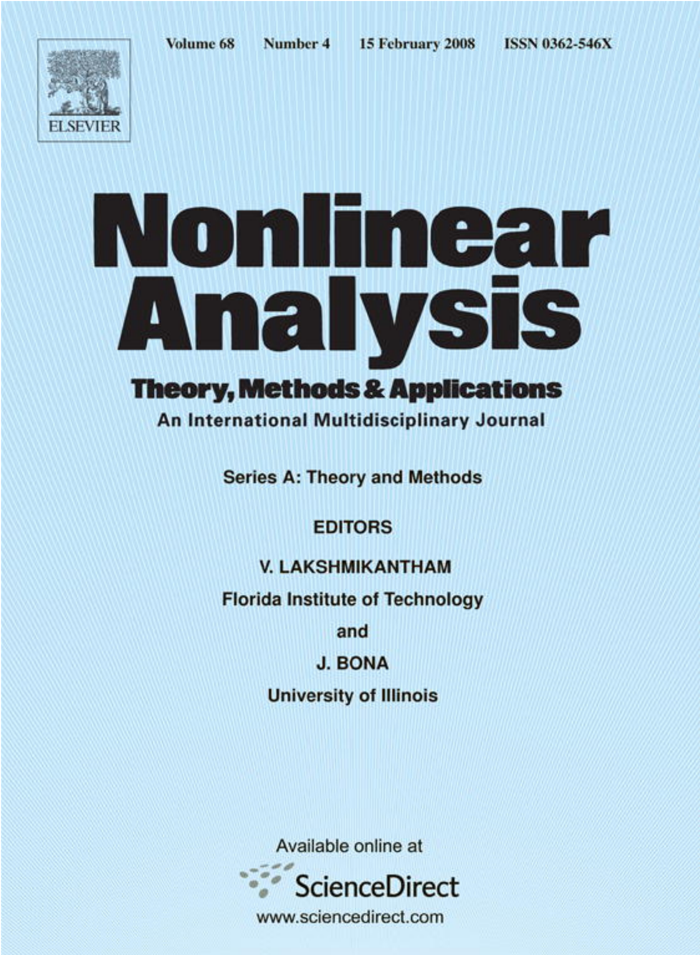
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On the solvability of one multidimensional version of the first Darboux problem for some nonlinear wave equations

S. Kharibegashvili*

A. Razmadze Mathematical Institute of the Georgian Academy of Sciences, 1 M. Aleksidze St., Tbilisi 0193, Georgia

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Abstract

In the present paper, for wave equations with power nonlinearity we investigate the problem of the existence or nonexistence of global solutions of a multidimensional version of the first Darboux problem in the conic domain.

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Keywords: Nonlinear wave equations; A multidimensional version of the first Darboux problem; Global solvability of the problem; Blow-up of solutions

1. Statement of the problem

Let us consider a nonlinear wave equation of the type

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u + mu = f(u) + F, \tag{1}$$

where f and F are given real functions; f is a nonlinear function, $f(0) = 0$, and u is an unknown real function, $m = \text{const} \geq 0$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

By $D : t > |x|, x_n > 0$, we denote a half of a light cone of the future which is bounded by a part $S^0 = D \cap \{x_n = 0\}$ of the hyperplane $x_n = 0$ and by a part $S : t = |x|, x_n \geq 0$, of the characteristic conoid $C : t = |x|$ of Eq. (1). Suppose $D_T = \{(x,t) \in D : t < T\}$, $S_T^0 = \{(x,t) \in S^0 : t \leq T\}$, $S_T = \{(x,t) \in S : t \leq T\}$, $T > 0$. In the case $T = \infty$, it is obvious that $D_\infty = D$, $S_\infty^0 = S^0$ and $S_\infty = S$.

For Eq. (1) we consider the problem on finding in the domain D_T a solution $u(x,t)$ of that equation under the boundary conditions

$$\left.\frac{\partial u}{\partial x_n}\right|_{S_T^0} = 0, \quad u|_{S_T} = g, \tag{2}$$

where g is the given real function on S_T.

* Tel.: +995 32 693501; fax: +995 32 364086.
E-mail address: khar@rmi.acnet.ge.

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The problem (1), (2) is by itself a multidimensional version of the first Darboux problem for the nonlinear Eq. (1), when one portion of the data support is the characteristic manifold and the other one is the manifold of time type [1, pp. 228,233].

The questions of existence or nonexistence of a global solution of the Cauchy problem for nonlinear equations of type (1) with the boundary conditions $u|_{t=0} = u_0$, $\frac{\partial u}{\partial t}\big|_{t=0} = u_1$ have been considered and studied in [2–17]. As for the multidimensional versions of the first Darboux problem for linear hyperbolic equations of the second order, they are formulated correctly and their global solvability arises in the corresponding functional spaces [18–20].

In the present work we distinguish particular cases of the nonlinear function $f = f(u)$, when the problem (1), (2) is globally solvable in one case and unsolvable in the other case.

2. Global solvability of the problem

We consider the case when $f(u) = -\lambda|u|^p u$, where $\lambda \neq 0$ and $p > 0$ are the given real numbers. In this case Eq. (1) takes the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u + mu = -\lambda|u|^p u + F. \tag{3}$$

Note that Eq. (3) emerges in relativistic quantum mechanics [21–24].

In this section we will restrict ourselves to the consideration of the case when the boundary conditions (2) are homogeneous, i.e.

$$\frac{\partial u}{\partial x_n}\bigg|_{S_T^0} = 0, \quad u|_{S_T} = 0. \tag{4}$$

Remark 1. The embedding operator $I : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_q(D_T)$ is the linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [25, p. 81]. At the same time, Nemytski's operator $K : L_q(D_T) \to L_2(D_T)$, acting by the formula $Ku := -\lambda|u|^p u$, is continuous and bounded if $q \geq 2(p+1)$ [26, p. 349], [27, pp. 66,67]. Thus if $p < \frac{2}{n-1}$, i.e. $2(p+1) < \frac{2(n+1)}{n-1}$, there exists a number q such that $1 < 2(p+1) \leq q < \frac{2(n+1)}{n-1}$, and hence the operator

$$K_0 = KI : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_q(D_T) \tag{5}$$

is continuous and compact. In addition, from $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ it all the more follows that $u \in L_{p+1}(D_T)$. As is mentioned above, we assume that here and in the sequel, $p > 0$.

If $u \in C^2(\overline{D}_T)$ is a classical solution of the problem (3), (4), then multiplying both parts of Eq. (3) by an arbitrary function $\varphi \in C^2(\overline{D}_T)$ which satisfies the condition $\varphi|_{t=T} = 0$, after integration by parts we obtain

$$\begin{aligned} &\int_{S_T^0 \cup S_T} \frac{\partial u}{\partial N}\varphi\,\mathrm{d}s - \int_{D_T} u_t\varphi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} \nabla_x u \nabla_x \varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} mu\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\lambda \int_{D_T} |u|^p u\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} F\varphi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{6}$$

where $\frac{\partial}{\partial N} = \nu_0 \frac{\partial}{\partial t} - \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal, $\nu = (\nu_1, \ldots, \nu_n, \nu_0)$ is the unit vector of the outer normal to ∂D_T, $\nabla_x = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Taking into account that $\frac{\partial u}{\partial N}\big|_{S_T^0} = \frac{\partial u}{\partial x_n}$, and S_T is the characteristic manifold on which $\frac{\partial}{\partial N}$ is the interior differential operator, by virtue of (4) we have $\frac{\partial u}{\partial N}\big|_{S_T^0 \cup S_T} = 0$. Therefore Eq. (6) takes the form

$$-\int_{D_T} u_t\varphi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} \nabla_x u \nabla_x \varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} mu\varphi\,\mathrm{d}x\,\mathrm{d}t = -\lambda \int_{D_T} |u|^p u\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} F\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{7}$$

Taking into account the fact that $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, by Remark 1, implies $|u|^p u \in L_2(D_T)$, we can consider equality (7) as the basis for finding a weak generalized solution of the problem (3), (4) of the class $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

Definition 1. Let $F \in L_2(D_T)$ and $0 < p < \frac{2}{n-1}$. The function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is said to be a weak generalized solution of the nonlinear problem (3), (4) in the domain D_T, if for any function $\varphi \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$ the integral equation (7) is fulfilled.

Remark 2. In a standard way [25, p. 113] we can prove that if a weak generalized solution u of the problem (3), (4) belongs to the space $W_2^2(D_T)$, then the homogeneous boundary conditions (4) for that solution will be fulfilled in the sense of the trace theory.

Suppose $\overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T) = \left\{ u \in C^2(\overline{D}_T) : \frac{\partial u}{\partial x_n}\Big|_{S_T^0} = 0, u|_{S_T} = 0 \right\}$.

Definition 2. Let $F \in L_2(D_T)$ and $0 < p < \frac{2}{n-1}$. The function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is said to be a strong generalized solution of the nonlinear problem (3), (4) in the domain D_T if there exists a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ such that $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, and $[Lu_k + \lambda|u_k|^p u_k] \to F$ in the space $L_2(D_T)$. Note that the convergence of the sequence $\{\lambda|u_k|^p u_k\}$ to the function $\lambda|u|^p u$ in the space $L_2(D_T)$, as $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, follows from Remark 1.

Remark 3. It can be easily verified that if $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a strong generalized solution of the problem (3), (4), then it will automatically be a weak generalized solution of that problem. Therefore if in addition the fact that $u \in W_2^2(D_T)$ is known, then the boundary conditions (4) for that solution are fulfilled in the sense of the trace theory.

Definition 3. Let $0 < p < \frac{2}{n-1}$, $F \in L_{2,\mathrm{loc}}(D)$ and $F \in L_2(D_T)$ for any $T > 0$. We say that the problem (3), (4) is globally solvable if for any $T > 0$ it has a strong generalized solution in the domain D_T from the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

Lemma 1. *Let $\lambda \geq 0$, $0 < p < \frac{2}{p-1}$ and $F \in L_2(D_T)$. Then for every strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (3), (4) in the domain D_T an a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{\frac{e}{2}} T \|F\|_{L_2(D_T)} \tag{8}$$

is valid.

Proof. Let $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ be the strong generalized solution of the problem (3), (4). By Definition 2, there exists a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \qquad \lim_{k\to\infty} \left\| Lu_k + \lambda|u_k|^p u_k - F \right\|_{L_2(D_T)} = 0. \tag{9}$$

Consider the function $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ in the capacity of a solution of the problem

$$Lu_k + \lambda|u_k|^p u_k = F_k, \tag{10}$$

$$\frac{\partial u_k}{\partial x_n}\Big|_{S_T^0} = 0, \qquad u_k|_{S_T} = 0. \tag{11}$$

Here

$$F_k = Lu_k + \lambda|u_k|^p u_k. \tag{12}$$

Multiplying both parts of Eq. (10) by $\frac{\partial u_k}{\partial t}$ and integrating with respect to the domain D_τ, $0 < \tau \leq T$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - \int_{D_\tau} \Delta u_k \frac{\partial u_k}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \frac{m}{2}\int_{D_\tau} \frac{\partial}{\partial t} u_k^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{\lambda}{p+2}\int_{D_\tau} \frac{\partial}{\partial t}|u_k|^{p+2}\,\mathrm{d}x\,\mathrm{d}t = \int_{D_\tau} F_k \frac{\partial u_k}{\partial t}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{13}$$

Let $\Omega_\tau := D_T \cap \{t = \tau\}$, $0 < \tau < T$. Obviously, $\partial D_\tau = S_\tau^0 \cup S_\tau \cup \Omega_\tau$. Taking into account (11) and equalities $\nu|_{\Omega_\tau} = (0, \dots, 0, 1)$, and $\nu|_{S_T^0} = (0, \dots, 0, -1, 0)$, integration by parts provides us with

$$\begin{aligned}
&\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t = \int_{\partial D_\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 \nu_0\,\mathrm{d}s = \int_{\Omega_\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 \mathrm{d}x + \int_{S_\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 \nu_0\,\mathrm{d}s,\\
&\int_{D_\tau} \frac{\partial}{\partial t}(u_k)^2\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial D_\tau} u_k^2 \nu_0\,\mathrm{d}s = \int_{\Omega_\tau} u_k^2\,\mathrm{d}x,\\
&\int_{D_\tau} \frac{\partial}{\partial t}|u_k|^{p+2}\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial D_\tau} |u_k|^{p+2}\nu_0\,\mathrm{d}s = \int_{\Omega_\tau} |u_k|^{p+2}\,\mathrm{d}x,\\
&\int_{D_\tau} \frac{\partial^2 u_k}{\partial x_i^2}\frac{\partial u_k}{\partial t}\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial D_\tau} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial t}\nu_i\,\mathrm{d}s - \frac{1}{2}\int_{D_\tau}\frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial x_i}\right)^2 \mathrm{d}x\,\mathrm{d}t\\
&\qquad = \int_{\partial D_\tau} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial t}\nu_i\,\mathrm{d}s - \frac{1}{2}\int_{\partial D_\tau}\left(\frac{\partial u_k}{\partial x_i}\right)^2 \nu_0\,\mathrm{d}s = \int_{S_\tau} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial t}\nu_i\,\mathrm{d}s\\
&\qquad\quad - \frac{1}{2}\int_{S_\tau}\left(\frac{\partial u_k}{\partial x_i}\right)^2 \nu_0\,\mathrm{d}s - \frac{1}{2}\int_{\Omega_\tau}\left(\frac{\partial u_k}{\partial x_i}\right)^2 \mathrm{d}x,
\end{aligned}$$

whence by virtue of (13) it follows that

$$\begin{aligned}
\int_{D_\tau} F_k \frac{\partial u_k}{\partial t}\,\mathrm{d}x\,\mathrm{d}t &= \int_{S_\tau} \frac{1}{2\nu_0}\left[\sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\nu_0 - \frac{\partial u_k}{\partial t}\nu_i\right)^2 + \left(\frac{\partial u_k}{\partial t}\right)^2\left(\nu_0^2 - \sum_{j=1}^n \nu_j^2\right)\right]\mathrm{d}s\\
&\quad + \frac{1}{2}\int_{\Omega_\tau}\left[m u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]\mathrm{d}x + \frac{\lambda}{p+2}\int_{\Omega_\tau}|u_k|^{p+2}\,\mathrm{d}x. \qquad (14)
\end{aligned}$$

Since S_τ is the characteristic manifold,

$$\left(\nu_0^2 - \sum_{j=1}^n \nu_j^2\right)\Bigg|_{S_\tau} = 0. \qquad (15)$$

Taking into account that $(\nu_0 \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial t})$, $i = 1, \dots, n$, is the interior differential operator on S_τ, by virtue of (11) we have

$$\left(\frac{\partial u_k}{\partial x_i}\nu_0 - \frac{\partial u_k}{\partial t}\nu_i\right)\Bigg|_{S_\tau} = 0, \quad i = 1, \dots, n. \qquad (16)$$

With regard for (15) and (16), from (14) we find that

$$\int_{\Omega_\tau}\left[m u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]\mathrm{d}x + \frac{2\lambda}{p+2}\int_{\Omega_\tau}|u_k|^{p+2}\,\mathrm{d}x = 2\int_{D_\tau} F_k\frac{\partial u_k}{\partial t}\,\mathrm{d}x\,\mathrm{d}t,$$

whence by virtue of $\lambda > 0$ it in turn follows that

$$\int_{\Omega_\tau}\left[m u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]\mathrm{d}x \le 2\int_{D_\tau} F_k\frac{\partial u_k}{\partial t}\,\mathrm{d}x\,\mathrm{d}t. \qquad (17)$$

With the notation $w(\delta) = \int_{\Omega_\delta}\left[m u_k^2 + (\frac{\partial u_k}{\partial t})^2 + \sum_{i=1}^n (\frac{\partial u_k}{\partial x_i})^2\right]\mathrm{d}x$, taking into account the inequality $2F_k\frac{\partial u_k}{\partial t} \le \varepsilon(\frac{\partial u_k}{\partial t})^2 + \frac{1}{\varepsilon}F_k^2$, valid for any $\varepsilon = \mathrm{const} > 0$, we find that inequality (17) yields

$$w(\delta) \le \varepsilon\int_0^\delta w(\sigma)\,\mathrm{d}\sigma + \frac{1}{\varepsilon}\|F_k\|^2_{L_2(D_\delta)}, \quad 0 < \delta \le T. \qquad (18)$$

From (18), bearing in mind that the value $\|F_k\|^2_{L_2(D_\delta)}$, as a function of δ, is nondecreasing, it follows by the Gronwall lemma [28, p. 13] that

$$\|w(\delta)\| \le \frac{1}{\varepsilon}\,\|F_k\|^2_{L_2(D_\delta)} \exp \delta\varepsilon.$$

The latter with regard to $\inf_{\varepsilon>0} \frac{\exp\delta\varepsilon}{\varepsilon} = e\delta$, which can be achieved for $\varepsilon = \frac{1}{\delta}$, results in

$$w(\delta) \le e\delta \|F_k\|^2_{L_2(D_\delta)}, \quad 0 < \delta \le T. \tag{19}$$

In turn, from (19) it follows that

$$\begin{aligned}\|u_k\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} &= \int_{D_T}\left[mu_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i}\right)^2\right] \mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T w(\delta)\,\mathrm{d}\delta \le \frac{e}{2}T^2 \|F_k\|^2_{L_2(D_\delta)}.\end{aligned} \tag{20}$$

Here we have used the fact that in the space $\overset{\circ}{W}{}^1_2(D_T, S_T)$ the expression $\{\int_{D_T}[mu^2 + (\frac{\partial u}{\partial t})^2 + \sum_{i=1}^n (\frac{\partial u_k}{\partial x_i})^2]\,\mathrm{d}x\,\mathrm{d}t\}^{1/2}$ provides us with one of the equivalent norms, irrespective of whether $m = 0$ or $m > 0$. Indeed, from equalities $u|_{S_T} = 0$ and $u(x,t) = \int_{\psi(x)}^t \frac{u(x,\tau)}{\partial t}\,\mathrm{d}\tau$, $(x,t) \in \overline{D}_T$, where $t - \psi(x) = 0$ is the equation of the conic manifold S_T, standard reasoning leads us to the inequality [25, p. 63]

$$\int_{D_T} u^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \le T^2 \int_{D_T}\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t.$$

Now, by (9) and (12), passing in inequality (20) to the limit as $k \to \infty$, we obtain (8), which proves our lemma. □

Theorem 1. *Let $\lambda > 0$, $0 < p < \frac{2}{n-1}$, $F \in L_{2,\mathrm{loc}}(D_T)$ and $F \in L_2(D_T)$ for any $T > 0$. Then the problem (3), (4) is globally solvable, i.e. for any $T > 0$ this problem has a strong generalized solution $u \in \overset{\circ}{W}{}^1_2(D_T, S_T)$ in the domain D_T.*

Proof. Before we pass to the question of the solvability of the nonlinear problem (3), (4), let us consider the same question for the linear case, when in Eq. (3) the parameter $\lambda = 0$, i.e. for the problem

$$\begin{aligned} &Lu(x,t) = F(x,t), \quad (x,t) \in D_T, \\ &\left.\frac{\partial u}{\partial x_n}\right|_{S_T^0} = 0, \quad u|_{S_T} = 0.\end{aligned} \tag{21}$$

In this case, for $F \in L_2(D_T)$ we analogously introduce the notion of a strong generalized solution $w \in \overset{\circ}{W}{}^1_2(D_T, S_T)$ of the problem (21) for which there exists the sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ such that $\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = 0$, $\lim_{k\to\infty} \|Lu_k - F\|_{L_2(D_T)} = 0$. Here it should be noted that by Lemma 1 for $\lambda = 0$ the a priori estimate (8) is valid for the strong generalized solution of the problem (21), as well.

Since the space $C_0^\infty(D_T)$ of finite functions infinitely differentiable in D_T is dense in $L_2(D_T)$, for the given $F \in L_2(D_T)$ there exists the sequence of functions $F_k \in C_0^\infty(D_T)$ such that $\lim_{k\to\infty} \|F_k - F\|_{L_2(D_T)} = 0$. For a fixed k, continuing evenly the function F_k with respect to the variable x_n into the domain $D_T^- := \{(x,t) \in R^{n+1} : x_n < 0, |x| < t < T\}$ and then by zero beyond the domain $D_T \cup D_T^-$, and retaining the same designation, we will have $F_k \in C^\infty(R_+^{n+1})$ for which the support $\operatorname{supp} F_k \subset D_\infty \cup D_\infty^-$, where $R_+^{n+1} = R^{n+1} \cap \{t \ge 0\}$. Denote by u_k a solution of the Cauchy problem

$$Lu_k = F_k, \quad u_k|_{t=0} = 0, \quad \left.\frac{\partial u_k}{\partial t}\right|_{t=0} = 0, \tag{22}$$

which, as is known, exists, is unique and belongs to the space $C^\infty(R_+^{n+1})$ [29, p. 192]. Moreover, since $\operatorname{supp} F_k \subset D_\infty \cup D_\infty^- \subset \{(x,t) \in R^{n+1} : t > |x|\}$ and $u_k|_{t=0} = 0$, $\left.\frac{\partial u_k}{\partial t}\right|_{t=0} = 0$, taking into account the geometry of the domain of dependence of a solution of the wave equation $Lu = F$, we will have $\operatorname{supp} u_k \subset \{(x,t) \in R^{n+1} : t > |x|\}$ [29, p. 191] and, in particular, $u_k|_{S_T} = 0$. On the other hand, the function $\widetilde{u}_k(x_1, \ldots, x_n, t) = u_k(x_1, \ldots, -x_n, t)$ is likewise a solution of the same Cauchy problem (22), since the function F_k is even with respect to the variable x_n. Therefore owing to the uniqueness of the solution of the Cauchy problem, we have $\widetilde{u}_k = u_k$, i.e. $u_k(x_1, \ldots, -x_n, t) = u_k(x_1, \ldots, x_n, t)$, and hence the function u_k is likewise even with respect to the variable x_n. This in turn results in $\frac{\partial u_k}{\partial x_n}|_{x_n=0} = 0$ which together with the condition $u_k|_{S_T} = 0$ implies that if for the narrowing of the function in the domain D_T we retain the same designation, then $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$. Next, by (8) and (22) we have the inequality

$$\|u_k - u_\ell\|_{\overset{\circ}{W}{}_2^1(D_T, S_T)} \le \sqrt{\frac{e}{2}}\, T \|F_k - F_\ell\|_{L_2(D_T)}, \tag{23}$$

because the a priori estimate (8) is valid for the strong generalized solution of the linear problem (21), as well.

Since the sequence $\{F_k\}$ is fundamental in $L_2(D_T)$, the sequence $\{u_k\}$ is, by virtue of (23), likewise fundamental in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, which is complete. Therefore there exists the function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ such that $\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T, S_T)} = 0$, and since $Lu_k = F_k \to F$ in the space $L_2(D_T)$, this function is, according to the definition, the strong generalized solution of the problem (21). The uniqueness of that solution from the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ follows from the a priori estimate (8). Consequently, for the solution u of the problem (21) we can write $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is the linear continuous operator, whose norm, by virtue of (8), admits the estimate

$$\|L^{-1}\|_{L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)} \le \sqrt{\frac{e}{2}}\, T. \tag{24}$$

Note that for $F \in L_2(D_T)$, $0 < p < \frac{2}{n-1}$, by (24) and Remark 1, the function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is the strong generalized solution of the problem (3), (4) if and only if u is the solution of the following functional equation:

$$u = L^{-1}(-\lambda |u|^p u + F) \tag{25}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

We rewrite Eq. (25) in the form

$$u = Au := L^{-1}(K_0 u + F), \tag{26}$$

where the operator $K_0 : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T)$ from (5) is, according to Remark 1, a continuous and compact one. Consequently, by virtue of (24), the operator $A : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is likewise continuous and compact. At the same time, by Lemma 1, for any parameter $\mu \in [0, 1]$ and for any solution of the equation with the parameter $u = \mu A u$ the a priori estimate $\|u\|_{\overset{\circ}{W}{}_2^1(D_T, S_T)} \le c\|F\|_{L_2(D_T)}$ with the positive constant c, independent of u, μ and F, is valid. Therefore by the Lere–Schauder theorem [30, p. 375], Eq. (26) and hence the problem (3), (4) has at least one solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$. Thus Theorem 1 is complete. □

3. Nonexistence of global solvability of the problem

Below we will consider the case when in the problem (1), (2) the coefficient $m = 0$ and $f(u) = \lambda |u|^{p+1}$, where λ and p are the given positive numbers, i.e. we will consider the problem

$$\Box u := \frac{\partial^2 u}{\partial t^2} - \Delta u = \lambda |u|^{p+1} + F, \tag{27}$$

$$\left.\frac{\partial u}{\partial x_n}\right|_{S_T^0} = 0, \quad u|_{S_T} = g \tag{28}$$

in the domain D_T, $T > 0$, where g is the given real function on S_T.

Remark 4. Under the assumption that $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ and $0 < p < \frac{2}{n-1}$, just analogously to Definitions 1 and 2 regarding a weak and a strong generalized solution of the problem (3), (4) in the domain D_T, and also taking into account Remark 1, we introduce the notions of a weak and of a strong generalized solution of the problem (27), (28):

(i) the function $u \in W_2^1(D_T)$ is said to be a weak generalized solution of the problem (27), (28) in the domain D_T if for any function $\varphi \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$ the integral equality

$$-\int_{D_T} u_t\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} \nabla_x u \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t = \lambda \int_{D_T} |u|^{p+1}\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} F\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_{S_T} \frac{\partial g}{\partial N}\varphi \,\mathrm{d}s \tag{29}$$

holds, where $\frac{\partial}{\partial N} = \nu_0 \frac{\partial}{\partial t} - \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal which is, in turn, the interior differential operator on S_T, since the conic manifold S_T is a characteristic one, and $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ is the unit vector of the outer normal to ∂D_T, $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$;

(ii) the function $u \in W_2^1(D_T)$ is said to be a strong generalized solution of the nonlinear problem (27), (28) in the domain D_T if there exists the sequence of functions $u_k \in \overset{\circ}{C}{}^2_*(\overline{D}_T, S_T^0) = \left\{ u \in C^2(\overline{D}_T) : \frac{\partial u}{\partial x_n}\Big|_{S_n^0} = 0 \right\}$ such that $u_k \to u$ in the space $W_2^1(D_T)$, $[\Box u_k - \lambda |u_k|^{p+1}] \to F$ in the space $L_2(D_T)$, and $u_k|_{S_T} \to g$ in the space $W_2^1(S_T)$.

Obviously, the strong generalized solution of the problem (27), (28) is likewise the weak generalized solution of that problem.

Remark 5. Note that the derivative with respect to the conormal $\frac{\partial}{\partial N}$, being the interior differential operator on the characteristic conic manifold S, coincides with the derivative $\frac{\partial}{\partial r}$ with respect to the spherical variable $r = (t^2 + |x|^2)^{1/2}$ taken with the minus sign.

There arises a theorem on the nonexistence of a global solution of the problem (27), (28).

Theorem 2. *Let $F \in L_{2,\mathrm{loc}}(D)$, $g \in W^1_{2,\mathrm{loc}}(S)$ and $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ for any $T > 0$. Then if $0 < p < \frac{2}{n-1}$ and*

$$F|_D \geq 0, \qquad g|_S \geq 0, \qquad \frac{\partial g}{\partial r}\Big|_S \geq 0, \tag{30}$$

there exists a positive number $T_0 = T_0(F, g)$ such that for $T > T_0$ the problem (27), (28) fails to have a weak generalized solution $w \in W_2^1(D_T)$ (which is nontrivial in case $F = 0$ and $g = 0$) in the domain D_T.

Proof. Let $G_T : |x| < t < T$, $G_T^- = G_T \cap \{x_n < 0\}$, $S_T^- : t = |x|$, $x_n \leq 0$, $t \leq T$. It is evident that $D_T = G_T^+ := G_T \cap \{x_n > 0\}$ and $G_T = G_T^- \cup S_T^0 \cup D_T$, where $S_T^0 = \partial D_T \cap \{x_n = 0\}$. We continue the functions u, F and g evenly with respect to the variable x_n into G_T^{-1} and S_T^{-1}, respectively. For the sake of simplicity, for the continued functions defined in G_T and $S_T^{-1} \cup S_T$ we retain the same designations u, F and g. Then if $u \in W_2^1(D_T)$ is a weak generalized solution of the problem (27), (28) in the domain D_T, for any function $\psi \in W_2^1(G_T)$ such that $\psi|_{t=T} = 0$ the equality

$$-\int_{G_T} u_t\psi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{G_T} \nabla_x u \nabla_x \psi \,\mathrm{d}x\,\mathrm{d}t = \lambda \int_{G_T} |u|^{p+1}\psi \,\mathrm{d}x\,\mathrm{d}t + \int_{G_T} F\psi \,\mathrm{d}x\,\mathrm{d}t - \int_{S_T^{-1}\cup S_T} \frac{\partial g}{\partial N}\psi \,\mathrm{d}s \tag{31}$$

holds.

Indeed, if $\psi \in W_2^1(G_T)$ and $\psi|_{t=T} = 0$, then it is obvious that $\psi|_{D_T} \in W_2^1(D_T)$ and $\widetilde{\psi} \in W_2^1(D_T)$, where by the definition, $\widetilde{\psi}(x_1, \dots, x_n, t) = \psi(x_1, \dots, -x_n, t)$, $(x_1, \dots, x_n, t) \in D_T$, where $\widetilde{\psi}\big|_{t=T} = 0$. Therefore according to equality (29), we have

$$-\int_{D_T} u_t\psi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} \nabla_x u \nabla_x \psi \,\mathrm{d}x\,\mathrm{d}t = \lambda \int_{D_T} |u|^{p+1}\psi \,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} F\psi \,\mathrm{d}x\,\mathrm{d}t - \int_{S_T} \frac{\partial g}{\partial N}\psi \,\mathrm{d}s, \tag{32}$$

$$-\int_{D_T} u_t\widetilde{\psi}_t \,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} \nabla_x u \nabla_x \widetilde{\psi} \,\mathrm{d}x\,\mathrm{d}t = \lambda \int_{D_T} |u|^{p+1}\widetilde{\psi} \,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} F\widetilde{\psi} \,\mathrm{d}x\,\mathrm{d}t - \int_{S_T} \frac{\partial g}{\partial N}\widetilde{\psi} \,\mathrm{d}s. \tag{33}$$

Taking now into account that u, F and g are the even functions with respect to the variable x_n, as well as the equality $\widetilde{\psi}(x_1,\dots,x_n,t)=\psi(x_1,\dots,-x_n,t)$, $(x_1,\dots,x_n,t)\in D_T$, we find that

$$-\int_{D_T} u_t\widetilde{\psi}_t\,\mathrm{d}x\,\mathrm{d}t+\int_{D_T}\nabla_x u\nabla_x\widetilde{\psi}\,\mathrm{d}x\,\mathrm{d}t=-\int_{G_T^-}u_t\psi_t\,\mathrm{d}x\,\mathrm{d}t+\int_{G_T^-}\nabla_x u\nabla_x\psi\,\mathrm{d}x\,\mathrm{d}t, \tag{34}$$

$$\begin{aligned}&\lambda\int_{D_T}|u|^{p+1}\widetilde{\psi}\,\mathrm{d}x\,\mathrm{d}t+\int_{D_T}F\widetilde{\psi}\,\mathrm{d}x\,\mathrm{d}t-\int_{S_T}\frac{\partial g}{\partial N}\widetilde{\psi}\,\mathrm{d}s\\&\quad=\lambda\int_{G_T^-}|u|^{p+1}\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{G_T^-}F\psi\,\mathrm{d}x\,\mathrm{d}t-\int_{S_T^-}\frac{\partial g}{\partial N}\widetilde{\psi}\,\mathrm{d}s.\end{aligned} \tag{35}$$

It follows from (33), (34) and (35) that

$$-\int_{G_T^-}u_t\psi_t\,\mathrm{d}x\,\mathrm{d}t+\int_{G_T^-}\nabla_x u\nabla_x\psi\,\mathrm{d}x\,\mathrm{d}t=\lambda\int_{G_T^-}|u|^{p+1}\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{G_T^-}F\psi\,\mathrm{d}x\,\mathrm{d}t-\int_{S_T^-}\frac{\partial g}{\partial N}\psi\,\mathrm{d}s. \tag{36}$$

Finally, summing equalities (32) and (36), we obtain (31).

Note that the inequality $\frac{\partial g}{\partial r}\big|_S\ge 0$ under the condition (30) should be understood in a generalized sense, i.e. by the assumption $g\in W^1_{2,\mathrm{loc}}(S)$, there exists the generalized derivative $\frac{\partial g}{\partial r}\in L_{2,\mathrm{loc}}(S)$ which is nonnegative, and hence for any function $\beta\in C(S)$, $\beta\ge 0$, finite with respect to the variable r, we have the inequality

$$\int_S\frac{\partial g}{\partial r}\beta\,\mathrm{d}s\ge 0. \tag{37}$$

Here we make use of the method of test functions [14, pp. 10–12]. In the capacity of such a function we take in equality (31) the function $\psi(x,t)=\psi_0[\frac{2}{T^2}(t^2+|x|^2)]$, where $\psi_0\in C^2((-\infty,+\infty))$, $\psi_0\ge 0$, $\psi_0'\le 0$; $\psi_0(\sigma)=1$ for $0\le\sigma\le 1$, and $\psi_0(\sigma)=0$ for $\sigma\ge 2$ [14, p. 22]. Obviously, $\psi|_{t=T}=0$ and $\psi\in C^2(\overline{G}_T)$, and furthermore, $\psi\in W_2^1(G_T)$. Integrating the left-hand side of (31) by parts, we obtain

$$\int_{G_T}u\Box\psi\,\mathrm{d}x\,\mathrm{d}t=\lambda\int_{G_T}|u|^{p+1}\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{G_T}F\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{S_T^-\cup S_T}g\frac{\partial\psi}{\partial N}\,\mathrm{d}s-\int_{S_T^-\cup S_T}\frac{\partial g}{\partial N}\psi\,\mathrm{d}s. \tag{38}$$

By Remark 5, owing to (30) and (37), we have

$$\int_{D_T}F\psi\,\mathrm{d}x\,\mathrm{d}t\ge 0,\qquad\int_{S_T^-\cup S_T}g\frac{\partial\psi}{\partial N}\,\mathrm{d}s\ge 0,\qquad\int_{S_T^-\cup S_T}\frac{\partial g}{\partial N}\psi\,\mathrm{d}s\le 0, \tag{39}$$

where ψ is the test function, introduced above.

Assuming that the functions F, g and ψ are fixed, we introduce into consideration the function of one variable T,

$$\gamma(T)=\int_{G_T}F\psi\,\mathrm{d}x\,\mathrm{d}t+\int_{S_T^-\cup S_T}g\frac{\partial\psi}{\partial N}\,\mathrm{d}s-\int_{S_T^-\cup S_T}\frac{\partial g}{\partial N}\psi\,\mathrm{d}s,\quad T>0. \tag{40}$$

Because of the absolute continuity both of the integral and of inequalities (39), the function $\gamma(T)$ from (40) is nonnegative, continuous, nondecreasing and $\lim_{T\to 0}\gamma(T)=0$.

Taking into account (40), we rewrite equality (38) in the form

$$\lambda\int_{G_T}|u|^{p+1}\psi\,\mathrm{d}x\,\mathrm{d}t=\int_{G_T}u\Box\psi\,\mathrm{d}x\,\mathrm{d}t-\gamma(T). \tag{41}$$

If in Young's inequality with the parameter $\varepsilon>0$ for $\alpha=p+1$

$$ab\le\frac{\varepsilon}{\alpha}a^\alpha+\frac{1}{\alpha'\varepsilon^{\alpha'-1}}b^{\alpha'},\quad a,b\ge 0,\ \alpha'=\frac{\alpha}{\alpha-1}=1+\frac{1}{p}$$

we take $a=|u|\psi^{1/\alpha}$, $b=\frac{|\Box\psi|}{\psi^{1/\alpha}}$, then bearing in mind that $\frac{\alpha'}{\alpha}=\alpha'-1=\frac{1}{p}$, we find that

$$|u\Box\psi|=|u|\psi^{1/\alpha}\cdot\frac{|\Box\psi|}{\psi^{1/\alpha}}\le\frac{\varepsilon}{\alpha}|u|^\alpha\psi+\frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|\Box\psi|^{\alpha'}}{\psi^{\alpha'-1}}. \tag{42}$$

Owing to (42), equality (41) yields

$$\left(\lambda - \frac{\varepsilon}{\alpha}\right) \int_{G_T} |u|^\alpha \psi \, dx \, dt \le \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \int_{G_T} \frac{|\Box \psi|^{\alpha'}}{\psi^{\alpha'-1}} \, dx \, dt - \gamma(T),$$

whence for $\varepsilon < \lambda\alpha$ we have

$$\int_{G_T} |u|^\alpha \psi \, dx \, dt \le \frac{\alpha}{(\lambda\alpha - \varepsilon)\alpha' \varepsilon^{\alpha'-1}} \int_{G_T} \frac{|\Box \psi|^{\alpha'}}{\psi^{\alpha'-1}} \, dx \, dt - \frac{\alpha}{\lambda\alpha - \varepsilon} \gamma(T). \tag{43}$$

Taking now into account that $\alpha' = \frac{\alpha}{\alpha-1}$, $\alpha = \frac{\alpha'}{\alpha'-1}$ and $\min_{0<\varepsilon<\lambda\alpha} \frac{\alpha}{(\lambda\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}}$, which can be achieved for $\varepsilon = \lambda$, from (43) it follows that

$$\int_{G_T} |u|^\alpha \psi \, dx \, dt \le \frac{1}{\lambda^{\alpha'}} \int_{G_T} \frac{|\Box \psi|^{\alpha'}}{\psi^{\alpha'-1}} \, dx \, dt - \frac{\alpha'}{\alpha} \gamma(T). \tag{44}$$

According to the properties of the function ψ_0, the test function $\psi(x,t) = \psi_0[\frac{2}{T^2}(t^2 + |x|^2)] = 0$ for $r = (t^2 + |x|^2)^{1/2} \ge T$. Therefore making the change of variable $t = \sqrt{2}T\xi_0$, $x = \sqrt{2}T\xi$, we can easily verify that

$$\int_{G_T} \frac{|\Box \psi|^{\alpha'}}{\psi^{\alpha'-1}} \, dx \, dt = \int_{r=(t^2+|x|^2)^{1/2} \le T} \frac{|\Box \psi|^{\alpha'}}{\psi^{\alpha'-1}} \, dx \, dt = (\sqrt{2}\, T)^{n+1-2\alpha'} \varkappa_0, \tag{45}$$

where, as is known [14, p. 23],

$$\varkappa_0 = \int_{1 \le |\xi_0|^2 + |\xi|^2 \le 2} \frac{|2(1-n)\psi_0' + 4(\xi_0^2 - |\xi|^2)\psi_0''|^{\alpha'}}{\psi_0^{\alpha'-1}} \, d\xi \, d\xi_0 < +\infty.$$

By (45), inequality (44) with regard for the fact that $\psi_0(\sigma) = 1$ for $0 \le \sigma \le 1$, we obtain

$$\int_{r \le T/\sqrt{2}} |u|^\alpha \, dx \, dt \le \int_{G_T} |u|^\alpha \psi \, dx \, dt \le \frac{(\sqrt{2}\, T)^{n+1-2\alpha'}}{\lambda^{\alpha'}} \varkappa_0 - \frac{\alpha'}{\lambda} \gamma(T). \tag{46}$$

If $p < \frac{2}{n-1}$, i.e. for $n + 1 - 2\alpha' < 0$, where $\alpha' = 1 + \frac{1}{p}$, the equation

$$\delta(T) = \frac{(\sqrt{2}\, T)^{n+1-2\alpha'}}{\lambda^{\alpha'}} \varkappa_0 - \frac{\alpha'}{\lambda} \gamma(T) = 0 \tag{47}$$

has the unique positive root $T = T_0 > 0$, since the function $\delta_1(T) = \frac{(\sqrt{2}\, T)^{n+1-2\alpha'}}{\lambda^{\alpha'}} \varkappa_0$ is positive, continuous and strictly decreasing on the interval $(0, +\infty)$, where $\lim_{T\to 0} \delta_1(T) = +\infty$ and $\lim_{T\to\infty} \delta_1(T) = 0$, while the function $\gamma(T)$ is, as is mentioned above, nonnegative, continuous and nondecreasing, and $\lim_{T\to+\infty} \gamma(T) > 0$, because we assume that at least one of the functions F and g is not trivial. Therefore if there exists a solution of the problem (27), (28) in the domain D_T, then without fail $T \le T_0 = T_0(F, g)$, which proves Theorem 2. $\square$

Remark 6. Making use of the reasoning of [14, p. 23], the conclusion of Theorem 2 remains also valid in the limiting case $p = \frac{2}{n-1}$. The conclusion of that theorem fails to be valid if $p > \frac{2}{n-1}$ and the second of the conditions (30), i.e. the condition $g|_S \ge 0$, is violated. Indeed, the function $u(x,t) = -\varepsilon(1 + t^2 - |x|^2)^{-\frac{1}{p}}$, $\varepsilon = \text{const} > 0$, is the global classical, and hence, generalized solution of the problem (27), (28) for $g = -\varepsilon \left(\frac{\partial g}{\partial r}\Big|_S = 0 \right)$ and $F = \left[2\varepsilon \frac{n+1}{p} - 4\varepsilon \frac{p+1}{p^2} \frac{t^2-|x|^2}{1+t^2-|x|^2} - \lambda\varepsilon^{p+1}\right] (1 + t^2 - |x|^2)^{\frac{p+1}{p}}$, where, as can be easily verified, $F|_D \ge 0$ if $p > \frac{2}{n-1}$ and $0 < \varepsilon \le \left\{ \frac{2}{\lambda} [\frac{n+1-\frac{2(p+1)}{p}}{p}] \right\}^{1/p}$. Note that inequality $n + 1 - \frac{2(p+1)}{p} > 0$ is equivalent to the inequality $p > \frac{2}{n-1}$.

Remark 7. The conclusion of Theorem 2 also fails to be valid if only the third of the conditions (30), i.e. the condition $\frac{\partial g}{\partial r}\Big|_S \geq 0$, is violated. Indeed, the function $u(x,t) = c_0[(t+1)^2 - |x|^2]^{-\frac{1}{p}}$, where $c_0 = \lambda^{-\frac{1}{p}}[\frac{4(p+1)}{p^2} - \frac{2(n+1)}{p}]^{-\frac{1}{p}}$, is the global classical solution of the problem (27), (28) for $F = 0$ and $g = u|_S = c_0[(t+1)^2 - t^2]^{-\frac{1}{p}} > 0$.

Remark 8. In the case $-1 < p < 0$, the problem (27), (28) may have more than one global solution. For example, for $F = 0$ and $g = 0$ the conditions (30) are fulfilled, but the above-mentioned problem may, besides a trivial solution, have an infinite set of global linearly independent solutions $u_\alpha(x,t)$, depending on the parameter $\alpha \geq 0$ and given by the formula

$$u_\alpha(x,t) = \begin{cases} c_0\left[(t-\alpha)^2 - |x|^2\right]^{-\frac{1}{p}}, & t > \alpha + |x|, \\ 0, & |x| \leq t \leq \alpha + |x|, \end{cases}$$

where $c_0 = \lambda^{-\frac{1}{p}}[\frac{4(p+1)}{p^2} - \frac{2(n+1)}{p}]^{-\frac{1}{p}}$. We can easily see that $u_\alpha(x,t) \in C^1(\overline{D})$ for $p < 0$, and for $-\frac{1}{2} < p < 0$ the function $u_\alpha(x,t) \in C^2(\overline{D})$.

4. Local solvability of the problem

Remark 9. It was shown in proving Theorem 1 that the linear problem (21), which for $m = 0$ coincides with the linear problem corresponding to (27) and (28) for $\lambda = 0$ and $g = 0$, has the unique solution $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is the linear continuous operator whose norm admits the estimate (24). Note also that analogously to Remark 1, for $0 < p < \frac{2}{n-1}$ the operator

$$K_1 : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T) \quad \left(K_1 u = \lambda |u|^{p+1}\right) \tag{48}$$

is a continuous and compact one. Therefore the nonlinear problem (27), (28) for $g = 0$ is equivalent to the functional equation

$$u = Au + u_0 \tag{49}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, where with regard for (48),

$$A = L^{-1}K_1, \qquad u_0 = L^{-1}F \in \overset{\circ}{W}{}_2^1(D_T, S_T). \tag{50}$$

Remark 10. Let $B(0,d) := \left\{u \in \overset{\circ}{W}{}_2^1(D_T, S_T) : \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq d\right\}$ be a closed (convex) ball in the Hilbert space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ of radius $d > 0$ with center in a zero element. Since by Remark 9, the operator $A : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ for $0 < p < \frac{2}{n-1}$ is continuous and compact; therefore by Schauder's principle, for the solvability of Eq. (49) it suffices to prove that the operator A_1, acting by the formula $A_1 u = Au + u_0$, transfers the ball $B(0,d)$ into itself for some $d > 0$ [30, p. 370]. Towards this end, we will give the necessary estimate for the value $\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$.

If $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, then we denote by $\widetilde{u}$ the function which by itself is the continuation of the function u_0 evenly through the planes $x_n = 0$ and $t = T$. Obviously, $\widetilde{u} \in \overset{\circ}{W}{}_2^1(D_T^*)$, where $D_T^* : |x| < t < 2T - |x|$.

Using the inequality [31, p. 258]

$$\int_\Omega |v|\,\mathrm{d}\Omega \leq (\mathrm{mes}\,\Omega)^{1-\frac{1}{q}} \|v\|_{q,\Omega}, \quad q \geq 1,$$

and taking into account the equalities

$$\|\widetilde{u}\|^q_{L_q(D_T^*)} = 4\|u\|^q_{L_q(D_T)}, \qquad \|\widetilde{u}\|^2_{\overset{\circ}{W}{}_2^1(D_T^*)} = 4\|u\|^2_{\overset{\circ}{W}{}_2^1(D_T,S_T)},$$

from the well-known multiplicative inequality [25, p. 78]

$$\|v\|_{q,\Omega} \le \beta \|\nabla v\|_{m,\Omega}^{\widetilde{\alpha}} \|v\|_{r,\Omega}^{1-\widetilde{\alpha}} \quad \forall v \in \mathring{W}_2^1(\Omega),\ \Omega \subset R^{n+1},$$

$$\widetilde{\alpha} = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{r} - \frac{1}{\widetilde{m}}\right)^{-1}, \quad \widetilde{m} = \frac{(n+1)m}{n+1-m},$$

for $\Omega = D_T^* \subset R^{n+1}$, $v = \widetilde{u}$, $r = 1$, $m = 2$ and $1 < q \le \frac{2(n+1)}{n-1}$, where $\beta = \text{const} > 0$ does not depend on v and T, we obtain the following inequality:

$$\|u\|_{L_q(D_T)} \le c_0 (\operatorname{mes} D_T)^{\frac{1}{q}+\frac{1}{n+1}-\frac{1}{2}} \|u\|_{\mathring{W}_2^1(D_T,S_T)} \quad \forall u \in \mathring{W}_2^1(D_T, S_T), \tag{51}$$

where $c_0 = \text{const} > 0$ does not depend on u.

Taking into account the fact that $\operatorname{mes} D_T = \frac{\omega_n}{2(n+1)} T^{n+1}$, where ω_n is the volume of the unit ball in R^n, for $q = 2(p+1)$ from (51) we find that

$$\|u\|_{L_{2(p+1)}(D_T)} \le c_0 \widetilde{\ell}_{p,n} T^{(n+1)(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2})} \|u\|_{\mathring{W}_2^1(D_T,S_T)} \quad \forall u \in \mathring{W}_2^1(D_T, S_T), \tag{52}$$

where $\widetilde{\ell}_{p,n} = (\frac{\omega_n}{2(n+1)})^{(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2})}$.

For $\|K_1 u\|_{L_2(D_T)}$, where $u \in \mathring{W}_2^1(D_T, S_T)$, and the operator K_1 is given by the equality from (48), by virtue of (52) the estimate

$$\begin{aligned}\|K_1 u\|_{L_2(D_T)} &\le \lambda \left[\int_{D_T} |u|^{2(p+1)}\,\mathrm{d}x\,\mathrm{d}t\right]^{1/2} = \lambda \|u\|_{L_{2(p+1)}(D_T)}^{p+1} \\ &\le \lambda \ell_{p,n} T^{(p+1)(n+1)(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2})} \|u\|_{\mathring{W}_2^1(D_T,S_T)}^{p+1}\end{aligned} \tag{53}$$

holds, where $\ell_{p,n} = [c_0 \widetilde{\ell}_{p,n}]^{p+1}$.

Now from (24) and (53), for the value $\|Au\|_{\mathring{W}_2^1(D_T,S_T)}$, where $Au = L^{-1} K_1 u$, the estimate

$$\begin{aligned}\|Au\|_{\mathring{W}_2^1(D_T,S_T)} &\le \|L^{-1}\|_{L_2(D_T)\to \mathring{W}_2^1(D_T,S_T)} \|K_1 u\|_{L_2(D_T)} \\ &\le \sqrt{\frac{e}{2}}\, \lambda \ell_{p,n} T^{1+(p+1)(n+1)(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2})} \|u\|_{\mathring{W}_2^1(D_T,S_T)}^{p+1} \quad \forall u \in \mathring{W}_2^1(D_T, S_T)\end{aligned} \tag{54}$$

is valid. Note that $\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2} > 0$ for $p < \frac{2}{n-1}$.

Consider the equation

$$a z^{p+1} + b = z \tag{55}$$

with respect to the unknown z, where

$$a = \sqrt{\frac{e}{2}}\, \lambda \ell_{p,n} T^{1+(p+1)(n+1)(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2})}, \qquad b = \sqrt{\frac{e}{2}}\, T \|F\|_{L_2(D_T)}. \tag{56}$$

For $T > 0$, it is evident that $a > 0$ and $b \ge 0$. A simple analysis, analogous to that of [30, pp. 373, 374] for $p = 2$, shows that: (i) for $b = 0$, along with the zero root $z_1 = 0$, Eq. (55) has the unique positive root $z_2 = a^{-\frac{1}{p}}$; (ii) if $b > 0$, then for $0 < b < b_0$, where

$$b_0 = \left[(p+1)^{-\frac{1}{p}} - (p+1)^{-\frac{p+1}{p}}\right] a^{-\frac{1}{p}}, \tag{57}$$

and Eq. (55) has two positive roots z_1 and z_2, $0 < z_1 < z_2$; these roots for $b = b_0$ merge, and we have one positive root

$$z_1 = z_2 = z_0 = [(p+1)a]^{-\frac{1}{p}};$$

(iii) for $b > b_0$, Eq. (55) has no nonnegative roots.

Note that for $0 < b < b_0$ there arise the inequalities

$$z_1 < z_0 = [(p+1)a]^{-\frac{1}{p}} < z_2.$$

By virtue of (56) and (57), the condition $b \le b_0$ is equivalent to the condition

$$\sqrt{\frac{e}{2}}\,T\|F\|_{L_2(D_T)} \le \left[\sqrt{\frac{e}{2}}\,\lambda\ell_{p,n}T^{1+(p+1)(n+1)(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2})}\right]^{-\frac{1}{p}}\left[(p+1)^{-\frac{1}{p}}-(p+1)^{-\frac{p+1}{p}}\right]$$

or

$$\|F\|_{L_2(D_T)} \le \gamma_{n,\lambda,p}T^{-\alpha_n}, \quad \alpha_n > 0, \tag{58}$$

where

$$\gamma_{n,\lambda,p} = \left[(p+1)^{-\frac{1}{p}}-(p+1)^{-\frac{p+1}{p}}\right](\lambda\ell_{p,n})^{-\frac{1}{p}}\exp\left[-\frac{1}{2}\left(1+\frac{1}{p}\right)\right],$$

$$\alpha_n = 1+\frac{1}{p}\left[1+(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)\right].$$

Owing to the absolute continuity of the Lebesgue integral, we have $\lim_{T\to 0}\|F\|_{L_2(D_T)} = 0$. At the same time, $\lim_{T\to 0} T^{-\alpha_n} = +\infty$. Therefore there exists a number $T_1 = T_1(F)$, $0 < T_1 < +\infty$, such that inequality (58) holds for

$$0 < T \le T_1(F). \tag{59}$$

Now let us show that if the condition (59) is fulfilled, the operator $A_1 : \mathring{W}_2^1(D_T, S_T) \to \mathring{W}_2^1(D_T, S_T)$, acting by the formula $A_1 u = Au + u_0$, transfers the ball $B(0, z_2)$, mentioned in Remark 10, into itself, where z_2 is the maximal positive root of Eq. (55). Indeed, if $u \in B(0, z_2)$, then by virtue of (54)–(56) we have

$$\|A_1 u\|_{\mathring{W}_2^1(D_T,S_T)} \le a\|u\|^{p+1}_{\mathring{W}_2^1(D_T,S_T)} + b \le a z_2^{p+1} + b = z_2.$$

Therefore by Remarks 9 and 10, the following theorem is valid.

Theorem 3. *Let $F \in L_{2,\mathrm{loc}}(D)$, $g = 0$, $0 < p < \frac{2}{n-1}$, and for the value T the condition (59) be fulfilled. Then the problem (27), (28) in the domain D_T has at least one strong generalized solution $u \in \mathring{W}_2^1(D_T, S_T)$.*

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