ISSN 0012-2661, Differential Equations, 2006, Vol. 42, No. 2, pp. 279–290. © Pleiades Publishing, Inc., 2006.
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PARTIAL DIFFERENTIAL EQUATIONS

On the Nonexistence of Global Solutions of the Characteristic Cauchy Problem for a Nonlinear Wave Equation in a Conical Domain

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Received July 2, 2004

DOI: 10.1134/S0012266106020157

1. STATEMENT OF THE PROBLEM

For the nonlinear wave equation

$$\Box u := u_{tt} - \Delta u = \lambda |u|^{\alpha} + F, \tag{1}$$

where λ and α are given positive constants, F is a given real function, and u is the unknown real function, we consider the characteristic Cauchy problem of finding a solution $u(x,t)$ of Eq. (1) in the future light cone $D\colon\ t > |x|$, $x = (x_1, \ldots, x_n)$, $n > 1$, with the boundary condition

$$u|_{\partial D} = f. \tag{2}$$

Here f is a given real function on the characteristic cone $\partial D\colon\ t = |x|$.

Note that existence and nonexistence issues for global solutions of the Cauchy problem for semilinear equations of the form (1) with the initial conditions $u|_{t=0} = u_0$ and $u_t|_{t=0} = u_1$ were considered in [1–17].

As to the characteristic problem in the linear case, that is, problem (1), (2) with $\lambda = 0$, this problem is known to be well posed and globaly solvable in appropriate function spaces [18–22].

In what follows, we show that, under certain conditions on the nonlinearity exponent α and the functions F and f, problem (1), (2) has no global solutions, although, as will be justified below, the problem is locally solvable.

Before introducing the notion of a weak generalized solution of problem (1), (2), note that if $u \in C^2\left(\bar{D}\right)$ is a classical solution of this problem, then, by multiplying both sides of Eq. (1) by an arbitrary function $\varphi \in C^1\left(\bar{D}\right)$ compactly supported with respect to the variable $r = \left(t^2 + |x|^2\right)^{1/2}$, i.e., vanishing for sufficiently large r, and by integrating by parts, we obtain

$$\int\limits_{\partial D} \frac{\partial u}{\partial N} \varphi \, ds - \int\limits_{D} u_t \varphi_t dx\, dt + \int\limits_{D} \nabla u \nabla \varphi \, dx\, dt = \lambda \int\limits_{D} |u|^{\alpha} \varphi \, dx\, dt + \int\limits_{D} F \varphi \, dx\, dt, \tag{3}$$

where

$$\frac{\partial}{\partial N} = \nu_{n+1} \frac{\partial}{\partial t} - \sum_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i} \tag{4}$$

is the conormal derivative, $\nu = (\nu_1, \nu_2, \ldots, \nu_n, \nu_{n+1})$ is the unit outward normal on ∂D, and

$$\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n).$$

Since the conormal derivative (4) is an intrinsic differential operator on the characteristic cone $\partial D\colon\ t = |x|$, it follows from (2) that relation (3) can be rewritten in the form

$$-\int\limits_{D} u_t\varphi_t dx\,dt + \int\limits_{D} \nabla u\nabla\varphi\,dx\,dt = \lambda\int\limits_{D} |u|^{\alpha}\varphi\,dx\,dt + \int\limits_{D} F\varphi\,dx\,dt - \int\limits_{\partial D}\frac{\partial f}{\partial N}\varphi\,ds. \qquad (5)$$

Relation (5) can be used as a basis of the definition of a weak generalized solution of problem (1), (2).

Definition 1. Let $F \in \tilde{L}_{2,\mathrm{loc}}(D)$ and $f \in \tilde{W}^1_{2,\mathrm{loc}}(\partial D)$. A function $u \in \tilde{L}_{\alpha,\mathrm{loc}}(D) \cap \tilde{W}^1_{2,\mathrm{loc}}(D)$ is called a *weak generalized solution* of problem (1), (2) if the integral relation (5) is valid for each function $\varphi \in C^1\left(\bar{D}\right)$ compactly supported with respect to the variable $r = \left(t^2 + |x|^2\right)^{1/2}$. Such a solution is also referred to as a *global solution* of problem (1), (2).

Here the space $\tilde{L}_{\alpha,\mathrm{loc}}(D)$ [respectively, $\tilde{W}^1_{2,\mathrm{loc}}(\partial D)$] consists of functions F (respectively, f) whose restriction to the set $D\cap\{t<\tau\}$ (respectively, $\partial D\cap\{t<\tau\}$) belongs to the space $L_{\alpha}(D\cap\{t<\tau\})$ [respectively, $W^1_2(\partial D\cap\{t<\tau\})$] for each $\tau > 0$. The spaces $\tilde{L}_{\alpha,\mathrm{loc}}(D)$ and $\tilde{W}^1_{2,\mathrm{loc}}(D)$ are defined in a similar way. The space $W^1_2(\Omega)$ is the well-known Sobolev space.

In a similar way, one can pose the characteristic problem for Eq. (1) in the finite domain $D_{\tau} = D\cap\{t<\tau\}$, $\tau = \mathrm{const} > 0$, i.e., $D_{\tau}\colon\ |x| < t < \tau$. We set $S_{\tau} = \partial D \cap \partial D_{\tau}$, i.e., $S_{\tau}\colon\ t = |x|$ for $t \le \tau$.

Definition 2. Let $F \in L_2\left(D_{\tau}\right)$ and $f \in W^1_2\left(S_{\tau}\right)$. A function $u \in L_{\alpha}\left(D_{\tau}\right) \cap W^1_2\left(D_{\tau}\right)$ is called a *weak generalized solution* of Eq. (1) in the domain D_{τ} with the boundary condition $u|_{S_{\tau}} = f$ instead of (2) if the integral relation

$$-\int\limits_{D_{\tau}} u_t\varphi_t dx\,dt + \int\limits_{D_{\tau}} \nabla u\nabla\varphi\,dx\,dt = \lambda\int\limits_{D_{\tau}} |u|^{\alpha}\varphi\,dx\,dt + \int\limits_{D_{\tau}} F\varphi\,dx\,dt - \int\limits_{S_{\tau}}\frac{\partial f}{\partial N}\varphi\,ds \qquad (6)$$

is valid for each function $\varphi \in C^1\left(\bar{D}_{\tau}\right)$ such that $\varphi|_{\partial D_{\tau}\setminus S_{\tau}} = 0$.

2. NONEXISTENCE OF A GLOBAL SOLUTION OF PROBLEM (1), (2)

Theorem 1. *Let*

$$F \in \tilde{L}_{2,\mathrm{loc}}(D), \qquad F|_D \ge 0, \qquad (7)$$

$$f \in \tilde{W}^1_{2,\mathrm{loc}}(\partial D), \qquad f|_{\partial D} \ge 0, \qquad \left.\frac{\partial f}{\partial r}\right|_{\partial D} \ge 0. \qquad (8)$$

If the nonlinearity exponent α in Eq. (1) *satisfies the inequalities*

$$1 < \alpha \le \frac{n+1}{n-1}, \qquad (9)$$

then, apart from the trivial solution for $F = f = 0$, problem (1), (2) *has no global weak generalized solution $u \in \tilde{L}_{\alpha,\mathrm{loc}}(D) \cap \tilde{W}^1_{2,\mathrm{loc}}(D)$.*

Proof. Note that the last inequality in condition (8) should be treated in the generalized sense: by virtue of the assumption that $f \in \tilde{W}^1_{2,\mathrm{loc}}(\partial D)$, there exists a generalized derivative $\partial f/\partial r$ belonging to $\tilde{L}_{2,\mathrm{loc}}(\partial D)$, which is nonnegative, and consequently, the inequality

$$\int\limits_{\partial D}\frac{\partial f}{\partial r}\psi\,ds \ge 0 \qquad (10)$$

is valid for each function $\psi \in C(\partial D)$, $\psi \ge 0$, compactly supported with respect to the variable r.

We use the method of test functions [14, pp. 10–12]. Suppose that, under the assumptions of the theorem, there exists a nontrivial global weak generalized solution $u \in \tilde{L}_{\alpha,\mathrm{loc}}(D) \cap \tilde{W}^1_{2,\mathrm{loc}}(D)$ of problem (1), (2).

By assuming that $\varphi \in C^2\left(\bar{D}\right)$ and $\operatorname{diam}\operatorname{supp}\varphi < +\infty$ in the integral relation (5), by integrating by parts on the left-hand side in this relation, and by taking into account the boundary condition (2), we obtain

$$\begin{aligned} -\int\limits_D u_t\varphi_t dx\,dt + \int\limits_D \nabla u \nabla\varphi\,dx\,dt &= \int\limits_D u\Box\varphi\,dx\,dt - \int\limits_{\partial D} u\frac{\partial\varphi}{\partial N}ds \\ &= \int\limits_D u\Box\varphi\,dx\,dt - \int\limits_{\partial D} f\frac{\partial\varphi}{\partial N}ds. \end{aligned} \tag{11}$$

Now, by using the fact that the conormal derivative $\partial/\partial N$ on ∂D coincides with minus the derivative with respect to the spherical variable $r = \left(t^2 + |x|^2\right)^{1/2}$ and by choosing the test function in the form $\varphi(x,t) = \varphi_0\left[R^{-2}\left(t^2 + |x|^2\right)\right]$, where $\varphi_0 \in C^2((-\infty,+\infty))$, $\varphi_0 \geq 0$, $\varphi_0' \leq 0$, $\varphi_0(\sigma) = 1$ for $0 \leq \sigma \leq 1$, and $\varphi_0(\sigma) = 0$ for $\sigma \geq 2$, $R = \mathrm{const} > 0$ [14, p. 22], from (7), (8), and (10), we obtain

$$\int\limits_D F\varphi\,dx\,dt \geq 0, \qquad \int\limits_{\partial D} f\frac{\partial\varphi}{\partial N}ds \geq 0, \qquad \int\limits_{\partial D} \frac{\partial f}{\partial N}\varphi\,ds \leq 0. \tag{12}$$

Relation (5), together with (11) and (12), implies that

$$\int\limits_D u\Box\varphi\,dx\,dt \geq \lambda\int\limits_D |u|^\alpha\varphi\,dx\,dt. \tag{13}$$

By using the Hölder inequality

$$\int\limits_D g_1 g_2 dx\,dt \leq \left(\int\limits_D |g_1|^\alpha\,dx\,dt\right)^{1/\alpha}\left(\int\limits_D |g_2|^{\alpha'}\,dx\,dt\right)^{1/\alpha'}, \qquad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1,$$

we obtain

$$\begin{aligned} \int\limits_D u\Box\varphi\,dx\,dt &\leq \int\limits_D \left(|u|\varphi^{1/\alpha}\right)\left(\varphi^{-1/\alpha}|\Box\varphi|\right)dx\,dt \\ &\leq \left(\int\limits_D |u|^\alpha\varphi\,dx\,dt\right)^{1/\alpha}\left(\int\limits_D \varphi^{-\alpha'/\alpha}|\Box\varphi|^{\alpha'}dx\,dt\right)^{1/\alpha'} \\ &= \left(\int\limits_D |u|^\alpha\varphi\,dx\,dt\right)^{1/\alpha}\left(\int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}dx\,dt\right)^{1/\alpha'}. \end{aligned} \tag{14}$$

It follows from (13) and (14) that

$$\lambda\int\limits_D |u|^\alpha\varphi\,dx\,dt \leq \left(\int\limits_D |u|^\alpha\varphi\,dx\,dt\right)^{1/\alpha}\left(\int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}dx\,dt\right)^{1/\alpha'},$$

which readily implies the inequality

$$\int\limits_D |u|^\alpha\varphi\,dx\,dt \leq \lambda^{-\alpha'}\int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}dx\,dt. \tag{15}$$

After the change of variables $t = R\xi_0$, $x = R\xi$, we have $\varphi(x,t) = \varphi_0\left(\xi_0^2 + |\xi|^2\right)$ and

$$\begin{aligned}\int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dx\,dt &= \int\limits_D \frac{\left|2(1-n)\varphi_0' + 4R^{-2}\left(t^2 - |x|^2\right)\varphi_0''\right|^{\alpha'}}{R^{2\alpha'}\varphi^{\alpha'-1}} dx\,dt \\ &= R^{n+1-2\alpha'} \int\limits_{\substack{1\le|\xi_0|^2+|\xi|^2\le 2,\\ \xi_0>|\xi|}} \frac{\left|2(1-n)\varphi_0' + 4\left(\xi_0^2 - |\xi|^2\right)\varphi_0''\right|^{\alpha'}}{\varphi_0^{\alpha'-1}} d\xi\,d\xi_0.\end{aligned} \tag{16}$$

The existence of a test function $\varphi(x,t) = \varphi_0\left[R^{-2}\left(t^2 + |x|^2\right)\right]$ with the above-mentioned properties for which the integrals on the right-hand sides in (15) and (16) are finite was proved in [14, p. 22].

From (15) and (16), we obtain the *a priori* estimate

$$\int\limits_D |u|^\alpha \varphi\,dx\,dt \le CR^{n+1-2\alpha'} \tag{17}$$

with a positive constant C independent of R. By passing to the limit as $R \to \infty$ in (17) for the case in which $n+1-2\alpha' < 0$ [if $n > 1$, then this is equivalent to the condition $\alpha < (n+1)/(n-1)$], we obtain $\int_D |u|^\alpha dx\,dt = 0$ and hence arrive at a contradiction with our assumption. The limit case $n+1-2\alpha' = 0$, i.e., $\alpha = (n+1)/(n-1)$, in condition (9) can be treated by analogy with [14, p. 23]. The proof of the theorem is complete.

Remark 1. Although, under the assumptions of Theorem 1, problem (1), (2) has no global solutions, there may exist a local solution of the characteristic problem in the domain D_τ in the sense of Definition 2, that is, of the problem

$$\Box u(x,t) = \lambda|u(x,t)|^\alpha + F(x,t), \qquad (x,t) \in D_\tau, \tag{18}$$

$$u(x,t) = f(x,t), \qquad (x,t) \in S_\tau. \tag{19}$$

Therefore, we naturally face the problem of estimating the number $t = T$ such that problem (18), (19) has a solution in the domain D_τ for $\tau < T$ but has no solution in the space $L_\alpha(D_\tau) \cap W_2^1(D_\tau)$ for $\tau \ge T$.

To this end, we suppose that $u \in L_\alpha(D_\tau) \cap W_2^1(D_\tau)$ is a solution of problem (18), (19) in the domain D_τ in the sense of the integral relation (6). For the test function in (6), we take the function $\varphi(x,t) = \varphi_0\left[(2/\tau^2)\left(t^2 + |x|^2\right)\right]$, where $\varphi_0 \in C^2((-\infty,+\infty))$ is the function introduced above in the proof of Theorem 1. Obviously, this function satisfies all assumptions in Definition 2. By integrating by parts on the left-hand side in (6), just as in (11), we obtain

$$\int\limits_{D_\tau} u\Box\varphi\,dx\,dt = \lambda\int\limits_{D_\tau} |u|^\alpha\varphi\,dx\,dt + \int\limits_{D_\tau} F\varphi\,dx\,dt + \int\limits_{S_\tau} f\frac{\partial\varphi}{\partial N}ds - \int\limits_{S_\tau} \frac{\partial f}{\partial N}\varphi\,ds. \tag{20}$$

By analogy with (12), by (7) and (8), we have the inequalities

$$\int\limits_{D_\tau} F\varphi\,dx\,dt \ge 0, \qquad \int\limits_{S_\tau} f\frac{\partial\varphi}{\partial N}ds \ge 0, \qquad \int\limits_{S_\tau} \frac{\partial f}{\partial N}\varphi\,ds \le 0. \tag{21}$$

We assume that F, f, and φ are given functions and introduce a function of one variable τ by setting

$$\gamma(\tau) = \int\limits_{D_\tau} F\varphi\,dx\,dt + \int\limits_{S_\tau} f\frac{\partial\varphi}{\partial N}ds - \int\limits_{S_\tau} \frac{\partial f}{\partial N}\varphi\,ds, \qquad \tau > 0. \tag{22}$$

By virtue of the absolute continuity of the integral and inequalities (21), the function $\gamma(\tau)$ given by (22) is nonnegative, continuous, and nondecreasing; moreover,

$$\lim_{\tau\to 0}\gamma(\tau) = 0. \tag{23}$$

Taking into account (22), we rewrite relation (20) in the form

$$\lambda \int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt = \int\limits_{D_\tau} u\Box\varphi \, dx \, dt - \gamma(\tau). \tag{24}$$

In the Young inequality $ab \le (\varepsilon/\alpha)a^\alpha + \left(\alpha'\varepsilon^{\alpha'-1}\right)^{-1} b^{\alpha'}$, $a, b \ge 0$, $\alpha' = \alpha/(\alpha-1)$, with parameter $\varepsilon > 0$, we set $a = |u|\varphi^{1/\alpha}$ and $b = |\Box\varphi|/\varphi^{1/\alpha}$. Then, by virtue of the relation $\alpha'/\alpha = \alpha' - 1$, we have

$$|u\Box\varphi| = |u|\varphi^{1/\alpha}\frac{|\Box\varphi|}{\varphi^{1/\alpha}} \le \frac{\varepsilon}{\alpha}|u|^\alpha\varphi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}. \tag{25}$$

Relation (24), together with (25), implies that

$$\left(\lambda - \frac{\varepsilon}{\alpha}\right) \int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt \le \frac{1}{\alpha'\varepsilon^{\alpha'-1}} \int\limits_{D_\tau} \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dx \, dt - \gamma(\tau),$$

whence it follows that

$$\int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt \le \frac{\alpha}{(\lambda\alpha - \varepsilon)\alpha'\varepsilon^{\alpha'-1}} \int\limits_{D_\tau} \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dx \, dt - \frac{\alpha}{\lambda\alpha - \varepsilon}\gamma(\tau) \tag{26}$$

for $\varepsilon < \lambda\alpha$.

By using the relations

$$\alpha' = \frac{\alpha}{\alpha - 1}, \qquad \alpha = \frac{\alpha'}{\alpha' - 1}, \qquad \min_{0<\varepsilon<\lambda\alpha} \frac{\alpha}{(\lambda\alpha - \varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}},$$

(where the minimum is attained at $\varepsilon = \lambda$), from (26), we obtain

$$\int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt \le \frac{1}{\lambda^{\alpha'}} \int\limits_{D_\tau} \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dx \, dt - \frac{\alpha'}{\lambda}\gamma(\tau). \tag{27}$$

By the properties of the function φ_0, we have

$$\varphi(x, t) = \varphi_0\left[2\tau^{-2}\left(t^2 + |x|^2\right)\right] = 0$$

for $r = \left(t^2 + |x|^2\right)^{1/2} \ge \tau$. Therefore, after the change of variables $t = \sqrt{2}\,\tau\xi_0$, $x = \sqrt{2}\,\tau\xi$, just as in the derivation of (16), one can readily see that

$$\int\limits_{D_\tau} \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dx \, dt = \int\limits_{r=(t^2+|x|^2)^{1/2}\le\tau} \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dx \, dt = \left(\sqrt{2}\,\tau\right)^{n+1-2\alpha'} \varkappa_0, \tag{28}$$

where

$$\varkappa_0 = \int\limits_{1\le|\xi_0|^2+|\xi|^2\le 2} \frac{\left|2(1-n)\varphi_0' + 4\left(\xi_0^2 - |\xi|^2\right)\varphi_0''\right|^{\alpha'}}{\varphi_0^{\alpha'-1}} d\xi \, d\xi_0 < +\infty.$$

Since $\varphi_0(\sigma) = 1$ for $0 \le \sigma \le 1$, it follows from (27) and (28) that

$$\int\limits_{r\le\tau/\sqrt{2}} |u|^\alpha dx \, dt \le \int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt \le \frac{\left(\sqrt{2}\,\tau\right)^{n+1-2\alpha'}}{\lambda^{\alpha'}}\varkappa_0 - \frac{\alpha'}{\lambda}\gamma(\tau). \tag{29}$$

For $\alpha < (n+1)/(n-1)$, i.e., for $n+1-2\alpha' < 0$, the equation

$$g(\tau) = \frac{\left(\sqrt{2}\,\tau\right)^{n+1-2\alpha'}}{\lambda^{\alpha'}}\varkappa_0 - \frac{\alpha'}{\lambda}\gamma(\tau) = 0 \tag{30}$$

has a unique positive root $\tau = \tau_0 > 0$, since

$$g_1(\tau) = \left(\left(\sqrt{2}\,\tau\right)^{n+1-2\alpha'}\Big/\lambda^{\alpha'}\right)\varkappa_0$$

is a positive continuous strictly decreasing function on the interval $(0, +\infty)$; moreover,

$$\lim_{\tau\to 0} g_1(\tau) = +\infty, \qquad \lim_{\tau\to+\infty} g_1(\tau) = 0$$

and, as was mentioned above, $\gamma(\tau)$ is a nonnegative continuous nondecreasing function. Since we assume that at least one of the functions F and f is not trivial, we have $\lim_{\tau\to+\infty}\gamma(\tau) > 0$. Furthermore, $g(\tau) < 0$ for $\tau > \tau_0$, and $g(\tau) > 0$ for $0 < \tau < \tau_0$. Consequently, if $\tau > \tau_0$, then the right-hand side of (29) is negative, which is impossible. Therefore, if problem (18), (19) has a solution in the domain D_τ, then necessarily $\tau \le \tau_0$ and hence the estimate

$$T \le \tau_0 \tag{31}$$

is valid for the number $\tau = T$ in Remark 1, where τ_0 is the unique positive root of Eq. (30).

In the limit case $\alpha = (n+1)/(n-1)$ for $n+1-2\alpha' = 0$, if

$$\lim_{\tau\to+\infty}\gamma(\tau) > \frac{\varkappa_0}{\alpha'\lambda^{\alpha'-1}}, \tag{32}$$

then we use exactly the same argument as in the case $\alpha < (n+1)/(n-1)$ and again obtain the estimate (31), where τ_0 is the least positive root of Eq. (30), whose existence is guaranteed by (32).

Remark 2. Since the right-hand sides in Eq. (1) and the boundary condition (2), as well as the derivative $\partial f/\partial r$, are nonnegative under conditions (7) and (8), it follows from well-known properties of the solution of a linear characteristic problem [19, p. 745 of the Russian translation; 22, p. 84] that the solution $u(x,t)$ of the nonlinear problem (1), (2) is also nonnegative for $n = 2$ and $n = 3$. But in this case, if $\alpha = 1$, then the above-mentioned solution satisfies the linear problem

$$\Box u = \lambda u + F, \qquad u|_{\partial D} = f,$$

which is globally solvable in the corresponding function spaces.

Remark 3. If $0 < \alpha < 1$, then problem (1), (2) can have more than one solution. For example, if $F = 0$ and $f = 0$, then conditions (7) and (8) are satisfied, but problem (1), (2) has (in addition to the trivial solution) infinitely many global linearly independent solutions $u_\sigma(x,t)$ depending on the parameter $\sigma \ge 0$ and given by the formula

$$u_\sigma(x,t) = \begin{cases} \beta\left[(t-\sigma)^2 - |x|^2\right]^{1/(1-\alpha)} & \text{if } \ t > \sigma + |x| \\ 0 & \text{if } \ |x| \le t \le \sigma + |x|, \end{cases}$$

where $\beta = \lambda^{1/(1-\alpha)}\left[4\alpha/(1-\alpha)^2 + 2(n+1)/(1-\alpha)\right]^{-1/(1-\alpha)}$. One can readily see that $u_\sigma(x,t) \in \tilde{L}_{\alpha,\mathrm{loc}}(D) \cap \tilde{W}^1_{2,\mathrm{loc}}(D)$. Moreover, $u_\sigma(x,t) \in C^1\left(\bar{D}\right)$, and if $1/2 \le \alpha < 1$, then $u_\sigma(x,t) \in C^2\left(\bar{D}\right)$.

Remark 4. The assertion of Theorem 1 becomes invalid if relation (9) is replaced by the inequality $\alpha > (n+1)/(n-1)$, and, at the same time, only the second condition in (8), i.e., the condition $f|_{\partial D} \ge 0$, is violated. Indeed, the function

$$u(x,t) = -\varepsilon\left(1 + t^2 - |x|^2\right)^{1/(1-\alpha)}, \qquad \varepsilon = \text{const} > 0,$$

is a global classical and hence generalized solution of problem (1), (2) with $f = -\varepsilon$ $(\partial f/\partial r|_{\partial D} = 0)$, and

$$F = \left[2\varepsilon\frac{n+1}{\alpha-1} - 4\varepsilon\frac{\alpha}{(\alpha-1)^2}\frac{t^2-|x|^2}{1+t^2-|x|^2} - \lambda\varepsilon^\alpha\right]\left(1+t^2-|x|^2\right)^{\alpha/(1-\alpha)};$$

moreover, one can readily see that $F|_D \geq 0$ if $\alpha > (n+1)/(n-1)$ and

$$0 < \varepsilon \leq \left\{\frac{2}{\lambda}\left[\frac{n+1}{\alpha-1} - \frac{2\alpha}{(\alpha-1)^2}\right]\right\}^{1/(\alpha-1)}.$$

Note that the inequality $n+1-2\alpha/(\alpha-1) > 0$ is equivalent to the inequality $\alpha > (n+1)/(n-1)$.

Remark 5. The assertion of Theorem 1 becomes invalid if only the third condition in (8), i.e., the condition $\partial f/\partial r|_{\partial D} \geq 0$, is violated. Indeed, the function $u(x,t) = \beta\left[(t+1)^2 - |x|^2\right]^{1/(1-\alpha)}$, where $\beta = \lambda^{1/(1-\alpha)}\left[4\alpha/(1-\alpha)^2 + 2(n+1)/(1-\alpha)\right]^{1/(\alpha-1)}$, is a global classical solution of problem (1), (2) for $F = 0$, and $f = u|_{\partial D:\, t=|x|} = \beta\left[(t+1)^2 - t^2\right]^{1/(1-\alpha)} > 0$.

3. LOCAL SOLVABILITY OF THE CHARACTERISTIC CAUCHY PROBLEM

In what follows, we restrict our considerations to problem (18), (19) in the domain D_τ with the homogeneous boundary condition (19):

$$u|_{S_\tau} = 0. \tag{33}$$

First, consider the linear case in which $\lambda = 0$ in Eq. (18), that is, the problem

$$Lu(x,t) = F(x,t), \qquad (x,t) \in D_\tau, \qquad u(x,t) = 0, \qquad (x,t) \in S_\tau, \tag{34}$$

where, for convenience, we have introduced the notation $L = \Box$ $(= \partial^2/\partial t^2 - \Delta)$.

Definition 3. Let $F \in L_2(D_\tau)$. A function

$$u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) = \left\{u \in W_2^1(D_\tau) :\ u|_{S_\tau} = 0\right\}$$

is called a *strong generalized solution* of problem (34) if there exists a sequence of functions $u_m = W_2^2(D_\tau) \cap \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{W_2^1(D_\tau)} = 0, \qquad \lim_{m\to\infty} \|Lu_m - F\|_{L_2(D_\tau)} = 0.$$

To derive the desired *a priori* estimate for a solution $u \in W_2^2(D_\tau)$ of problem (34), we use the argument in [23]. By multiplying both sides of Eq. (34) by $2u_t$, by integrating the resulting relation over the domain D_δ, $0 < \delta \leq \tau$, and by performing simple transformations with the use of integration by parts, we obtain

$$\int_{\Omega_\delta}\left[u_t^2 + \sum_{i=1}^n u_{x_i}^2\right]dx = 2\int_{D_\delta} Fu_t\,dx\,dt, \tag{35}$$

where $\Omega_\delta = D_\tau \cap \{t = \delta\}$. By setting $w(\delta) = \int_{\Omega_\delta}\left[u_t^2 + \sum_{i=1}^n u_{x_i}^2\right]dx$ and by using the inequality $2Fu_t \leq \varepsilon u_t^2 + \varepsilon^{-1}F^2$, from (35), we obtain

$$w(\delta) \leq \varepsilon\int_0^\delta w(\sigma)d\sigma + \frac{1}{\varepsilon}\|F\|^2_{L_2(D_\delta)}, \qquad 0 < \delta \leq \tau, \tag{36}$$

for each $\varepsilon = \text{const} > 0$.

Since $\|F\|^2_{L_2(D_\delta)}$ is a nondecreasing function of δ, it follows from (36) and the Gronwall lemma [24, p. 13 of the Russian translation] that $w(\delta) \le \varepsilon^{-1}\|F\|^2_{L_2(D_\delta)} \exp\delta\varepsilon$; therefore, by virtue of the relation $\inf_{\varepsilon>0}(\exp\delta\varepsilon)/\varepsilon = e\delta$ and the fact that this greatest lower bound is attained at $\varepsilon = 1/\delta$, the last inequality acquires the form $w(\delta) \le e\delta\|F\|^2_{L_2(D_\delta)}$. In turn, this implies that

$$\int\limits_{D_\tau}\left[u_t^2 + \sum_{i=1}^n u_{x_i}^2\right] dx\,dt = \int\limits_0^\tau w(\sigma)d\sigma \le e\tau^2\|F\|^2_{L_2(D_\tau)},$$

and consequently,

$$\|u\|_{\overset{\circ}{W}{}^1_2(D_\tau,S_\tau)} \le \sqrt{e}\,\tau\|F\|_{L_2(D_\tau)}. \tag{37}$$

Here we have used the fact that the norm

$$\|u\|_{W_2^1(D_\tau)} = \left\{\int\limits_{D_\tau}\left[u^2 + u_t^2 + \sum_{i=1}^n u_{x_i}^2\right] dx\,dt\right\}^{1/2}$$

in the space $\overset{\circ}{W}{}^1_2(D_\tau, S_\tau)$ is equivalent to the norm

$$\|u\| = \left\{\int\limits_{D_\tau}\left[u_t^2 + \sum_{i=1}^n u_{x_i}^2\right] dx\,dt\right\}^{1/2}.$$

Since the space $C_0^\infty(D_\tau)$ is dense in $L_2(D_\tau)$, it follows that, for a given $F \in L_2(D_\tau)$, there exists a sequence of functions $F_m \in C_0^\infty(D_\tau)$ such that $\lim_{m\to\infty}\|F_m - F\|_{L_2(D_\tau)} = 0$. For a given m, we continue the function F_m by zero outside D_τ and retain the same notation for the continued function; then we have the inclusion $F_m \in C^\infty\left(R_+^{n+1}\right)$, and for the support of this function, we have $\operatorname{supp} F_m \subset D$, where $R_+^{n+1} = R^{n+1}\cap\{t \ge 0\}$. By u_m we denote the solution of the Cauchy problem $Lu_m = F_m$, $u_m|_{t=0} = 0$, $\partial u_m/\partial t|_{t=0} = 0$. We know that u_m exists, is unique, and belongs to the space $C^\infty\left(R_+^{n+1}\right)$; moreover, since $\operatorname{supp} F_m \subset D$, $u_m|_{t=0} = 0$, and $\partial u_m/\partial t|_{t=0} = 0$, it follows from the geometric properties of the dependence domain of the solution of the wave equation that [25, p. 191 of the Russian translation] $\operatorname{supp} u_m \subset D$: $t > |x|$. Using the same notation for the restriction of u_m to D_τ, one can readily see that $u_m \in W_2^2(D_\tau) \cap \overset{\circ}{W}{}^1_2(D_\tau, S_\tau)$; by (37),

$$\|u_m - u_{m_1}\|_{\overset{\circ}{W}{}^1_2(D_\tau,S_\tau)} \le \sqrt{e}\,\tau\,\|F_m - F_{m_1}\|_{L_2(D_\tau)}. \tag{38}$$

Since $\{F_m\}$ is a Cauchy sequence in $L_2(D_\tau)$, it follows from (38) that $\{u_m\}$ is also a Cauchy sequence in the complete space $\overset{\circ}{W}{}^1_2(D_\tau, S_\tau)$. Therefore, there exists a function $u \in \overset{\circ}{W}{}^1_2(D_\tau, S_\tau)$ such that $\lim_{m\to\infty}\|u_m - u\|_{\overset{\circ}{W}{}^1_2(D_\tau,S_\tau)} = 0$; since $Lu_m = F_m \to F$ in $L_2(D_\tau)$, it follows from Definition 3 that u is a strong generalized solution of problem (34). The uniqueness of a strong generalized solution of problem (34) in the space $\overset{\circ}{W}{}^1_2(D_\tau, S_\tau)$ follows from the *a priori* estimate (37). Consequently, we can represent the solution u of problem (34) in the form $u = L^{-1}F$, where $L^{-1} : L_2(D_\tau) \to \overset{\circ}{W}{}^1_2(D_\tau, S_\tau)$ is a linear continuous operator, whose norm, by virtue of (37), can be estimated as

$$\left\|L^{-1}\right\|_{L_2(D_\tau)\to\overset{\circ}{W}{}^1_2(D_\tau,S_\tau)} \le \sqrt{e}\,\tau. \tag{39}$$

Remark 6. The embedding operator $I : \overset{\circ}{W}{}^1_2(D_\tau, S_\tau) \to L_q(D_\tau)$ is a linear continuous compact operator for $1 < q < 2(n+1)/(n-1)$ and $n > 1$ [26, p. 81]. At the same time, the Nemytskii operator $T : L_q(D_\tau) \to L_2(D_\tau)$ given by the formula $Tu = \lambda|u|^\alpha$ is continuous and bounded if $q \ge 2\alpha$

[27, p. 349; 28, pp. 66–67 of the Russian translation]. Therefore, if $\alpha < (n+1)/(n-1)$, then there exists a number q such that $1 < 2\alpha \le q < 2(n+1)/(n-1)$ and hence the operator

$$T_0 = TI : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to L_2(D_\tau) \tag{40}$$

is a continuous compact operator. Moreover, the inclusion $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ implies that so much the more $u \in L_\alpha(D_\tau)$. Throughout the preceding considerations, we have assumed that $\alpha > 1$.

Definition 4. Let $F \in L_2(D_\tau)$ and $1 < \alpha < (n+1)/(n-1)$. A function $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ is called a *strong generalized solution* of the nonlinear problem (18), (33) if there exists a sequence of functions $u_m \in W_2^2(D_\tau) \cap \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ and $[Lu_m - \lambda |u_m|^\alpha] \to F$ in the space $L_2(D_\tau)$. In this case, the convergence of the sequence $\{\lambda |u_m|^\alpha\}$ to the function $\lambda|u|^\alpha$ in the space $L_2(D_\tau)$ as $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ follows from Remark 6; moreover, since $|u|^\alpha \in L_2(D_\tau)$, it follows from the boundedness of the domain D_τ that so much the more $u \in L_\alpha(D_\tau)$.

Remark 7. One can readily see that, by Remark 6, if $1 < \alpha < (n+1)/(n-1)$, then a strong generalized solution u of problem (18), (33) in the sense of Definition 4 is a weak generalized solution of this problem for $f = 0$ in the sense of Definition 2, i.e., in the sense of the integral identity (6).

Remark 8. Note that if $F \in L_2(D_\tau)$ and $1 < \alpha < (n+1)/(n-1)$, then a function u belonging to $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ is a strong generalized solution of problem (18), (33) if and only if u is a solution of the functional equation

$$u = L^{-1}(\lambda|u|^\alpha + F) \tag{41}$$

in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$.

We rewrite Eq. (41) in the form

$$u = Au + u_0, \tag{42}$$

where, by virtue of (39) and (40) and by Remark 6, $A = L^{-1}T_0 : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ is a continuous compact operator in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ and $u_0 = L^{-1}F \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$.

Remark 9. Let

$$B(0, z_2) := \left\{ u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) : \ \|u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} \le z_2 \right\}$$

be the closed (convex) ball of radius $z_2 > 0$ in the Hilbert space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ centered at zero. Since $A : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$, $1 < \alpha < (n+1)/(n-1)$, is a continuous compact operator, it follows from the Schauder principle that, to prove the solvability of Eq. (42), it suffices to show that the operator A_1 given by the formula $A_1 u = Au + u_0$ maps the ball $B(0, z_2)$ into itself for some $z_2 > 0$ [29, p. 370]. To this end, below we represent the desired estimate for $\|Au\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)}$.

If $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$, then by $\tilde{u}$ we denote the function that is the continuation of u as an even function around the plane $t = \tau$ into the domain D_τ^*: $\tau < t < 2\tau - |x|$; i.e.,

$$\tilde{u}(t, x) = \begin{cases} u(x, t) & \text{for} \quad (x, t) \in D_\tau \\ u(x, 2\tau - t) & \text{for} \quad (x, t) \in D_\tau^*, \end{cases}$$

and $\tilde{u}(x, t) = u(x, t)$ for $t = \tau$, $|x| < \tau$ in the sense of the trace theory. Obviously, $\tilde{u} \in \overset{\circ}{W}{}_2^1\left(\tilde{D}_\tau\right)$, where $\tilde{D}_\tau :\ |x| < t < 2\tau - |x|$. Moreover, $\tilde{D}_\tau = D_\tau \cup \{(x, t) :\ t = \tau,\ |x| < \tau\} \cup D_\tau^*$.

By using the inequality [30, p. 258] $\int_\Omega |v| d\Omega \le (\operatorname{mes}\Omega)^{1-1/p} \|v\|_{p,\Omega}$, $p \ge 1$, and by taking into account the relations $\|\tilde{u}\|^p_{L_p(\tilde{D}_\tau)} = 2\|u\|^p_{L_p(D_\tau)}$ and $\|\tilde{u}\|^2_{\overset{\circ}{W}{}_2^1(\tilde{D}_\tau)} = 2\|u\|^2_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)}$, from the well-known multiplicative inequality [26, p. 78] $\|v\|_{p,\Omega} \le \beta \|v_x\|^{\tilde{\alpha}}_{m,\Omega} \|v\|^{1-\tilde{\alpha}}_{r,\Omega}$, $v \in \overset{\circ}{W}{}_2^1(\Omega)$, $\Omega \subset R^{n+1}$,

$\tilde{\alpha} = (1/r - 1/p)\,(1/r - 1/\tilde{m})^{-1}$, $\tilde{m} = (n+1)m/(n+1-m)$, for $\Omega = \tilde{D}_\tau \subset R^{n+1}$, $v = \tilde{u}$, $r = 1$, $m = 2$, and $1 < p \le 2(n+1)/(n-1)$, where $\beta = \text{const} > 0$ is independent of v and τ, we obtain

$$\|u\|_{L_p(D_\tau)} \le c_0 \,(\operatorname{mes} D_\tau)^{1/p + 1/(n+1) - 1/2}\, \|u\|_{\mathring{W}_2^1(D_\tau, S_\tau)} \quad \forall u \in \mathring{W}_2^1\,(D_\tau, S_\tau)\,, \tag{43}$$

where $c_0 = \text{const} > 0$ is independent of u.

Since $\operatorname{mes} D_\tau = (\omega_n/(n+1))\,\tau^{n+1}$, where ω_n is the volume of the unit ball in R^n, it follows from (43) with $p = 2\alpha$ that

$$\|u\|_{L_{2\alpha}(D_\tau)} \le c_0 \tilde{\ell}_{\alpha,n} \tau^{\delta_n} \|u\|_{ringW_2^1(D_\tau, S_\tau)} \quad \forall u \in \mathring{W}_2^1\,(D_\tau, S_\tau)\,, \tag{44}$$

where $\delta_n = (n+1)(1/(2\alpha) + 1/(n+1) - 1/2)$ and $\tilde{\ell}_{\alpha,n} = (\omega_n/(n+1))^{\delta_n/(n+1)}$.

By virtue of (44), the number $\|T_0 u\|_{L_2(D_\tau)}$, where $u \in \mathring{W}_2^1\,(D_\tau, S_\tau)$ and T_0 is the operator given by (40), satisfies the estimate

$$\|T_0 u\|_{L_2(D_\tau)} \le \lambda \left[\, \int\limits_{D_\tau} |u|^{2\alpha} dx\, dt \right]^{1/2} = \lambda \|u\|^{\alpha}_{L_{2\alpha}(D_\tau)} \le \lambda \ell_{\alpha,n} \tau^{\alpha\delta_n} \|u\|^{\alpha}_{\mathring{W}_2^1(D_\tau, S_\tau)}, \tag{45}$$

where $\ell_{\alpha,n} = \left[c_0 \tilde{\ell}_{\alpha,n}\right]^{\alpha}$.

Now from (39) and (45), we find that the number $\|Au\|_{\mathring{W}_2^1(D_\tau, S_\tau)}$, where $Au = L^{-1} T_0 u$, admits the estimate

$$\begin{aligned} \|Au\|_{\mathring{W}_2^1(D_\tau, S_\tau)} &\le \left\|L^{-1}\right\|_{L_2(D_\tau) \to \mathring{W}_2^1(D_\tau, S_\tau)} \|T_0 u\|_{L_2(D_\tau)} \\ &\le \sqrt{e}\, \lambda \ell_{\alpha,n} \tau^{1+\alpha\delta_n} \|u\|^{\alpha}_{\mathring{W}_2^1(D_\tau, S_\tau)} \quad \forall u \in \mathring{W}_2^1\,(D_\tau, S_\tau)\,. \end{aligned} \tag{46}$$

Note that $\delta_n > 0$ for $\alpha < (n+1)/(n-1)$.

Consider the equation

$$a z^{\alpha} + b = z \tag{47}$$

for the unknown z, where

$$a = \sqrt{e}\, \lambda \ell_{\alpha,n} \tau^{1+\alpha\delta_n}, \qquad b = \sqrt{e}\, \tau \|F\|_{L_2(D_\tau)}. \tag{48}$$

If $\tau > 0$, then, obviously, $a > 0$ and $b \ge 0$. Arguing by analogy with the case in which $\alpha = 3$ [29, pp. 373–374], one can show that (1) if $b = 0$, then, along with the zero root $z_1 = 0$, Eq. (47) has the unique positive root $z_2 = a^{-1/(\alpha-1)}$; (2) if $b > 0$, then Eq. (47) has two positive roots z_1 and z_2, $0 < z_1 < z_2$, for $0 < b < b_0$, where

$$b_0 = \left[\alpha^{-1/(\alpha-1)} - \alpha^{-\alpha/(\alpha-1)}\right] a^{-1/(\alpha-1)}; \tag{49}$$

moreover, these roots merge for $b = b_0$, and we obtain the single positive root $z_1 = z_2 = z_0 = (\alpha a)^{-1/(\alpha-1)}$; (3) if $b > b_0$, then Eq. (47) does not have a nonnegative root.

Note that if $0 < b < b_0$, then $z_1 < z_0 = (\alpha a)^{-1/(\alpha-1)} < z_2$. By (48) and (49), the condition $b \le b_0$ is equivalent to the condition

$$\sqrt{e}\, \tau \|F\|_{L_2(D_\tau)} \le \left[\sqrt{e}\, \lambda \ell_{\alpha,n} \tau^{1+\alpha\delta_n}\right]^{-1/(\alpha-1)} \left[\alpha^{-1/(\alpha-1)} - \alpha^{\alpha/(\alpha-1)}\right],$$

or

$$\|F\|_{L_2(D_\tau)} \le \gamma_{n,\lambda,\alpha} \tau^{-\alpha_n}, \qquad \alpha_n > 0, \tag{50}$$

where

$$\gamma_{n,\lambda,\alpha} = \left[\alpha^{-1/(\alpha-1)} - \alpha^{\alpha/(\alpha-1)}\right] (\lambda \ell_{\alpha,n})^{-1/(\alpha-1)} \exp\left[-\frac{1}{2}\left(1 + \frac{1}{\alpha-1}\right)\right],$$

$$\alpha_n = 1 + \frac{1}{\alpha-1}\left[1 + \alpha\delta_n\right].$$

By virtue of the absolute continuity of the Lebesgue integral, we have $\lim_{\tau\to 0}\|F\|_{L_2(D_\tau)} = 0$. At the same time, $\lim_{\tau\to 0}\tau^{-\alpha_n} = +\infty$. Therefore, there exists a number $\tau_1 = \tau_1(F)$, $0 < \tau_1 < +\infty$, such that inequality (50) is valid for

$$0 < \tau \le \tau_1(F). \tag{51}$$

Now let us show that if condition (51) is satisfied, then the operator

$$A_1 u = Au + u_0 : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$$

maps the ball $B(0, z_2)$, where z_2 is the maximum positive root of Eq. (47) (see Remark 9), into itself. Indeed, if $u \in B(0, z_2)$, then, by (46)–(48), we have

$$\|A_1 u\|_{\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} \le a\|u\|^{\alpha}_{\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} + b \le a z_2^{\alpha} + b = z_2.$$

Therefore, by Remarks 7–9, the following assertion is valid.

Theorem 2. *Let $F \in \tilde{L}_{2,\mathrm{loc}}(D)$, $1 < \alpha < (n+1)/(n-1)$, and let τ satisfy condition* (51). *Then problem* (18), (33) *in the domain D_τ has at least one strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ in the sense of Definition* 4, *which is also a weak generalized solution of this problem in the sense of Definition* 2.

Remark 10. Note that if $1 < \alpha < (n+1)/(n-1)$, then the uniqueness of the solution of problem (18), (33) in the domain D_τ can be proved in the narrower function space

$$\overset{\circ}{E}{}_2^1 = \left\{ u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) : \operatorname*{ess\,sup}_{0<\sigma\le\tau} \int\limits_{\Omega_\sigma = D\cap\{t=\sigma\}} \left[u_t^2 + \sum_{i=1}^{n} u_{x_i}^2 \right] dx < +\infty \right\}$$

than $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$.

Remark 11. It follows from the preceding assertions that, by virtue of the estimates (31) and (51), the number $t = T$ considered in Remark 1 lies in the interval $[\tau_1, \tau_0]$.

ACKNOWLEDGMENTS

The work was financially supported by INTAS (project no. 03-51-5007).

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