1 - \alpha(0)(\lambda^{(1)}(0))^i$, $i = 0, 1, \dots, k-1$, since by the assumption $\lambda(0) = 0$. Therefore the system of equations (70) is uniquely solvable if and only if

$$\alpha(0)(\lambda^{(1)}(0))^i \neq 1, \quad i = 0, 1, \dots, k-1. \tag{71}$$

Now we differentiate k times both parts of equation (8) with respect to ξ and in the obtained equation we replace derivatives $\psi^{(j)}(\xi)$ for $j = 0, 1, \dots, k-1$, by the right-hand sides of equalities (69). As a result, with respect to the unknown vector quantity (Ψ, ψ_0), where $\Psi = (\psi(0), \psi^{(1)}(0), \dots, \psi^{(k-1)}(0))$ and $\psi_0(\xi) = \psi^{(n)}(\xi) \in C([0,1])$ we obtain the system of equations, consisting of the system (70) itself and the following integro-functional equation

$$\psi_0(\xi) - \alpha_0(\xi)\big(\lambda^{(1)}(\xi)\big)^k \psi_0\big(\lambda(\xi)\big) =$$
$$= \int\limits_0^{\lambda(\xi)} K_1(\xi,\sigma)\psi(\sigma)\,d\sigma + \sum_{i=0}^{k-1} d_i \psi^{(i)}(0) + \widetilde{f}(\xi), \quad 0 \le \xi \le 1, \tag{72}$$

where $K_1(\xi,\sigma)$ and $\widetilde{f}(\xi)$ are entirely definite functions of the class $C([0,1] \times [0,1])$ and $C([0,1])$, respectively, and d_i, $i = 0, \dots, k-1$, are entirely definite numbers.

Remark 6. Under the above assumptions regarding the data of the problem (1), (2), (3), this problem in the class $C^k(\overline{D})$ is equivalent to the system of equations (70), (72) with respect to the vector quantity (Ψ, ψ_0), where $\Psi \in R^k$, $\psi_0 \in C([0,1])$ and since the system (70) is Fredholmian, therefore the Fredholmity of the problem (1), (2), (3) is equivalent to that of the integral functional part of equation (72), i.e., of equation

$$\psi_0(\xi) - \alpha_0(\xi)\big(\lambda^{(1)}(\xi)\big)^k \psi_0\big(\lambda(\xi)\big) = \int\limits_0^{\lambda(\xi)} K_1(\xi,\sigma)\psi(\sigma)\,d\sigma + \widetilde{f}(\xi), \tag{73}$$
$$0 \le \xi \le 1,$$

in the class $C([0,1])$. Since the conditions of Fredholmity of the integro-functional equation (8) in the class $C([0,1])$ are given in Theorem 1, and equation (73) itself differs from equation (8) only by the factor $(\lambda^{(1)}(\xi))^k$ appearing in front of the functional term $\psi_0(\lambda(\xi))$ in the left-hand side of that equation, the following theorem is valid.

Theorem 8. *Let the conditions* (66), (67) *be fulfilled, and* $\lambda(0) = 0$. *The problem* (1), (2), (3) *is Fredholmian if at least one of the following conditions is fulfilled:*

1. $|\alpha(\xi)(\lambda^{(1)}(\xi))^k| < \exp \int\limits_0^{\lambda(\xi)} a(\lambda(\xi),\tau)d\tau$, $0 \le \xi \le 1$;

2. *The continuous mapping* $\lambda : [0,1] \to [0,1]$ *is strictly monotonically increasing, and the inequality*

$$\big|\alpha(\xi)\big(\lambda^{(1)}(\xi)\big)^k\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)d\tau \quad \forall\, \xi \in I_1 := \big\{\xi \in [0,1] : \lambda(\xi) = \xi\big\}$$

is fulfilled;

3. *The continuous mapping* $\lambda : [0,1] \to [0,1]$ *is strictly monotonically increasing, and the condition*

$$\lambda(\xi) < \xi \quad \textit{for} \quad 0 < \xi \le 1, \quad \big|\alpha(0)\big(\lambda^{(1)}(0)\big)^k\big| < 1$$

are fulfilled;

4. *The mapping* $\lambda : [0,1] \to [0,1]$ *is a continuous homeomorphism of the segment* $[0,1]$ *into itself, and*

$$\big|\alpha(\xi)\big(\lambda^{(1)}(\xi)\big)^k\big| \ne \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)d\tau, \quad 0 \le \xi \le 1$$

holds;

5. *The mapping* $\lambda : [0,1] \to [0,1]$ *is a continuous homeomorphism of the segment* $[0,1]$ *into itself, leaving the ends of that segment fixed, and the inequality*

$$\big|\alpha(\xi)\big(\lambda^{(1)}(\xi)\big)^k\big| > \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)d\tau \quad \forall\, \xi \in I_1$$

is fulfilled;

6. *The continuous mapping* $\lambda : [0,1] \to [0,1]$ *is strictly monotonically increasing,* $\lambda(\xi_0) = \xi_0$ *for some number* $\xi_0 \in (0,1)$*, where* $\lambda(\xi) < \xi$ *if* $\xi_0 < \xi < 1$ *and*

$$\big|\alpha(\xi)\big(\lambda^{(1)}(\xi)\big)^k\big| \ne \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)d\tau, \quad 0 \le \xi \le \xi_0,$$

$$\big|\alpha(\xi_0)\big(\lambda^{(1)}(\xi_0)\big)^k\big| < \exp \int\limits_0^{\lambda(\xi_0)} a\big(\lambda(\xi),\tau\big)d\tau;$$

7. *Let* $\Lambda_n = \{\xi \in [0,1] : \lambda_k(\xi) \ne \xi,\ k = 1,\dots,n-1,\ \lambda_n(\xi) = \xi\} = \varnothing$ *for* $n > n_0$*. Denote by* m *the least common multiple of numbers* $k_1, k_2, \dots, k_{n_0}$*, where* $k_i = i$ *and* $k_i = 1$ *for* $\Lambda_i \ne \varnothing$ *and* $\Lambda_i = \varnothing$*, respectively. Let* $E_{n_0} =$

$[0,1]\setminus\bigcup\limits_{k=1}^{n_0}\Lambda_k$ *and the conditions*

$$\alpha_0(\xi)\big(\lambda^{(1)}(\xi)\big)^k\alpha_0\big(\lambda(\xi)\big)\big(\lambda^{(1)}(\lambda(\xi))\big)^k\cdots\alpha_0\big(\lambda_{m-1}(\xi)\big)\big(\lambda^{(1)}(\lambda_{m-1}(\xi))\big)^k\neq 1$$
$$\forall\,\xi\in\bigcup_{k=1}^{n_0}\Lambda_k,$$
$$\big|\alpha(\xi)\big(\lambda^{(1)}(\xi)\big)^k\big|<\exp\int\limits_0^{\lambda(\xi)}a\big(\lambda(\xi),\tau\big)d\tau\quad\forall\,\xi\in\overline{E}_{n_0}$$

be fulfilled.

Remark 7. If the condition $\lambda(0)=0$ violates, i.e., the point $\xi=0$ of the mapping $\lambda:[0,1]\to[0,1]$ is not fixed, then the reasoning above and all the results remain valid, if instead of $\xi=0$ we take any other fixed point $\xi_0\in I_1\neq\varnothing$ of the mapping λ.

3.2. In this section we consider the case in which the problem (1), (2), (3) is uniquely solvable in the class $C^k(\overline{D})$.

Remark 8. Let the conditions (71) be fulfilled. Then the linear algebraic system (70) is uniquely solvable with respect to the vector $\Psi=(\psi(0),\psi^{(1)}(0),\dots,\psi^{(k-1)}(0))$. Substituting components $\psi^{(i)}(0)$, $i=0,\dots,k-1$, of that vector in the right-hand side of equation (72), we obtain the integro-functional equation

$$\psi_0(\xi)-\alpha_0(\xi)\big(\lambda^{(1)}(\xi)\big)^k\psi_0\big(\lambda(\xi)\big)=\int\limits_0^{\lambda(\xi)}K_1(\xi,\sigma)\psi(\sigma)\,d\sigma+\widetilde{f}(\xi),\qquad(74)$$
$$0\leq\xi\leq 1,$$

where $\widetilde{f}\in C([0,1])$ is the entirely definite function. If $\psi_0\in C([0,1])$ is a solution of equation (74), then the corresponding solution $\psi\in C^k([0,1])$ of equation (8) is restored by formula (68), i.e.,

$$\psi(\xi)=\sum_{i=0}^{k-1}\frac{\psi^{(i)}(0)}{i!}\xi^i+\frac{1}{(k-1)!}\int\limits_0^{\xi}(\xi-\tau)^{k-1}\psi_0(\tau)\,d\tau,\quad 0\leq\xi\leq 1.$$

Thus owing to the fact that problem (1), (2), (3) in the class $C^k(\overline{D})$ and the system (70), (72) are equivalent with respect to the unknowns $\Psi\in R^k$ and $\psi_0\in C([0,1])$, when the conditions (71) are fulfilled, the unique solvability of the problem (1), (2), (3) in $C^k(\overline{D})$ is equivalent to that of equation (74) with respect to the function ψ_0 in the class $C([0,1])$. Since the conditions for the unique solvability of the integro-functional equation in the class $C([0,1])$ are given in Theorems 4–5, and equation (74) itself differs from equation

(8) only by the factor $(\lambda^{(1)}(\xi))^k$ appearing in front of the functional term $\psi_0(\lambda(\xi))$ in the left-hand side of that equation, the following theorem is valid.

Theorem 9. *Let the conditions* (66), (67), (71) *be fulfilled, and* $\lambda(0) = 0$. *Moreover, let the continuous mapping* $\lambda : [0,1] \to [0,1]$ *be strictly monotonically increasing, satisfy the condition* $\lambda(\xi) \le \xi \;\forall\, \xi \in [0,1]$ *and the inequality*

$$\big|\alpha(\xi)\big(\lambda^{(1)}(\xi)\big)^k\big| < \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau \quad \forall\, \xi \in I_1 := \big\{\xi \in [0,1] : \lambda(\xi) = \xi\big\}$$

be fulfilled. Then the problem (1), (2), (3) *for any* $a_4 \in C^{k-1}(\overline{D})$, $\varphi \in C^k(OB)$, $\beta \in C^k(OA)$ *has the unique solution* $u \in C^k(\overline{D})$ *for which the estimate*

$$\|u\|_{C^k(\overline{D}_{x,y})} \le c_0\Big(\|\varphi\|_{C^k(OB_{x,y})} + \|\beta\|_{C^k(OA_{x,y})} + \|a_4\|_{C^{k-1}(\overline{D}_{x,y})}\Big) \quad (75)$$
$$\forall\,(x,y) \in \overline{D},$$

with the positive constant c_0, *independent of* φ, β, a_4 *and the point* $(x,y) \in \overline{D}$, *is valid; the sets* $D_{x,y}$ *and* $OA_{x,y}$, $OB_{x,y}$ *were introduced when formulating Theorem* 3.

Theorem 10. *Let the conditions* (66), (67), (71) *be fulfilled, the continuous mapping* $\lambda : [0,1] \to [0,1]$ *be strictly monotonically increasing and* $\lambda(0) = 0$, $\lambda(\xi) < \xi$ *for* $0 < \xi \le 1$. *If* $|\alpha(0)(\lambda^{(1)}(0))^k| < 1$, *then the problem* (1), (2), (3) *for any* $a_4 \in C^{k-1}(\overline{D})$, $\varphi \in C^k(OB)$, $\beta \in C^k(OA)$ *has the unique solution* $u \in C^k(\overline{D})$ *for which estimate* (75) *is valid.*

3.3. In this section we will present the conditions under the fulfilment of which the homogeneous problem, corresponding to the problem (1), (2), (3), has an infinite set of linearly independent solutions. We will also establish the conditions when, in spite of the fact that the smoothness conditions (66), (67) are fulfilled, the problem (1), (2), (3) has a continuous solution $u \in C(\overline{D})$, not admitting raising of the smoothness.

Since by Remark 6 the problem (1), (2), (3) in the class $C^k(\overline{D})$ under the conditions (66), (67) is equivalent to the system of equations (70), (72), the system (70) is Fredholmian, equation (72) differs in the left-hand side from equation (8) only by the factor $(\lambda^{(1)}(\xi))^k$ appearing in front of the functional term $\psi_0(\lambda(\xi))$, and the conditions, ensuring the normal Hausdorf solvability and the index $\varkappa = +\infty$ for equation (11) are given in Theorems 6 and 7, we have the following results which are valid.

Theorem 11. *Let the conditions* (66), (67) *be fulfilled, the continuous mapping* $\lambda : [0,1] \to [0,1]$ *be strictly monotonically increasing and* $\lambda(0) = 0$, $\lambda(\xi) < \xi$ *for* $0 < \xi \le 1$. *If* $|\alpha(0)(\lambda^{(1)}(0))^k| > 1$, *then the problem* (1), *(2), (3) in the class $C^k(\overline{D})$ is normally Hausdorf solvable and its index $\varkappa = +\infty$. In particular, the homogeneous problem in the class $C^k(\overline{D})$, corresponding to the problem (1), (2), (3), has an infinite set of linearly independent solutions.*

Theorem 12. *Let the conditions (66), (67) be fulfilled, the continuous mapping $\lambda : [0,1] \to [0,1]$ be strictly monotonically increasing and $\lambda(0) = 0$, $\lambda(\xi_0) = \xi_0$ for some number $\xi_0 \in (0,1)$, where $\lambda(\xi) < \xi$ for $\xi_0 < \xi \leq 1$. Moreover, let the condition*

$$\big|\alpha(\xi)\big(\lambda^{(1)}(\xi)\big)^k\big| \neq \exp \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi),\tau\big)\,d\tau, \quad 0 \leq \xi \leq \xi_0$$

be fulfilled. If

$$\big|\alpha(\xi_0)(\big(\lambda^{(1)}(\xi)\big)^k\big| > \exp \int\limits_0^{\lambda(\xi_0)} a\big(\lambda(\xi_0),\tau\big)\,d\tau,$$

then the problem (1), (2), (3) in the class $C^k(\overline{D})$ is normally Hausdorf solvable and its index $\varkappa = +\infty$. In particular, the homogeneous problem in the class $C^k(\overline{D})$, corresponding to the problem (1), (2), (3), has an infinite set of linearly independent solutions.

Remark 9. Despite the fact that the conditions (66), (67) are fulfilled, we can cite the cases when the solution $u \in C(\overline{D})$ does not belong to the class $C^k(\overline{D})$, $k \geq 1$. Indeed, let the conditions (66), (67) be fulfilled and the continuous mapping $\lambda : [0,1] \to [0,1]$ be strictly monotonically increasing, where $\lambda(0) = 0$, $\lambda(\xi) < \xi$ for $0 < \xi \leq 1$ and $|\alpha(0)| > 1$, but $|\lambda^{(1)}(0)| < 1$. Thus for some k we will have $|\alpha(0)(\lambda^{(1)}(0))^k| < 1$. In this case, according to Theorem 6, the homogeneous problem, corresponding to the problem (1), (2), (3), has an infinite set of linearly independent solutions of the class $C(\overline{D})$. The fact that not every continuous solution is of the class $C^k(\overline{D})$ follows from Condition 3 of Theorem 8, which is, by our assumptions, fulfilled. Since by Theorem 8 and Fredholmity of the problem (1), (2), (3) in the class $C^k(\overline{D})$, the set of solutions of the homogeneous problem of the class $C^k(\overline{D})$, corresponding to the problem (1), (2), (3), is finite dimensional and the kernel of the problem (1), (2), (3) of the class $C(\overline{D})$ is infinite dimensional, there exists a continuous solution $u \in C(\overline{D})$ of the homogeneous problem, corresponding to the problem (1), (2), (3), and this solution does not belong to the class $C^k(\overline{D})$. If, in addition, the condition (71) is fulfilled, then by Theorem 10, the homogeneous problem of the class $C^k(\overline{D})$, corresponding to the problem (1), (2), (3) has only a trivial solution, and in this case every nontrivial solution of the infinite dimensional kernel of the problem (1), (2), (3) of the class $C(\overline{D})$ does not belong to the class $C^k(\overline{D})$.

Remark 10. The problem (1), (2), (3) can also be considered in the case when $u = (u_1, \dots, u_n)$, $n > 1$, is a vector function, a_4 is the given vector function, a_i, $i = 1, 2, 3$, and α and β are the given $(n \times n)$-matrices satisfying the same smoothness conditions as in the scalar case. In this case, representation (7) in which $R(\xi_1, \eta_1; \xi, \eta)$ is the Riemann function for the hyperbolic system (4), is also valid ([30], p. 66). In addition, (8) represents the system of integro-functional equations with respect to an unknown vector function $\psi = (\psi_1, \dots, \psi_n)$. Here

$$\alpha_0(\xi) = \alpha(\xi) R\big(\lambda(\xi), 0; \lambda(\xi), \lambda(\xi)\big), \tag{76}$$

and taking into account the co-factor order, the matrix $R(\lambda(\xi), 0; \lambda(\xi), \lambda(\xi))$ with respect to the second argument η_1 of the Riemann matrix $R(\xi_1, \eta_1; \xi, \eta)$ is the solution of the following Cauchy problem:

$$\begin{gathered} \frac{\partial}{\partial \eta_1} R(\xi, \eta_1; \xi, \eta) - R(\xi, \eta_1; \xi, \eta) a(\xi, \eta_1) = 0, \\ R(\xi, \eta; \xi, \eta) = E. \end{gathered} \tag{77}$$

Unlike the scalar case, the matrix $R(\lambda(\xi), 0; \lambda(\xi), \lambda(\xi))$, being the unique solution of the Cauchy problem (77), ceases to be representable in the general case in the form

$$R\big(\lambda(\xi), 0; \lambda(\xi), \lambda(\xi)\big) = \exp\bigg[- \int\limits_0^{\lambda(\xi)} a\big(\lambda(\xi), \tau\big)\, d\tau \bigg]. \tag{78}$$

Representation (78) holds if, for example, the matrix $a(\xi, \eta)$ does not depend on the second argument η.

Let $\sigma_i = \sigma_i(\xi)$, $i = 1, \dots, n$, be eigen numbers with regard for the multiplicity of the $(n \times n)$-matrix $\alpha_0(\xi)$, and let $r(\xi) = \max\limits_{1 \le i \le n} |\sigma_i(\xi)|$ be spectral radius of that matrix $\alpha_0(\xi)$. Here we present some results on the solvability of the problem (1), (2), (3) in case $n > 1$:

1. Let the continuous mapping $\lambda : [0, 1] \to [0, 1]$ satisfy the condition $\lambda(\xi) \le \xi \ \forall \xi \in [0, 1]$, and $\sup\limits_{0 \le \xi \le 1} r(\xi) < 1$. Then the problem (1), (2), (3) for any n-dimensional vector functions $a_4 \in C(\overline{D})$, $\varphi \in C(OB)$, $\beta \in C(OA)$ has a unique solution $u = (u_1, \dots, u_n) \in C(\overline{D})$ for which estimate (59) with the corresponding norms for continuous vector functions, is valid.

2. Let the mapping $\lambda : [0, 1] \to [0, 1]$ be a continuous homeomorphism of the segment $[0, 1]$ into itself, and either $\sigma_i(\xi) \ne 0$, $i = 1, \dots, n$, $\sup\limits_{0 \le \xi \le 1} r(\xi) < 1$ or $\inf\limits_{0 \le \xi \le 1} |\sigma_i(\xi)| > 1$. Then the problem (1), (2), (3) is Fredholmian in the class $C(\overline{D})$. In case $\inf\limits_{0\le\xi\le 1}|\sigma_i(\xi)|>1$, if it is additionally known that $\lambda(\xi)\ge\xi$ $\forall\,\xi\in[0,1]$, then for any n-dimensional continuous vector functions a_4, φ and β the problem (1), (2), (3) has a unique solution $u=(u_1,\dots,u_n)\in C(\overline{D})$ for which estimate (59) is valid.

3. Let the continuous mapping $\lambda:[0,1]\to[0,1]$ be strictly monotonically increasing and $\lambda(0)=0$, $\lambda(\xi)<\xi$ for $0<\xi\le 1$. If $r(0)<1$, then the problem (1), (2), (3) for any n-dimensional continuous vector functions a_4, φ and β has the unique solution $u=(u_1,\dots,u_n)\in C(\overline{D})$ for which estimate (59) is valid. If, however, $r(0)>1$ and $|\sigma_i|\ne 1$, $i=1,\dots,n$, then the problem (1), (2), (3) in the class $C(\overline{D})$-is normally Hausdorf solvable and its index $\varkappa=+\infty$. In particular, the homogeneous problem, corresponding to the problem (1), (2), (3), has in the class $C(\overline{D})$ an infinite set of linearly independent solutions.

The proof of these results is, to certain extent, analogous to that for the scalar case. The same remark is likewise true when the question is on the solvability of the problem (1), (2), (3) in the class $C^k(\overline{D})$ for $k\ge 1$.

4. Consideration of Some Special Cases in the Problem (1), (2), (3)

Consider the case when in the problem (1), (2), (3)

$$\lambda(\xi)=\xi,\quad \alpha(\xi)=1,\quad \beta(\xi)=0\quad \forall\,\xi\in[0,1]. \tag{79}$$

If the conditions (79) are fulfilled, equation (8) takes the form

$$\mu(\xi)\psi(\xi)-\int\limits_0^{\xi}K(\xi,\sigma)\psi(\sigma)\,d\sigma=f(\xi),\quad 0\le\xi\le 1. \tag{80}$$

Here

$$\mu(\xi)=1-R\big(\xi,0;\xi,\xi\big)=1-\exp\bigg[-\int\limits_0^{\xi}a(\xi,\tau)\,d\tau\bigg],\quad 0\le\xi\le 1. \tag{81}$$

Note that equation (80) is, generally speaking, the Volterra type equation of the third kind, since by (81), the coefficient $\mu(\xi)$ is at least at the point $\xi=0$ equal to zero. Therefore the condition

$$\big|\alpha(\xi)\big|<\exp\int\limits_0^{\lambda(\xi)}a\big(\lambda(\xi),\tau\big)\,d\tau\quad \forall\,\xi\in I_1:=\{\xi\in[0,1]:\lambda(\xi)=\xi\}$$

established in Theorem 4 and ensuring the correctness of the problem (1), (2), (3) in the class $C(\overline{D})$, is violated. Note also that the case for $a=b=0$, $c=\text{const}\ne 0$ has been considered in [22]. In this case $\mu(\xi)\equiv 0\ \forall\,\xi\in[0,1]$, and equation (80) is the Volterra type equation of the first kind.

Below, we will consider first the case when the coefficient $\mu(\xi) \neq 0$ for $0 < \xi \leq 1$, which is, owing to (81), equivalent to the condition

$$\int_0^{\xi} a(\xi,\eta)\, d\eta \neq 0, \qquad 0 < \xi \leq 1. \tag{82}$$

Remark 11. It can be easily seen that if the condition

$$\frac{K(\xi,\sigma)}{\mu(\xi)} \in C(\overline{\Omega}), \quad \Omega : 0 < \xi < 1, \quad 0 < \sigma < 1 \tag{83}$$

is fulfilled, then under the assumption that

$$\frac{f(\xi)}{\mu(\xi)} \in C([0,1]), \tag{84}$$

equation (80), and hence the problem (1), (2), (3) is uniquely solvable in the class $C(\overline{D})$.

Here we impose some restrictions on the coefficients a, b and c of equation (4) which ensure the fulfilment of the condition (83) and consider the cases when the conditions (84) are fulfilled.

4.1. In equation (4), let coefficients $a, b, c = \text{const}$. In this case, the Riemann function for that equation has the form ([32], p. 18)

$$R(\xi_1,\eta_1;\xi,\eta) = \\ = J_0\big(2\sqrt{(c-ab)(\xi-\xi_1)(\eta-\eta_1)}\big) \exp\big(-\big[b(\xi-\xi_1)+a(\eta-\eta_1)\big]\big). \tag{85}$$

Here

$$J_0\big(2\sqrt{(c-ab)(\xi-\xi_1)(\eta-\eta_1)}\big) = \\ = \sum_{k=0}^{\infty} (-1)^k \frac{(c-ab)^k}{(k!)^2} (\xi-\xi_1)^k (\eta-\eta_1)^k. \tag{86}$$

By (81), (85) and (86), we have

$$\mu(\xi) = 1 - R(\xi,0;\xi,\xi) = 1 - e^{-a\xi} = -\xi \sum_{k=0}^{\infty} \frac{(-a)^{k+1}\xi^k}{(k+1)!}, \tag{87}$$

$$\frac{\partial R(\xi_1,\eta_1;\xi,\eta)}{\partial \xi_1} = bR(\xi_1,\eta_1;\xi,\eta) + \\ + \Big[\sum_{k=1}^{\infty} (-1)^k \frac{k(c-ab)^k}{(k!)^2} (-1)(\xi-\xi_1)^{k-1}(\eta-\eta_1)^k\Big] \times \\ \times \exp\big(-\big[b(\xi-\xi_1)+a(\eta-\eta_1)\big]\big) = bR(\xi_1,\eta_1;\xi,\eta) + \\ + (\eta-\eta_1)\Big[(c-ab) + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{k(c-ab)^k}{(k!)^2} (\xi-\xi_1)^{k-1}(\eta-\eta_1)^{k-1}\Big] \times$$

$$\times \exp\big(-\big[b(\xi-\xi_1)+a(\eta-\eta_1)\big]\big). \tag{88}$$

For the kernel $K(\xi,\sigma)$ from (8), by virtue of (79) and (88), we find that

$$K(\xi,\sigma)=b(\sigma,0)R(\sigma,0;\xi,\xi)-\frac{\partial R(\sigma,0;\xi,\xi)}{\partial\sigma}=bR(\sigma,0;\xi,\xi)-$$
$$-bR(\sigma,0;\xi,\xi)-\xi\Big[(c-ab)+\sum_{k=2}^{\infty}(-1)^{k+1}\frac{k(c-ab)^k}{(k!)^2}(\xi-\sigma)^{k-1}\xi^{k-1}\Big]\times$$
$$\times\exp\big(-[b(\xi-\sigma)+a\xi]\big)=-\xi\Big[(c-ab)+$$
$$+\sum_{k=2}^{\infty}(-1)^{k+1}\frac{k(c-ab)^k}{(k!)^2}(\xi-\sigma)^{k-1}\xi^{k-1}\Big]\exp\big(-[b(\xi-\sigma)+a\xi]\big). \tag{89}$$

From (89) and (87) for $a\neq 0$, which is, due to the fact that $a=\text{const}$, equivalent to the condition (82), we obtain

$$\frac{K(\xi,\sigma)}{\mu(\xi)}=$$
$$=\frac{[(c-ab)+\sum\limits_{k=2}^{\infty}(-1)^{k+1}\frac{k(c-ab)^k}{(k!)^2}(\xi-\sigma)^{k-1}\xi^{k-1}]\exp\big(-[b(\xi-\sigma)+a\xi]\big)}{\sum\limits_{k=0}^{\infty}\frac{(-a)^{k+1}\xi^k}{(k+1)!}}\in$$
$$\in C^n(\overline{\Omega}),\quad n=0,1,2,\dots. \tag{90}$$

Here we have taken into account that

$$-\frac{\mu(\xi)}{\xi}=\sum_{k=0}^{\infty}\frac{(-a)^{k+1}\xi^k}{(k+1)!}=\frac{e^{-a\xi}-1}{\xi}\in C^\infty([0,1]),$$

where $\xi^{-1}(e^{-a\xi}-1)\neq 0$ for $0\le\xi\le 1$. Thus

$$\frac{\mu(\xi)}{\xi}\in C^\infty([0,1]),\quad \frac{\mu(\xi)}{\xi}\neq 0\quad \forall\,\xi\in[0,1],\quad \lim_{\xi\to 0}\frac{\mu(\xi)}{\xi}=-a. \tag{91}$$

Now let us reveal under which additional restrictions on the function $f\in C([0,1])$, and hence on the initial functions φ and g, the condition (84) is fulfilled.

By (10), (85) and (87) we have

$$\frac{f(\xi)}{\mu(\xi)}=\frac{1}{\mu(\xi)}\Big\{\varphi(\xi)\exp(-b\xi)-J_0\big(2\sqrt{(c-ab)\xi^2}\big)\varphi(0)\exp\big(-(a+b)\xi\big)+$$
$$+\int_0^{\xi}\Big[a(0,\tau)R(0,\tau;\xi,\xi)-\frac{\partial R(0,\tau;\xi,\xi)}{\partial\tau}\Big]\varphi(\tau)\,d\tau+$$

$$+\int_0^\xi dt\int_0^\xi R(t,\tau;\xi,\xi)g(t,\tau)\,d\tau\Big\}. \tag{92}$$

If $\varphi\in C([0,1])$ and $g\in C(\overline{\Omega})$, then by (91) and (92), for the condition (84) to be fulfilled, it is necessary and sufficient that

$$\frac{\varphi(\xi)-\varphi(0)}{\xi}\in C([0,1]), \tag{93}$$

which undoubtedly will be fulfilled if, in particular, $\varphi\in C^1([0,\varepsilon])$ for an arbitrarily small positive ε. Thus the following statement is valid: let the conditions (79) be fulfilled, a, b, $c=\text{const}$, $a\neq 0$, i.e., in equation (1) the quantity $a_1+a_2\neq 0$. Then for any $a_4\in C(\overline{D})$ and function φ, satisfying condition (93), the problem (1), (2), (3) is uniquely solvable in the class $C(\overline{D})$.

Under the above given conditions the question on the solvability of the problem (1), (2), (3) in the class $C^k(\overline{D})$, $k\geq 1$, is considered analogously.

4.2. As is mentioned above, the case $a=b=0$ with $c=\text{const}\neq 0$ has been considered in [22]. Let now $a=0$, but $bc\neq 0$, b, $c=\text{const}$. Then by virtue of (81), the function $\mu(\xi)\equiv 0$ and equation (80) takes the form

$$\int_0^\xi K_1(\xi,\sigma)\psi(\sigma)\,d\sigma=\frac{f(\xi)}{\xi},\quad 0<\xi\leq 1, \tag{94}$$

where, according to (89),

$$K_1(\xi,\sigma)=\Big[c+\sum_{k=2}^{\infty}(-1)^{k+1}\frac{kc^k}{(k!)^2}(\xi-\sigma)^{k-1}\xi^{k-1}\Big]\exp\big[-b(\xi-\sigma)\big]. \tag{95}$$

If equation (94) is solvable in the class $C([0,1])$, we have $f_1(\xi)=\frac{f(\xi)}{\xi}\in C^1([0,1])$, and differentiating equation (94) with respect to ξ, by virtue of (95) and the fact that $K_1(\xi,\xi)=c$, we obtain

$$c\psi(\xi)+\int_0^\xi\frac{\partial K_1(\xi,\sigma)}{\partial\xi}\,\psi(\sigma)\,d\sigma=f_1'(\xi),\quad 0\leq\xi\leq 1. \tag{96}$$

Since $c\neq 0$, equation (96) is of Volterra type second kind equation, and thus we have the unique continuous solution $\psi\in C([0,1])$. Therefore it remains to reveal the conditions imposed on the function φ from which it would follow that $f_1(\xi)=\frac{f(\xi)}{\xi}\in C^1([0,1])$.

Similarly to the representation (92), by (10), (85) and the condition $a = 0$, we will have

$$f_1(\xi) = \frac{f(\xi)}{\xi} = \frac{1}{\xi}\Big\{\Big(\varphi(\xi) - J_0\big(2\sqrt{c}\xi\big)\varphi(0)\Big)\exp(-b\xi) - \\ - \int_0^\xi \frac{\partial R(0,\tau;\xi,\xi)}{\partial \tau}\,\varphi(\tau)\,d\tau + \int_0^\xi dt \int_0^\xi R(t,\tau;\xi,\xi) g(t,\tau)\,d\tau\Big\}. \tag{97}$$

As far as

$$\int_0^\xi dt \int_0^\xi R(t,\tau;\xi,\xi) g(t,\tau)\,d\tau = \xi^2 \int_0^1 d\widetilde{t} \int_0^1 R(\widetilde{t}\xi,\widetilde{\tau}\xi;\xi,\xi) g(\widetilde{t}\xi,\widetilde{\tau}\xi)\,d\widetilde{\tau},$$

the condition $f_1(\xi) \in C^1([0,1])$ with regard for (97) will be fulfilled if we require

$$\frac{\varphi(\xi) - \varphi(0)}{\xi} \in C^1([0,1]). \tag{98}$$

Thus we have shown that if the conditions (79) are fulfilled, $a = 0$, $bc \neq 0$, b, $c = \text{const}$, then for any $a_4 \in C(\overline{D})$ and for the function φ, satisfying the condition (98), the problem (1), (2), (3) is uniquely solvable in the class $C(\overline{D})$.

4.3. Let now $a = c = 0$, but $b = \text{const} \neq 0$. Then by (85) we have $R(\xi_1,\eta_1;\xi,\eta) = \exp[-b(\xi - \xi_1)]$. In this case, in equation (80) we have $\mu(\xi) \equiv 0$, $K(\xi,\sigma) \equiv 0$. Therefore, by (10) and (79), the problem (1), (2), (3) is solvable in the class $C(\overline{D})$ if and only if

$$f(\xi) = \big(\varphi(\xi) - \varphi(0)\big)\exp(-b\xi) + \\ + \int_0^\xi dt \int_0^\xi g(t,\tau)\exp\big[-b(\xi - t)\big]\,d\tau = 0, \quad 0 \le \xi \le 1. \tag{99}$$

If the conditions (99) are fulfilled, the problem (1), (2), (3) has an infinite set of linearly independent solutions of the class $C(\overline{D})$ which by virtue of (7) are given by the formula

$$u(x,y) = \psi(\xi) + \big(\varphi(\eta) - \varphi(0)\big)\exp(-b\xi) + \\ + \int_0^\xi d\sigma \int_0^\eta g(\sigma,\tau)\exp\big[-b(\xi - \sigma)\big]\,d\tau,$$

where ψ is an arbitrary function of the class $C([0,1])$, $\psi(0) = \varphi(0)$ and $\xi = \frac{1}{2}(t + x)$, $\eta = \frac{1}{2}(t - x)$.

We have above taken into account that $R(\xi_1,\eta_1;\xi,\eta)=\exp[-b(\xi-\xi_1)]$ and, respectively, in the representation (7)

$$b(\sigma,0)R(\sigma,0;\xi,\xi)-\frac{\partial R(\sigma,0;\xi,\xi)}{\partial\sigma}=0,$$
$$a(0,\tau)R(0,\tau;\xi,\eta)-\frac{\partial R(0,\tau;\xi,\xi)}{\partial\tau}=0.$$

4.4. Consider the case of continuous coefficients a, b and c of equation (4), when one of the Laplace invariants is equal to zero. Let, for example, $h=a_\xi+ab-c=0$. In this case as is known, the Riemann function for equation (4) has the form ([32], p. 16)

$$R(\xi_1,\eta_1;\xi,\eta)=\exp\Bigg[\int\limits_\eta^{\eta_1}a(\xi,t)\,dt+\int\limits_\xi^{\xi_1}b(\tau,\eta_1)\,d\tau\Bigg]. \tag{100}$$

By virtue of (100), we have

$$\begin{aligned}\frac{\partial R(\xi_1,\eta_1;\xi,\eta)}{\partial\xi_1}&=b(\xi_1,\eta_1)R(\xi_1,\eta_1;\xi,\eta),\\ \frac{\partial R(\xi_1,\eta_1;\xi,\eta)}{\partial\eta_1}&=a(\xi,\eta_1)R(\xi_1,\eta_1;\xi,\eta).\end{aligned} \tag{101}$$

Under the conditions (79), for the kernel $K(\xi,\sigma)$ and for the function $f(\xi)$ from equation (80) and by virtue of (101), we find that

$$\begin{gathered}K(\xi,\sigma)=b(\sigma,0)R(\sigma,0;\xi,\xi)-\frac{\partial R(\sigma,0;\xi,\xi)}{\partial\sigma}=\\ =b(\sigma,0)R(\sigma,0;\xi,\xi)-b(\sigma,0)R(\sigma,0;\xi,\xi),\ \ 0\le\xi,\sigma\le1,\end{gathered} \tag{102}$$

$$\begin{gathered}f(\xi)=\varphi(\xi)\exp\Bigg[-\int\limits_0^\xi b(\tau,\xi)\,d\tau\Bigg]-\varphi(0)\exp\Bigg[-\int\limits_0^\xi b(\tau,0)\,d\tau-\int\limits_0^\xi a(\xi,t)\,dt\Bigg]+\\ +\int\limits_0^\xi\Bigg\{a(0,\tau)\exp\Bigg[-\int\limits_0^\xi b(t,\tau)\,dt-\int\limits_\tau^\xi a(\xi,t)\,dt\Bigg]-\\ -\Bigg(a(\xi,\tau)-\int\limits_0^\xi b_\eta(t,\tau)\,dt\Bigg)\exp\Bigg[-\int\limits_0^\xi b(t,\tau)\,dt-\int\limits_\tau^\xi a(\xi,t)\,dt\Bigg]\Bigg\}\varphi(\tau)\,d\tau+\\ +\int\limits_0^\xi d\sigma\int\limits_0^\xi g(\sigma,\tau)\exp\Bigg[-\int\limits_\sigma^\xi b(t,\tau)\,dt-\int\limits_\tau^\xi a(\xi,t)\,dt\Bigg]d\tau,\ \ 0\le\xi\le1.\end{gathered} \tag{103}$$

By virtue of (91) and (102), equation (80) takes the form

$$\Big(1-\exp\Big[-\int\limits_0^{\xi} a(\xi,t)\,dt\Big]\Big)\psi(\xi)=f(\xi),\quad 0\le\xi\le 1. \tag{104}$$

It follows from (104) that for the problem (1), (2), (3) in the class $C(\overline{D})$ to be solvable, it is necessary and sufficient that the condition

$$f(\xi)\Big(1-\exp\Big[-\int\limits_0^{\xi} a(\xi,t)\,dt\Big]\Big)^{-1}\in C([0,1]) \tag{105}$$

be fulfilled.

Introduce into consideration the set $S:=\{\xi\in[0,1]:\int_0^{\xi}a(\xi,t)dt=0\}$. Obviously, $S\neq\varnothing$ because $\{0\}\in S$. When $a\big|_{\overline{\Omega}}\neq 0$, we have $S=\{0\}$, and in this case, proceeding from the representation (103) for the function f and relying on the de L'Hospital rule, it is not difficult to show that if one additionally requires of the function $\varphi\in C([0,1])$ that $\varphi\in C^1([0,\varepsilon])$ for arbitrarily small positive ε, then the condition (105) will be fulfilled, and the problem (1), (2), (3) uniquely solvable in the class $C(\overline{D})$.

Proceeding from equation (104), it is easy to show that if the set S has interior points, i.e., there exists the interval (d_1,d_2), $0\le d_1<d_2\le 1$ such that $(d_1,d_2)\subset S$, then the homogeneous problem, corresponding to the problem (1), (2), (3) has an infinite set of linearly independent solutions.

4.5. Let now the second Laplace invariant be equal to zero, i.e., $k=b_\eta+ab-c=0$. In this case, the Riemann function for equation (4) is written in the form ([32], p. 15)

$$R(\xi,\eta_1;\xi,\eta)=\exp\Big[\int\limits_{\eta}^{\eta_1} a(\xi_1,t)\,dt+\int\limits_{\xi}^{\xi_1} b(t,\eta)\,dt\Big]. \tag{106}$$

From (106) we have

$$\begin{aligned}\frac{\partial R(\xi_1,\eta_1;\xi,\eta)}{\partial\xi_1}&=b(\xi_1,\eta)R(\xi_1,\eta_1;\xi,\eta),\\ \frac{\partial R(\xi_1,\eta_1;\xi,\eta)}{\partial\eta_1}&=a(\xi_1,\eta_1)R(\xi_1,\eta_1;\xi,\eta).\end{aligned} \tag{107}$$

Under the conditions (79), for the kernel $K(\xi,\sigma)$ and for the function $f(\xi)$ from equation (80) by virtue of (107), we have

$$\begin{aligned}K(\xi,\sigma)&=b(\sigma,0)R(\sigma,0;\xi,\xi)-\frac{\partial R(\sigma,0;\xi,\xi)}{\partial\sigma}=\\ &=b(0,0)R(\sigma,0;\xi,\xi)-b(\sigma,\xi)R(\sigma,0;\xi,\xi)=\\ &=\big(b(\sigma,0)-b(\sigma,\xi)\big)R(\sigma,0;\xi,\xi)=\end{aligned}$$

$$= \big(b(\sigma,0) - b(\sigma,\xi)\big) \exp\bigg[-\int\limits_0^\xi a(\sigma,t)\,dt - \int\limits_\sigma^\xi b(t,\xi)\,dt\bigg], \tag{108}$$

$$f(\xi) = \varphi(\xi)\exp\bigg[-\int\limits_0^\xi b(t,\xi)\,dt\bigg] - \varphi(0)\exp\bigg[-\int\limits_0^\xi a(0,t)\,dt - \int\limits_0^\xi b(t,\xi)\,dt\bigg] +$$

$$+\int\limits_0^\xi \bigg\{a(0,\tau)\exp\bigg[-\int\limits_\tau^\xi a(0,t)\,dt - \int\limits_0^\xi b(t,\xi)\,dt\bigg] -$$

$$-a(0,\tau)\exp\bigg[-\int\limits_\tau^\xi a(0,t)\,dt - \int\limits_0^\xi b(t,\xi)\,dt\bigg]\bigg\}\varphi(\tau)\,d\tau +$$

$$+\int\limits_0^\xi d\sigma \int\limits_0^\xi g(\sigma,\tau)\exp\bigg[-\int\limits_\tau^\xi a(\sigma,t)\,dt - \int\limits_\sigma^\xi b(t,\xi)\,dt\bigg]\,d\tau =$$

$$= \bigg(\varphi(\xi) - \varphi(0)\exp\bigg[-\int\limits_0^\xi a(0,t)\,dt\bigg]\bigg)\exp\bigg[-\int\limits_0^\xi b(t,\xi)\,dt\bigg] +$$

$$+\int\limits_0^\xi d\sigma \int\limits_0^\xi g(\sigma,\tau)\exp\bigg[-\int\limits_\tau^\xi a(\sigma,t)\,dt - \int\limits_\sigma^\xi b(t,\xi)\,dt\bigg]\,d\tau. \tag{109}$$

Assuming that the condition (82) is fulfilled, to prove that the condition (83) is valid, it is sufficient to show that there exists the limit of the ratio $\frac{K(\xi,\sigma)}{\mu(\xi)}$ as $\xi \to 0+$ uniformly with respect to the parameter $\sigma \in [0,1]$. Since by our assumption the coefficients a, $b \in C^1(\overline{\Omega})$ we have

$$b(\sigma,0) - b(\sigma,\xi) = -\int\limits_0^\xi b_\eta(\sigma,\eta)\,d\eta = -\xi\int\limits_0^1 b_\eta(\sigma,\xi t)\,dt. \tag{110}$$

Next, assuming that $\mu(\xi) = \mu_0(0) - \mu_0(\xi)$, where

$$\mu_0(\xi) = \exp\bigg[-\int\limits_0^\xi a(\xi,\tau)\,d\tau\bigg],$$

analogously to (110) we have

$$\mu(\xi) = -\int\limits_0^\xi \frac{d\mu_0(\sigma)}{d\sigma} = -\int\limits_0^\xi \bigg[-\int\limits_0^\sigma a_\xi(\sigma,\tau)\,d\tau - a(\sigma,\sigma)\bigg]\mu_0(\sigma)\,d\sigma =$$

$$= \xi \int_0^1 \left[\int_0^{\xi t} a_\xi(\xi t, \tau)\, d\tau \right] \mu_0(\xi t)\, dt. \tag{111}$$

Equation (111) results in

$$\lim_{\xi\to 0} \frac{\mu(\xi)}{\xi} = -a(0,0). \tag{112}$$

Under the assumption that

$$a(0,0) \neq 0 \tag{113}$$

from (108), (110)–(113) it immediately follows that

$$\lim_{\xi\to 0} \frac{K(\xi,\sigma)}{\mu(\xi)} = \frac{b_\eta(\sigma,0) \exp\big[\int_0^\sigma b(t,0)\, dt\big]}{a(0,0)}, \tag{114}$$

where the limit (114) exists uniformly with respect to the parameter $\sigma \in [0,1]$.

Reasoning analogously, from (109), (112) and (113) we obtain

$$\lim_{\xi\to 0} \frac{f(\xi)}{\mu(\xi)} = \lim_{\xi\to 0} \frac{f(\xi)}{\xi} \lim_{\xi\to 0} \frac{\xi}{\mu(\xi)} = -\frac{\varphi'(0) + \varphi(0)a(0,0)}{a(0,0)},$$

from which in its turn it follows that $\frac{f(\xi)}{\mu(\xi)} \in C([0,1])$.

Thus if the conditions (79), (82), (93) and (113) are fulfilled for $k = b_\eta + ab - c = 0$, the conditions (83) and (84) are likewise fulfilled, and for $a_4 \in C(\overline{D})$ the problem (1), (2), (3) is uniquely solvable in the class $C(\overline{D})$.

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