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PARTIAL DIFFERENTIAL EQUATIONS

On the Solvability of the Characteristic Cauchy Problem for Some Nonlinear Wave Equations in the Future Light Cone

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Abstract—We consider some multidimensional versions of the sine-Gordon equation and the Liouville equation. For these equations, the existence or absence of a global solution of the characteristic Cauchy problem in the future light cone is studied.

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1. STATEMENT OF THE PROBLEM

Consider the nonlinear wave equation

$$\square u := \frac{\partial^2 u}{\partial t^2} - \Delta u = f(u) + F, \tag{1}$$

where f and F are given real functions; moreover, f is a nonlinear function, u is the desired unknown function, and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$, $n > 1$.

For Eq. (1), we consider the characteristic Cauchy problem on finding a solution $u(x,t)$ of this equation with the boundary condition

$$u|_{S_T} = g \tag{2}$$

in the truncated future light cone D_T: $|x| < t < T$, $x = (x_1, \ldots, x_n)$, $T = \text{const} > 0$, where g is a given real function on the characteristic conic surface S_T, $t = |x|$, $t \le T$. If $T = +\infty$, then $D_\infty :$ $t > |x|$ and $S_\infty = \partial D_\infty :\ t = |x|$.

Numerous publications (e.g., see the bibliography in [1–20]) deal with the local or global solvability of the Cauchy problem for nonlinear equations of the form (1) with the initial conditions $u|_{t=0} = u_0$ and $u_t|_{t=0} = u_1$. As to the characteristic Cauchy problem in the linear case, that is problem (1), (2) with $f = 0$, it is known that this problem is well posed and global solvability takes place in the corresponding function spaces [21–25].

In what follows, we consider special cases of a nonlinear function f corresponding to many-dimensional variants of the sine-Gordon equation and the Liouville equation. For these equations, we analyze the existence or absence of a global solution of the characteristic Cauchy problem (1), (2). Note that this problem for Eq. (1) with a power-law nonlinearity was considered in [26].

2. MANY-DIMENSIONAL VERSION OF THE SINE-GORDON EQUATION

Consider the case in which $f(u) = \lambda \sin \mu u$, where λ and μ are given nonzero real numbers. In this case, Eq. (1) acquires the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u = \lambda \sin \mu u + F, \tag{3}$$

where we have set $L = \square$ for convenience.

To simplify the exposition, in what follows, we assume that the boundary condition (2) is homogeneous; i.e.,

$$u|_{S_T} = 0. \tag{4}$$

We set $\overset{\circ}{W}{}^1_2(D_T, S_T) := \{u \in W^1_2(D_T) : \ u|_{S_T} = 0\}$, where $W^1_2(D_T)$ is the usual Sobolev space [27, p. 56] and condition (4) is treated in the sense of the theory of traces [27, p. 70].

Remark 1. Since the Nemytskii operator $N : L_2(D_T) \to L_2(D_T)$ acting by the formula $Nu = \lambda \sin \mu u$ is continuous and bounded [28, p. 349; 29, pp. 66–67 of the Russian translation] and the embedding $I : \overset{\circ}{W}{}^1_2(D_T, S_T) \to L_2(D_T)$ is a continuous compact operator [27, p. 81], it follows that

$$N_0 = NI : \overset{\circ}{W}{}^1_2(D_T, S_T) \to L_2(D_T) \tag{5}$$

is also a continuous compact operator.

Definition 1. Let $F \in L_2(D_T)$. A function $u \in \overset{\circ}{W}{}^1_2(D_T, S_T)$ is referred to as a *strong generalized solution* of the nonlinear problem (3), (4) in the domain D_T if there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T) := \{u \in C^2(\bar{D}_T) : \ u|_{S_T} = 0\}$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}^1_2(D_T, S_T)$ and $[Lu_m - \lambda \sin \mu u_m] \to F$ in the space $L_2(D_T)$. In this case, the convergence of the sequence $\{\lambda \sin \mu u_m\}$ to the function $\lambda \sin \mu u$ in the space $L_2(D_T)$ as $u_m \to u$ in the space $\overset{\circ}{W}{}^1_2(D_T, S_T)$ follows from Remark 1.

Lemma 1. *Let $F \in L_2(D_T)$. Then any strong generalized solution $u \in \overset{\circ}{W}{}^1_2(D_T, S_T)$ of problem* (3), (4) *in the domain D_T admits the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \le \sqrt{\frac{e}{2}}\, T \|F\|_{L_2(D_T)} + c(\lambda, \mu, T), \tag{6}$$

where $c(\lambda, \mu, T) = \left(\dfrac{3e|\lambda|\omega_n}{|\mu| n(n+1)} T^{n+1} \right)^{1/2}$ *and ω_n is the area of the unit sphere in R^n.*

Proof. Let $u \in \overset{\circ}{W}{}^1_2(D_T, S_T)$ be a strong generalized solution of problem (3), (4). By Definition 1 and Remark 1, there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T)$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = 0, \qquad \lim_{m\to\infty} \|Lu_m - \lambda \sin \mu u_m - F\|_{L_2(D_T)} = 0. \tag{7}$$

Consider the function $u_m \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T)$ that is found from the problem

$$Lu_m - \lambda \sin \mu u_m = F_m, \tag{8}$$

$$u_m|_{S_T} = 0. \tag{9}$$

Here

$$F_m = Lu_m - \lambda \sin \mu u_m. \tag{10}$$

By multiplying both sides of Eq. (8) by $\partial u_m / \partial t$ and by integrating the resulting relation over the domain D_τ, $0 < \tau \le T$, we obtain

$$\frac{1}{2} \int\limits_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_m}{\partial t} \right)^2 dx\,dt - \int\limits_{D_\tau} \Delta u_m \frac{\partial u_m}{\partial t} dx\,dt + \frac{\lambda}{\mu} \int\limits_{D_\tau} \frac{\partial}{\partial t} (\cos \mu u_m)\, dx\,dt = \int\limits_{D_\tau} F_m \frac{\partial u_m}{\partial t} dx\,dt. \tag{11}$$

We set $\Omega_\tau := D \cap \{t = \tau\}$, and by $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ we denote the unit outward normal to $S_T \backslash \{(0, \dots, 0)\}$. By performing integration by parts and by taking into account (9) and the relations $\nu|_{\Omega_\tau} = (0,\dots,0,1)$ and $\nu_0|_{S_\tau} = -\sqrt{2}/2$, one can readily obtain the relations

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial t}\right)^2 dx\,dt = \int\limits_{\partial D_\tau} \left(\frac{\partial u_m}{\partial t}\right)^2 \nu_0 ds = \int\limits_{\Omega_\tau} \left(\frac{\partial u_m}{\partial t}\right)^2 dx + \int\limits_{S_\tau} \left(\frac{\partial u_m}{\partial t}\right)^2 \nu_0 ds,$$

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}(\cos\mu u_m)\,dx\,dt = \int\limits_{\partial D_\tau} (\cos\mu u_m)\,\nu_0 ds = \int\limits_{\Omega_\tau} \cos\mu u_m dx - \frac{\sqrt{2}}{2}\int\limits_{S_\tau} \cos\mu u_m ds$$
$$= \int\limits_{\Omega_\tau} \cos\mu u_m dx - \frac{1}{2}\int\limits_{\Omega_\tau} dx,$$

$$\int\limits_{D_\tau} \frac{\partial^2 u_m}{\partial x_i^2}\frac{\partial u_m}{\partial t} dx\,dt = \int\limits_{\partial D_\tau} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial t}\nu_i ds - \frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_m}{\partial x_i}\right)^2 dx\,dt$$
$$= \int\limits_{\partial D_\tau} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial t}\nu_i ds - \frac{1}{2}\int\limits_{\partial D_\tau} \left(\frac{\partial u_m}{\partial x_i}\right)^2 \nu_0 ds$$
$$= \int\limits_{\partial D_\tau} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial t}\nu_i ds - \frac{1}{2}\int\limits_{S_\tau} \left(\frac{\partial u_m}{\partial x_i}\right)^2 \nu_0 ds - \frac{1}{2}\int\limits_{\Omega_\tau} \left(\frac{\partial u_m}{\partial x_i}\right)^2 dx.$$

Then relation (11) can be represented in the form

$$\int\limits_{D_\tau} F_m \frac{\partial u_m}{\partial t} dx\,dt = \int\limits_{S_\tau} \frac{1}{2\nu_0}\left[\sum_{i=1}^n \left(\frac{\partial u_m}{\partial x_i}\nu_0 - \frac{\partial u_m}{\partial t}\nu_i\right)^2 + \left(\frac{\partial u_m}{\partial t}\right)^2 \left(\nu_0^2 - \sum_{j=1}^n \nu_j^2\right)\right] ds$$
$$+ \frac{1}{2}\int\limits_{\Omega_\tau} \left[\left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_m}{\partial x_i}\right)^2\right] dx + \frac{\lambda}{\mu}\int\limits_{\Omega_\tau} \left(\cos\mu u_m - \frac{1}{2}\right) dx. \qquad (12)$$

Since $\left(\nu_0 \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial t}\right)$, $i = 1,\dots,n$, is an interior differential operator on S_τ, it follows from (9) that

$$\left.\left(\frac{\partial u_m}{\partial x_i}\nu_0 - \frac{\partial u_m}{\partial t}\nu_i\right)\right|_{S_\tau} = 0, \qquad i = 1,\dots,n.$$

By virtue of the relation

$$\left.\left(\nu_0^2 - \sum_{j=1}^n \nu_j^2\right)\right|_{S_\tau} = 0$$

valid on the characteristic surface S_τ, relation (12) acquires the form

$$\int\limits_{\Omega_\tau} \left[\left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_m}{\partial x_i}\right)^2\right] dx + \frac{2\lambda}{\mu}\int\limits_{\Omega_\tau} \left(\cos\mu u_m - \frac{1}{2}\right) dx = 2\int\limits_{D_\tau} F_m \frac{\partial u_m}{\partial t} dx\,dt. \qquad (13)$$

We use the notation

$$w(\delta) = \int\limits_{\Omega_\delta} \left[\left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial u_m}{\partial x_i}\right)^2\right] dx.$$

Then, by virtue of the inequality

$$2F_m \frac{\partial u_m}{\partial t} \le \varepsilon \left(\frac{\partial u_m}{\partial t}\right)^2 + \frac{1}{\varepsilon} F_m^2$$

valid for any $\varepsilon = \text{const} > 0$, from (13), we have

$$w(\delta) \le \varepsilon \int\limits_0^\delta w(\sigma)d\sigma + \frac{1}{\varepsilon}\|F_m\|^2_{L_2(D_\delta)} + \frac{3|\lambda|\omega_n}{|\mu|n}\delta^n, \qquad 0 < \delta \le T. \tag{14}$$

Since $\|F_m\|^2_{L_2(D_\delta)}$ is a nondecreasing function of δ, it follows from (14) and the Gronwall lemma [30, p. 13 of the Russian translation] that

$$w(\delta) \le \left[\frac{1}{\varepsilon}\|F_m\|^2_{L_2(D_\delta)} + \frac{3|\lambda|\omega_n}{|\mu|n}\delta^n\right]\exp\delta\varepsilon.$$

If $\varepsilon = 1/\delta$, then we obtain the inequality

$$w(\delta) \le e\left[\delta\|F_m\|^2_{L_2(D_\delta)} + \frac{3|\lambda|\omega_n}{|\mu|n}\delta^n\right], \tag{15}$$

which implies that

$$\begin{aligned}\|u_m\|^2_{\mathring{W}_2^1(D_T,S_T)} &= \int\limits_{D_T}\left[\left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_m}{\partial x_i}\right)^2\right]dx\,dt = \int\limits_0^T w(\delta)d\delta \\ &\le e\left[\frac{1}{2}T^2\|F_m\|^2_{L_2(D_T)} + \frac{3|\lambda|\omega_n}{|\mu|n(n+1)}T^{n+1}\right]\end{aligned}$$

and hence

$$\|u_m\|_{\mathring{W}_2^1(D_T,S_T)} \le \sqrt{\frac{e}{2}}T\|F_m\|_{L_2(D_T)} + c(\lambda,\mu,T), \tag{16}$$

where $c(\lambda,\mu,T)$ is the function defined in Lemma 1.

By using (7) and (10) and by passing in inequality (16) to the limit as $m \to \infty$, we obtain the estimate (6), which completes the proof of the lemma.

Remark 2. Note that in the linear case [$\lambda = 0$ in (3)], that is, in the case of the problem

$$Lu(x,t) = F(x,t), \qquad (x,t) \in D_T, \qquad u(x,t) = 0, \qquad (x,t) \in S_T, \tag{17}$$

for $F \in L_2(D_T)$, one can introduce the notion of a strong generalized solution $u \in \mathring{W}_2^1(D_T,S_T)$ of problem (17) in a similar way. In this case, by definition, there exists a sequence of functions $u_m \in \mathring{C}^2(\bar{D}_T,S_T)$ satisfying relation (7). It follows from the proof of Lemma 1 that the a priori estimate (6) with $\lambda = 0$, that is, the estimate

$$\|u\|_{\mathring{W}_2^1(D_T,S_T)} \le \sqrt{\frac{e}{2}}T\|F\|_{L_2(D_T)} \tag{18}$$

is also valid for a strong generalized solution of problem (17). By [26], problem (17) has a strong generalized solution in the class $\mathring{W}_2^1(D_T,S_T)$, whose uniqueness follows from the estimate (18). Consequently, the solution u of problem (17) can be represented in the form $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \mathring{W}_2^1(D_T,S_T)$ is a linear continuous operator, whose norm, by (18), admits the estimate

$$\|L^{-1}\|_{L_2(D_T)\to\mathring{W}_2^1(D_T,S_T)} \le \sqrt{\frac{e}{2}}T. \tag{19}$$

Remark 3. By virtue of (19) and Remark 1, one can readily see that if $F \in L_2(D_T)$, then for the function $u \in \mathring{W}_2^1(D_T,S_T)$ to be a strong generalized solution of problem (3), (4), it is necessary and sufficient that u is a solution of the functional equation

$$u = L^{-1}(\lambda\sin\mu u + F) \tag{20}$$

in the space $\mathring{W}_2^1(D_T,S_T)$.

We rewrite Eq. (20) in the form

$$u = Au := L^{-1}\left(N_0 u + F\right), \tag{21}$$

where, by Remark 1, the operator $N_0 : \mathring{W}_2^1\left(D_T, S_T\right) \to L_2\left(D_T\right)$ given by (5) is a continuous compact operator. At the same time, by Lemma 1, the a priori estimate $\|u\|_{\mathring{W}_2^1(D_T,S_T)} \le c_0$ with a positive constant $c_0 = c_0(\lambda, \mu, T, F)$ independent of u and τ is valid for any value of the parameter τ in the interval $[0, 1]$ and for any solution u of the parametric equation $u = \tau Au$. Therefore, by the Leray–Schauder theorem [31, p. 375], Eq. (21) and hence problem (3), (4) have at least one solution $u \in \mathring{W}_2^1\left(D_T, S_T\right)$.

Let us show that problem (3), (4) has at most one strong generalized solution in the class $\mathring{W}_2^1\left(D_T, S_T\right)$. Indeed, let u_1 and u_2 be strong generalized solutions of problem (3), (4) in the class $\mathring{W}_2^1\left(D_T, S_T\right)$; i.e., there exist sequences of functions $u_{1m}, u_{2m} \in \mathring{C}^2\left(\bar{D}_T, S_T\right)$ such that

$$\lim_{m\to\infty} \|u_{im} - u_i\|_{\mathring{W}_2^1(D_T,S_T)} = 0, \qquad \lim_{m\to\infty} \|Lu_{im} - \lambda \sin \mu u_{im} - F\|_{L_2(D_T)} = 0, \qquad i = 1, 2. \tag{22}$$

By setting $v = u_{2m} - u_{1m}$ and by using the obvious relation

$$f\left(u_{2m}\right) - f\left(u_{1m}\right) = \left(u_{2m} - u_{1m}\right) \int\limits_0^1 f'\left(\tau u_{2m} + (1-\tau)u_{1m}\right) d\tau \quad \text{for} \quad f(u) = \lambda \sin \mu u,$$

one can readily see that $v \in \mathring{C}^2\left(\bar{D}_T, S_T\right)$ is a solution of the problem

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = g_m v + F_{2m} - F_{1m}, \qquad v|_{S_T} = 0. \tag{23}$$

Here $F_{im} = Lu_{im} - \lambda \sin \mu u_{im}$, $i = 1, 2$, and

$$g_m = g_m\left(u_{1m}, u_{2m}\right) = \lambda\mu \int\limits_0^1 \cos\left(\mu\tau u_{2m} + \mu(1-\tau)u_{1m}\right) d\tau, \qquad |g_m| \le |\lambda\mu|. \tag{24}$$

By analogy with the derivation of (13), for the solution v of problem (23), one can derive the relation

$$\int\limits_{\Omega_\tau} \left[\left(\frac{\partial v}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2\right] dx = 2\int\limits_{D_\tau} g_m v \frac{\partial v}{\partial t} dx\, dt + 2\int\limits_{D_\tau} \left(F_{2m} - F_{1m}\right) \frac{\partial v}{\partial t} dx\, dt. \tag{25}$$

By taking into account the inequality in (24), one can readily see that

$$\left|2\int\limits_{D_\tau} g_m v \frac{\partial v}{\partial t} dx\, dt\right| \le \int\limits_{D_\tau} v^2 dx\, dt + \int\limits_{D_\tau} \left(\frac{\partial v}{\partial t}\right)^2 dx\, dt. \tag{26}$$

In a similar way, we obtain

$$\left|2\int\limits_{D_\tau} \left(F_{2m} - F_{1m}\right) \frac{\partial v}{\partial t} dx\, dt\right| \le \int\limits_{D_\tau} \left|F_{2m} - F_{1m}\right|^2 dx\, dt + \int\limits_{D_\tau} \left|\frac{\partial v}{\partial t}\right|^2 dx\, dt. \tag{27}$$

Since

$$v|_{S_T} = 0, \qquad v(x,t) = \int\limits_{|x|}^t \frac{\partial v(x,\tau)}{\partial t} d\tau, \qquad (x,t) \in \bar{D}_T$$

for a function $v \in \overset{\circ}{C}^2\left(\bar{D}_T, S_T\right)$, it follows that, by using standard considerations, one can obtain the inequality [27, p. 63]

$$\int\limits_{D_\tau} v^2(x,t)dx\,dt \le \tau^2 \int\limits_{D_\tau} \left(\frac{\partial v}{\partial t}\right)^2 dx\,dt. \tag{28}$$

By setting

$$w_0(\delta) = \int\limits_{\Omega_\delta} \left[(\partial v/\partial t)^2 + \sum_{i=1}^n (\partial v/\partial x_i)^2\right] dx$$

and by using (26)–(28), from (25), we obtain the inequality

$$w_0(\delta) \le \left(2+\delta^2\right) \int\limits_0^\delta w_0(\sigma)d\sigma + \|F_{2m} - F_{1m}\|^2_{L_2(D_\delta)}, \qquad 0 < \delta \le T,$$

which, together with the Gronwall lemma, implies that

$$w_0(\delta) \le \|F_{2m} - F_{1m}\|^2_{L_2(D_T)} \exp\left(2+T^2\right)T, \qquad 0 < \delta \le T. \tag{29}$$

In turn, it follows from (29) that

$$\|u_{2m} - u_{1m}\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = \|v\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = \int\limits_{D_T} \left[\left(\frac{\partial v}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2\right] dx\,dt$$

$$= \int\limits_0^T w_0(\delta)d\delta \le T\,\|F_{2m} - F_{1m}\|^2_{L_2(D_T)} \exp\left(2+T^2\right)T. \tag{30}$$

By virtue of (30) and the second inequality in (22), we have $\lim_{m\to\infty} \|u_{2m} - u_{1m}\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = 0$. This, together with the first relation in (22), implies that $\|u_2 - u_1\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = 0$; i.e., $u_2 = u_1$. This completes the proof of the uniqueness of the strong generalized solution of problem (3), (4) in the class $\overset{\circ}{W}{}^1_2\left(D_T, S_T\right)$.

Therefore, the following assertion is valid.

Theorem 1. *Let $F \in L_2\left(D_T\right)$. Then problem* (3), (4) *has a unique strong generalized solution $u \in \overset{\circ}{W}{}^1_2\left(D_T, S_T\right)$ in the domain D_T.*

Remark 4. Let $F \in L_{2,\mathrm{loc}}\left(D_\infty\right)$ and $F \in L_2\left(D_T\right)$ for any $T > 0$. We say that problem (3), (4) with $T = \infty$ has a global solution of the class $\overset{\circ}{W}{}^1_2$ in the future light cone D_∞ if there exists a function $u \in L_{2,\mathrm{loc}}\left(D_\infty\right)$ that is a strong generalized solution of problem (3), (4) of the class $\overset{\circ}{W}{}^1_2\left(D_T, S_T\right)$ in the domain D_T for any finite $T > 0$. It follows from Theorem 1 that problem (3), (4) has a unique global solution of the class $\overset{\circ}{W}{}^1_2$ in the future light cone D_∞.

3. MANY-DIMENSIONAL VERSION OF THE LIOUVILLE EQUATION

If $f(u) = \lambda \exp \mu u$, where λ and μ are given nonzero real numbers, then Eq. (1) acquires the form

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u = \lambda \exp \mu u + F. \tag{31}$$

Definition 2. Let $F \in L_2(D_T)$. A function u is called a *weak generalized solution* of problem (31), (4) in the class $\overset{\circ}{W}{}_2^1$ in the domain D_T if $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, $\exp \mu u \in L_2(D_T)$, and the integral relation

$$\int\limits_{D_T} [-u_t\varphi_t + \nabla_x u \nabla_x \varphi]\, dx\, dt = \lambda \int\limits_{D_T} \varphi \exp \mu u\, dx\, dt + \int\limits_{D_T} F\varphi\, dx\, dt \tag{32}$$

is valid for any function $\varphi \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$.

By performing integration by parts, one can readily show that a classical solution $u \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T)$ of problem (31), (4) is also a weak generalized solution of problem (31), (4) in the class $\overset{\circ}{W}{}_2^1$ in the sense of Definition 2. Conversely, if a weak generalized solution of this problem belongs to the class $C^2(\bar{D}_T)$, then it is also classical.

Definition 3. Let $F \in L_2(D_T)$. A function u is called a *strong generalized solution* of problem (31), (4) in the class $\overset{\circ}{W}{}_2^1$ in D_T if $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, $\exp \mu u \in L_2(D_T)$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\bar{D}_T, S_T)$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ and $\exp \mu u_m \to \exp \mu u$, $[Lu_m - \lambda \exp \mu u_m] \to F$ in the space $L_2(D_T)$.

Obviously, a classical solution of problem (31), (4) in the space $\overset{\circ}{C}{}^2(\bar{D}_T, S_T)$ is a strong generalized solution of this problem in the class $\overset{\circ}{W}{}_2^1$, and, in turn, the latter is a weak generalized solution of problem (31), (4) in the class $\overset{\circ}{W}{}_2^1$.

Theorem 2. *Let $F \in L_2(D_T)$. If $\lambda\mu < 0$ and $\lambda F|_{D_T} \geq 0$, then problem* (31), (4) *with $n = 2$ has a weak generalized solution in the class $\overset{\circ}{W}{}_2^1$ in the domain D_T.*

Theorem 2 is a consequence of the following assertions. First, let $\lambda > 0$ and $\mu < 0$. Instead of (31), we consider Eq. (1) with $f(u) = \lambda \exp \mu|u|$, that is,

$$Lu := \frac{\partial^2 u}{\partial t^2} - \Delta u = \lambda \exp \mu|u| + F. \tag{33}$$

By analogy with Definition 3, for problem (33), (4), we introduce the notion of a strong generalized solution in the class $\overset{\circ}{W}{}_2^1$ in the domain D_T.

Lemma 2. *Let $F \in L_2(D_T)$. Then any strong generalized solution u of problem* (33), (4) *in the class $\overset{\circ}{W}{}_2^1$ in the domain D_T satisfies the estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{\frac{e}{2}}T\|F\|_{L_2(T)} + c_1(\lambda, \mu, T), \qquad c_1(\lambda, \mu, T) = \left(\frac{e|\lambda|\omega_n}{|\mu|n(n+1)}T^{n+1}\right)^{1/2}.$$

Proof. The proof reproduces the proof of Lemma 1 almost literally. One should only use the relation

$$\begin{aligned} I_\tau = \lambda \int\limits_{D_\tau} \frac{\partial u_m}{\partial t} \exp \mu\, |u_m|\, dx\, dt &= \frac{\lambda}{\mu} \int\limits_{D_\tau} \frac{\partial}{\partial t} [(\exp \mu\, |u_m| - 1) \operatorname{sgn} u_m]\, dx\, dt \\ &= \frac{\lambda}{\mu} \int\limits_{\Omega_\tau} (\exp \mu\, |u_m| - 1) \operatorname{sgn} u_m dx \end{aligned}$$

with regard of the boundary condition (4) instead of the relation

$$\lambda \int\limits_{D_\tau} \frac{\partial u_m}{\partial t} \sin \mu u_m dx\, dt = -\frac{\lambda}{\mu} \int\limits_{\Omega_\tau} \left(\cos \mu u_m - \frac{1}{2}\right) dx$$

used in the proof of Lemma 1. Since $\mu < 0$ and $0 < \exp\mu|u_m| \le 1$, it follows that

$$|I_\delta| \le \left|\frac{\lambda}{\mu}\right| \int\limits_{\Omega_\delta} dx = \left|\frac{\lambda}{\mu}\right| \frac{\omega_n}{n}\delta^n.$$

Remark 5. Since $\mu < 0$ and $0 < \exp\mu|u| \le 1$, it follows that Remark 1 is also valid for the Nemytskii operator $\tilde{N} : L_2(D_T) \to L_2(D_T)$ acting by the formula $\tilde{N}u = \lambda\exp\mu|u|$. Therefore, by analogy with Remark 3, which implies Eq. (21), problem (33), (4) is equivalent to the operator equation $u = \tilde{A}u := L^{-1}\left(\tilde{N}Iu + F\right)$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, where $I : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T)$ is the embedding operator and $\tilde{N}I : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T)$ is a continuous compact operator. Further, by Lemma 2, each solution u of the equation $u = \tau\tilde{A}u$ with any value of the parameter τ in the interval $[0, 1]$ admits the a priori estimate $\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le c_2$ with a positive constant $c_2 = c_2(\lambda, \mu, T, F)$ independent of u and τ. Therefore, by the Leray–Schauder theorem [31, p. 375], the equation $u = \tilde{A}u$ and hence problem (33), (4) has at least one strong generalized solution u in the class $\overset{\circ}{W}{}_2^1$ in the domain D_T. Obviously, this solution is also a weak generalized solution of problem (33), (4) in the class $\overset{\circ}{W}{}_2^1$ in the domain D_T; i.e., the integral relation

$$\int\limits_{D_T} [-u_t\varphi_t + \nabla_x u\nabla_x\varphi]\,dx\,dt = \lambda\int\limits_{D_T} \varphi\exp\mu|u|dx\,dt + \int\limits_{D_T} F\varphi\,dx\,dt \tag{34}$$

is valid for any function $\varphi \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$.

Therefore, Theorem 2 will be proved for the case in which $\lambda > 0$ and $\mu < 0$ once, for $n = 2$, we show that the above-mentioned solution u of problem (33), (4) is nonnegative, i.e., $u|_{D_T} \ge 0$, since, in this case, relation (34) implies (32).

Indeed, by setting $g = \lambda\exp\mu|u| + F$, we rewrite relation (34) in the form

$$\int\limits_{D_T} [-u_t\varphi_t + \nabla_x u\nabla_x\varphi]\,dx\,dt = \int\limits_{D_T} g\varphi\,dx\,dt \quad \forall\varphi \in W_2^1(D_T), \qquad \varphi|_{t=T} = 0. \tag{35}$$

Let G_T: $|x| < T$, $0 < t < T$, be a cylinder containing D_T. We continue the functions u and g by zero outside D_T into the domain G_T and keep the previous notation for the resulting functions. Obviously, $g \in L_2(G_T)$, and since $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, we have $u \in W_{2,0}^1(G_T) := \left\{u \in W_2^1(G_T) : u|_{|x|=T} = 0\right\}$ and $u|_{t=0} = 0$. By taking into account the relations $u|_{G_T\setminus D_T} = 0$ and $g|_{G_T\setminus D_T} = 0$ and formula (35), we obtain

$$\int\limits_{G_T} [-u_t\psi_t + \nabla_x u\nabla_x\psi]\,dx\,dt = \int\limits_{G_T} g\psi\,dx\,dt \qquad \forall\psi \in \hat{W}_{2,0}^1(G_T), \tag{36}$$

where $\hat{W}_{2,0}^1(G_T) := \left\{\psi \in W_{2,0}^1(G_T) : \psi|_{t=T} = 0\right\}$. Relation (36) implies that u is a generalized solution of the mixed problem [27, p. 210]

$$\Box u = g, \qquad u|_{t=0} = 0, \qquad u_t|_{t=0} = 0, \qquad u|_{|x|=T} = 0 \tag{37}$$

in the class $W_2^1(G_T)$ in the domain G_T.

On the other hand, assuming that the function g is continued by zero outside G_T in R^3, by $\tilde{u} \in D'(R^3)$ we denote the generalized function that is a solution of the Cauchy problem

$$\Box\tilde{u} = g, \qquad \tilde{u}|_{t<0} = 0 \tag{38}$$

in R^3. Since $g|_{t<0} = 0$, we find that the solution of problem (38) in the space $D'(R^3)$ of generalized functions exists, is unique, and can be represented in the form [32, p. 225]

$$\tilde{u} = \mathscr{E}_2 * g, \tag{39}$$

where $\mathscr{E}_2(x_1, x_2, t) = \theta(t-|x|)/\left(2\pi\sqrt{t^2-|x|^2}\right)$, and θ is the Heaviside function. Since $g \in L_2(R^3)$ and $n = 2$, it follows that the function $\tilde{u}$ given by (39) is a locally integrable function that can be represented by the Poisson formula [32, p. 214] for $t > 0$:

$$\tilde{u}(x_1, x_2, t) = \frac{1}{2\pi}\int\limits_0^t \int\limits_{|\xi-x|<t-\tau} \frac{g(\xi, t)}{\sqrt{(t-\tau)^2-|\xi-x|^2}}d\xi\, d\tau, \quad t > 0 \qquad \left(\tilde{u}|_{t<0} = 0\right). \tag{40}$$

Now, by taking into account the inequalities $\lambda > 0$, $F \geq 0$, and hence by virtue of the above-performed construction, $g|_{D_T} = (\lambda \exp \mu|u|+F)|_{D_T} \geq 0$ and $g|_{R^3\setminus D_T} = 0$, and by using relation (40), we obtain $\tilde{u} \geq 0$ in R^3. Since $\tilde{u}$ is a solution of the Cauchy problem (38), where $g \in L_2(R^3)$, it follows from [33, p. 189 of the Russian translation] that $\tilde{u} \in W^1_{2,\mathrm{loc}}(R^3)$; moreover, by virtue of the relation $g|_{R^3\setminus D_T} = 0$, from (40), we have $\tilde{u}|_{G_T\setminus D_T} = 0$. Therefore, the function $\tilde{u}|_{G_T} \in W^1_2(G_T)$ is a solution of the mixed problem (37) in the sense of the integral relation (36). Now, by virtue of the uniqueness theorem for the mixed problem (37) in the class $W^1_2(G_T)$ [27, p. 210], we have $u = \tilde{u}$ in the domain G_T and hence $u = \tilde{u} \geq 0$ in the domain D_T. This completes the proof of Theorem 2 for the case in which $\lambda > 0$ and $\mu < 0$. If $\lambda < 0$ and $\mu > 0$, then the considerations are performed in a similar way.

Remark 6. Let $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$. We say that problem (31), (4) is globally solvable in the weak sense if, for any $T > 0$, this problem has a weak generalized solution of the class $\overset{\circ}{W}{}^1_2$ in the domain D_T. It follows from Theorem 2 that if the function F satisfies the inclusions $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$, then, for $\lambda\mu < 0$, $\lambda F|_{D_T} \geq 0$, and $n = 2$, problem (31), (4) is globally solvable in the weak sense. In what follows, we show that if $\lambda\mu > 0$, then problem (31), (4) is not necessarily globally solvable in the weak sense.

Theorem 3. *Let $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$. In addition, let $\lambda F|_{D_\infty} \geq 0$ and $F \not\equiv 0$; i.e., $F \neq 0$ on a set of a positive Lebesgue measure. If $\lambda\mu > 0$ and $n = 2$, then problem* (31), (4) *is not globally solvable in the weak sense*; *i.e., there exists a $T_0 = T_0(F) > 0$ such that for $T > T_0$ problem* (31), (4) *has no weak generalized solutions of the class $\overset{\circ}{W}{}^1_2$ in the domain D_T.*

Proof. We restrict our considerations to the case in which $\lambda > 0$ and $\mu > 0$, since the case in which $\lambda < 0$ and $\mu < 0$ can be considered in a similar way. Let u be a weak generalized solution of problem (31), (4) in the class $\overset{\circ}{W}{}^1_2$ in the domain D_T. Since $\lambda > 0$ and hence, by the assumptions of Theorem 3, the right-hand side $(\lambda \exp \mu u + F)$ of Eq. (31) is nonnegative in the domain D_T, it follows from the considerations performed in the proof of Theorem 2 in the case $n = 2$ that the solution u of problem (31), (4) is nonnegative in the domain D_T as well. Therefore, by using the formula $\exp \mu u = \sum_{k=0}^{\infty}(\mu u)^k/k!$, we obtain

$$\exp \mu u(x,t) > \frac{\mu^2}{2}u^2(x,t), \qquad (x,t) \in D_T, \tag{41}$$

for the case in which $\mu > 0$.

By performing integration by parts on the left-hand side in (32) and by taking into account the boundary conditions (4) and $\varphi|_{t=T} = 0$, we obtain the relation

$$\int\limits_{D_T} u\Box\varphi\, dx\, dt = \lambda \int\limits_{D_T} \varphi \exp \mu u\, dx\, dt + \int\limits_{D_T} F\varphi\, dx\, dt. \tag{42}$$

We assume that $\varphi \geq 0$ in (42) and use the Cauchy inequality with the parameter $\varepsilon > 0$:

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \qquad a \geq 0, \qquad b \geq 0,$$

where $a = |u|\varphi^{1/2}$ and $b = |\Box\varphi|/\varphi^{1/2}$; then we obtain

$$|u\Box\varphi| = |u|\varphi^{1/2}\frac{|\Box\varphi|}{\varphi^{1/2}} \leq \frac{\varepsilon}{2}|u|^2\varphi + \frac{1}{2\varepsilon}\frac{|\Box\varphi|^2}{\varphi}. \tag{43}$$

By (41), (43) and the inequality $\varphi \geq 0$, from (42), we have

$$\left(\frac{1}{2}\lambda\mu^2 - \frac{\varepsilon}{2}\right)\int\limits_{D_T} u^2\varphi\,dx\,dt \leq \frac{1}{2\varepsilon}\int\limits_{D_T}\frac{|\Box\varphi|^2}{\varphi}dx\,dt - \int\limits_{D_T} F\varphi\,dx\,dt. \tag{44}$$

If $\varepsilon = \lambda\mu^2/2$, then inequality (44) acquires the form

$$\int\limits_{D} u^2\varphi\,dx\,dt \leq \frac{4}{\lambda^2\mu^4}\int\limits_{D_T}\frac{|\Box\varphi|^2}{\varphi}dx\,dt - \frac{4}{\lambda\mu^2}\int\limits_{D_T} F\varphi\,dx\,dt. \tag{45}$$

We use the method of trial functions [12, pp. 10–12]. For the trial function in (45), we take the function

$$\varphi(x,t) = \varphi_0\left[\frac{2}{T^2}\left(t^2 + |x|^2\right)\right],$$

where $\varphi_0 \in C^2((-\infty,+\infty))$, $\varphi_0 \geq 0$, $\varphi_0' \leq 0$, $\varphi_0(\sigma) = 1$ for $0 \leq \sigma \leq 1$, and $\varphi_0(\sigma) = 0$ for $\sigma \geq 2$ [12, p. 22]. Since $\varphi(x,t) = 0$ for $r = \left(t^2 + |x|^2\right)^{1/2} \geq T$, after the change of variables $t = \sqrt{2}\,T\xi_0$, $x = \sqrt{2}\,T\xi$ $[n = 2,\ x = (x_1, x_2),\ \xi = (\xi_1, \xi_2)]$, for the first integral on the right-hand side in inequality (45), we have

$$\int\limits_{D_T}\frac{|\Box\varphi|^2}{\varphi}dx\,dt = \int\limits_{\substack{r=(t^2+|x|^2)^{1/2}\leq T,\\ t>|x|}}\frac{|\Box\varphi|^2}{\varphi}dx\,dt = \frac{1}{\sqrt{2}\,T}\varkappa_0, \tag{46}$$

where

$$\varkappa_0 = \int\limits_{\substack{1\leq|\xi_0|^2+|\xi|^2\leq 2,\\ \xi_0>|\xi|}}\frac{\left|-2\varphi_0' + 4\left(\xi_0^2 - |\xi|^2\right)\varphi_0''\right|^2}{\varphi_0}d\xi\,d\xi_0$$

is a finite quantity [12, p. 22] for an appropriate choice of the function φ_0.

Since $\varphi_0(\sigma) = 1$ for $0 \leq \sigma \leq 1$, it follows from (46) and (45) that

$$\int\limits_{\substack{r\leq T/\sqrt{2},\\ t>|x|}} u^2 dx\,dt \leq \int\limits_{D_T} u^2\varphi\,dx\,dt \leq \frac{4}{\lambda^2\mu^4}\frac{1}{\sqrt{2}\,T}\varkappa_0 - \frac{4}{\lambda\mu^2}\int\limits_{D_T} F\varphi\,dx\,dt. \tag{47}$$

Since $\lambda > 0$ and, by assumption, $F \geq 0$, $F \not\equiv 0$, and $F \in L_2(D_T)$ for any $T > 0$, it follows from the absolute continuity of the integral that $\gamma(T) = \int_{D_T} F\varphi\,dx\,dt$ is a nonnegative continuous nondecreasing function; moreover,

$$\lim_{T\to 0}\gamma(T) = 0, \qquad \lim_{T\to+\infty}\gamma(T) > 0. \tag{48}$$

One can readily see that the equation

$$\psi(T) = \frac{4}{\lambda^2\mu^4}\frac{1}{\sqrt{2}\,T}\varkappa_0 - \frac{4}{\lambda\mu^2}\gamma(T) = 0 \tag{49}$$

has a unique positive root $T = T_0 > 0$, since

$$\psi_1(T) = \frac{4}{\lambda^2 \mu^4} \frac{1}{\sqrt{2}\,T} \varkappa_0$$

is a positive continuous strictly decreasing function on the interval $(0, \infty)$; moreover,

$$\lim_{T \to 0} \psi_1(T) = +\infty, \qquad \lim_{T \to +\infty} \psi_1(T) = 0,$$

and, as was mentioned above, $\gamma(T)$ is a nonnegative continuous nondecreasing function satisfying condition (48). In addition, $\psi(T) < 0$ for $T > T_0$ and $\psi(T) > 0$ for $0 < T < T_0$. Consequently, if $T > T_0$, then the right-hand side of inequality (47) is negative, which is impossible. This completes the proof of Theorem 3 for the case in which $\lambda > 0$ and $\mu > 0$. If $\lambda < 0$ and $\mu < 0$, then, as was mentioned above, considerations are performed in a similar way.

Remark 7. It follows from the proof of Theorem 3 that if there exists a weak generalized solution of problem (31), (4) of the class $\overset{\circ}{W}{}_2^1$ in the domain D_T, then, under the assumptions of this theorem, $T \le T_0$, where T_0 is the unique positive root of Eq. (49).

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