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On the solvability of the Cauchy characteristic problem for a nonlinear equation with iterated wave operator in the principal part ☆

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Abstract

We consider one multidimensional version of the Cauchy characteristic problem in the light cone of the future for a hyperbolic equation with power nonlinearity with iterated wave operator in the principal part. Depending on the exponent of nonlinearity and spatial dimension of equation, we investigate the problem on the nonexistence of global solutions of the Cauchy characteristic problem. The question on the local solvability of that problem is also considered.
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1. Statement of the problem

Consider the nonlinear equation of the type

$$Lu := \square^2 u = \lambda |u|^\alpha + F, \tag{1}$$

where λ and α are the given positive constants, F is the given and u is an unknown real functions, $\square = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $n > 1$.

For Eq. (1), we consider the Cauchy characteristic problem on finding in the truncated light cone of the future D_T: $|x| < t < T$, $x = (x_1, \dots, x_n)$, $T = \text{const} > 0$, a solution $u(x,t)$ of that equation by the boundary conditions

$$u|_{S_T} = 0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{S_T} = 0, \tag{2}$$

where S_T: $t = |x|$, $t \leqslant T$, is the characteristic manifold which is, in fact, a conic portion of the boundary of the domain D_T, $\frac{\partial}{\partial \nu}$ is the derivative in the direction of the outer normal to ∂D_T. For the case $T = +\infty$ we assume that D_∞: $t > |x|$ and $S_\infty = \partial D_\infty$: $t = |x|$.

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Note that for nonlinear hyperbolic type equations the question of local and global solvability of the Cauchy problem with initial conditions for $t = 0$ has been considered by various authors (see, for e.g., [1–20]). For linear second order hyperbolic equations the characteristic problem in a conic domain is, as is known, well-posed and there takes place the global solvability in the corresponding function spaces [21–25].

In the case of one second order wave equation with power nonlinearity the problems of existence or nonexistence of a global solution have been considered in [26].

Below we will show that under certain conditions, imposed on the exponent of nonlinearity α and on the function F, the problem (1), (2) has no global solution, although, as it will be proved below, this problem is locally solvable.

Assume $\overset{\circ}{W}{}_2^2(D_T, S_T) = \{u \in W_2^2(D_T)\colon\ u|_{S_T} = 0,\ \frac{\partial u}{\partial \nu}|_{S_T} = 0\}$, where $W_2^2(D_T)$ is the Sobolev space [27, p. 56] consisting of elements $L_2(D_T)$ with generalized derivatives from $L_2(D_T)$ up to the second order, inclusive, and the conditions (2) are understood in the sense of the trace theory [27, p. 70].

Definition 1. Let $F \in L_2(D_T)$. The function u is said to be a weak generalized solution of the problem (1), (2) of the class W_2^2 in D_T, if $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, $|u|^\alpha \in L_2(D_T)$, and for every function $\varphi \in W_2^2(D_T)$, such that $\varphi|_{t=T} = 0$, $\frac{\partial \varphi}{\partial t}|_{t=T} = 0$, the integral equality

$$\int\limits_{D_T} \Box u \Box \varphi \, dx\, dt = \lambda \int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt + \int\limits_{D_T} F\varphi \, dx\, dt \tag{3}$$

is valid.

Integration by parts shows that the classical solution $u \in \overset{\circ}{C}{}^4(\overline{D}_T, S_T) = \{u \in C^4(\overline{D}_T)\colon\ u|_{S_T} = 0,\ \frac{\partial u}{\partial \nu}|_{S_T} = 0\}$ of the problem (1), (2) is also a weak generalized solution of that problem of the class W_2^2 in the sense of Definition 1, and vice versa, if a weak generalized solution of the problem (1), (2) of the class W_2^2 belongs to the space $C^4(\overline{D}_T)$, then this solution will be classical as well. Here we have used the fact that if $u \in C^4(\overline{D}_T)$ and the conditions (2) are fulfilled, then taking into account that S_T is the characteristic manifold, the equality $\Box u|_{S_T} = 0$ is valid. In addition, since the derivative with respect to the conormal $\frac{\partial}{\partial N} = \nu_{n+1}\frac{\partial}{\partial t} - \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$ ($\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$) is the interior differential operator on the characteristic manifold S_T, then $\frac{\partial}{\partial N}\Box u|_{S_T} = 0$, and likewise $\frac{\partial u}{\partial N}|_{S_T} = 0$, since $u|_{S_T} = 0$.

Definition 2. Let $F \in L_2(D_T)$. The function u is said to be a strong generalized solution of the problem (1), (2) of the class W_2^2 in D_T, if $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, $|u|^\alpha \in L_2(D_T)$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^4(\overline{D}_T, S_T)$, such that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$ and $|u_m|^\alpha \to |u|^\alpha$, $[Lu_m - \lambda|u_m|^\alpha] \to F$ in the space $L_2(D_T)$.

Obviously, the classical solution of the problem (1), (2) from $\overset{\circ}{C}{}^4(\overline{D}_T, S_T)$ is a strong generalized solution of that problem of the class W_2^2. In its turn, a strong generalized solution of the problem (1), (2) of the class W_2^2 is a weak generalized solution of that problem of the class W_2^2.

Definition 3. Let $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$. We say that the problem (1), (2) is globally solvable in a weak (strong) sense in the class W_2^2, if for any $T > 0$ this problem has a weak (strong) generalized solution of the class W_2^2 in the domain D_T.

Remark 1. It is easy to see that if the problem (1), (2) is not globally solvable in a weak sense, then it fails to be globally solvable in a strong sense as well in the class W_2^2. It is also evident that the global solvability of the problem (1), (2) in a strong sense implies global solvability of that problem in a weak sense in the class W_2^2.

2. The nonexistence of global solvability of the problem (1), (2)

Theorem 1. *Let $F \in L_{2,\mathrm{loc}}(D_\infty)$, $F \geqslant 0$, $F \not\equiv 0$ and $F \in L_2(D_T)$ for any $T > 0$. Then if the exponent of nonlinearity α in Eq.* (1) *satisfies the inequalities*

$$\begin{cases} 1 < \alpha < \dfrac{n+1}{n-3}, & n > 3, \\ 1 < \alpha < \infty, & n = 2, 3, \end{cases} \tag{4}$$

and in the limiting case $\alpha = \frac{n+1}{n-3}$ *for* $n > 3$ *the function* F *satisfies the condition*

$$\lim_{T\to\infty} \int_{D_T} F\,dx\,dt = \infty, \tag{5}$$

then the problem (1), (2) *is not globally solvable in a weak sense in the class* W_2^2*, i.e. there exists a number* $T_0 = T_0(F) > 0$*, such that for* $T > T_0$ *the problem* (1), (2) *fails to have a weak generalized solution of the problem* (1), (2) *of the class* W_2^2 *in the domain* D_T.

Proof. Assume that u is a weak generalized solution of the problem (1), (2) of the class W_2^2 in the domain D_T, i.e. the integral equality (3) is valid for every function $\varphi \in W_2^2(D_T)$, such that $\varphi|_{t=T} = 0$, $\frac{\partial\varphi}{\partial t}|_{t=T} = 0$. Integrating the left-hand side of equality (3) by parts, we obtain

$$\begin{aligned}\int_{D_T} \Box u \Box\varphi\,dx\,dt &= \int_{\partial D_T} \frac{\partial u}{\partial N}\Box\varphi\,ds - \int_{D_T} \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\Box\varphi\,dx\,dt + \int_{D_T} \nabla_x u \nabla_x(\Box\varphi)\,dx\,dt \\ &= \int_{\partial D_T} \frac{\partial u}{\partial N}\Box\varphi\,ds - \int_{\partial D_T} u\frac{\partial}{\partial N}\Box\varphi\,ds + \int_{D_T} u\Box^2\varphi\,dx\,dt,\end{aligned} \tag{6}$$

where $\frac{\partial}{\partial N}$ is the derivative with respect to the conormal, $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$.

Let the function $\varphi_0 = \varphi_0(\sigma)$ of one real variable σ be such that

$$\varphi_0 \in C^4\big((-\infty,+\infty)\big), \quad \varphi_0 \geqslant 0, \quad \varphi_0' \leqslant 0, \quad \varphi_0(\sigma) = \begin{cases} 1, & 0 \leqslant \sigma \leqslant 1, \\ 0, & \sigma \geqslant 2. \end{cases} \tag{7}$$

Using the method of test functions [12, pp. 10–12], in the capacity of a test function in equality (3) we take the function $\varphi(x,t) = \varphi_0[\frac{2}{T^2}(t^2 + |x|^2)]$.

Taking into account that $u|_{S_T} = 0$, and hence $\frac{\partial u}{\partial N}|_{S_T} = 0$, since $\frac{\partial}{\partial N} = \nu_{n+1}\frac{\partial}{\partial t} - \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$ is the inner differential operator on S_T, as well as by virtue of (7), the equalities $\frac{\partial^i\varphi}{\partial t^i}|_{t=T} = 0$, $0 \leqslant i \leqslant 4$, $\Box\varphi|_{t=T} = \frac{\partial}{\partial N}\Box\varphi|_{t=T} = 0$, it follows from (6) that

$$\int_{D_T} \Box u\Box\varphi\,dx\,dt = \int_{D_T} u\Box^2\varphi\,dx\,dt.$$

Therefore equality (3) can be rewritten in the form

$$\lambda\int_{D_T} |u|^\alpha\varphi\,dx\,dt = \int_{D_T} u\Box^2\varphi\,dx\,dt - \int_{D_T} F\varphi\,dx\,dt. \tag{8}$$

If in the Young's inequality with the parameter $\varepsilon > 0$,

$$ab \leqslant \frac{\varepsilon}{\alpha}a^\alpha + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}b^{\alpha'}, \quad a,b \geqslant 0,\ \alpha' = \frac{\alpha}{\alpha-1},$$

we take $a = |u|\varphi^{1/\alpha}$, $b = \frac{|\Box\varphi|^2}{\varphi^{1/\alpha}}$, then taking into account that $\frac{\alpha'}{\alpha} = \alpha' - 1$, we have

$$\big|u\Box^2\varphi\big| = |u|\varphi^{1/\alpha}\frac{|\Box^2\varphi|}{\varphi^{1/\alpha}} \leqslant \frac{\varepsilon}{\alpha}|u|^\alpha\varphi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}. \tag{9}$$

By virtue of (9), from (8) we obtain the inequality

$$\left(\lambda - \frac{\varepsilon}{\alpha}\right)\int_{D_T} |u|^\alpha\varphi\,dx\,dt \leqslant \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt - \int_{D_T} F\varphi\,dx\,dt,$$

whence

$$\int\limits_{D_T} |u|^\alpha \varphi\,dx\,dt \leqslant \frac{\alpha}{(\lambda\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt - \frac{\alpha}{\lambda\alpha-\varepsilon} \int\limits_{D_T} F\varphi\,dx\,dt \tag{10}$$

for $\varepsilon < \lambda\alpha$.

Bearing in mind the equalities

$$\alpha' = \frac{\alpha}{\alpha-1}, \qquad \alpha = \frac{\alpha'}{\alpha'-1} \quad \text{and} \quad \min_{0<\varepsilon<\lambda\alpha} \frac{\alpha}{(\lambda\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}},$$

which is achieved for $\varepsilon = \lambda$, it follows from (10) that

$$\int\limits_{D_T} |u|^\alpha \varphi\,dx\,dt \leqslant \frac{1}{\lambda^{\alpha'}} \int\limits_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt - \frac{\alpha'}{\lambda} \int\limits_{D_T} F\varphi\,dx\,dt. \tag{11}$$

According to the properties (7) of the function φ_0, the test function $\varphi(x,t) = \varphi_0[\frac{2}{T^2}(t^2+|x|^2)] = 0$ for $r = (t^2+|x|^2)^{1/2} \geqslant T$. Therefore after the change of variables $t = T\xi_0$ and $x = T\xi$ we have

$$\begin{aligned}\int\limits_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt &= \int\limits_{\substack{r=(t^2+|x^2|)^{1/2}<T\\ t>|x|}} \frac{|c_1T^{-4}\varphi_0'' + (c_2t^2+c_3|x|^2)T^{-6}\varphi_0''' + c_4T^{-8}(t^2-|x|^2)^2\varphi_0''''|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt \\ &= T^{n+1-4\alpha'} \int\limits_{\substack{1<2(\xi_0^2+|\xi|^2)<2\\ \xi_0>|\xi|}} \frac{|c_1\varphi_0'' + (c_2\xi_0^2+c_3|\xi|^2)\varphi_0''' + c_4(\xi_0^2-|\xi|^2)^2\varphi_0''''|^{\alpha'}}{\varphi^{\alpha'-1}}\,d\xi_0\,d\xi,\end{aligned} \tag{12}$$

where $c_i = c_i(n)$, $i = 1,\dots,4$, are some integers.

As is known, the test function $\varphi(x,t) = \varphi_0[\frac{2}{T^2}(t^2+|x|^2)]$ with the above-mentioned properties for which the integrals in the right-hand sides of (11) and (12) are finite, does exist [12, p. 28].

Owing to (12) and the fact that $\varphi_0(\sigma) = 1$ for $0 \leqslant \sigma \leqslant 1$, from inequality (11) we obtain

$$\int\limits_{\substack{r=(t^2+|x^2|)^{1/2}<\frac{T}{\sqrt{2}}\\ t>|x|}} |u|^\alpha\,dx\,dt \leqslant \int\limits_{D_T} |u|^\alpha\varphi\,dx\,dt \leqslant \frac{T^{n+1-4\alpha'}}{\lambda^{\alpha'}}\varkappa_0 - \frac{\alpha'}{\lambda}\gamma(T), \tag{13}$$

where

$$\gamma(T) = \int\limits_{D_T} F\varphi\,dx\,dt,$$

$$\varkappa_0 = \int\limits_{\substack{1<2(\xi_0^2+|\xi|^2)<2\\ \xi_0>|\xi|}} \frac{|c_1\varphi_0'' + (c_2\xi_0^2+c_3|\xi|^2)\varphi_0''' + c_4(\xi_0^2-|\xi|^2)^2\varphi_0''''|^{\alpha'}}{\varphi^{\alpha'-1}}\,d\xi_0\,d\xi < +\infty.$$

Let us consider first the case $q = n+1-4\alpha' < 0$, which in accordance with the condition (4) means that $\alpha < \frac{n+1}{n-3}$ for $n > 3$, and $\alpha < \infty$ for $n = 2, 3$. In this case the equation

$$g(T) = \frac{T^{n+1-4\alpha'}}{\lambda^{\alpha'}}\varkappa_0 - \frac{\alpha'}{\lambda}\gamma(T) = 0 \tag{14}$$

has the unique positive root $T = T_0 > 0$, since the function $g_1(T) = \frac{T^{n+1-4\alpha'}}{\lambda^{\alpha'}}\varkappa_0$ is positive, continuous and strictly decreasing on the interval $(0,+\infty)$, and also $\lim_{T\to 0} g_1(T) = +\infty$ and $\lim_{T\to+\infty} g_1(T) = 0$, while the function $\gamma(T) = \int_{D_T} F\varphi\,dx\,dt$ is, by virtue of $F \geqslant 0$ and (7), nonnegative, nondecreasing and, owing to the absolute continuity of the integral, is likewise continuous. Moreover, $\lim_{T\to\infty}\gamma(T) > 0$, since $F \geqslant 0$ and $F \not\equiv 0$, i.e. $F \neq 0$ on some set of the positive Lebesgue measure. Thus $g(T) < 0$ for $T > T_0$, and $g(T) > 0$ for $0 < T < T_0$. Consequently, for $T > T_0$ the right-hand side of inequality (13) is negative, but this is impossible.

Consider now the limiting case $q = n + 1 - 4\alpha' = 0$, i.e. when $\alpha = \frac{n+1}{n-3}$ for $n > 3$. In this case equation (14) takes the form $\frac{1}{\lambda^{\alpha'}}\varkappa_0 - \frac{\alpha'}{\lambda}\gamma(T) = 0$, which by the obvious equality $\lim_{T\to 0}\gamma(T) = 0$ and conditions (5), (7) has likewise the unique positive root $T = T_0 > 0$. For $T > T_0$, the right-hand side of inequality (13) is negative, and this again leads to the contradiction. Thus the proof of the theorem is complete. $\square$

Remark 2. From the proof of Theorem 1 it follows that when the conditions of the theorem are fulfilled, if there exists a weak generalized solution of the problem (1), (2) of the class W_2^2 in the domain D_T, then the estimate

$$T \leqslant T_0 \tag{15}$$

is valid, where T_0 is the unique positive root of Eq. (14).

3. Local solvability of the problem (1), (2)

Consider first the linear case, when in Eq. (1) the parameter $\lambda = 0$, i.e. we consider the problem

$$Lu(x,t) = F(x,t), \quad (x,t) \in D_T, \tag{16}$$

$$u|_{S_T} = 0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{S_T} = 0 \tag{17}$$

in which for the sake of convenience we introduce the notation $L = \Box^2$.

Definition 4. Let $F \in L_2(D_T)$. The function u is said to be a strong generalized solution of the problem (16), (17) of the class W_2^2 in the domain D_T, if $u \in \mathring{W}_2^2(D_T, S_T)$ and there exists a sequence of functions $u_m \in \mathring{C}^4(\overline{D}_T, S_T)$, such that $u_m \to u$ in the space $\mathring{W}_2^2(D_T, S_T)$, and $Lu_m \to F$ in the space $L_2(D_T)$.

Obviously, the classical solution $u \in \mathring{C}^4(\overline{D}_T, S_T)$ of the problem (16), (17) is a strong generalized solution of that problem of the class W_2^2 in the domain D_T.

Lemma 1. *For a strong generalized solution u of the problem* (16), (17) *of the class W_2^2 in the domain D_T the estimate*

$$\|u\|_{\mathring{W}_2^2(D_T,S_T)} \leqslant c_n(1+T)^4\|F\|_{L_2(D_T)}, \tag{18}$$

where $c_n = \frac{e}{2}\sqrt{n+2}$, holds.

Proof. Let us first show that

$$\|v\|_{W_2^1(D_T)} \leqslant \sqrt{\frac{e}{2}}(1+T)^2\|\Box v\|_{L_2(D_T)} \quad \forall v \in \mathring{C}^2(\overline{D}_T, S_T), \tag{19}$$

where $\mathring{C}^2(\overline{D}_T, S_T) = \{v \in C^2(\overline{D}_T):\ v|_{S_T} = 0\}$.

Indeed, assume that $\Omega_\tau := \partial D_\tau \cap \{t = \tau\}$ and denote by $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ the unit vector of the outer normal to ∂D_T. Taking into account the equalities $v|_{S_T} = 0$ and $\nu|_{\Omega_\tau} = (0, \dots, 0, 1)$, integration by parts yields

$$\int_{D_\tau} \frac{\partial^2 v}{\partial t^2}\frac{\partial v}{\partial t}\,dx\,dt = \frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right)^2 dx\,dt = \frac{1}{2}\int_{\partial D_\tau}\left(\frac{\partial v}{\partial t}\right)^2 \nu_{n+1}\,ds = \frac{1}{2}\int_{\Omega_\tau}\left(\frac{\partial v}{\partial t}\right)^2 dx + \frac{1}{2}\int_{S_\tau}\left(\frac{\partial v}{\partial t}\right)^2 \nu_{n+1}\,ds, \tag{20}$$

$$\int_{D_\tau} \frac{\partial^2 v}{\partial x_i^2}\frac{\partial v}{\partial t}\,dx\,dt = \int_{\partial D_\tau} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial t}\nu_i\,ds - \frac{1}{2}\int_{D_\tau}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\,dt = \int_{\partial D_\tau} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial t}\nu_i\,ds - \frac{1}{2}\int_{\partial D_\tau}\left(\frac{\partial v}{\partial x_i}\right)^2 \nu_{n+1}\,ds$$

$$= \int_{\partial D_\tau} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial t}\nu_i\,ds - \frac{1}{2}\int_{S_\tau}\left(\frac{\partial v}{\partial x_i}\right)^2 \nu_{n+1}\,ds - \frac{1}{2}\int_{\Omega_\tau}\left(\frac{\partial v}{\partial x_i}\right)^2 dx, \quad \tau \leqslant T. \tag{21}$$

It easily follows from (20) and (21) that

$$\int_{D_\tau} \Box v \frac{\partial v}{\partial t}\, dx\, dt = \int_{S_\tau} \frac{1}{2\nu_{n+1}} \left[\sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \nu_{n+1} - \frac{\partial v}{\partial t} \nu_i \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \left(\nu_{n+1}^2 - \sum_{j=1}^{n} \nu_j^2 \right) \right] ds$$
$$+ \frac{1}{2} \int_{\Omega_\tau} \left[\left(\frac{\partial v}{\partial t} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \right] dx, \quad \tau \leqslant T. \tag{22}$$

Since $v|_{S_T} = 0$, and $(\nu_{n+1} \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial t})$, $1 \leqslant i \leqslant n$, is the interior differential operator on S_T, there take place the equalities

$$\left(\frac{\partial v}{\partial x_i} \nu_{n+1} - \frac{\partial v}{\partial t} \nu_i \right) \bigg|_{S_\tau} = 0, \quad i = 1, \ldots, n. \tag{23}$$

Therefore taking into account that $\nu_{n+1}^2 - \sum_{j=1}^{n} \nu_j^2 = 0$ on the characteristic manifold S_T, by (22) and (23) we have

$$\int_{\Omega_\tau} \left[\left(\frac{\partial v}{\partial t} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \right] dx = 2 \int_{D_\tau} \Box v \frac{\partial v}{\partial t}\, dx\, dt, \quad \tau \leqslant T. \tag{24}$$

Assuming $w(\delta) = \int_{\Omega_\delta} [(\frac{\partial v}{\partial t})^2 + \sum_{i=1}^{n} (\frac{\partial v}{\partial x_i})^2]\, dx$ and using the inequality $2 \Box v \frac{\partial v}{\partial t} \leqslant \varepsilon (\frac{\partial v}{\partial t})^2 + \frac{1}{\varepsilon} |\Box v|^2$, valid for any $\varepsilon = \text{const} > 0$, from (24) we get

$$w(\delta) \leqslant \varepsilon \int_0^\delta w(\sigma)\, d\sigma + \frac{1}{\varepsilon} \|\Box v\|_{L_2(D_\delta)}^2, \quad 0 < \delta \leqslant T. \tag{25}$$

From (25), taking into account that $\|\Box v\|_{L_2(D_\delta)}^2$, being the function of δ, is nondecreasing, by the Gronwall lemma [28, p. 13] it follows that

$$w(\delta) \leqslant \frac{1}{\varepsilon} \|\Box v\|_{L_2(D_\delta)}^2 \exp \delta\varepsilon.$$

Taking into account that $\inf_{\varepsilon > 0} \frac{1}{\varepsilon} \exp \delta\varepsilon = e\delta$ is achieved for $\varepsilon = \frac{1}{\delta}$, from the above inequality we obtain

$$w(\delta) \leqslant e\delta \|\Box v\|_{L_2(D_\delta)}^2, \quad 0 < \delta \leqslant T. \tag{26}$$

In its turn, (26) yields

$$\int_{D_T} \left[\left(\frac{\partial v}{\partial t} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \right] dx\, dt = \int_0^T w(\delta)\, d\delta \leqslant \frac{e}{2} T^2 \|\Box v\|_{L_2(D_T)}^2. \tag{27}$$

Using the equalities $v|_{S_T} = 0$ and $v(x,t) = \int_{|x|}^{t} \frac{\partial v(x,\tau)}{\partial t}\, d\tau$, $(x,t) \in \overline{D}_T$ which are valid for any function $v \in \mathring{C}^2(\overline{D}_T, S_T)$, standard reasoning [27, p. 63] leads to the inequality

$$\int_{D_T} v^2(x,t)\, dx\, dt \leqslant T^2 \int_{D_T} \left(\frac{\partial v}{\partial t} \right)^2 dx\, dt. \tag{28}$$

By (27) and (28) we have

$$\|v\|_{W_2^1(D_T)}^2 = \int_{D_T} \left[v^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \right] dx\, dt \leqslant \frac{e}{2} (1+T)^4 \|\Box v\|_{L_2(D_T)}^2,$$

which results in inequality (19).

By the definition, if u is a strong generalized solution of the problem (16), (17) of the class W_2^2 in the domain D_T, then there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^4(\overline{D}_T, S_T)$, such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)} = 0, \qquad \lim_{m\to 0} \left\|\Box^2 u_m - F\right\|_{L_2(D_T)} = 0. \tag{29}$$

Since $u_m \in \overset{\circ}{C}{}^4(\overline{D}_T, S_T)$ satisfies the homogeneous boundary conditions (17), and S_T is the characteristic manifold, corresponding to the operator $\Box$, therefore, as is known [22, p. 546],

$$\Box u_m|_{S_T} = 0. \tag{30}$$

By virtue of (30), the function $v = \Box u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ according to (19) satisfies the inequalities

$$\begin{aligned}
&\|\Box u_m\|^2_{L_2(D_T)} \leqslant \frac{e}{2}(1+T)^4 \left\|\Box^2 u_m\right\|^2_{L_2(D_T)},\\
&\left\|\Box \frac{\partial u_m}{\partial t}\right\|^2_{L_2(D_T)} \leqslant \frac{e}{2}(1+T)^4 \left\|\Box^2 u_m\right\|^2_{L_2(D_T)},\\
&\left\|\Box \frac{\partial u_m}{\partial x_i}\right\|^2_{L_2(D_T)} \leqslant \frac{e}{2}(1+T)^4 \left\|\Box^2 u_m\right\|^2_{L_2(D_T)}, \quad i = 1,\dots,n.
\end{aligned} \tag{31}$$

Since $\frac{\partial u_m}{\partial t}, \frac{\partial u_m}{\partial x_i} \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, by (19) and (31) we have

$$\begin{aligned}
\|u_m\|^2_{\overset{\circ}{W}{}_2^2(D_T,S_T)} &= \int_{D_T} \left[u_m^2 + \left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_m}{\partial x_i}\right)^2 + \left(\frac{\partial^2 u_m}{\partial t^2}\right)^2 + \sum_{i=1}^n \left(\frac{\partial^2 u_m}{\partial t \partial x_i}\right)^2 + \sum_{i,j=1}^n \left(\frac{\partial^2 u_m}{\partial x_i \partial x_j}\right)^2\right] dx\,dt\\
&\leqslant \|u_m\|^2_{W_2^1(D_T)} + \left\|\frac{\partial u_m}{\partial t}\right\|^2_{W_2^1(D_T)} + \sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|^2_{W_2^1(D_T)}\\
&\leqslant \frac{e^2}{4}(1+T)^8 \left\|\Box^2 u_m\right\|^2_{L_2(D_T)} + \frac{e^2}{4}(1+T)^8 \left\|\Box^2 u_m\right\|^2_{L_2(D_T)} + \sum_{i=1}^n \frac{e^2}{4}(1+T)^8 \left\|\Box^2 u_m\right\|^2_{L_2(D_T)}\\
&= (n+2)\frac{e^2}{4}(1+T)^8 \left\|\Box^2 u_m\right\|^2_{L_2(D_T)},
\end{aligned}$$

whence

$$\|u_m\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)} \leqslant c_n (1+T)^4 \left\|\Box^2 u_m\right\|_{L_2(D_T)}, \quad c_n = \frac{e}{2}\sqrt{n+2}. \tag{32}$$

In (29), passing in inequality (32) to the limit as $m \to \infty$, we obtain (18) which shows that the proof of the lemma is complete. $\Box$

Lemma 2. *For every $F \in L_2(D_T)$ there exists a unique strong generalized solution u of the problem* (16), (17) *of the class W_2^2 in the domain D_T for which the estimate* (18) *is valid.*

Proof. As far as the space $C_0^\infty(D_T)$ of finite infinitely differentiable in D_T functions is dense in $L_2(D_T)$, for the given $F \in L_2(D_T)$ there exists the sequence of functions $F_m \in C_0^\infty(D_T)$, such that $\lim_{m\to\infty} \|F_m - F\|_{L_2(D_T)} = 0$. For m fixed, continuing the function F_m by zero outside the limits of D_T and leaving for it the same notation, we have $F_m \in C^\infty(R_+^{n+1})$ for which the support $\operatorname{supp} F_m \subset D_\infty$, where $R_+^{n+1} = R^{n+1} \cap \{t \geqslant 0\}$. Denote by u_m a solution of the Cauchy problem: $Lu_m = F_m$, $\frac{\partial^i u}{\partial t^i}|_{t=0} = 0$, $0 \leqslant i \leqslant 3$, which, as is known, exists, is unique and belongs to the space $C^\infty(R_+^{n+1})$ [29, p. 192]. In addition, since $\operatorname{supp} F_m \subset D_\infty$, $\frac{\partial^i u}{\partial t^i}|_{t=0} = 0$, $0 \leqslant i \leqslant 3$, taking into account the geometry of the domain of dependence of a solution of the linear equation $Lu_m = F_m$ of hyperbolic type, we have $\operatorname{supp} u_m \subset D_\infty$ [29, p. 191]. Leaving for the restriction of the function u_m on the domain D_T the same notation, we can easily see that $u_m \in \overset{\circ}{C}{}^4(\overline{D}_T, S_T)$, and by virtue of (18) the inequality

$$\|u_m - u_k\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)} \leqslant c_n (1+T)^4 \|F_m - F_k\|_{L_2(D_T)} \tag{33}$$

is valid.

Since the sequence $\{F_m\}$ is fundamental in $L_2(D_T)$, the sequence $\{u_m\}$ by virtue of (33) is likewise fundamental in the complete space $\overset{\circ}{W}{}_2^2(D_T, S_T)$. Therefore there exists the function $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)} = 0,$$

and as far as $Lu_m = F_m \to F$ in the space $L_2(D_T)$, the function u is, by Definition 4, a strong generalized solution of the problem (16), (17) of the class W_2^2 in the domain D_T, for which the estimate (18) is valid. The uniqueness of that solution follows from the estimate (18). Thus the proof of the Lemma 2 is complete. □

Remark 3. By Lemma 2, for the strong generalized solution u of the problem (16), (17) of the class W_2^2 in the domain D_T we can write $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^2(D_T, S_T)$ is the linear continuous operator whose norm, by virtue of (18), admits the estimate

$$\|L^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}_2^2(D_T,S_T)} \leqslant c_n(1+T)^4. \tag{34}$$

Remark 4. The embedding operator $I : \overset{\circ}{W}{}_2^2(D_T, S_T) \to L_2(D_T)$ is linear continuous compact for $1 < q < \frac{2(n+1)}{n-3}$, when $n > 3$, and $1 < q < \infty$, when $n = 2, 3$ [27, p. 84]. At the same time, the Nemytski's operator $T : L_q(D_T) \to L_2(D_T)$, acting by the formula $Tu = \lambda|u|^\alpha$, is continuous and bounded, if $q \geqslant 2\alpha$ [30, p. 349], [31, pp. 66, 67]. Thus if the exponent α of nonlinearity in Eq. (1) satisfies inequalities (4), then putting $q = 2\alpha$, we obtain that the operator

$$T_0 = TI : \overset{\circ}{W}{}_2^2(D_T, S_T) \to L_2(D_T) \tag{35}$$

is continuous and compact one. In addition, from $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ it all the more follows that $|u|^\alpha \in L_2(D_T)$, and in Definition 2, relying on the fact that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$, it automatically follows that $|u_m|^\alpha \to |u|^\alpha$ in the space $L_2(D_T)$, as well.

Remark 5. If $F \in L_2(D_T)$ and the exponent α of nonlinearity satisfies inequalities (4), then according to Definition 2 and Remarks 3 and 4, the function $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ is a strong generalized solution of the problem (1), (2) of the class W_2^2 in the domain D_T, if and only if u is a solution of the functional equation

$$u = L^{-1}\big(\lambda|u|^\alpha + F\big) \tag{36}$$

in the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$.

We rewrite Eq. (36) as follows:

$$u = Ku + u_0, \tag{37}$$

where the operator $K = L^{-1}T_0 : \overset{\circ}{W}{}_2^2(D_T, S_T) \to \overset{\circ}{W}{}_2^2(D_T, S_T)$ is, by virtue of (34), (35) and Remark 4, continuous and compact, acting in the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$, and $u_0 = L^{-1}F \in \overset{\circ}{W}{}_2^2(D_T, S_T)$.

Remark 6. Let

$$B(0, z_2) := \big\{u \in \overset{\circ}{W}{}_2^2(D_T, S_T):\ \|u\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)} \leqslant z_2\big\}$$

be a closed (convex) ball in the Hilbert space $\overset{\circ}{W}{}_2^2(D_T, S_T)$ of radius $z_2 > 0$ with center at a zero element. Since the operator $K : \overset{\circ}{W}{}_2^2(D_T, S_T) \to \overset{\circ}{W}{}_2^2(D_T, S_T)$ is continuous and compact when inequalities (4) are fulfilled, by the Schauder's principle, for Eq. (37) to be solvable, is sufficient to show that the operator K_1, acting by the formula $K_1u = Ku + u_0$ transfers the ball $B(0, z_2)$ into itself for some $z_2 > 0$ [32, p. 370]. Towards this end, we cite the needed estimate for $\|Ku\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)}$.

As is known, if $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, then the inequality [27, p. 83]

$$\|u\|_{L_p(D_T)} \leqslant \tilde{c}\|u\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)} \tag{38}$$

is valid, where $p = \frac{2(n+1)}{n-3}$ if $n > 3$ and $1 < p < \infty$ for $n = 2, 3$, while $\tilde{c} = \tilde{c}(n, p)$ is some positive constant. Consider first the case $n > 3$ and $p = \frac{2(n+1)}{n-3}$. If for $n > 3$ the exponent α satisfies inequalities (4), then $2\alpha < p = \frac{2(n+1)}{n-3}$, and we can take advantage of the well-known inequality [33, p. 258]

$$\|u\|_{L_{2\alpha}(D_T)} \leqslant (\operatorname{mes} D_T)^{\frac{p-2\alpha}{2\alpha p}} \|u\|_{L_p(D_T)},$$

from which with regard for the fact that $\operatorname{mes} D_T = \frac{\omega_n}{n+1} T^{n+1}$, where ω_n is the volume of the unit ball in R^n, by virtue of (38) we have

$$\|u\|_{L_{2\alpha}(D_T)} \leqslant \tilde{c}_1 T^{\delta_{n,\alpha}} \|u\|_{\mathring{W}_2^2(D_T, S_T)}, \quad \delta_{n,\alpha} = (n+1)\left(\frac{1}{2\alpha} - \frac{n-3}{2(n+1)}\right), \tag{39}$$

where $\tilde{c}_1 = \tilde{c}_1(n, \alpha) = \tilde{c}(\frac{\omega_n}{n+1})^{\frac{1}{2\alpha} - \frac{n-3}{2(n+1)}}$.

For $\|T_0 u\|_{L_2(D_T)}$, where $u \in \mathring{W}_2^2(D_T, S_T)$, and the operator T_0 acts by formula (35), owing to (39) we have the estimate

$$\|T_0 u\|_{L_2(D_T)} \leqslant \lambda \left[\int_{D_T} |u|^{2\alpha} \, dx \, dt \right]^{1/2} = \lambda \|u\|_{L_{2\alpha}(D_T)}^{\alpha} \leqslant \lambda \tilde{c}_1^{\alpha} T^{\alpha \delta_{n,\alpha}} \|u\|_{\mathring{W}_2^2(D_T, S_T)}^{\alpha}. \tag{40}$$

Next, from (34) and (40) for $\|Ku\|_{\mathring{W}_2^2(D_T, S_T)}$, where $Ku = L^{-1} T_0 u$, the estimate

$$\|Ku\|_{\mathring{W}_2^2(D_T, S_T)} \leqslant \|L^{-1}\|_{L_2(D_T) \to \mathring{W}_2^2(D_T, S_T)} \|T_0 u\|_{L_2(D_T)} \leqslant \lambda c_n \tilde{c}_1^{\alpha} (1+T)^4 T^{\alpha \delta_{n,\alpha}} \|u\|_{\mathring{W}_2^2(D_T, S_T)}^{\alpha}$$
$$\forall u \in \mathring{W}_2^2(D_T, S_T) \tag{41}$$

is valid. Note that $\delta_{n,\alpha} = (n+1)(\frac{1}{2\alpha} - \frac{n-3}{2(n+1)}) > 0$ for $\alpha < \frac{n+1}{n-3}$.

Consider now the equation

$$az^{\alpha} + b = z \tag{42}$$

with respect to an unknown z, where

$$a = \lambda c_n \tilde{c}_1^{\alpha} (1+T)^4 T^{\alpha \delta_{n,\alpha}}, \qquad b = c_n (1+T)^4 \|F\|_{L_2(D_T)}. \tag{43}$$

For $T > 0$, it is evident that $a > 0$ and $b \geqslant 0$. A simple analysis, similar to that carried out for $\alpha = 3$ in [32, pp. 373–374], shows that:

(1) in case $b = 0$ Eq. (42) has, along with the zero root $z_1 = 0$, the unique positive root $z_2 = a^{-\frac{1}{\alpha-1}}$;
(2) if $b > 0$, then for $0 < b < b_0$, where

$$b_0 = \left[\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right] a^{-\frac{1}{\alpha-1}}, \tag{44}$$

Eq. (42) has two positive roots z_1 and z_2, $0 < z_1 < z_2$, and for $b = b_0$ these roots merge, and we have one positive root

$$z_1 = z_2 = z_0 = (\alpha a)^{-\frac{1}{\alpha-1}};$$

(3) when $b > b_0$, Eq. (42) has no nonnegative roots.

Note that for $0 < b < b_0$ there take place the inequalities

$$z_1 < z_0 = (\alpha a)^{-\frac{1}{\alpha-1}} < z_2.$$

By (43) and (44), the condition $b \leqslant b_0$ is equivalent to the condition

$$c_n (1+T)^4 \|F\|_{L_2(D_T)} \leqslant \left(\lambda c_n \tilde{c}_1^{\alpha} (1+T)^4 T^{\alpha \delta_{n,\alpha}}\right)^{-\frac{1}{\alpha-1}} \left[\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right]$$

or

$$\|F\|_{L_2(D_T)} \leqslant \ell_{n,\lambda,\alpha} (1+T)^{-\frac{4\alpha}{\alpha-1}} T^{-\sigma_n}, \quad \sigma_n > 0, \tag{45}$$

where

$$\ell_{n,\lambda,\alpha} = \left[\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right] c_n^{-1} \left(\lambda c_n \tilde{c}_1^{\alpha}\right)^{-\frac{1}{\alpha-1}},$$

$$\sigma_n = \frac{\alpha}{\alpha-1}\delta_{n,\alpha} > 0, \quad \delta_{n,\alpha} = (n+1)\left(\frac{1}{2\alpha} - \frac{n-3}{2(n+1)}\right).$$

According to the absolute continuity of Lebesgue integral, we have $\lim_{T\to 0}\|F\|_{L_2(D_T)} = 0$. At the same time, $\lim_{T\to 0}(1+T)^{-\frac{4\alpha}{\alpha-1}} T^{-\sigma_n} = +\infty$. Therefore there exists $T_1 = T_1(F)$, $0 < T_1 < +\infty$, such that inequality (45) holds for

$$0 < T \leqslant T_1(F). \tag{46}$$

Let us now show that if the condition (46) is fulfilled, the operator $K_1 u = (Ku + u_0) : \overset{\circ}{W}{}_2^2(D_T, S_T) \to \overset{\circ}{W}{}_2^2(D_T, S_T)$ transforms the ball $B(0, z_2)$, mentioned in Remark 6, into itself, where z_2 is the maximal positive root of Eq. (42). Indeed, if $u \in B(0, z_2)$, then by virtue of (41)–(43) we have

$$\|K_1 u\|_{\overset{\circ}{W}{}_2^2(D_T,S_T)} \leqslant a\|u\|^{\alpha}_{\overset{\circ}{W}{}_2^2(D_T,S_T)} + b \leqslant a z_2^{\alpha} + b = z_2.$$

Therefore by Remark 6, Eq. (37) has a solution in the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$. The cases $n = 2, 3$ for $1 < \alpha < \infty$ are considered analogously.

Thus by Remarks 4–6, the following theorem is valid.

Theorem 2. *Let $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$. Then if the exponent α of nonlinearity in Eq.* (1) *satisfies inequalities* (4)*, and for T the condition* (46) *is fulfilled, then the problem* (1), (2) *has at least one strong generalized solution of the class W_2^2 in the domain D_T in a sense of Definition* 2*, which at the same time is, in fact, a weak generalized solution of that problem of the class W_2^2 in the domain D_T in a sense of Definition* 1.

Remark 7. In case $0 < \alpha < 1$, the problem (1), (2) may have more than one global solution. For example, for $F = 0$, the problem (1), (2) in the domain D_∞ has, besides a trivial solution, an infinite set of global linearly independent solutions $u_\sigma \in \overset{\circ}{C}{}^2(\overline{D}_\infty, S_\infty)$ depending on the parameter $\sigma \geqslant 0$ and given by formula

$$u_\sigma(x,t) = \begin{cases} \beta[(t-\sigma)^2 - |x|^2]^{\frac{2}{1-\alpha}}, & t > \sigma + |x|, \\ 0, & |x| \leqslant t \leqslant \sigma + |x|, \end{cases}$$

where $\beta = \lambda^{\frac{1}{1-\alpha}}[4k(k-1)(n+2k-1)(n+2k-3)]^{-\frac{1}{1-\alpha}}$, $k = \frac{2}{1-\alpha}$, and for $1/2 < \alpha < 1$ the function $u_\sigma \in C^4(\overline{D}_\infty)$.

Remark 8. Note that for $n = 2$ and $n = 3$, by the well-known properties [22, p. 745], [25, p. 84] of solution of the linear characteristic problem: $\Box v = g$ in D_∞, $v|_{S_\infty} = 0$, if $g \geqslant 0$, then $v \geqslant 0$, as well. Therefore for $n = 2, 3$, if $F \geqslant 0$, then the classical solution u of the problem (1), (2) satisfying, analogously to (30), the condition $\Box u|_{S_\infty} = 0$ will likewise be nonnegative. But in this case, this solution for $\alpha = 1$ will satisfy the following linear problem:

$$\Box^2 u = \lambda u + F,$$

$$u|_{S_\infty} = 0, \qquad \frac{\partial u}{\partial \nu}\Big|_{S_\infty} = 0,$$

which is globally solvable in the corresponding functional spaces.

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