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ON SOME PROBLEMS WITH INTEGRAL RESTRICTIONS FOR HYPERBOLIC SECOND ORDER EQUATIONS AND SYSTEMS ON A PLANE

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Abstract. Some problems with integral restrictions to hyperbolic second order equations and systems with two independent variables are formulated and investigated. The conditions on the data of the problems are found which guarantee their correctness. When these conditions violate, we distinguish the cases in which the corresponding homogeneous problem has an infinite set of linearly independent solutions.

რეზიუმე. მეორე რიგის ჰიპერბოლური განტოლებებისა და სისტემებისათვის ორი დამოუკიდებელი ცვლადის შემთხვევაში დასმულია და გამოკვლეული ზოგიერთი ამოცანა ინტეგრალური შეზღუდვებით. ნაპოვნია პირობები ამოცანის მონაცემებზე, რომლებიც უზრუნველყოფენ მათ კორექტულობას. ამ პირობათა დარღვევისას გამოყოფილია შემთხვევები, როდესაც შესაბამის ერთგვაროვან ამოცანას გააჩნია წრფივად დამოუკიდებელ ამონახსნთა უსასრულო რაოდენობა.

The problems with integral restrictions to partial differential equations arise in mathematical modelling of some physical processes and represent certain class of nonlocal problems (see for e.g., [1]–[13]). Nonlocal problems, free from integral restrictions to parabolic, elliptic and hyperbolic equations have been studied in [14]–[29].

In the present work we formulate and investigate some problems with integral restrictions to hyperbolic second order equations and systems with two independent variables.

1. Statement of the Problems with Integral Restrictions to the Equation of Forced Oscillations of a String, and Their Investigation

1^0. In this section, for the equation of forced oscillations of a string

$$\Box\, u := u_{tt} - u_{xx} = F \tag{1}$$

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in a triangular domain $D: 0<t<T$, $0<x<t$ with vertices at the points $O(0,0)$, $A(0,T)$, $B(T,T)$ we consider the problem which is formulated as follows: find a regular in the domain D and continuous in $\overline{D}$ solution $u(x,t)$ of equation (1), satisfying both the boundary condition

$$u(0,t)=f_1(t), \quad 0\le t\le T, \tag{2}$$

and the integral condition of the type

$$\int\limits_{I_t} K(x,t)u(x,t)dx=f_2(t), \quad 0\le t\le T. \tag{3}$$

Here $I_\tau=\overline{D}\cap\{t=\tau\}$, i.e. $I_\tau: t=\tau,\ 0\le x\le\tau$; $F(x,t)\in C(\overline{D})$, $K(x,t)\in C^1(\overline{D})$ and $f_1(t)\in C([0,T])$, $f_2(t)\in C^1([0,T])$ are the given functions, and the function $f_2(t)$ satisfies the necessary condition $f_2(0)=0$ of solvability of the problem (1),(2),(3).

Remark 1. Below, we will first restrict ourselves to the investigation of the problem (1), (2), (3) in the class of generalized solutions $u(x,t)$ of equation (1) of the class $C(\overline{D})$, i.e. when $u\in C(\overline{D})$, and there exists a sequence of functions $u_n\in C^2(\overline{D})$ such that $u_n\to u$ and $\Box u_n\to F$ as $n\to\infty$ in the norm of the space $C^2(\overline{D})$.

Let the point $P=P(x,t)\in D$. By $D_{x,t}=PP_1P_0P_2$ we denote a characteristic quadrangle of equation (1), whose vertices P_0 and P_2 lie on the segment $OB\in\partial D$, while the vertex P_1 lies on the segment $OA\subset\partial D$. Obviously, $P_1=P_1(0,t-x)$, $P_2=P_2\Big(\frac{t+x}{2};\frac{t+x}{2}\Big)$ and $P_0=P_0\Big(\frac{t-x}{2};\frac{t-x}{2}\Big)$. Suppose that the function $u_n\in C^2(\overline{D})$, and we consider this function in the characteristic quadrangle $D_{x,t}=PP_1P_0P_2$ as a solution of the Goursat problem

$$\begin{gathered}\Box\, u=F_n,\\ u\big|_{P_0P_2}=u_n\big|_{P_0P_2},\quad u\big|_{P_0P_1}=u_n\big|_{P_0P_1},\end{gathered} \tag{4}$$

where $F_n=\Box u_n$. Since u_n is a solution of the problem (4), from the uniqueness of a solution and from the formula allowing one to solve this problem ([30], p.172), for the function u_n in the closed domain $\overline{D}_{x,t}$ the representation

$$u_n(P)=u_n(P_1)+u_n(P_2)-u_n(P_0)+\frac{1}{2}\iint\limits_{D_{x,t}}F_n dxdt, \quad F_n=\Box\, u_n, \quad P\in\overline{D}_{x,t},$$

or

$$\begin{aligned}u_n(x,t)=u_n(0,t-x)+u_n\Big(\frac{t+x}{2},\frac{t+x}{2}\Big)-u_n\Big(\frac{t-x}{2},\frac{t-x}{2}\Big)+\\ +\frac{1}{2}\iint\limits_{D_{x,t}}\Box\, u_n dxdt \quad \forall(x,t)\in\overline{D}\end{aligned} \tag{5}$$

holds.

Remark 2. By Remark 1 and representation (5), it is evident that every generalized solution $u(x,t)$ of equation (1) of the class $C(\overline{D})$ can be represented in the form

$$u(x,t) = u(0,t-x) + u\Big(\frac{t+x}{2},\frac{t+x}{2}\Big) - u\Big(\frac{t-x}{2},\frac{t-x}{2}\Big) + \\ + \frac{1}{2}\iint\limits_{D_{x,t}} F dxdt \quad \forall (x,t)\in\overline{D}. \tag{6}$$

Conversely, since the spaces $C^2(\overline{D})$ and $C^2([0,T])$ are dense respectively in the spaces $C(\overline{D})$ and $C([0,T])$, for any $\widetilde{f}$, $\widetilde{\varphi}\in C([0,T])$ and $\widetilde{F}\in C(\overline{D})$ the function $u(x,t)$ represented by the equality

$$u(x,t) = \widetilde{f}(t-x) + \widetilde{\varphi}\Big(\frac{t+x}{2}\Big) - \widetilde{\varphi}\Big(\frac{t-x}{2}\Big) + \frac{1}{2}\iint\limits_{D_{x,t}} \widetilde{F} dxdt, \quad (x,t)\in\overline{D},$$

is a generalized solution of equation (1) of the class $C(\overline{D})$.

With the notation $\varphi(t) = u(t,t)$, by means of the boundary condition (2), the representation (6) for the generalized solution of the problem (1), (2), (3) of the class $C(\overline{D})$ can be rewritten in the form

$$u(x,t) = \varphi\Big(\frac{t+x}{2}\Big) - \varphi\Big(\frac{t-x}{2}\Big) + f_1(t-x) + g_0(x,t) \quad \forall (x,t)\in\overline{D}. \tag{7}$$

Here, the function $g_0(x,t) = \frac{1}{2}\iint\limits_{D_{x,t}} F d\xi d\tau \in C(\overline{D})$. Therefore, taking is equal to into account that the area of the rectangle $D_{x,t} = PP_1P_0P_2$ is equal to $x(t-x)$, the estimate

$$|g_0(x,t)| \le \frac{1}{2} x(t-x)\|F\|_{C(\overline{D})}, \quad (x,t)\in\overline{D}, \tag{8}$$

is valid.

It is easy to see that

$$\begin{gathered}\int\limits_{I_t} K(x,t)\varphi\Big(\frac{t+x}{2}\Big)dx = \\ = \int\limits_0^t K(x,t)\varphi\Big(\frac{t+x}{2}\Big)dx = 2\int\limits_{\frac{1}{2}t}^{t} K(2\xi-t,t)\varphi(\xi)d\xi, \\ \int\limits_{I_t} K(x,t)\varphi\Big(\frac{t-x}{2}\Big)dx = \\ = \int\limits_0^t K(x,t)\varphi\Big(\frac{t-x}{2}\Big)dx = 2\int\limits_0^{\frac{1}{2}t} K(t-2\xi,t)\varphi(\xi)d\xi.\end{gathered} \tag{9}$$

Substituting the representation (7) for the solution $u(x,t)$ of equation (1) into the integral condition (3) and taking into account equalities (9), with respect to the unknown function $\varphi(t)$ we obtain the following equation:

$$\int\limits_{\frac{1}{2}t}^{t} K(2\xi-t,t)\varphi(\xi)d\xi - \int\limits_0^{\frac{1}{2}t} K(t-2\xi,t)\varphi(\xi)d\xi = g_1(t), \quad 0\le t\le T. \tag{10}$$

Here,

$$\begin{aligned} g_1(t) =& \frac{1}{2}f_2(t) - \frac{1}{2}\int\limits_0^t K(x,t)[f_1(t-x)+g_0(x,t)]dx = \\ =& \frac{1}{2}f_2(t) - \frac{1}{2}\int\limits_0^t K(t-\xi,t)f_1(\xi)d\xi - \frac{1}{2}\int\limits_0^t K(x,t)g_0(x,t)dx. \end{aligned} \tag{11}$$

By (11) and the above assumptions, both parts of equation (10) are the continuously differentiable functions. Therefore, differentiating both parts of equation (10) with respect to the variable t, we obtain

$$K(t,t)\varphi(t) - K(0,t)\varphi\Big(\frac{1}{2}t\Big) + \int\limits_0^t K_1(\xi,t)\varphi(\xi)d\xi = g_2(t), \quad 0\le t\le T. \tag{12}$$

Here $g_2(t) = g_1^{(1)}(t)$ and

$$K_1(\xi,t) = \begin{cases} -K_x(t-2\xi,t) - K_t(t-2\xi,t), & 0\le\xi\le\frac{1}{2}t, \\ -K_x(2\xi-t,t) + K_t(2\xi-t,t), & \frac{1}{2}t\le\xi\le t. \end{cases} \tag{13}$$

Obviously, the continuity of the function $K_1(\xi,t)$ from (13) in the domain of its definition is equivalent to the condition $K|_{OA} = \text{const}$.

Let the condition

$$K\big|_{\overline{OB}} \neq 0, \quad \text{i.e.} \quad K(t,t) \neq, \quad 0 \le t \le T, \tag{14}$$

be fulfilled.

Then equation (12) can be rewritten as

$$\varphi(t) - a(t)\varphi\Big(\frac{1}{2}t\Big) + \int\limits_0^t K_2(\xi,t)\varphi(\xi)d\xi = g_3(t), \quad 0 \le t \le T, \tag{15}$$

where

$$K_2(\xi,t) = K^{-1}(t,t)K_1(\xi,t), \quad g_3(t) = K^{-1}(t,t)g_2(t), \tag{16}$$

$$a(t) = K^{-1}(t,t)K(0,t) \in C^1([0,T]). \tag{17}$$

By (17), it is obvious that

$$a(0) = 1. \tag{18}$$

First, we will restrict ourselves to the consideration only of the functional part of equation (15), i.e. of the equation

$$L\psi(t) := \psi(t) - a(t)\psi\Big(\frac{1}{2}t\Big) = g(t), \quad 0 \le t \le T, \tag{19}$$

in the class $C([0,T])$.

Lemma 1. *The infinite product*

$$\psi_0(t) = \prod_{i=0}^{\infty} a\Big(\frac{1}{2^i}t\Big), \quad 0 \le t \le T, \tag{20}$$

converging uniformly on the segment $[0,T]$*, is by itself a continuous function on* $[0,T]$ *and a solution of the homogeneous equation*

$$\psi(t) - a(t)\psi\Big(\frac{1}{2}t\Big) = 0, \quad 0 \le t \le T, \tag{21}$$

corresponding to (19).

Every solution of the homogeneous equation (21) in the class $C([0,T])$ is representable in the form

$$\psi(t) = \psi(0)\psi_0(t). \tag{22}$$

Proof. Since $a(t) \in C^1([0,T])$, by virtue of (18) we have

$$a(t) = 1 + t\lambda(t), \quad \lambda(t) = \int\limits_0^t a^{(1)}(t\xi)d\xi \in C([0,1]). \tag{23}$$

By (23) and the continuity of the function $\lambda(t)$, there exists a positive number $t_0 \in (0,T)$ such that

$$|t\lambda(t)| < 1 \quad \text{and} \quad a(t) > 0 \quad \text{for} \quad 0 \le t \le t_0. \tag{24}$$

Since for $i \geq i_0 = \Big[\dfrac{\log Tt_0^{-1}}{\log 2}\Big]$ the inequality $\dfrac{1}{2^i}t \leq t$ holds for $0 \leq t \leq T$, where $[\alpha]$ denotes the integral part of number α, therefore by (24)

$$a\Big(\frac{1}{2^i}t\Big) > 0, \quad 0 \leq t \leq T, \quad i \geq i_0. \tag{25}$$

The uniform convergence of the infinite product (20) on the segment $[0,T]$ is equivalent to that of the residual infinite product

$$\prod_{i=i_0}^{\infty} a\Big(\frac{1}{2^i}t\Big) = \prod_{i=i_0}^{\infty} \Big(1 + \frac{1}{2^i}t\lambda\Big(\frac{1}{2^i}t\Big)\Big), \tag{26}$$

whose terms are positive by virtue of (25).

In turn, the series

$$\sum_{i=i_0}^{\infty} \frac{1}{2^i}t\lambda\Big(\frac{1}{2^i}t\Big)$$

is uniformly and absolutely convergent on the segment $[0,T]$. Therefore by the well-known result from the theory of an infinite product ([31], p. 356), the product (26), and hence the product (20), is uniformly converging on the segment $[0,T]$ to some continuous function $\psi_0(t)$ which is, as is easily seen, positive for $0 \leq t \leq t_0$.

The fact that the function $\psi_0(t)$ from (20) is the solution of the homogeneous equation (21), can be easily verified. Thus we have proved the first part of Lemma 1.

Let now $\psi(t) \in C([0,T])$ be a solution of the homogeneous equation (21). Then, as is seen, for any natural n the equality

$$\psi(t) = \Big[\prod_{i=0}^{n-1} a\Big(\frac{1}{2^i}t\Big)\Big]\psi\Big(\frac{1}{2^n}t\Big), \quad 0 \leq t \leq T, \tag{27}$$

is valid. Since $\lim\limits_{n\to\infty} \prod\limits_{i=0}^{n-1} a\Big(\frac{1}{2^i}t\Big) = \psi_0(t)$ and $\lim\limits_{n\to\infty} \psi\Big(\frac{1}{2^n}t\Big) = \psi(0)$, passing in (27) to the limit as $n \to \infty$, we obtain (22). This proves the second part of Lemma 1. □

Remark 3. By (18), it is not difficult to see that the necessary condition for the solvability of equation (19) in the class $C([0,T])$ is the fulfilment of the equality

$$g(0) = 0. \tag{28}$$

Lemma 2. *Let* $g(t) \in C([0,T])$, *and let the necessary condition* (28) *for the solvability of equation* (19) *in the class* $C([0,T])$ *be fulfilled. Denote by* $L_0 : C([0,T]) \to C([0,T])$ *the operator acting by the formula*

$$L_0\, g(t) = a(t)g\Big(\frac{1}{2}t\Big), \quad 0 \leq t \leq T. \tag{29}$$

Then the necessary and sufficient condition for the solvability of equation (19) *in the class* $C([0,T])$ *is the uniform convergence on the segment* $[0,T]$ *of the following functional series:*

$$\psi_1(t) = \Big[\sum_{i=0}^{\infty} L_0^i\Big] g(t), \quad 0 \le t \le T. \tag{30}$$

In case the series (30) is uniformly convergent on $[0,T]$, for any constant number c there exists the unique solution $\psi(t) \in C([0,T])$ of equation (19) satisfying the condition

$$\psi(0) = C \tag{31}$$

and this solution is representable in the form

$$\psi(t) = c\psi_0(t) + \psi_1(t), \quad 0 \le t \le T, \tag{32}$$

where the functions $\psi_0(t)$ and $\psi_1(t)$ are given by equalities (20) and (30).The uniform convergence of the series (30) will be automatically fulfilled if the function $g(t)$ for any arbitrarily small positive α satisfies the following supplementary condition:

$$g(t) \in C_\alpha((0,T]) := \{f(t) \in C([0,T]) : \|f\|_{C_\alpha((0,T])} = \\ = \sup_{0<t\le T} |t^{-\alpha} f(t)| < +\infty\}. \tag{33}$$

In this case the estimate

$$|\psi_1(t)| = |\psi(t) - c\psi_0(t)| \le c_1 t^\alpha \|g\|_{C_\alpha((0,T])}, \quad 0 < t \le T, \tag{34}$$

with the positive constant c_1, independent of $g(t)$, is valid.

Proof. If $\psi(t) \in C([0,T])$ is the solution of equation (19), then taking into account (30), from equality (19) for any natural number n we immediately find that

$$\psi(t) - \Big[\prod_{i=0}^{n-1} a\Big(\frac{1}{2^i}t\Big)\Big]\psi\Big(\frac{1}{2^n}t\Big) = \sum_{i=0}^{n-1} L_0^i g(t), \quad 0 \le t \le T. \tag{35}$$

By Lemma 1, we have

$$\psi_0(t) = \lim_{n\to\infty} \Big[\prod_{i=0}^{n-1} a\Big(\frac{1}{2^i}t\Big)\Big], \quad 0 \le t \le T, \tag{36}$$

and

$$\psi(0) = \lim_{n\to\infty} \psi\Big(\frac{1}{2^n}t\Big), \quad 0 \le t \le T. \tag{37}$$

Taking into account the fact that the convergence in (30) and (37) is uniform on the segment $[0,T]$, from equality (35), as $n \to \infty$, follows uniform convergence of the series (36) on $[0,T]$. Therefore passing in (35) to the limit, as $n \to \infty$, by virtue of (20), (30) and (31) we obtain (32). That in case of uniform convergence of the series (30) on the segment $[0,T]$ the function $\psi(t) \in C([0,T])$ from (32) is the solution of equation (19), can be easily verified.

It remains only to show that if the condition (33) is fulfilled, then the series (30) converges uniformly on the segment $[0,T]$, and for the function $\psi(t)$ from (30) and (32) the estimate (34) is valid. Indeed, it follows from (33) that

$$g(t) = t^\alpha \widetilde{g}(t), \quad |\widetilde{g}(t)| \le \|g\|_{C_\alpha((0,t])}, \quad 0 < t \le T. \tag{38}$$

By (29) and (38) we have

$$|L_0^i g(t)| = \Big|\prod_{j=0}^{i-1} a\Big(\frac{1}{2^j}t\Big)\Big|\Big|g\Big(\frac{1}{2^i}t\Big)\Big| \le M\Big(\frac{1}{2^\alpha}\Big)^i t^\alpha \|g\|_{C_\alpha((0,t])}, \quad i > 0,$$
$$|L_0^i g(t)| = |g(t)| \le t^\alpha |\widetilde{g}(t)| \le t^\alpha \|g\|_{C_\alpha((0,t])} \quad \text{for} \quad i = 0,$$

where by Lemma 1 we use the fact that

$$\sup_{i>0} \max_{0\le t\le T} \Big|\prod_{j=0}^{i-1} a\Big(\frac{1}{2^j}t\Big)\Big| = M < +\infty.$$

This implies that the series (30) is uniformly convergent on the segment $[0,T]$, and by (30) the estimate

$$|\psi_1(t)| \le t^\alpha \|g\|_{C_\alpha((0,T])} + M\|g\|_{C_\alpha((0,T])} t^\alpha \sum_{i=1}^{\infty} \Big(\frac{1}{2^\alpha}\Big)^i =$$
$$= \Big(1 + M\frac{\frac{1}{2^\alpha}}{1-\frac{1}{2^\alpha}}\Big) t^\alpha \|g\|_{C_\alpha((0,t])}$$

as well.

Thus we have proved the estimate (34) in which $c_1 = 1 + M\frac{\frac{1}{2^\alpha}}{1-\frac{1}{2^\alpha}} < +\infty$. □

Remark 4. Since $\varphi(t) = u(t,t)$, therefore by (2) the equality

$$\varphi(0) = f_1(0) \tag{39}$$

holds.

The necessary condition for the solvability of (28) applied to equation (15) takes by virtue of (11), (13) and (16) the form

$$f_2^{(1)}(0) - K(0,0)f_1(0) = 0. \tag{40}$$

The condition (40) can likewise be obtained directly by differentiating (2) with respect to t and then putting $t = 0$.

Remark 5. Taking into account (40) and the requirements for the smoothness of the functions $K(x,t)$ and $f_i(t)$, $i=1,2$, by virtue of (16) the condition $g_3(t)\in C_\alpha((0,T])$, for which $\alpha\in(0,1)$, will be fulfilled if

$$f_1(t)-f_1(0)\in C_\alpha((0,T]),\quad f_2^{(1)}(t)-f_2^{(1)}(0)\in C_\alpha((0,T]),\quad 0<\alpha<1. \tag{41}$$

Lemma 3. *Let the necessary condition* (40) *for the solvability of equation* (15) *in the class* $C([0,T])$ *be fulfilled. Then if the functions* $f_1(t)$, $f_2(t)$ *satisfy additionally the condition* (41) *for some arbitrarily small positive* $\alpha\in(0,1)$, *then equation* (15) *has the unique solution* $\varphi(t)\in C([0,T])$ *satisfying the initial condition* (39). *For this solution the estimate*

$$|\varphi(t)-f_1(0)\psi_0(t)|\le c_2t^\alpha(\|g_3\|_{C_\alpha((0,t])}+|f_1(0)|,\quad 0<t\le T, \tag{42}$$

with the positive constant c_2, *independent of* $g_3(t)$, *is valid, and the function* $\psi_0(t)$ *is given by equality* (20).

Proof. We solve equation (15) in the class $C([0,T])$ by the method of successive approximations using the following scheme:

$$\varphi_0(t)=f_1(0)\psi_0(t),\quad 0\le t\le T, \tag{43}$$

$$\varphi_n(t)-a(t)\varphi_n\Big(\frac{1}{2}t\Big)=-\int\limits_0^t K_2(\xi,t)\varphi_{n-1}(\xi)d\xi+g_3(t), \tag{44}$$

$$0\le t\le T,\quad n\ge 1,$$

$$\varphi_n(0)=f_1(0). \tag{45}$$

If approximation $\varphi_{n-1}(t)$ is known, then the approximation $\varphi_n(t)$ is defined by Lemma 2, as a solution of functional equation (44) with respect to the function $\varphi_n(t)$, satisfying the initial condition (45).

By (43), (44), (45) and Lemma 1, using the notation from (19), we have

$$L(\varphi_1(t)-\varphi_0(t))=-\int\limits_0^t K_2(\xi,t)\varphi_0(\xi)d\xi+g_3(t),\quad \varphi_1(0)-\varphi_0(0)=0,$$

whence, according to Remark 5 and estimate (34) from Lemma 2, we have

$$|\varphi_1(t)-\varphi_0(t)|\le c_1t^\alpha\|-\int\limits_0^\tau K_2(\xi,\tau)\varphi_0(\xi)d\xi+g_3(\tau)\|_{C_{\alpha((0,t])}}\le$$

$$\le c_1t^\alpha\|g_3\|_{C_\alpha((0,t])}+c_1t^\alpha\sup_{0<\tau\le t}|\tau^{-\alpha}\int\limits_0^\tau K_2(\xi,\tau)\varphi_0(\xi)d\xi|\le c_1t^\alpha\|g_3\|_{C_\alpha((0,t])}+$$

$$+c_1t^\alpha\sup_{0<\tau\le t}|\tau^{-\alpha}\int\limits_0^\tau K_3|f_1(0)|Md\tau|\le$$

$$\leq c_1 t^\alpha \Big[\|g_3\|_{C_\alpha((0,t])} + t^{1-\alpha} K_3 M |f_1(0)|\Big], \tag{46}$$

where $K_3 = \|K_2\|_{C(\overline{D})}$, and $M = \sup\limits_{i>0} \max\limits_{0\leq t\leq T} \Big|\prod_{j=0}^{i-1} a\Big(\frac{1}{2^j}t\Big)\Big| \leq +\infty$ is the number used when proving Lemma 2, $0<\alpha<1$, $0<t\leq T$.

Note that if $f_1(0)=0$, then the estimate (46) takes the form

$$|\varphi_1(t)| \leq c_1 t^\alpha \|g_3\|_{C_\alpha((0,t])}, \tag{47}$$

which is valid for any positive α, where $g_3 \in C((0,T])$.

Reasoning analogously, we obtain

$$L(\varphi_2(t)-\varphi_1(t)) = -\int\limits_0^t K_2(\xi,t)[\varphi_1(\xi)-\varphi_0(\xi)]d\xi, \quad \varphi_2(0)-\varphi_1(0)=0,$$

and hence with regard for (45), we have

$$|\varphi_2(t)-\varphi_1(t)| \leq c_1 t^\alpha \| -\int\limits_0^\tau K_2(\xi,\tau)[\varphi_1(\xi)-\varphi_0(\xi)]d\xi\|_{C_\alpha((0,t])} \leq$$

$$\leq c_1 t^\alpha \sup_{0<\tau\leq t} |t^{-\alpha} \int\limits_0^\tau K_2(\xi,\tau)[\varphi_1(\xi)-\varphi_0(\xi)]d\xi| \leq$$

$$\leq c_1 t^\alpha \sup_{0<\tau\leq t} \tau^{-\alpha} \int\limits_0^\tau K_3 c_1 \xi^\alpha [\|g_3\|_{C((0,T])} +$$

$$+t^{1-\alpha} K_3 M |f_1(0)|] d\xi \leq c_1^2 K_3 M_1 \frac{t^{\alpha+1}}{\alpha+1}. \tag{48}$$

Here,

$$M_1 = [\|g_3\|_{C((0,t])} + t^{1-\alpha} K_3 M |f_1(0)|]. \tag{49}$$

Continuing this process, we obtain

$$|\varphi_n(t)-\varphi_{n-1}(t)| \leq$$

$$\leq M_1 c_1^n K_3^{n-1} \frac{t^{\alpha+n-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}, \quad 0<t\leq T, \tag{50}$$

whose particular cases for $n=1$ and $n=2$ are, respectively inequalities (46) and (48), where M_1 is given by equality (49).

From (50) follows uniform convergence on the segment $[0,T]$ of the series

$$\varphi(t) = \varphi_0(t) + \sum_{n=1}^\infty (\varphi_n(t)-\varphi_{n-1}(t)),$$

whose sum $\varphi(t) \in C([0,T])$ is, as it can be easily verified, a solution of equation (15), satisfying both the initial condition (39) and the estimate (42).

Reasoning just as when deducing inequality (50), we can prove the uniqueness of the solution $\varphi(t) \in C([0,T])$ of equation (15), satisfying the initial condition (39). □

Remark 6. Analogously to inequality (47), the estimate (42) for $f_1(0)=0$ takes the form

$$|\varphi(t)| \le c_2 t^{\alpha} \|g_3\|_{C((0,t])}, \tag{51}$$

which is valid for any positive α, where $g_3(t) \in C((0,T])$.

Under our assumptions regarding the smoothness of the data of the problem (1), (2), (3), and if the conditions (14) and (40) are fulfilled, this problem in the class $C(\overline{D})$ will be equivalent to the integro-functional equation (15). Therefore taking into account representation (7) and estimates (8) and (42), from Lemma 3 we arrive at

Theorem 1. *Let $F \in C(\overline{D})$, $f_i \in C^{i-1}([0,T])$, $i=1,2$, and let the condition* (14) *be fulfilled. Then, if the necessary conditions of solvability $f_2(0)=0$ and* (40) *of the problem* (1), (2), (3) *in the class $C(\overline{D})$ and the supplementary condition* (41) *imposed on the function f_2 are fulfilled, then the problem* (1), (2), (3) *has the unique solution $u \in C(\overline{D})$ for which the estimate*

$$\|u\|_{C(\overline{D})} \le c[\|f_1\|_{C([0,T])}] + \|f_1(t)-f_1(0)\|_{C_\alpha((0,T])} + \|f_2\|_{C^1([0,T])} + \\ + \|f_2^{(1)}(t) - f_2^{(1)}(0)\|_{C_\alpha((0,T])} + \|F\|_{C(\overline{D})} \tag{52}$$

with the positive constant c, independent of f_1, f_2 and F, where $0<\alpha<1$, is valid.

Remark 7. Let $P=P(x,t)\in D$, and let $D_{x,t}$ be the characteristic rectangle $PP_1P_0P_2$ appearing in the statement of the Goursat problem (4). Then under the conditions of Theorem 1, by virtue of the representation (7) and inequality (42), for the solution $u \in C(\overline{D})$ of the problem (1), (2), (3) there takes place along with (52) the following improved estimate:

$$|u(x,t)| \le c\Big[\|f_1\|_{C([0,t-x])} + \|f_1(t)-f_1(0)\|_{C_\alpha((0,t-x])} + \\ + \|f_2\|_{C^1([0,\frac{t+x}{2}])} + \|f_2^{(1)}(t)-f_2^{(1)}(0)\|_{C_\alpha((0,\frac{t+x}{2}])} + \|F\|_{C(\overline{D}_{x,t})}\Big] \quad \forall (x,t)\in\overline{D},$$

with the positive constant c, independent both of the point $(x,t)\in D$ and of the functions f_1, f_2 and F.

Lemma 4. *Let $f_i=0$, $i=1,2$. Then the function $g_3(t)=K^{-1}(t,t)g_1^{(1)}(t)$ from* (15), *where $g_1(t)$ is given by equality* (11), *is representable in the form*

$$g_3(t) = -\frac{1}{2}K^{-1}(t,t)\bigg[\int_0^t K_t(x,t)g_0(x,t)\,dx\bigg] +$$

$$+\frac{1}{2\sqrt{2}}\int\limits_0^t K(x,t)\Big[\int\limits_{P_2P} Fds-\int\limits_{P_0P_1} Fds+\int\limits_{P_1P} F\Big]dx. \tag{53}$$

Here $g_0(x,t)=\frac{1}{2}\iint\limits_{D_{x,t}} Fd\xi d\tau$, and $D_{x,t}$ is the characteristic rectangle $PP_1P_0P_2$ with vertices at the points $P=P(x,t)$, $P_1=P_1(0,t-x)$, $P_0=P_0\Big(\frac{t-x}{2},\frac{t-x}{2}\Big)$, $P_2=P_2\Big(\frac{t+x}{2},\frac{t+x}{2}\Big)$, $(x,t)\in\overline{D}$.

Proof. As a result of the linear orthogonal transformation $\xi=\frac{1}{\sqrt{2}}(t+x)$, $\eta=\frac{1}{\sqrt{2}}(t-x)$ the rectangle $PP_1P_0P_2$ transforms into the rectangle with vertices $\widetilde{p}\Big(\frac{1}{\sqrt{2}}(t+x),\frac{1}{\sqrt{2}}(t-x)\Big)$, $\widetilde{P}_1\Big(\frac{1}{\sqrt{2}}(t-x),\frac{1}{\sqrt{2}}(t-x)\Big)$, $\widetilde{P}_0\Big(\frac{1}{\sqrt{2}}(t-x),0\Big)$ and $\widetilde{P}_2\Big(\frac{1}{\sqrt{2}}(t+x),0\Big)$. Therefore we have

$$g_0(x,t)=\frac{1}{2}\iint\limits_{PP_1P_0P_2} Fdxdt=$$

$$=\frac{1}{2}\iint\limits_{\widetilde{P}\widetilde{P}_1\widetilde{P}_0\widetilde{P}_2}\widetilde{F}d\xi d\eta=\frac{1}{2}\int\limits_{\frac{1}{\sqrt{2}}(t-x)}^{\frac{1}{\sqrt{2}}(t+x)} d\xi\int\limits_0^{\frac{1}{\sqrt{2}}(t-x)}\widetilde{F}(\xi,\eta)d\eta, \tag{54}$$

where $\widetilde{F}(\xi,\eta)=F(x,t)$, $x=\frac{1}{\sqrt{2}}(\xi-\eta)$, $y=\frac{1}{\sqrt{2}}(\xi+\eta)$. By (54), it can be easily verified that

$$\frac{\partial}{\partial t}g_0(x,t)=\frac{1}{2\sqrt{2}}\int\limits_0^{\frac{1}{\sqrt{2}}(t-x)}\widetilde{F}\Big(\frac{1}{\sqrt{2}}(t+x),\eta\Big)d\eta-$$

$$-\frac{1}{2\sqrt{2}}\int\limits_0^{\frac{1}{\sqrt{2}}(t-x)}\widetilde{F}\Big(\frac{1}{\sqrt{2}}(t-x),\eta\Big)d\eta+$$

$$+\frac{1}{2\sqrt{2}}\int\limits_{\frac{1}{\sqrt{2}}(t-x)}^{\frac{1}{\sqrt{2}}(t+x)}\widetilde{F}(\xi,\frac{1}{\sqrt{2}}(t-x))d\xi=$$

$$=\frac{1}{2\sqrt{2}}\iint\limits_{P_2P} Fds-\frac{1}{2\sqrt{2}}\int\limits_{P_0P_1} Fds+\frac{1}{2\sqrt{2}}\int\limits_{P_1P} Fds. \tag{55}$$

From (11) for $f_i = 0$, $i = 1,2$, with regard for (55) and the fact that $g_0(t,t) = 0$, we obtain

$$g_1^{(1)}(t) = -\frac{1}{2}\frac{d}{dt}\Big[\int\limits_0^t K(x,t)g_0(x,t)dx\Big] = -\frac{1}{2}K(t,t)g_0(t,t)-$$

$$-\frac{1}{2}\int\limits_0^t K_t(x,t)g_0(x,t)dx - \frac{1}{2}\int\limits_0^t K(x,t)\frac{\partial}{\partial t}g_0(x,t)dx = -\frac{1}{2}\int\limits_0^t K_t(x,t)g_0(x,t)dx-$$

$$-\frac{1}{4\sqrt{2}}\int\limits_0^t K(x,t)\Big[\int\limits_{P_2P} Fds - \int\limits_{P_0P_1} Fds + \int\limits_{P_1P} Fds\Big]dx. \tag{56}$$

Owing to (56), from the equality $g_3(t) = K^{-1}(t,t)g_1^{(1)}(t)$ it follows that the representation (53) is valid. □

Lemma 5. *Let $F \in C(\overline{D})$, $f_i = 0$, $i = 1,2$, and let the condition* (14) *be fulfilled. Then if for $(x,t) \in \overline{D}$*

$$|F(x,t)| \le M_F t^\alpha, \quad M_F = \text{const} \ge 0, \quad \alpha = \text{const} \ge 0, \tag{57}$$

then the problem (1), (2), (3) *has a unique solution $u \in C(\overline{D})$ for which the estimate*

$$|u(x,t)| \le c_3 M_F \frac{t^{\alpha+2}}{\alpha+2} \quad \forall (x,t) \in \overline{D}, \tag{58}$$

with the positive constant c_3, independent of F, M_F and α, is valid.

Proof. First of all it should be noted that by virtue of (57),

$$|g_0(x,t)| \le \frac{1}{2}M_F\frac{t^{\alpha+2}}{\alpha+2};$$
$$\max\Big(\Big|\int\limits_{P_2P} Fds\Big|, \Big|\int\limits_{P_0P_1} Fds\Big|, \Big|\int\limits_{P_1P} Fds\Big|\Big) \le \sqrt{2}M_F\frac{t^{\alpha+1}}{\alpha+1}. \tag{59}$$

Therefore, from (56) by virtue of (59) we have

$$|g_3(t)| = |K^{-1}(t,t)g_1^{(1)}(t)| \le \widetilde{c}M_F\frac{t^{\alpha+2}}{\alpha+2} \quad \forall t \in [0,T], \tag{60}$$

with the positive constant $\widetilde{c}$, independent of F, M_F and α.

Now, from (60), by Lemma 3, estimate (51) and representation (7) it follows that the unique solution $u \in C(\overline{D})$ of the problem (1), (2), (3) exists for which the estimate (58) is valid. □

Remark 8. Investigation of the integro-functional equation (15) shows that if the necessary conditions are fulfilled at the point $O(0,0)$, then the rise of smoothness of the data of the problem (1), (2), (3) implies the corresponding rise of smoothness of a solution of that problem.

2^0. In this section, for a more general than (1) inhomogeneous equation with the lowest term

$$u_{tt} - u_{xx} + \lambda u = F \tag{61}$$

we consider in the domain $D : 0 < t < T$, $0 < x < t$ the problem of finding a solution $u(x,t) \in C(\overline{D})$ of equation (61), satisfying the homogeneous, corresponding to (2) and (3), conditions

$$u(0,t) = 0, \quad 0 \le t \le T, \tag{62}$$

$$\int_{I_t} K(x,t)u(x,t)dx = 0, \quad 0 \le t \le T, \tag{63}$$

where in $\overline{D}$ the coefficient λ in equation (61) is the given continuous function, $F \in C(\overline{D})$.

If the condition (14) is fulfilled, the problem (61), (62), (63) is solved by the method of successive approximations by the following scheme:

$$u_0 = 0, \quad \square\, u_n = -\lambda u_{n-1} + F, \quad n \ge 1, \tag{64}$$

$$u_n(0,t) = 0, \quad 0 \le t \le T, \tag{65}$$

$$\int_{I_t} K(x,t)u_n(x,t)dx = 0, \quad 0 \le t \le T. \tag{66}$$

By Theorem 1, the solution $u_n \in C(\overline{D})$ of the problem (64), (65), (66) exists and is unique. Now, using the method of mathematical induction, we can show that the estimate

$$|u_n(x,t) - u_{n-1}(x,t)| \le M c_3^n \lambda_0^{n-1} \frac{t^{2n}}{2^n n!}, \quad n \ge 1, \quad (x,t) \in \overline{D}, \tag{67}$$

with the positive constant c_3 from (58), independent of F and n, where $M = \|F\|_{C(\overline{D})}$, and $\lambda_0 = \|\lambda\|_{C(\overline{D})}$, is valid.

Indeed, by virtue of (64), the validity of the estimate (67) for $n = 1$ follows from the estimate (58) of Lemma 5 for $\alpha = 0$. Suppose now that the estimate (67) is valid for $n = m$, and let us prove that it is valid for $n = m + 1$. By (64), (65), (66) we have

$$\square\,(u_{m+1} - u_m) = -\lambda(u_m - u_{m-1}),$$

$$(u_{m+1} - u_m)(0,t) = 0, \quad 0 \le y \le T,$$

$$\int_{I_t} K(x,t)(u_{m+1} - u_m)(x,t)dx = 0, \quad 0 \le y \le T,$$

whence for the difference $(u_{m+1} - u_m)$ by the estimate (67) for $n = m$ and inequality (58) we obtain

$$|u_{m+1}(x,t) - u_m(x,t)| \le$$

$$\leq c_3\lambda_0 M c_3^m \lambda_0^{m-1} \frac{t^{2m+2}}{2^m m!\,(2m+2)} = M c_3^{m+1}\lambda_0^m \frac{t^{2(m+1)}}{2^{m+1}(m+1)!}.$$

Thus we have proved the estimate (67) for any $n \geq 1$.

From (67) follows uniform convergence in $\overline{D}$ of the following functional series:

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty}(u_n(x,t) - u_{n-1}(x,t)),$$

whose sum $u(x,t)$ is, as it easily seen, a generalized solution of equation (61) of the class $C(\overline{D})$, satisfying the conditions (62) and (63).

The uniqueness of the generalized solution of the problem (61), (62), (63) of the class $C(\overline{D})$ is proved analogously.

Thus the following theorem is valid.

Theorem 2. *Let the condition* (14) *be fulfilled. Then for any* $F \in C(\overline{D})$ *the problem* (61), (62), (63) *has the unique generalized solution of the class* $C(\overline{D})$ *for which the estimate*

$$|u(x,t)| \leq c\|F\|_{C(\overline{D}_{x,t})} \quad \forall (x,t) \in \overline{D}$$

is valid.

In the case of equation with the lowest terms of the type

$$u_{tt} - u_{xx} + \lambda_1 u_t + \lambda_2 u_x + \lambda u = F \tag{68}$$

when considering the problem (68), (2), (3) in the domain $D : 0 < t < T$, $0 < x < t$ it is required that the functions $K(x,t)$, $f_1(t)$ and $f_2(t)$ should have additional smoothness, namely

$$K(x,t) \in C^2(\overline{D}), \quad f_i(t) \in C^i([0,T]), \quad i = 1,2. \tag{69}$$

Here λ_1, λ_2 and λ_3 are the given continuous functions, $F \in C(\overline{D})$.

We present a brief scheme of investigation of the problem (68), (2), (3) which is based on the method of successive approximations. This method in turn requires investigation of integro-functional equation (15) in the class $C^1([0,T])$. In this case, taking into account (39) and the equality $\varphi(t) = f_1(0) + \int\limits_0^t \psi(\tau)d\tau$, where $\psi(\tau) = \varphi^{(1)}(\tau)$, equation (15) in the class $C^1([0,T])$ is equivalent to the equation

$$\psi(t) - \frac{1}{2}a(t)\psi\Big(\frac{1}{2}\Big) - a^{(1)}(t)\int\limits_0^{\frac{1}{2}t}\psi(\tau)d\tau + K_2(t,t)\int\limits_0^t \psi(\tau)d\tau +$$

$$+\int\limits_0^t K_{2t}(\xi,t)\Big[\int\limits_0^\xi \psi(\tau)d\tau\Big]d\xi = g_4(t), \quad 0 \leq t \leq T, \tag{70}$$

with respect to the unknown function ψ in the class $C([0,T])$, where $g_4(t) = g_3^{(1)}(t) + (a^{(1)}(t) - K_2(t,t))f_1(0)$. Note that the one-to-one correspondence between $\varphi \in C^1([0,T])$ and $\psi \in C([0,T])$ is given by the equality $\varphi(t) = f_1(0) + \int\limits_0^t \psi(\tau)d\tau$.

Since by (69), (56) and $g_3(t) = K^{-1}(t,t)g_1^{(1)}(t)$ we have $g_4(t) \in C([0,T])$ and $\left|\frac{1}{2}a(0)\right| < 1$, the integro-functional equation (70) is, as is known [32], uniquely solvable in the class $C([0,T])$ for the solution ψ of which the estimate

$$|\psi(t)| \leq c_4 \|g_4\|_{C([0,t])}, \quad 0 \leq t \leq T, \tag{71}$$

with the positive constant c_4, independent of $g_4(t)$, is valid.

By the estimate (71), under the conditions of Lemma 5, we prove analogously that a solution u of the problem (1), (2), (3) belongs to the class $C^1(\overline{D})$ and together with the estimate (58) satisfies likewise the estimates

$$|u_t(x,t)| \leq \widetilde{c}_3 M_F \frac{t^{\alpha+1}}{\alpha+1}, \quad |u_x(x,t)| \leq \widetilde{c}_3 M_F \frac{t^{\alpha+1}}{\alpha+1} \quad \forall (x,t) \in \overline{D}, \tag{72}$$

with the positive constant $\widetilde{c}_3$, independent of F, M_F and α.

We solve the problem (68), (2), (3) by using the method of successive approximations by the scheme:

$$u_0 = 0, \quad \Box\, u_n = -\lambda_1 \frac{\partial}{\partial t} u_{n-1} - \lambda_2 \frac{\partial}{\partial x} u_{n-1} - \lambda u_{n-1} + F, \quad n \geq 1, \tag{73}$$

$$u_n(0,t) = f_1(t), \quad 0 \leq t \leq T, \tag{74}$$

$$\int\limits_{I_t} K(x,t) u_n(x,t)dx = f_2(t), \quad 0 \leq t \leq T. \tag{75}$$

In a similar way, just as in obtaining inequality (67), using the estimates (72), we prove that the successive approximations from (73), (74), (75) satisfy the inequalities

$$|\partial^{i,j} u_n(x,t) - \partial^{i,j} u_{n-1}(x,t)| \leq M_1 M_2^n \frac{t^n}{n!},$$

$$n \geq 1, \partial^{i,j} = \frac{\partial^{i+j}}{\partial x^i \partial t^j}, \quad 0 \leq i+j \leq 1, \quad \forall (x,t) \in \overline{D}, \tag{76}$$

with the positive constants $M_1 = M_1(F, f_1, f_2)$ and $M_2 = M_2(\lambda_1, \lambda_2, \lambda, K)$, independent of n.

From (76) it follows that the functional series

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} (u_n(x,t) - u_{n-1}(x,t))$$

converges in the space $C^1(\overline{D})$, and its sum $u(x,t)$ is a generalized solution of equation (68) of the class $C^1(\overline{D})$ which satisfies the conditions (2) and (3). The uniqueness of that solution is proved analogously. Thus the following theorem is valid.

Theorem 3. *Let $K \in C^2(\overline{D})$; λ_1; λ_2; $\lambda \in C(\overline{D})$; $f_i \in C^i([0,T])$, $i=1,2$, $F \in C(\overline{D})$. If the condition* (14) *and the necessary conditions of solvability $f_2(0)=0$ and* (40) *of the problem* (68), (2), (3) *are fulfilled, then this problem has the unique generalized solution u of the class $C^1(\overline{D})$ for which the estimate*

$$\|u\|_{C^1(\overline{D})} \leq c\Big[\|f_1\|_{C^1([0,T])} + \|f_2\|_{C^2([0,T])} + \|F\|_{C(\overline{D})}\Big] \tag{77}$$

with the positive constant c, independent of f_1, f_2 and F, is valid.

Analogously to what is said in Remark 7, along with (77), for the solution $u \in C^1(\overline{D})$ of the problem (68), (2), (3) the following improved estimate

$$|u(x,t)| \leq c\Big[\|f_1\|_{C^1([0,t-x])} + \|f_2\|_{C^2([0,\frac{t+x}{2}])} + \|F\|_{C(\overline{D}_{x,t})}\Big] \quad \forall (x,t) \in \overline{D}$$

is valid.

3⁰. Consider now the case in which the boundary condition (2) in the problem (1), (2), (3) is replaced by the condition

$$u(t,t) = f_1(t), \quad 0 \leq t \leq T, \tag{78}$$

where f_1 is the given continuous function on the segment $[0,T]$.

The representation (6) given in terms of $\psi(t)=u(0,t)$ takes by virtue of (78) the form

$$u(x,t) = \psi(t-x) + f_1\Big(\frac{t+x}{2}\Big) - \\ -f_1\Big(\frac{t-x}{2}\Big) + \frac{1}{2}\iint\limits_{D_{x,t}} F\,dxdxt \quad \forall (x,t) \in \overline{D}. \tag{79}$$

Substituting the representation (79) for $u(x,t)$ into (3) and taking into account the equality

$$\int\limits_{I_t} K(x,t)\psi(t-x)dx = \int\limits_0^t K(x,t)\psi(t-x)dx = \int\limits_0^t K(t-\xi,t)\psi(\xi)d\xi,$$

we obtain

$$\int\limits_0^t K(t-\xi,t)\psi(\xi)d\xi = \widetilde{f}(t), \quad 0 \leq t \leq T. \tag{80}$$

With regard for equations of the type (9) we have

$$\widetilde{f}(t)=f_2(t)-\int\limits_{I_t}K(x,t)\Big[f_1\Big(\frac{t+x}{2}\Big)-f_1\Big(\frac{t-x}{2}\Big)+\frac{1}{2}\int\limits_{D_{x,t}}Fdxdt\Big]dx=$$

$$=f_2(t)-2\int\limits_{\frac{1}{2}t}^{t}K(2\xi-t,t)f_1(\xi)d\xi+$$

$$+2\int\limits_{0}^{\frac{1}{2}t}K(t-2\xi,t)f_1(\xi)d\xi-\frac{1}{2}\int\limits_{0}^{t}K(x,t)g_0(x,t)dx, \tag{81}$$

where $g_0(x,t)=\frac{1}{2}\iint_{D_{x,t}}Fd\xi d\tau$. By (81) it is clear that $\widetilde{f}(t)\in C^1([0,T])$.

Differentiation of equality (80) with respect to t yields

$$K(0,t)\psi(t)+\int\limits_{0}^{t}K_0(\xi,t)\psi(\xi)d\xi=f(t),\quad 0\le t\le T, \tag{82}$$

where $K_0(\xi,t)=K_x(t-\xi,t)+K_t(t-\xi,t)$, $f(t)=\widetilde{f}^{(1)}(t)$.

If $K(0,t)\neq 0$, $0\le t\le T$, then dividing the both parts of (82) by $K(0,t)$, we obtain the linear integral second kind Volterra equation which is, as is known, solvable in the class $C([0,T])$. In this connection, the following theorem holds.

Theorem 4. *Let $K\in C^1(\overline{D})$, $K\Big|_{\overline{OA}}\neq 0$, i.e. $K(0,t)\neq 0$, $0\le t\le T$, and $F\in C(\overline{D})$, $f_i\in C^{i-1}([0,T])$, $i=1,2$. Then if the necessary conditions of solvability $f_0(0)=0$ and (40) of the problem (1), (78), (3) are fulfilled in the class $C(\overline{D})$, then this problem has the unique solution $u\in C(\overline{D})$ for which the estimate*

$$|u(x,t)|\le c\Big[\|f_1\|_{C([0,t-x])}+\|f_2\|_{C^1([0,\frac{t+x}{2}])}+\|F\|_{C(\overline{D}_{x,t})}\Big]\quad\forall(x,t)\in\overline{D},$$

with the positive constant c, independent of f_1, f_2 and F, is valid.

4^0. In this section we consider the case in which instead of (3) we take the integral condition of the type

$$\int\limits_{I_t}[K_1(x,t)u_t(x,t)+K_2(x,t)u_x(x,t)+K_0(x,t)u(x,t)]dx=f_2(t), \tag{83}$$

$$0\le t\le T,$$

where $K_i(x,t)\in C^1(\overline{D})$, $i=0,1,2$, and $f_2\in C^1([0,T])$ are the given functions, and $K_1^2(x,t)+K_2^2(x,t)\neq 0$, $(x,t)\in\overline{D}$ and $f_2(0)=0$.

Substituting the representation (7) for $u(x,t)$ into (83) and taking into account equalities of the type (9), for the unknown function $\varphi(t)=u(t,t)$ we obtain the equality:

$$\int\limits_{\frac{1}{2}t}^{t} K_1(2\xi-t,t)\varphi^{(1)}(\xi)d\xi-$$
$$-\int\limits_{0}^{\frac{1}{2}t} K_1(t-2\xi,t)\varphi^{(1)}(\xi)d\xi+\int\limits_{\frac{1}{2}t}^{t} K_2(2\xi-t,t)\varphi^{(1)}(\xi)d\xi+$$
$$+\int\limits_{0}^{\frac{1}{2}t} K_2(t-2\xi,t)\varphi^{(1)}(\xi)d\xi+2\int\limits_{\frac{1}{2}t}^{t} K_0(2\xi-t,t)\varphi(\xi)d\xi-$$
$$-2\int\limits_{0}^{\frac{1}{2}t} K_0(t-2\xi,t)\varphi(\xi)d\xi=$$
$$=\widetilde{g}(t),\quad 0\le t\le T. \tag{84}$$

Here,

$$\widetilde{g}(t)=f_2(t)-\int\limits_{0}^{t}[K_1(x,t)f_1^{(1)}(t-x)-K_2(x,t)f_1^{(1)}(t-x)+K_0(x,t)f_1(t-x)]dx-$$
$$-\int\limits_{0}^{1}[K_1(x,t)g_{0t}(x,t)+K_2(x,t)g_{0x}(x,t)+K_0(x,t)g_0(x,t)]dx=$$
$$=f_2(t)+(K_1(t,t)+K_2(t,t))f_1(0)-(K_1(0,t)+K_2(0,t))f_1(t)+$$
$$+\int\limits_{0}^{t}\Big[\frac{\partial}{\partial t}(K_1(t-\xi,t)+K_2(t-\xi,t))-K_0(t-\xi,t)\Big]f_1(\xi)d\xi-$$
$$-\int\limits_{0}^{t}[K_1(x,t)g_{0t}(x,t)+K_2(x,t)g_{0x}(x,t)+K_0(x,t)g_0(x,t)]dx. \tag{85}$$

Integrating by parts the summands in the left-hand side of equality (84), we obtain

$$K_1(t,t)\varphi(t)-K_1(0,t)\varphi\Big(\frac{1}{2}t\Big)-\int\limits_{\frac{1}{2}t}^{t}\Big[\frac{\partial}{\partial t}K_1(2\xi-t,t)\Big]\varphi(\xi)d\xi-K_1(0,t)\varphi\Big(\frac{1}{2}t\Big)+$$

$$+K_1(t,t)\varphi(0)+\int\limits_0^{\frac12 t}\Big[\frac{\partial}{\partial t}K_1(t-2\xi,t)\Big]\varphi(\xi)d\xi+K_2(t,t)\varphi(t)-K_2(0,t)\varphi\Big(\frac12 t\Big)-$$

$$-\int\limits_{\frac12 t}^{t}\Big[\frac{\partial}{\partial t}K_2(2\xi-t,t)\Big]\varphi(\xi)d\xi+K_2(0,t)\varphi\Big(\frac12 t\Big)-K_2(t,t)\varphi(0)-$$

$$-\int\limits_0^{\frac12 t}\Big[\frac{\partial}{\partial t}K_2(t-2\xi,t)\Big]\varphi(\xi)d\xi+2\int\limits_{\frac12 t}^{t}K_0(2\xi-t,t)\varphi(\xi)d\xi-$$

$$-2\int\limits_0^{\frac12 t}K_0(t-2\xi,t)\varphi(\xi)d\xi=\widetilde{g}(t),\quad 0\le t\le T. \tag{86}$$

Since by (2) there takes place the equality $\varphi(0)=u(0,0)=f_1(0)$, equation (86) takes the form

$$(K_1(t,t)+K_2(t,t))\varphi(t)-2K_1(0,t)\varphi\Big(\frac12 t\Big)+\int\limits_0^t K_3(\xi,t)\varphi(\xi)d\xi=$$

$$=\widetilde{g}(t)+(K_2(t,t)-K_1(t,t))f_1(0),\quad 0\le t\le T, \tag{87}$$

where $K_3(\xi,t)$ is the completely definite, piecewise continuously differentiable function of its arguments.

Under the assumption that

$$K_1(t,t)+K_2(t,t)\neq 0,\quad 0\le t\le T, \tag{88}$$

we can rewrite equation (87) as follows:

$$\varphi(t)-a_0(t)\varphi\Big(\frac12 t\Big)+\int\limits_0^t K_4(\xi,t)\varphi(\xi)d\xi=g(t),\quad 0\le t\le T. \tag{89}$$

Here,

$$a_0(t)=\frac{2K_1(0,t)}{K_1(t,t)+K_2(t,t)}, \tag{90}$$

and the functions $K_4(\xi,t)$ and $g(t)$ are defined from (87).

Remark 9. By the representation (7), equality (85) and the expression for $\frac{\partial}{\partial x}g_0(x,t)$ which, just as the expression (55) for $\frac{\partial}{\partial x}g_0(x,t)$, does not contain derivatives of the functions F, the condition (83) is rewritten in the form (89), where the derivatives of the functions φ, f_1, f_2 and F are omitted. Therefore the problem (1), (2), (83) in the class $C(\overline{D})$ is, by virtue of Remarks 1 and 2, equivalent to the integro-functional equation (89) in the class $C([0,T])$, and hence by equality (85) it is sufficient to require that the functions f_1 and f_2 be only continuous on the segment $[0,T]$.

Remark 10. According to the results presented in [29], if the condition

$$|a_0(0)| < 1, \quad \text{i.e.} \quad 2|K_1(0,0)| < |K_1(0,0) + K_2(0,0)| \tag{91}$$

is fulfilled, then equation (89) for any continuous function $g(t)$ has a unique continuous solution $\varphi(t)$ for which the estimate

$$|\varphi(t)| \le c\|g\|_{C([0,t])}, \quad 0 \le t \le T, \tag{92}$$

with the positive constant c, independent of $g(t)$, is valid. However, if

$$|a_0(0)| > 1, \quad \text{i.e.} \quad 2|K_1(0,0)| > |K_1(0,0) + K_2(0,0)|, \tag{93}$$

then equation (89) is normally Hausdorff solvable in the class $C([0,T])$, and the homogeneous problem, corresponding to (89), has in $C([0,T]$ an infinite set of linearly independent solutions. In case

$$a_0(0) = 1, \quad \text{i.e.} \quad K_1(0,0) = K_2(0,0), \tag{94}$$

by virtue of $f_2(0) = 0$ and (85) the necessary condition $g(0) = 0$ of solvability of the equation will be fulfilled, and by Lemma 3, if the supplementary conditions

$$f_i(t) - f_i(0) \in C_\alpha((0,T]), \quad i = 1,2; \quad 0 < \alpha < 1, \tag{95}$$

are also fulfilled, then equation (89) has the unique solution $\varphi(t) \in C([0,T])$, satisfying the initial condition (39). If, however,

$$a_0(0) = -1, \quad \text{i.e.} \quad 3K_1(0,0) + K_2(0,0) = 0, \tag{96}$$

then similarly to the proof of Lemma 3, if the conditions (95) are fulfilled, then equation (89) has the unique continuous solution on the segment $[0,T]$.

In accordance with Remarks 9 and 10, as well as owing to the representation (7), for the problem (1), (2), (83) in the class $C(\overline{D})$ the following theorem is valid.

Theorem 5. *Let* $K_i \in C^1(\overline{D})$, $i = 1,2$; $f_i \in C([0,T])$, $i = 1,2$, $f_2(0) = 0$; $F \in C(\overline{D})$ *and let the condition* (88) *be fulfilled. Then:*

(i) *if inequality* (91) *is fulfilled, the problem* (1), (2), (83) *has the unique solution* $u \in C(\overline{D})$ *for which the estimate*

$$|u(x,t)| \le c\Big[\|f_1\|_{C([0,t-x])} + \|f_2\|_{C([0,\frac{t+x}{2}])} + \|F\|_{C(\overline{D}_{x,t})}\Big] \quad \forall (x,t) \in \overline{D},$$

with the positive constant c*, independent of* f_1, f_2 *and* F*, is valid;*

(ii) *if inequality* (93) *is fulfilled, the problem* (1), (2), (83) *is normally Hausdorff solvable in the class* $C(\overline{D})$*, and the homogeneous problem, corresponding to* (1), (2), (3)*, has in* $C(\overline{D})$ *an infinite set of linearly independent solutions;*

(iii) *in fulfilling equality* (94), *or* (96), *if of the functions* f_1 *and* f_2 *it is required that the supplementary conditions* (95) *be fulfilled, then the problem* (1), (2), (83) *has the unique solution* $u \in C(\overline{D})$ *for which the estimate*

$$|u(x,t)| \le c\Big[\|f_1\|_{C([0,t-x])} + \|f_1(t)-f_1(0)\|_{C_\alpha((0,t-x])} + \|f_2\|_{C([0,\frac{t+x}{2}])} + \\ +\|f_2(t)-f_2(0)\|_{C_\alpha((0,\frac{t+x}{2}])} + \|F\|_{C(\overline{D}_{x,t})}\Big], \quad 0<\alpha<1,$$

with the positive constant c, *independent of* f_1, f_2 *and* F, *is valid.*

Consider now the question on the solvability of the problem (1), (2), (83) in the class $C^1(\overline{D})$ which is, by the representation (7), equivalent to the solvability of equation (84) with respect to the unknown function $\mu(t)=\varphi^{(1)}(t)\in C([0,T])$ connection of the latter with the function $\varphi(t)\in C^1([0,T])$ by virtue of $\varphi(t)=u(t,t)$ and the equality $\varphi(0)=f_1(0)$ from (2) is given by the relation $\varphi(t)=f_1(0)+\int_0^t \mu(\tau)d\tau$.

Below it will be assumed that $K_i\in C^2(\overline{D})$, $i=1,2$; $f_i\in C^1([0,T])$, $i=1,2$, $f_2(0)=0$; $F\in C^1(\overline{D})$ and the condition (88) is fulfilled. Under these requirements, the function $\widetilde{g}(t)$ from (84) belongs by virtue of (85) to the class $C^1([0,T])$. Therefore differentiating both parts of equation (84) with respect to the variable t and taking into account the notation $\mu(t)=\varphi^{(1)}(t)$ and the equality $\varphi(t)=f_1(0)+\int_0^t \mu(\tau)d\tau$, we obtain

$$(K_1(t,t)+K_2(t,t))\mu(t)-K_1(0,t)\mu\Big(\frac{1}{2}t\Big)+\int\limits_0^t K_5(\xi,t)\mu(\xi)d\xi=\widetilde{g}^{(1)}, \quad (97)$$
$$0\le t\le T,$$

where $K_5(\xi,t)$ is the completely definite piecewise continuous function of its arguments.

In the assumption that the condition (88) is fulfilled, we can rewrite equation (97) in the form

$$\mu(t)-\frac{1}{2}a_0(t)\mu\Big(\frac{1}{2}t\Big)+\int\limits_0^t K_5(\xi,t)\mu(\xi)d\xi=\widetilde{g}^{(1)}(t), \quad 0\le t\le T, \qquad (98)$$

where the function $a_0(t)$ is defined by equality (90).

Applying Remark 10 to equation (98), we arrive at the following

Theorem 6. *Let* $K_i\in C^2(\overline{D})$, $i=1,2$; $f_i\in C^1([0,T])$, $i=1,2$; $f_2(0)=0$; $F\in C^1(\overline{D})$ *and let the condition* (88) *be fulfilled. Then:*

(i) *if the inequality* $\Big|\frac{1}{2}a_0(0)\Big|<1$, *i.e.* $|K_1(0,0)|<|K_1(0,0)+K_2(0,0)|$, *is fulfilled, then the problem (1), (2), (83) has the unique solution* $u\in C^1(\overline{D})$

for which the estimate

$$\|u\|_{C^1(\overline{D}_{x,t})} \leq c\|f_1\|_{C^1([0,t-x])} + \|f_2\|_{C^1([0,\frac{t+x}{2}])} + \|F\|_{C^1(\overline{D}_{x,t})}\Big] \quad \forall (x,t) \in \overline{D},$$

with the positive constant c, independent of f_1, f_2 and F, is valid;

(ii) *if the inequality* $\left|\frac{1}{2}a_0(0)\right| > 1$, *i.e.* $|K_1(0,0)| > |K_1(0,0) + K_2(0,0)|$ *is fulfilled, then the problem* (1), (2), (83) *is normally Hausdorff solvable in the class* $C^1(\overline{D})$, *and the homogeneous problem, corresponding to* (1), (2), (83), *has in the class* $C^1(\overline{D})$ *an infinite set of linearly independent solutions;*

(iii) *in fulfilling the equality* $\frac{1}{2}a_0(0) = 1$, *i.e.* $K_2(0,0) = 0$, *if the necessary condition of solvability of equation* (98),

$$\widetilde{g}^{(1)}(0) = f_2^{(1)}(0) + 2\sum_{i=1}^{2}(K_{ix}(0,0) + K_{it}(0,0))f_1(0) -$$
$$-(K_{1t}(0,0) + K_{2t}(0,0))f_1(0) - K_1(0,0)f_2^{(1)}(0) + K_0(0,0)f_1(0) = 0,$$

is fulfilled, and of the functions f_1 and f_2 it is required that the supplementary conditions

$$f_i^{(1)}(t) - f_i^{(1)}(0) \in C_\alpha((0,T]), \quad i = 1,2; \quad 0 < \alpha < 1, \tag{99}$$

be fulfilled, then the problem (1), (2), (83) *has the unique solution* $u \in C^1(\overline{D})$ *for which the estimate*

$$\|u\|_{C^1(\overline{D}_{x,t})} \leq c\Big[\|f_1\|_{C^1([0,t-x])} + \|f_1^{(1)}(t) - f_1^{(1)}(0)\|_{C_\alpha((0,t-x])} + \|f_2\|_{C^1((0,\frac{t+x}{2}])} +$$
$$+\|f_2^{(1)}(t) - f_2^{(1)}(0)\|_{C_\alpha((0,\frac{t+x}{2}])} + \|F\|_{C^1(\overline{D}_{x,t})}\Big] \quad \forall (x,t) \in \overline{D}, \tag{100}$$

with the positive constant c, independent of f_1, f_2 and F, is valid;

(iiii) *if the equality* $\frac{1}{2}a_0(0) = -1$, *i.e.* $2K_1(0,0) + K_2(0,0) = 0$, *is fulfilled and the functions f_1 and f_2 satisfy the supplementary conditions* (99), *then the problem* (1), (2), (83) *has the unique solution* $u \in C^1(\overline{D})$ *for which the estimate* (100) *is valid.*

According to Remark 9 applied to the problem (1), (2), (83), the following theorem is valid.

Theorem 7. *Let* $K_i \in C^{k+1}(\overline{D})$, $i = 1,2$; $f_i \in C^k([0,T])$, $i = 1,2$; $f_2(0) = 0$; $F \in C^k(\overline{D})$, $k \geq 2$; $\frac{1}{2^i}a_0(0) \neq 1$, $i = 1,\dots,k-1$, *and let the condition* (88) *be fulfilled. Then:*

(i) *if the inequality* $\left|\frac{1}{2^k}a_0(0)\right| < 1$ *is fulfilled, the problem* (1), (2), (83) *has the unique solution* $u \in C^k(\overline{D})$ *for which the estimate*

$$\|u\|_{C^k(\overline{D})} \leq c\Big[\|f_1\|_{C^k([0,t-x])} + \|f_2\|_{C^k([0,\frac{t+x}{2}])} + \|F\|_{C^k(\overline{D}_{x,t})}\Big] \quad (x,t) \in \overline{D},$$

with the positive constant c, independent of f_1, f_2 and F, is valid;

(ii) *if the equality* $\left|\frac{1}{2^k}a_0(0)\right| > 1$ *is fulfilled, then the problem* (1), (2), (83) *is normally Hausdorff solvable in the class* $C^k(\overline{D})$, *and the homogeneous problem, corresponding to* (1), (2), (83), *has in the class* $C^k(\overline{D})$ *an infinite set of linearly independent solutions;*

(iii) *in fulfilling the equality* $\frac{1}{2^k}a_0(0) = 1$, *if the necessary condition of solvability* $\widetilde{g}^{(k-1)}(0) = 0$ *of equation (98) in the class* $C^k([0,T])$ *is fulfilled, and of the functions* f_1 *and* f_2 *it is required that the supplementary conditions*

$$f_i^{(k)}(t) - f_i^{(k)}(0) \in C_\alpha((0,T]), \quad i = 1,2; \quad 0 < \alpha < 1, \tag{101}$$

be fulfilled, then the problem (1), (2), (83) *has the unique solution* $u \in C^k(\overline{D})$ *for which the estimate*

$$\|u\|_{C^k(\overline{D}_{x,t})} \leq c\Big[\|f_1\|_{C^k([0,t-x])} + \|f_1^{(k)}(t) - f_1^{(k)}(0)\|_{C_\alpha((0,t-x])} + \tag{102}$$

$$+\|f_2\|_{C^k([0,\frac{t+x}{2}])} + \|f_2^{(k)}(t) - f_2^{(k)}(0)\|_{C_\alpha([0,\frac{t+x}{2}])} + \|F\|_{C^k(\overline{D}_{x,t})}\Big] \quad \forall (x,t) \in \overline{D},$$

with the positive constant c, *independent of* f_1, f_2, *and* F, *is valid;*

(iiii) *if the equality* $\frac{1}{2^k}a_0(0) = -1$ *is fulfilled, and the functions* f_1 *and* f_2 *satisfy the supplementary conditions* (101), *then the problem* (1), (2), (83) *has the unique solution* $u \in C^k(\overline{D})$ *for which the estimate* (102) *is valid.*

2. Problems with Integral Restrictions for Some Classes of Hyperbolic Systems of Second Order

1^0. In this section it will be assumed that (1), (61) and (68) are the systems of equations with respect to an unknown vector function $u = (u_1, \dots, u_n)$, $n \geq 2$; K, K_i, λ, λ_j $(i = 0,1,2;\ j = 1,2)$ are the given square $(n \times n)$-matrices appearing in the boundary conditions (3) and (83) as well as in equations (61) and (68); $F = (F_1, \dots, F_n)$, $f_i = (f_{i1}, \dots, f_{in})$, $i = 1,2$, are the given n-dimensional vectors. In this case, the above-proven theorems remain valid if one makes some changes in the conditions of these theorems. For example: 1) in the conditions of Theorems 1-3, instead of inequality (14) it is required that $\det K(t,t) \neq 0$, $0 \leq t \leq T$; 2) in Theorem 4, instead of the condition $K(0,t) \neq 0$, $0 \leq t \leq T$, it is required that $\det K(0,t) \neq 0$; $0 \leq t \leq T$ 3) in Theorem 5, instead of the condition (88) it is required that

$$\det[K_1(t,t) + K_2(t,t)] \neq 0, \quad 0 \leq t \leq T, \tag{103}$$

and in item (i) of that theorem instead of (91) it is required that the eigen values of the matrix $A_0 = 2K_1(0,0)[K_1(0,0)+K_2(0,0)]^{-1}$ be less than unity in modulo; in item (ii) of the above-mentioned theorem instead of (93) it is required that $\sigma_i \neq 1$, $i = 1, \dots, n$, and $\max\limits_{1\leq i\leq n} |\sigma_i| > 1$, where σ_i, $i = 1, \dots, n$, are eigen values with regard for the multiplicity of the $(n \times n)$-matrix A_0; in item (iii) of Theorem 5 instead of equalities (94), or (96) it is required that $\max_{1\le i\le n}|\sigma_i|=1$; (4) in Theorem 7, whose particular case is Theorem 6 for $k=1$, instead of the condition $\frac{1}{2^i}a_0(0)\neq 1$, $i=1,\dots,k-1$, it is required that $\det\left[\frac{1}{2^i}A_0-E\right]\neq 0$, $i=1,\dots,k-1$, where E is the unit $(n\times n)$-matrix; in item (i) of that theorem instead of the inequality $\frac{1}{2^k}a_0(0)\neq 1$ it is required that $\max_{1\le i\le n}|\sigma_i|<2^k$; in item (ii) of Theorem 7 instead of the inequality $\left|\frac{1}{2^k}a_0(0)\right|>1$ it is required that the inequalities $\sigma_i\neq 2^k$, $i=1,\dots,n$, and $\max_{1\le i\le n}|\sigma_i|>1$ be fulfilled; in items (iii) and (iiii) of the theorem instead of the equalities $\frac{1}{2^k}a_0(0)=\pm 1$ it is required that $\max_{1\le i\le n}|\sigma_i|=2^k$, and in case $\max_{1\le i\le n}|\sigma_i|=\sigma_{i_0}=2^k$ for some index i_0, $1\le i_0\le n$, instead of $\widetilde{g}^{(k-1)}(0)=0$ we have to consider the corresponding condition of solvability of that problem.

2^0**.** In this section we consider the linear system of differential second order equations of the type

$$Au_{xx}+2Bu_{xt}+Cu_{tt}=0, \tag{104}$$

where A, B and C are the given real constants $(n\times n)$-matrices, $u=(u_1,\dots,u_n)$ is the unknown n-dimensional real vector, $n\ge 2$.

Below, it will be assumed that $\det C\neq 0$ and the system (104) is strictly hyperbolic, i.e. the characteristic polynomial

$$p(\lambda)=\det(A+2B\lambda+C\lambda^2)$$

of the system (104) has exactly $2n$ real simple roots $\lambda_1,\lambda_2,\dots,\lambda_{2n}$.

Since the roots of the polynomial $p(\lambda)$ are simple, $\mathrm{Ker}(A+2B\lambda_i+C\lambda_i^2)=1$, $i=1,\dots,2n$ [32], and if $\nu_i\in R^n$, $\nu_i\in\mathrm{Ker}(A+2B\lambda_i+C\lambda_i^2)$, $\|\nu_i\|_{R^n}\neq 0$, $i=1,\dots,2n$, then a general solution of the system (104) of the class $C^k(\overline{D})$ where $D:0<t<T$, $0<x<t$; $k\ge 0$, is given by the equality [33]

$$u(x,t)=u(0,0)+\sum_{i=1}^{2n}\nu_i\varphi_i(x+\lambda_i t)\quad\forall(x,t)\in\overline{D}. \tag{105}$$

Here, φ_i is an arbitrary function of the class $C^k([\alpha_i,\beta_i])$, $\alpha_i=\min\limits_{(x,t)\in\overline{D}}(x+\lambda_i t)$, $\beta_i=\max\limits_{(x,t)\in\overline{D}}(x+\lambda_i t)$ satisfying the normalizing condition

$$\varphi_i(0)=0,\quad i=1,\dots,2n, \tag{106}$$

and between the solutions $u(x,t)\in C^k(\overline{D})$, $k\ge 0$, of the system (104) and the functions $\varphi_i\in C^k([\alpha_i,\beta_i])$, satisfying equalities (106), there takes place the one-to-one correspondence.

Below, it will be assumed that the closed domain $\overline{D}:0\le t\le T$, $0\le x\le t$, does not contain the characteristic straight lines $l_i:x+\lambda_i t=0$,

$i = 1, \dots, 2n$, of the system (104) passing through the origin, i.e. $l_i \cap \overline{D} = \{(0,0)\}$. This is, in turn, equivalent to the condition

$$\lambda_i \notin [-1, 0], \quad i = 1, \dots, 2n. \tag{107}$$

By (107), one of the numbers α_i and β_i appearing in the domain of definition $[\alpha_i, \beta_i]$ of the function φ_i is equal to zero.

Taking into account (107), without restriction of generality, we assume that the roots $\lambda_1, \dots, \lambda_{2n}$ of the characteristic polynomial $p(\lambda)$ are enumerated in such a way that

$$\lambda_1 < \lambda_2 < \dots < \lambda_{s_0} < -1, \quad 0 < \lambda_{s_0+1} < \dots < \lambda_{2n}, \tag{108}$$

where it is assumed that $0 < s_0 < 2n$.

In accordance with (108), if from the function $\varphi_i(\xi)$ appearing in (105) with the domain of definition $[\alpha_i, \beta_i]$ we pass to the function $\psi_i(\eta)$ using the rule $\psi_i\Big(\dfrac{\eta}{\lambda_i}\Big) = \varphi(\eta)$ for $1 \le i \le s_0$ and to the function $\psi_i(\eta)$ using the equality $\psi_i\Big(\dfrac{\eta}{1+\lambda_i}\Big) = \varphi_i(\eta)$ for $s_0 < i \le 2n$, then the domain of definition for the function $\psi_i(\eta)$ is the segment $[0, T]$, and the representation (105) takes the form

$$u(x,t) = u(0,0) + \sum_{i=1}^{s_0} \nu_i \psi_i\Big(\frac{x+\lambda_i t}{\lambda_i}\Big) + \\ + \sum_{i=s_0+1}^{2n} \nu_i \psi_i\Big(\frac{x+\lambda_i t}{1+\lambda_i}\Big) \quad \forall (x,t) \in D, \tag{109}$$

where $\psi_i \in C^k([0,T])$, and

$$\psi_i(0) = 0, \quad i = 1, \dots, 2n. \tag{110}$$

For the system (104) in the domain D we consider the problem which is formulated as follows: find in D a solution $u(x,t)$ of the system (104), satisfying both the boundary condition

$$N(t) u(0,t) = f_1(t), \quad 0 \le t \le T, \tag{111}$$

and the integral condition of the type

$$\int\limits_{I_t} K(x,t) u(x,t) dx := \int\limits_0^t K(x,t) u(x,t) dx = f_2(t), \quad 0 \le t \le T, \tag{112}$$

where $I_\tau : t = \tau$; $0 \le x \le \tau$; $N(t) \in C([0,T])$ and $K(x,t) \in C^1(\overline{D})$ are the given matrices of order $s_0 \times n$ and $(2n - s_0) \times n$, respectively; $f_1(t) \in C([0,T])$ is the given s_0-dimensional vector function, and $f_2(t) \in C^1([0,T])$ is the given $(2n - s_0)$-dimensional vector function, where $f_2(0) = 0$.

It can be easily verified that

$$\int\limits_0^t K(x,t)\nu_i\psi_i\Big(\frac{x+\lambda_i t}{\lambda_i}\Big)dx = \tag{113}$$

$$= -\lambda_i \int\limits_{\frac{1+\lambda_i}{\lambda_i}t}^{t} K(\lambda_i(\xi - t),t)\nu_i\psi_i(\xi)d\xi, \quad 1 \le i \le s_0,$$

$$\int\limits_0^t K(x,t)\nu_i\psi_i\Big(\frac{x+\lambda_i t}{1+\lambda_i}\Big)dx = \tag{114}$$

$$= (1+\lambda_i) \int\limits_{\frac{\lambda_i}{1+\lambda_i}t}^{t} K((1+\lambda_i)\xi - \lambda_i t,t)\nu_i\psi_i(\xi)d\xi, \quad s_0+1 \le i \le 2n.$$

By (108) we have

$$0 < \tau_i = \frac{1+\lambda_i}{\lambda_i} < 1, \quad i = 1,\dots,s_0;$$

$$0 < \tau_j = \frac{\lambda_i}{1+\lambda_j} < 1, \quad j = s_0+1,\dots,2n. \tag{115}$$

Substituting the representation (109) for the solution $u(x,t)$ of the system (104) into the integral condition (112) and taking into account (113), (114) and (115), we obtain

$$\int\limits_0^t K(x,t)u(0,0)dx - \sum_{i=1}^{s_0}\lambda_i \int\limits_{\tau_i t}^{t} K(\lambda_i(\xi - t),t)\nu_i\psi_i(\xi)d\xi +$$

$$+ \sum_{i=s_0+1}^{2n}(1+\lambda_i)\int\limits_{\tau_i t}^{t} K((1+\lambda_i)\xi - \lambda_i t,t)\nu_i\psi_i(\xi)d\xi = f_2(t), \quad 0 \le t \le T. \tag{116}$$

Differentiation of both parts of equality (116) with respect to the variable t yields

$$K(t,t)u(0,0) - \sum_{i=1}^{s_0}\lambda_i K(0,t)\nu_i\psi_i(t) + \sum_{i=1}^{s_0}\lambda_i\tau_i K(\lambda_i(\tau_i-1)t,t)\nu_i\psi_i(\tau_i t) +$$

$$+ \sum_{i=s_0+1}^{2n}(1+\lambda_i)K(t,t)\nu_i\psi_i(t) - \sum_{i=s_0+1}^{2n}(1+\lambda_i)\tau_i K([(1+\lambda_i)\tau_i - \lambda_i]t,t)\nu_i\psi_i(\tau_i t) -$$

$$-\sum_{i=1}^{s_0}\lambda_i\int\limits_{\tau_i t}^{t}\frac{\partial}{\partial t}[K(\lambda_i(\xi-t),t)]\nu_i\psi_i(\xi)d\xi+$$

$$+\sum_{i=s_0+1}^{2n}(1+\lambda_i)\int\limits_{\tau_i t}^{t}\frac{\partial}{\partial t}[K((1+\lambda_i)\xi-\lambda_i t,t)]\nu_i\psi_i(\xi)d\xi=$$

$$=f_2^{(1)}(t),\quad 0\le t\le T. \tag{117}$$

Regarding (109) and (110), from equalities (111) and (117) we obtain the following overdetermined linear system of $2n$ equations with respect to $u_1(0,0),\dots,u_n(0,0)$ unknowns:

$$\begin{pmatrix}N(0)\\K(0,0)\end{pmatrix}u(0,0)=\begin{pmatrix}f_1(0)\\f_2^{(1)}(0)\end{pmatrix},\quad u(0,0)=(u_1(0,0),\dots,u_n(0,0)). \tag{118}$$

As is known, for the unique solvability of the overdetermined system (118) it is necessary and sufficient that the following two conditions

$$\operatorname{rank}\begin{pmatrix}N(0)\\K(0,0)\end{pmatrix}=n, \tag{119}$$

and

$$\operatorname{rank}\begin{pmatrix}N(0) & f_1(0)\\K(0,0) & f_2^{(1)}(0)\end{pmatrix}=\operatorname{rank}\begin{pmatrix}N(0)\\K(0,0)\end{pmatrix}. \tag{120}$$

be fulfilled. In addition, we can consider condition (120) as the necessary condition of solvability of the problem (104), (111), (112) in the class $C(\overline{D})$.

With regard for the representation (109) and notation of (115), we can rewtite the boundary condition (111) in the form

$$\sum_{i=1}^{s_0}N(t)\nu_i\psi_i(t)+\sum_{i=s_0+1}^{2n}N(t)\nu_i\psi_i(\tau_i t)=$$

$$=f_1(t)-N(t)u(0,0),\quad 0\le t\le T. \tag{121}$$

Introduce now into consideration the matrices Λ_1, Λ_2, Λ_3 and K_0 of orders $n\times s_0$, $n\times(2n-s_0)$, $n\times s_0$ and $2n\times 2n$, respectively, as follows:

$$\Lambda_1=(\nu_1,\dots,\nu_{s_0}),\quad \Lambda_2=((1+\lambda_{s_0+1})\nu_{s_0+1},\dots,(1+\lambda_{2n})\nu_{2n}),$$

$$\Lambda_3=(\lambda_1\nu_1,\quad,\lambda_{s_0}\nu_{s_0}),\quad K_0(t)=\begin{pmatrix}N(t)\Lambda_1 & 0\\-K(0,t)\Lambda_3 & K(t,t)\Lambda_2\end{pmatrix}, \tag{122}$$

where, for example, the columns in the matrix Λ_1 are the vectors $\nu_1,\nu_2,\dots,\nu_{s_0}$.

Taking into account (122), we can write the system of integro-functional equations (117), (121) with respect to the unknown vector function $\psi(t)=$

$(\psi_1(t), \quad \psi_{2n}(t))$ in the form

$$K_0(t)\psi(t)+\sum_{i=1}^{2n}K_i(t)\psi_i(\tau_i t)+$$
$$+\sum_{i=1}^{2n}\int\limits_{\tau_i t}^{t}L_i(\xi,t)\psi(\xi)d\xi=f(t), \quad 0\le t\le T. \tag{123}$$

Here, $K_i(t)$ and $L_i(\xi,t)$ are completely definite continuous $2n\times 2n$ matrices of their arguments, and the right-hand side of the system (123) defined by the equality

$$f(t)=\begin{pmatrix} f_1(t)\\ f_2^{(1)}(t)\end{pmatrix}-\begin{pmatrix} N(t)\\ K(t,t)\end{pmatrix}u(0,0)$$

can likewise be assumed as given, if the conditions (119) and (120) are fulfilled. In this case, owing to (110) and (118), the necessary condition

$$f(0)=0 \tag{124}$$

of solvability of the system (123) in the class $C([0,T])$ is fulfilled automatically.

Remark 11. By (122), it is easy to see that the condition $\det K_0(t)\neq 0$, $0\le t\le T$, is equivalent to the fulfilment of the following two conditions:

$$\det N(t)\Lambda_1\neq 0, \quad \det K(t,t)\Lambda_2\neq 0, \quad 0\le t\le T. \tag{125}$$

If the conditions (125) are fulfilled, and if in addition, the inequality

$$\sum_{i=1}^{2n}\|K_0^{-1}(0)K_i(0)\|_{R^{2n}\to R^{2n}}<1, \tag{126}$$

is fulfilled, then, as is known, the system (123) has the unique solution in the class $C([0,T])$ for which the estimate [33]

$$\|\psi(t)\|_{R^n}\le c\Big(\|f_1\|_{C([0,t])}+\|f_2\|_{C^1([0,t])}\Big), \quad 0\le t\le T,$$

with the positive constant c, independent of f_1 and f_2, is valid. Moreover, by (124), the conditions (110) will likewise be fulfilled, i.e. $\psi(0)=0$.

Remark 12. In fulfilling the conditions (119) and (120), since the problem (104), (111), (112) in the class $C([0,T])$ is equivalently reduced to the system (123) in the class $C([0,T])$, according to Remark 11, the following theorem is valid.

Theorem 8. *Let* $N(t)\in C([0,T])$, $K(x,t)\in C^1(\overline{D})$, $f_1(t)\in C([0,T])$, $f_2(t)\in C^1([0,T])$, $f_2(0)=0$ *and let the conditions* (108), (119), (120),

(125) *and* (126) *be fulfilled. Then the problem* (104), (111), (112) *has the unique solution* $u(x,t)$ *in the class* $C(\overline{D})$ *for which the estimate*

$$\|u(x,t)\|_{R^n} \leq c\Big[\|f_1\|_{C[0,\frac{x+\lambda_1 t}{\lambda_1}]} + \|f_2\|_{C[0,\frac{x+\lambda_{2n} t}{1+\lambda_{2n}}]}\Big] \quad \forall (x,t) \in \overline{D},$$

with the positive constant c*, independent of* f_1 *and* f_2*, is valid.*

Note that if the condition (125), or (126) is violated, then the homogeneous system of integro-functional equations, corresponding to (123), may have an infinite set of linearly independent solutions ([32]), and hence the homogeneous problem, corresponding to (104), (111), (112) will have the same property.

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