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PARTIAL DIFFERENTIAL EQUATIONS

On the Existence and Absence of Global Solutions of the First Darboux Problem for Nonlinear Wave Equations

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Abstract—For the one-dimensional wave equation with a power-law nonlinearity, we consider the first Darboux problem, for which we study issues related to the existence and absence of local and global solutions.

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1. STATEMENT OF THE PROBLEM

In the plane of the independent variables x and t, consider the nonlinear wave equation

$$L_\lambda u := u_{tt} - u_{xx} + \lambda |u|^\alpha u = f(x,t), \tag{1}$$

where λ and α are given real constants, $\lambda\alpha \neq 0$, $\alpha > -1$, f is a given real function, and u is the unknown real function.

By $D_T := \{(x,t) : \; -kt < x < t,\ 0 < t < T;\ 0 \le k = \mathrm{const} < 1\}$, $T \le \infty$, we denote the triangular domain lying inside the characteristic angle $\{(x,t) \in \mathbb{R}^2 : \; t > |x|\}$ and bounded by the characteristic segment $\gamma_{1,T}$: $x = t$, $0 \le t \le T$ and by the segments $\gamma_{2,T}$: $x = -kt$, $0 \le t \le T$, and $\gamma_{3,T}$: $t = T$, $-kT \le x \le T$, of time and space type, respectively (Fig. 1).

For Eq. (1), we consider the first Darboux problem of finding a solution $u(x,t)$ of this equation in the domain D_T on the basis of the boundary conditions (e.g., see [1, p. 228])

$$u|_{\gamma_{i,T}} = 0, \qquad i = 1, 2. \tag{2}$$

Note that numerous papers (e.g., see [2–11]) deal with the existence or absence of global solutions of various problems (initial value problems, mixed problems, and nonlocal problems of various forms, including periodic ones) for nonlinear equations of the hyperbolic type. In the linear case, i.e., for $\lambda\alpha = 0$, problem (1), (2) is known to be well posed, and global solvability takes place in appropriate function spaces [1, 12–15].

We show that if the nonlinearity α and the parameter λ satisfy certain conditions, then problem (1), (2) is globally solvable in some cases and has no global solution in other cases, although, as shown below, this problem is locally solvable.

Definition 1. Let $f \in C\left(\bar D_T\right)$. A function u is called a *strong generalized solution* of problem (1), (2) of the class C in the domain D_T if $u \in C\left(\bar D_T\right)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2\left(\bar D_T, \Gamma_T\right)$ such that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C\left(\bar D_T\right)$ as $n \to \infty$, where $\overset{\circ}{C}{}^2\left(\bar D_T, \Gamma_T\right) := \left\{u \in C^2\left(\bar D_T\right) : \; u|_{\Gamma_T} = 0\right\}$ and $\Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

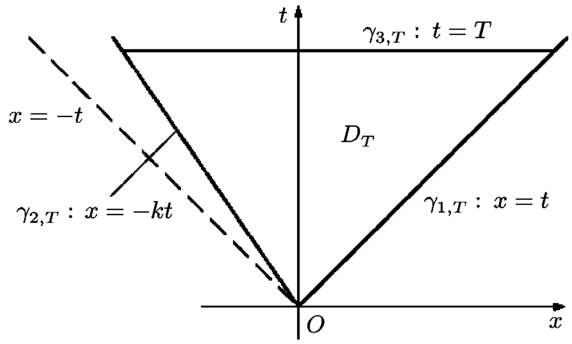

Fig. 1.

Remark 1. Obviously, a classical solution of problem (1), (2) in the space $\mathring{C}^2\left(\bar{D}_T, \Gamma_T\right)$ is a strong generalized solution of this problem of the class C in the domain D_T. In turn, if a strong generalized solution of problem (1), (2) of the class C in the domain D_T belongs to the space $C^2\left(\bar{D}_T\right)$, then it is also a classical solution of this problem.

Definition 2. Let $f \in C\left(\bar{D}_\infty\right)$. We say that problem (1), (2) is *globally solvable* in the class C if, for any finite $T > 0$, this problem has a strong generalized solution of the class C in the domain D_T.

2. A PRIORI ESTIMATE FOR THE SOLUTION OF PROBLEM (1), (2)

Lemma 1. *Let $-1 < \alpha < 0$. If $\alpha > 0$, then it is additionally assumed that $\lambda > 0$. Then any strong generalized solution of problem* (1), (2) *of the class C in the domain D_T satisfies the a priori estimate*

$$\|u\|_{C(\bar{D}_T)} \leq c_1 \|f\|_{C(\bar{D}_T)} + c_2 \tag{3}$$

with positive constants $c_i(T, \alpha, \lambda)$, $i = 1, 2$, independent of u and f.

Proof. First, consider the case in which $\alpha > 0$ and $\lambda > 0$. Let u be a strong generalized solution of problem (1), (2) of the class C in the domain D_T. Then, by Definition 1, there exists a sequence of functions $u_n \in \mathring{C}^2\left(\bar{D}_T, \Gamma_T\right)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\bar{D}_T)} = 0, \qquad \lim_{n\to\infty} \|L_\lambda u_n - f\|_{C(\bar{D}_T)} = 0, \tag{4}$$

and consequently,

$$\lim_{n\to\infty} \left\|\lambda \left|u_n\right|^\alpha u_n - \lambda |u|^\alpha u\right\|_{C(\bar{D}_T)} = 0. \tag{5}$$

Let us treat the function $u_n \in \mathring{C}^2\left(\bar{D}_T, \Gamma_T\right)$ as a solution of the problem

$$L_\lambda u_n = f_n, \tag{6}$$

$$u_n|_{\Gamma_T} = 0, \qquad \Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}. \tag{7}$$

Here

$$f_n := L_\lambda u_n. \tag{8}$$

By multiplying both sides of Eq. (6) by $\partial u_n/\partial t$ and by integrating the resulting relation over the domain $D_\tau := \{(x,t) \in D_T : \ 0 < t < \tau\}$, $0 < \tau \leq T$, we obtain

$$\frac{1}{2} \int\limits_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt - \int\limits_{D_\tau} \frac{\partial^2 u_n}{\partial x^2} \frac{\partial u_n}{\partial t} dx\,dt + \frac{\lambda}{\alpha + 2} \int\limits_{D_\tau} \frac{\partial}{\partial t} \left|u_n\right|^{\alpha+2} dx\,dt = \int\limits_{D_\tau} f_n \frac{\partial u_n}{\partial t} dx\,dt. \tag{9}$$

Set $\Omega_\tau := \bar{D}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. Then, by taking into account (7) and by integrating by parts on the left-hand side in relation (9) in the case $k \in (0,1)$, we obtain

$$\int\limits_{D_\tau} f_n \frac{\partial u_n}{\partial t} dx\, dt = \int\limits_{\Gamma_\tau} \frac{1}{2\nu_t} \left[\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x \right)^2 + \left(\frac{\partial u_n}{\partial t} \right)^2 \left(\nu_t^2 - \nu_x^2\right) \right] ds$$
$$+ \frac{1}{2} \int\limits_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial t} \right)^2 + \left(\frac{\partial u_n}{\partial x} \right)^2 \right] dx + \frac{\lambda}{\alpha+2} \int\limits_{\Omega_\tau} |u_n|^{\alpha+2}\, dx, \qquad (10)$$

where $\nu := (\nu_x, \nu_t)$ is the unit outward normal vector on ∂D_τ and $\Gamma_\tau := \Gamma_T \cap \{t \le \tau\}$.

Since $\nu_t \dfrac{\partial}{\partial x} - \nu_x \dfrac{\partial}{\partial t}$ is an interior differential operator on Γ_T, it follows from (7) that

$$\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x \right)\Bigg|_{\Gamma_\tau} = 0. \qquad (11)$$

Since D_τ: $-kt < x < t$, $0 < t < \tau$, we obviously have

$$\left(\nu_t^2 - \nu_x^2\right)\big|_{\Gamma_\tau} \le 0, \qquad \nu_t|_{\Gamma_\tau} < 0. \qquad (12)$$

(One can readily see that $(\nu_t^2 - \nu_x^2)|_{\gamma_{1,\tau}} = 0$, $(\nu_t^2 - \nu_x^2)|_{\gamma_{2,\tau}} < 0$.) If $\lambda > 0$, then, by (11) and (12), from (10), we obtain the estimate

$$w_n(\tau) := \int\limits_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial t} \right)^2 + \left(\frac{\partial u_n}{\partial x} \right)^2 \right] dx \le 2 \int\limits_{D_\tau} f_n \frac{\partial u_n}{\partial t} dx\, dt. \qquad (13)$$

One can readily see that the estimate (13) is valid for $k = 0$. By using the inequality

$$2 f_n \frac{\partial u_n}{\partial t} \le \varepsilon \left(\frac{\partial u_n}{\partial t} \right)^2 + \frac{1}{\varepsilon} f_n^2,$$

valid for any $\varepsilon = \text{const} > 0$, we obtain

$$w_n(\tau) \le \varepsilon \int\limits_0^\tau w_n(\sigma) d\sigma + \frac{1}{\varepsilon} \left\| f_n \right\|^2_{L_2(D_\tau)}, \qquad 0 < \tau \le T. \qquad (14)$$

Since, by (14), the quantity $\|f_n\|^2_{L_2(D_\tau)}$ is a nondecreasing function of τ, it follows from the Gronwall lemma (e.g., see [16, p. 13 of the Russian translation]) that

$$w_n(\tau) \le \frac{1}{\varepsilon} \left\| f_n \right\|^2_{L_2(D_\tau)} \exp(\tau\varepsilon).$$

This, together with the relation $\inf\limits_{\varepsilon>0} \dfrac{\exp(\tau\varepsilon)}{\varepsilon} = e\tau$ attained for $\varepsilon = \dfrac{1}{\tau}$, implies the estimate

$$w_n(\tau) \le e\tau \left\| f_n \right\|^2_{L_2(D_\tau)}, \qquad 0 < \tau \le T. \qquad (15)$$

If $(x,t) \in \bar{D}_T$, then, by (7), we have

$$u_n(x,t) = u_n(x,t) - u_n(-kt,t) = \int\limits_{-kt}^{x} \frac{\partial u_n(\sigma,t)}{\partial x} d\sigma,$$

which, together with (150, implies the inequalities

$$|u_n(x,t)|^2 \le \int_{-kt}^{x} d\sigma \int_{-kt}^{x} \left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \le (x+kt) \int_{\Omega_t} \left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \le (x+kt) w_n(t)$$
$$\le (1+k) t w_n(t) \le (1+k) e t^2 \|f_n\|^2_{L_2(D_t)} \le (1+k) e t^2 \|f_n\|^2_{C(\bar{D}_t)} \operatorname{mes} D_t$$
$$\le \frac{1}{2}(1+k)^2 e t^4 \|f_n\|^2_{C(\bar{D}_T)}. \tag{16}$$

It follows from (16) that

$$\|u_n\|_{C(\bar{D}_T)} \le \sqrt{\frac{e}{2}}(1+k)T^2 \|f_n\|_{C(\bar{D}_T)}. \tag{17}$$

By using (4)–(8) and by passing in inequality (17) to the limit as $n \to \infty$, we obtain the estimate

$$\|u\|_{C(\bar{D}_T)} \le \sqrt{\frac{e}{2}}(1+k)T^2 \|f\|_{C(\bar{D}_T)}. \tag{18}$$

From (18), we find that the estimate (3) is valid for $\alpha > 0$ and $\lambda > 0$.

Now consider the case in which $-1 < \alpha < 0$ for an arbitrary λ. If $-1 < \alpha < 0$, i.e., $1 < \alpha+2 < 2$, then we use the well-known inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \qquad \left(a = |u_n|^{\alpha+2}, \quad b = 1, \quad p = \frac{2}{\alpha+2} > 1, \quad q = -\frac{2}{\alpha} > 1, \quad \frac{1}{p} + \frac{1}{q} = 1\right)$$

and obtain the relations

$$\int_{\Omega_\tau} |u_n|^{\alpha+2} dx \le \int_{\Omega_\tau} \left[\frac{\alpha+2}{2}|u_n|^2 - \frac{\alpha}{2}\right] dx = \frac{\alpha+2}{2} \int_{\Omega_\tau} |u_n|^2 dx + \frac{|\alpha|}{2}(1+k)\tau. \tag{19}$$

By virtue of (11), (12), and (19), it follows from (10) that

$$\frac{1}{2}\int_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right] dx \le \frac{|\lambda|}{2} \int_{\Omega_\tau} |u_n|^2 dx + \frac{|\lambda\alpha|}{2(\alpha+2)}(1+k)\tau + \int_{D_\tau} f_n \frac{\partial u_n}{\partial t} dx\, dt. \tag{20}$$

By the trace theory, we have the estimate (e.g., see [17, pp. 77, 86])

$$\|u_n\|_{L_2(\Omega_\tau)} \le c_0(\tau) \|u_n\|_{\mathring{W}_2^1(D_\tau,\Gamma_\tau)}, \tag{21}$$

where $\mathring{W}_2^1(D_\tau,\Gamma_\tau) := \{u \in W_2^1(D_\tau) : \; u|_{\Gamma_\tau} = 0\}$, $W_2^1(D_\tau)$ is the well-known Sobolev space, and

$$\|u_n\|_{\mathring{W}_2^1(D_\tau,\Gamma_\tau)} := \int_{D_\tau} \left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right] dx\, dt.$$

Here $c_0(\tau)$ is a positive constant independent of u_n, which can be estimated from above as follows:

$$c_0(\tau) \le \sqrt{\tau}, \qquad 0 < \tau \le T. \tag{22}$$

Since $2 f_n \dfrac{\partial u_n}{\partial t} \le f_n^2 + \left(\dfrac{\partial u_n}{\partial t}\right)^2$, it follows from (21), (22), and (20) that

$$\int_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right] dx \le |\lambda|\tau \int_{D_\tau} \left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right] dx\, dt$$
$$+ \int_{D_\tau} \left(\frac{\partial u_n}{\partial t}\right)^2 dx\, dt + \int_{D_\tau} f_n^2 dx\, dt + \frac{|\lambda\alpha|}{\alpha+2}(1+k)\tau. \tag{23}$$

By using the form of the function $w_n(\tau)$, from (23), we obtain

$$w_n(\tau) \le (|\lambda|\tau + 1)\int\limits_0^\tau w_n(\sigma)d\sigma + \|f_n\|^2_{L_2(D_\tau)} + \frac{|\lambda\alpha|}{\alpha+2}(1+k)\tau,$$

which, together with the Gronwall lemma (e.g., see [16, p. 13 of the Russian translation]), implies the inequality

$$w_n(\tau) \le \left[\|f_n\|^2_{L_2(D_T)} + \frac{|\lambda\alpha|}{\alpha+2}(1+k)T\right]\exp(|\lambda|T+1)\tau. \tag{24}$$

By analogy with the derivation of (16) from (15), from (24), we obtain

$$\begin{aligned}|u_n(x,t)|^2 &\le (1+k)tw_n(t)\\ &\le (1+k)T\left[\|f_n\|^2_{C(\bar D_T)}\operatorname{mes} D_T + \frac{|\lambda\alpha|}{\alpha+2}(1+k)T\right]\exp(|\lambda|T+1)T\\ &= (1+k)T\left[\frac{1}{2}(1+k)T^2\|f_n\|^2_{C(\bar D_T)} + \frac{|\lambda\alpha|}{\alpha+2}(1+k)T\right]\exp\left(|\lambda|T^2+T\right).\end{aligned} \tag{25}$$

It follows from (25) that

$$\|u_n\|_{C(\bar D_T)} \le \left[\sqrt{\frac{T}{2}}T(1+k)\|f_n\|_{C(\bar D_T)} + \sqrt{\frac{|\lambda\alpha|}{\alpha+2}}(1+k)T\right]\exp\left\{\frac{1}{2}\left(|\lambda|T^2+T\right)\right\},$$

which, together with (4)–(8) and after the passage to the limit as $n \to \infty$, implies the estimate

$$\begin{aligned}\|u\|_{C(\bar D_T)} \le \sqrt{\frac{T}{2}}T(1+k)\exp\left\{\frac{1}{2}\left(|\lambda|c_0^2(1)T^3+T\right)\right\}\|f\|_{C(\bar D_T)}\\ + \sqrt{\frac{|\lambda\alpha|}{\alpha+2}}(1+k)T\exp\left\{\frac{1}{2}\left(|\lambda|T^2+T\right)\right\}.\end{aligned} \tag{26}$$

The proof of the estimate (3) is complete.

Remark 2. It follows from (18) and (26) that the constants c_1 and c_2 occurring in the estimate (3) have the form

$$c_1 = \sqrt{\frac{e}{2}}(1+k)T^2, \qquad c_2 = 0 \quad \text{if} \quad \alpha > 0, \quad \lambda > 0, \tag{27}$$

$$\begin{aligned}c_1 &= \sqrt{\frac{T}{2}}T(1+k)\exp\left\{\frac{1}{2}\left(|\lambda|T^2+T\right)\right\},\\ c_2 &= \sqrt{\frac{|\lambda\alpha|}{\alpha+2}}(1+k)T\exp\left\{\frac{1}{2}\left(|\lambda|T^2+T\right)\right\} \quad \text{if} \quad -1<\alpha<0, \quad -\infty<\lambda<+\infty.\end{aligned} \tag{28}$$

3. EQUIVALENT REDUCTION OF PROBLEM (1), (2) TO A NONLINEAR INTEGRAL VOLTERRA EQUATION

Let $P_0 := P_0(x_0,t_0)$ be an arbitrary point of the domain D_T. By G_{x_0,t_0} we denote the characteristic quadrangle with vertices at the point $P_0(x_0,t_0)$ and at the points P_1, P_2, and P_3 lying on the supports of the data $\gamma_{2,T}$ and $\gamma_{1,T}$, respectively; i.e., (see Fig. 2)

$$\begin{aligned}P_1 &:= P_1\left(\frac{k(x_0-t_0)}{k+1}, \frac{t_0-x_0}{k+1}\right),\\ P_2 &:= P_2\left(\frac{(1-k)(t_0-x_0)}{2(1+k)}, \frac{(1-k)(t_0-x_0)}{2(1+k)}\right),\\ P_3 &:= P_3\left(\frac{x_0+t_0}{2}, \frac{x_0+t_0}{2}\right).\end{aligned}$$

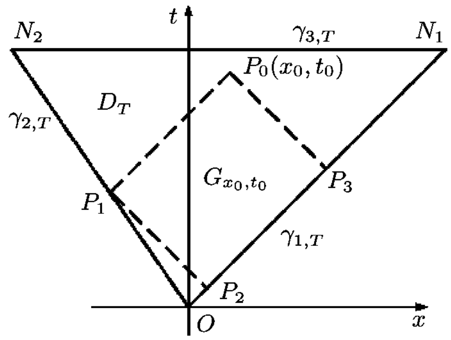

Fig. 2.

Let $u \in C^2\left(\bar{D}_T\right)$ be a classical solution of problem (1), (2). By integrating Eq. (1) over the domain G_{x_0,t_0}, which is the characteristic quadrangle of this equation, by using the homogeneous boundary conditions (2), and by returning to the original variables x and t, one can readily obtain the relation

$$u(x,t) + \frac{\lambda}{2} \int\limits_{G_{x,t}} |u|^\alpha u \, dx' dt' = \frac{1}{2} \int\limits_{G_{x,t}} f\left(x', t'\right) dx' dt', \qquad (x,t) \in \bar{D}_T. \tag{29}$$

Remark 3. Relation (29) can be treated as a nonlinear integral Volterra equation, which can be represented in the form

$$u(x,t) + \lambda \left(L_0^{-1} |u|^\alpha u\right)(x,t) = F(x,t), \qquad (x,t) \in \bar{D}_T. \tag{30}$$

Here L_0^{-1} is the linear operator acting by the formula

$$\left(L_0^{-1} v\right)(x,t) := \frac{1}{2} \int\limits_{G_{x,t}} v\left(x', t'\right) dx' dt', \qquad (x,t) \in \bar{D}_T, \tag{31}$$

and

$$F(x,t) := \left(L_0^{-1} f\right)(x,t), \qquad (x,t) \in \bar{D}_T. \tag{32}$$

Lemma 2. *A function $u \in C\left(\bar{D}_T\right)$ is a strong generalized solution of problem* (1), (2) *of the class C in the domain D_T if and only if it is a continuous solution of the nonlinear integral equation* (30).

Proof. Indeed, let $u \in C\left(\bar{D}_T\right)$ be a solution of Eq. (30). Since $f \in C\left(\bar{D}_T\right)$, and the space $C^2\left(\bar{D}_T\right)$ is dense in $C\left(\bar{D}_T\right)$ (e.g., see [18, p. 37 of the Russian translation]), it follows that there exists a function sequence $f_n \in C^2\left(\bar{D}_T\right)$ such that $f_n \to f$ in the space $C\left(\bar{D}_T\right)$ as $n \to \infty$. Likewise, since $u \in C\left(\bar{D}_T\right)$, it follows that there exists a function sequence $w_n \in C^2\left(\bar{D}_T\right)$ such that $w_n \to u$ in the space $C\left(\bar{D}_T\right)$ as $n \to \infty$. We set

$$u_n := -\lambda \left(L_0^{-1} \left|w_n\right|^\alpha w_n\right) + L_0^{-1} f_n, \qquad n = 1, 2, \ldots$$

One can readily see that $u_n \in \overset{\circ}{C}{}^2\left(\bar{D}_T, \Gamma_T\right)$; and since L_0^{-1} is a linear continuous operator acting in the space $C\left(\bar{D}_T\right)$, and moreover, $\lim_{n\to\infty} \|w_n - u\|_{C(\bar{D}_T)} = 0$, $\lim_{n\to\infty} \|f_n - f\|_{C(\bar{D}_T)} = 0$, we have $u_n \to -\lambda\left(L_0^{-1}|u|^\alpha u\right) + L_0^{-1} f$ in the space $C\left(\bar{D}_T\right)$ as $n \to \infty$. But it follows from (30) that $-\lambda\left(L_0^{-1}|u|^\alpha u\right) + L_0^{-1} f = u$. Therefore, we have $\lim_{n\to\infty} \|u_n - u\|_{C(\bar{D}_T)} = 0$. On the other

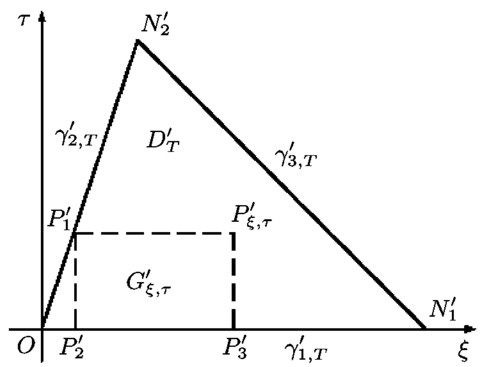

Fig. 3.

hand, $L_0 u_n = -\lambda |w_n|^\alpha w_n + f_n$, which, together with the relations $\lim_{n\to\infty} \|u_n - u\|_{C(\bar D_T)} = 0$, $\lim_{n\to\infty} \|w_n - u\|_{C(\bar D_T)} = 0$, and $\lim_{n\to\infty} \|f_n - f\|_{C(\bar D_T)} = 0$, implies that

$$\begin{aligned} L_\lambda u_n &= L_0 u_n + \lambda |u_n|^\alpha u_n = -\lambda |w_n|^\alpha w_n + f_n + \lambda |u_n|^\alpha u_n \\ &= -\lambda \left[|w_n|^\alpha w_n - |u|^\alpha u\right] + \lambda \left[|u_n|^\alpha u_n - |u|^\alpha u\right] + f_n \to f \end{aligned}$$

in the space $C\left(\bar D_T\right)$ as $n \to \infty$. The converse is obvious.

4. THE CASE OF GLOBAL SOLVABILITY OF PROBLEM (1), (2) IN THE CLASS OF CONTINUOUS FUNCTIONS

As was mentioned above, the operator L_0^{-1} occurring in (31) is a linear continuous operator acting in the space $C\left(\bar D_T\right)$. Let us now show that this operator is actually a linear continuous mapping of the space $C\left(\bar D_T\right)$ into the space $C^1\left(\bar D_T\right)$ of continuously differentiable functions. To this end, we use the linear nonsingular transformation $t = \xi + \tau$, $x = \xi - \tau$ of independent variables and pass into the plane of the variables ξ, τ. After that the triangular domain D_T becomes the triangle D_T' with vertices at the points $O(0,0)$, $N_1'(T,0)$, and $N_2'\left(\frac{1-k}{2}T, \frac{1+k}{2}T\right)$, and the characteristic quadrangle $G_{x,t}$ occurring in Section 3 becomes the rectangle $G_{x,t}'$ with vertices

$$P'\left(\frac{t+x}{2}, \frac{t-x}{2}\right), \quad P_1'\left(\frac{(1-k)(t-x)}{2(1+k)}, \frac{t-x}{2}\right), \quad P_2'\left(\frac{(1-k)(t-x)}{2(1+k)}, 0\right), \quad P_3'\left(\frac{t+x}{2}, 0\right),$$

i.e., in the variables ξ and τ, the rectangle $G_{\xi,\tau}'$ $(= G_{x,t}')$ with vertices

$$P'(\xi,\tau), \quad P_1'\left(\frac{1-k}{1+k}\tau, \tau\right), \quad P_2'\left(\frac{1-k}{1+k}\tau, 0\right),$$

and $P_3'(\xi, 0)$ (Fig. 3). Moreover, the operator L_0^{-1} occurring in (31) becomes the operator $\left(L_0^{-1}\right)'$ acting in the space $C\left(\bar D_T'\right)$ by the formula

$$\left(\left(L_0^{-1}\right)' w\right)(\xi,\tau) = \int\limits_{G_{\xi,\tau}'} w\left(\xi', \tau'\right) d\xi' d\tau' = \int\limits_{(1-k)\tau/(1+k)}^{\xi} d\xi' \int\limits_0^\tau w\left(\xi', \tau'\right) d\tau', \qquad (\xi,\tau) \in \bar D_T'. \tag{33}$$

If $w \in C\left(\bar D_T'\right)$, then it readily follows from (33) that

$$\frac{\partial}{\partial \xi}\left(\left(L_0^{-1}\right)' w\right)(\xi,\tau) = \int\limits_0^\tau w\left(\xi, \tau'\right) d\tau', \qquad (\xi,\tau) \in \bar D_T', \tag{34}$$

$$\frac{\partial}{\partial \tau}\left(\left(L_0^{-1}\right)' w\right)(\xi,\tau) = \frac{k-1}{1+k}\int\limits_0^\tau w\left(\xi', \frac{1-k}{1+k}\tau\right) d\xi' + \int\limits_{(1-k)\tau/(1+k)}^{\xi} w\left(\xi',\tau\right) d\xi', \qquad (\xi,\tau)\in \bar{D}'_T. \tag{35}$$

Now, since $0 \le \xi \le T$ and $0 \le \tau \le \dfrac{1+k}{2}T$ for $(\xi,\tau) \in \bar{D}'_T$, it follows from (33)–(35) and the inequality $0 \le k < 1$ that

$$\begin{aligned}\left\|\left(L_0^{-1}\right)' w\right\|_{C(\bar{D}'_T)} &+ \left\|\frac{\partial}{\partial \xi}\left(L_0^{-1}\right)' w\right\|_{C(\bar{D}'_T)} + \left\|\frac{\partial}{\partial \tau}\left(L_0^{-1}\right)' w\right\|_{C(\bar{D}'_T)} \\ &\le \left(\xi - \frac{1-k}{1+k}\tau\right)\tau\|w\|_{C(\bar{D}'_T)} + \tau\|w\|_{C(\bar{D}'_T)} + \frac{1-k}{1+k}\tau\|w\|_{C(\bar{D}'_T)} \\ &\quad + \left(\xi - \frac{1-k}{1+k}\tau\right)\|w\|_{C(\bar{D}'_T)} \le \left(T^2+2T\right)\|w\|_{C(\bar{D}'_T)};\end{aligned}$$

i.e.,

$$\left\|\left(L_0^{-1}\right)'\right\|_{C(\bar{D}'_T)\to C^1(\bar{D}'_T)} \le T^2+2T, \tag{36}$$

which completes the proof.

Further, since the space $C^1\left(\bar{D}'_T\right)$ is compactly embedded in the space $C\left(\bar{D}'_T\right)$ (e.g., see [19, p. 135 of the Russian translation]), it follows from (36) that the operator $\left(L_0^{-1}\right)' : C\left(\bar{D}'_T\right) \to C\left(\bar{D}'_T\right)$ is a linear compact operator. Therefore, by returning from the variables ξ and τ to the variables x and t for the operator L_0^{-1} in (31), we obtain the following assertion.

Lemma 3. *The operator $L_0^{-1} : C\left(\bar{D}_T\right) \to C\left(\bar{D}_T\right)$ given by* (31) *is a linear compact operator and maps the space $C\left(\bar{D}_T\right)$ into the space $C^1\left(\bar{D}_T\right)$.*

By using (32), we rewrite Eq. (30) in the form

$$u = Au := L_0^{-1}\left(-\lambda|u|^\alpha u + f\right), \tag{37}$$

where $A : C\left(\bar{D}_T\right) \to C\left(\bar{D}_T\right)$ is a continuous compact operator, since the nonlinear operator $K : C\left(\bar{D}_T\right) \to C\left(\bar{D}_T\right)$ acting by the formula $Ku := -\lambda|u|^\alpha u + f$ is bounded and continuous for $\alpha > -1$, and, by virtue of Lemma 3, the linear operator $L_0^{-1} : C\left(\bar{D}_T\right) \to C\left(\bar{D}_T\right)$ is a compact operator. At the same time, by Lemmas 1 and 2 and relations (27) and (28), the a priori estimate

$$\|u\|_{C(\bar{D}_T)} \le c\|f\|_{C(\bar{D}_T)} + \tilde{c}$$

with positive constants c and $\tilde{c}$ independent of u, τ, and f is valid for any parameter $\tau \in [0,1]$ and any solution $u \in C\left(\bar{D}_T\right)$ of the equation $u = \tau Au$. Therefore, by the Leray–Schauder theorem (e.g., see [20, p. 375]), Eq. (37) has at least one solution $u \in C\left(\bar{D}_T\right)$ under the assumptions of Lemma 1. Therefore, by virtue of Lemma 2, we have justified the following assertion.

Theorem 1. *Let $-1 < \alpha < 0$, and let $\lambda > 0$ if $\alpha > 0$. Then problem* (1), (2) *is globally solvable in the class C in the sense of Definition* 2; *i.e., if $f \in C\left(\bar{D}_\infty\right)$, then for any $T > 0$, problem* (1), (2) *has a strong generalized solution of the class C in the domain D_T.*

5. SMOOTHNESS AND UNIQUENESS OF THE SOLUTION OF PROBLEM (1), (2). THE EXISTENCE OF A GLOBAL SOLUTION IN D_∞

Relations (30)–(32), together with Lemmas 2 and 3, readily imply the following assertion.

Lemma 4. *Let u be a strong generalized solution of problem* (1), (2) *of the class C in the domain D_T in the sense of Definition* 1. *If $\alpha > 0$ and $f \in C^1(\bar{D}_T)$, then $u \in C^2(\bar{D}_T)$; therefore, it is a classical solution of this problem.*

Lemma 5. *If $\alpha > 0$, then problem* (1), (2) *has at most one strong generalized solution of the class C in the domain D_T.*

Proof. Indeed, suppose that problem (1), (2) has two strong generalized solutions u_1 and u_2 of the class C in the domain D_T. By Definition 1, there exists a sequence of functions $u_{in} \in \mathring{C}^2(\bar{D}_T, \Gamma_T)$, $i = 1, 2$, such that

$$\lim_{n\to\infty} \|u_{in} - u_i\|_{C(\bar{D}_T)} = 0, \qquad \lim_{n\to\infty} \|L_\lambda u_{in} - f\|_{C(\bar{D}_T)} = 0, \qquad i = 1, 2. \tag{38}$$

We set $\omega_{nm} := u_{2n} - u_{1m}$. One can readily see that the function $\omega_{nm} \in \mathring{C}^2(\bar{D}_T, \Gamma_T)$ is a classical solution of the problem

$$L_0\omega_{nm} + g_{nm}\omega_{nm} = f_{nm}, \tag{39}$$

$$\omega_{nm}|_{\Gamma_T} = 0, \qquad \Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}. \tag{40}$$

Here

$$g_{nm} := \lambda(1+\alpha)\int_0^1 |u_{1m} + t(u_{2n} - u_{1m})|^\alpha\, dt, \tag{41}$$

$$f_{nm} := L_\lambda u_{2n} - L_\lambda u_{1m}, \tag{42}$$

where we have used the obvious relation

$$\varphi(x_2) - \varphi(x_1) = (x_2 - x_1)\int_0^1 \varphi'(x_1 + t(x_2 - x_1))\, dt$$

for the function $\varphi(x) := |x|^\alpha x$ with $x_2 = u_{2n}$, $x_1 = u_{1m}$, and $\alpha > 0$. By virtue of the first relation in (38), there exists a number $M := \text{const} > 0$, independent of indices i and n, such that $\|u_{in}\|_{C(\bar{D}_T)} \le M$, which, in turn, by (41), implies the estimate

$$\|g_{n,m}\|_{C(\bar{D}_T)} \le |\lambda|(1+\alpha)M^\alpha \qquad \forall n, m. \tag{43}$$

By (42), it follows from the second relation in (38) that

$$\lim_{n,m\to\infty} \|f_{nm}\|_{C(\bar{D}_T)} = 0. \tag{44}$$

If we multiply both sides of Eq. (39) by $\partial\omega_{nm}/\partial t$, integrate the resulting relation over the domain $D_\tau := \{(x,t) \in D_T : \ 0 < t < \tau\}$, $0 < \tau \le T$, use the boundary conditions (40), and follow the derivation of inequality (13) from (6) and (7), then we obtain

$$\int_{\Omega_\tau} \left[\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2 + \left(\frac{\partial\omega_{nm}}{\partial x}\right)^2\right] dx \le 2\int_{D_\tau} (f_{nm} - g_{nm}\omega_{nm})\frac{\partial\omega_{nm}}{\partial t} dx\, dt, \tag{45}$$

where $\Omega_\tau := \bar{D}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$.

By virtue of the estimate (43) and the Cauchy inequality, we obtain the relations

$$
\begin{aligned}
2\int\limits_{D_\tau}(f_{nm}-g_{nm}\omega_{nm})\frac{\partial\omega_{nm}}{\partial t}dx\,dt &\le \int\limits_{D_\tau}\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2dx\,dt+\int\limits_{D_\tau}(f_{nm}-g_{nm}\omega_{nm})^2\,dx\,dt\\
&\le \int\limits_{D_\tau}\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2dx\,dt+2\int\limits_{D_\tau}f_{nm}^2dx\,dt+2\int\limits_{D_\tau}g_{nm}^2\omega_{nm}^2dx\,dt\\
&\le \int\limits_{D_\tau}\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2dx\,dt+2\int\limits_{D_\tau}f_{nm}^2dx\,dt+2\lambda^2(1+\alpha)^2M^{2\alpha}\int\limits_{D_\tau}\omega_{nm}^2dx\,dt. \qquad (46)
\end{aligned}
$$

Further, by applying standard considerations to the relations

$$
\omega_{nm}|_{\Gamma_T}=0,\qquad \omega_{nm}(x,t)=\int\limits_{\psi(x)}^{t}\frac{\partial\omega_{nm}(x,\tau)}{\partial t}d\tau,\qquad (x,t)\in\bar{D}_T,
$$

where $t-\psi(x)=0$ for $k\ne 0$ is an equation of the support $\Gamma_T:=\gamma_{1,T}\cup\gamma_{2,T}$ of data of problem (1), (2), we obtain the inequality (e.g., see [17, p. 63])

$$
\int\limits_{D_\tau}\omega_{nm}^2dx\,dt\le\tau^2\int\limits_{D_\tau}\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2dx\,dt. \qquad (47)
$$

If $k=0$, then, in a similar way, we obtain (47). By setting

$$
w_{nm}(\tau):=\int\limits_{\Omega_\tau}\left[\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2+\left(\frac{\partial\omega_{nm}}{\partial x}\right)^2\right]dx
$$

and by using (46) and (47), from inequality (45), we obtain

$$
\begin{aligned}
w_{nm}(\tau)&\le\left[1+2\lambda^2(1+\alpha)^2M^{2\alpha}\tau^2\right]\int\limits_{D_\tau}\left(\frac{\partial\omega_{nm}}{\partial t}\right)^2dx\,dt+2\int\limits_{D_\tau}f_{nm}^2dx\,dt\\
&\le\left[1+2\lambda^2(1+\alpha)^2M^{2\alpha}T^2\right]\int\limits_0^\tau w_{nm}(\sigma)d\sigma+2\int\limits_{D_T}f_{nm}^2dx\,dt. \qquad (48)
\end{aligned}
$$

This, together with the Gronwall lemma (e.g., see [16, p. 13 of the Russian translation]), implies the estimate

$$
w_{nm}(\tau)\le c\,\|f_{nm}\|^2_{L_2(D_T)},\qquad 0<\tau\le T, \qquad (49)
$$

where $c:=2\exp\left[1+2\lambda^2(1+\alpha)^2M^{2\alpha}T^2\right]T$.

By performing the same considerations as those used in the derivation of inequality (16), by taking into account the obvious inequality

$$
\|f_{nm}\|^2_{L_2(D_T)}\le\|f_{nm}\|^2_{C(\bar{D}_T)}\operatorname{mes}D_T
$$

and by using (49), we obtain

$$
\begin{aligned}
|\omega_{nm}(x,t)|^2&\le(1+k)tw_{nm}(t)\le(1+k)Tc\operatorname{mes}D_T\,\|f_{nm}\|^2_{C(\bar{D}_T)}\\
&=\frac{c}{2}(1+k)^2T^3\,\|f_{nm}\|^2_{C(\bar{D}_T)},\qquad (x,t)\in\bar{D}_T. \qquad (50)
\end{aligned}
$$

It follows from (50) that

$$\|\omega_{nm}\|_{C(\bar{D}_T)} \leq \sqrt{\frac{cT}{2}}T(1+k)\,\|f_{nm}\|_{C(\bar{D}_T)}\,. \tag{51}$$

Since $\omega_{nm} := u_{2n} - u_{1m}$, it follows from the first relation in (38) that

$$\lim_{n,m\to\infty} \|\omega_{nm}\|_{C(\bar{D}_T)} = \|u_2 - u_1\|_{C(\bar{D}_T)}\,. \tag{52}$$

By using (44) and (52) and by passing in inequality (51) to the limit as $n, m \to \infty$, we obtain the relation $\|u_2 - u_1\|_{C(\bar{D}_T)} = 0$, i.e., $u_1 = u_2$, which completes the proof of Lemma 5.

Theorem 2. *Let $\alpha > 0$ and $\lambda > 0$. Then, for any function $f \in C^1\left(\bar{D}_\infty\right)$, problem* (1), (2) *has a unique global classical solution $u \in \overset{\circ}{C}{}^2\left(\bar{D}_\infty, \Gamma_\infty\right)$ in the domain D_∞.*

Proof. If $\alpha > 0$, $\lambda > 0$, and $f \in C^1\left(\bar{D}_\infty\right)$, then, by Theorem 1 and Lemmas 4 and 5, in the domain D_T with $T = n$, there exists a unique classical solution $u_n \in \overset{\circ}{C}{}^2\left(\bar{D}_n, \Gamma_n\right)$ of problem (1), (2). Since u_{n+1} is a classical solution of problem (1), (2) in the domain D_n as well, it follows from Lemma 5 that $u_{n+1}|_{D_n} = u_n$. Therefore, the function u constructed in the domain D_∞ by the rule $u(x,t) = u_n(x,t)$ for $n = [t] + 1$, where $[t]$ is the integer part of the number t and $(x,t) \in D_\infty$, is the unique classical solution of problem (1), (2) of the class $\overset{\circ}{C}{}^2\left(\bar{D}_\infty, \Gamma_\infty\right)$ in the domain D_∞. The proof of Theorem 2 is complete.

6. THE CASE OF ABSENCE OF A GLOBAL SOLUTION OF PROBLEM (1), (2)

In what follows, we consider the case in which $\lambda < 0$ in Eq. (1) and the nonlinearity exponent satisfies $\alpha > 0$.

Lemma 6. *Let u be a strong generalized solution of problem* (1), (2) *of the class C in the domain D_T in the sense of Definition* 1. *Then one has the integral relation*

$$\int\limits_{D_T} u\Box\varphi\,dx\,dt = -\lambda\int\limits_{D_T} |u|^\alpha u\varphi\,dx\,dt + \int\limits_{D_T} f\varphi\,dx\,dt \tag{53}$$

for any function φ such that

$$\varphi \in C^2\left(\bar{D}_T\right), \qquad \varphi|_{t=T} = 0, \qquad \varphi_t|_{t=T} = 0, \qquad \varphi|_{\gamma_{2,T}} = 0, \tag{54}$$

where $\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$.

Proof. By the definition of a strong generalized solution u of problem (1), (2) of the class C in the domain D_T, $u \in C\left(\bar{D}_T\right)$, and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2\left(\bar{D}_T, \Gamma_T\right)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\bar{D}_T)} = 0, \qquad \lim_{n\to\infty} \|L_\lambda u_n - f\|_{C(\bar{D}_T)} = 0. \tag{55}$$

We set $f_n := L_\lambda u_n$. We multiply both sides of the relation $L_\lambda u_n = f_n$ by the function φ and integrate the resulting relation over the domain D_T. After the integration of the left-hand side of this relation by parts in view of (54) and the boundary conditions $u_n|_{\gamma_i,T} = 0$, $i = 1, 2$, we obtain

$$\int\limits_{D_T} u_n\Box\varphi\,dx\,dt = -\lambda\int\limits_{D_T} |u_n|^\alpha\,u_n\varphi\,dx\,dt + \int\limits_{D_T} f_n\varphi\,dx\,dt. \tag{56}$$

By using (55) and by passing to the limit as $n \to \infty$ in (56), we obtain (53). The proof of Lemma 6 is complete.

Lemma 7. *Let $\lambda < 0$ and $\alpha > 0$, and let the function $u \in C\left(\bar{D}_T\right)$ be a strong generalized solution of problem* (1), (2) *of the class C in the domain D_T. If $f \geq 0$ in the domain D_T, then $u \geq 0$ in the domain D_T.*

Proof. By Lemma 2 and relations (30)–(32), the function u is a solution of the Volterra integral equation

$$u(x,t) = \int\limits_{G_{x,t}} K\left(x',t'\right) u\left(x',t'\right) dx'dt' + F(x,t), \quad (x,t) \in \bar{D}_T. \tag{57}$$

Here $K(x,t) := -(\lambda/2)|u(x,t)|^\alpha \in C\left(\bar{D}_T\right)$, $F(x,t) := (1/2)\int_{G_{x,t}} f\left(x',t'\right) dx'dt'$, and, by virtue of the assumptions of Lemma 7,

$$K(x,t) \geq 0, \qquad F(x,t) \geq 0 \qquad \forall (x,t) \in \bar{D}_T. \tag{58}$$

We assume that $K(x,t)$ is a given function and consider the linear integral Volterra equation

$$v(x,t) = \int\limits_{G_{x,t}} K\left(x',t'\right) v\left(x',t'\right) dx'dt' + F(x,t), \qquad (x,t) \in \bar{D}_T, \tag{59}$$

for the unknown function $v(x,t)$ in the class $C\left(\bar{D}_T\right)$. It is known (e.g., see [15]) that, in the class $C\left(\bar{D}_T\right)$, Eq. (59) has a unique continuous solution $v(x,t)$, which can be obtained by the successive approximation method

$$\begin{aligned} v_0(x,t) &= 0, \\ v_{n+1}(x,t) &= \int\limits_{G_{x,t}} K\left(x',t'\right) v_n\left(x',t'\right) dx'dt' + F(x,t), \qquad n \geq 1, \qquad (x,t) \in \bar{D}_T. \end{aligned} \tag{60}$$

By virtue of (58), from (60), we have $v_n(x,t) \geq 0$ in $\bar{D}_T$ for all $n = 0, 1, \ldots$ But $v_n \to v$ in the class $C\left(\bar{D}_T\right)$ as $n \to \infty$. Therefore, the limit function satisfies $v \geq 0$ in the domain D_T. It remains to note that, by virtue of relation (57), the function u is also a solution of Eq. (59); therefore, by virtue of the uniqueness of the solution of this equation, we finally obtain $u = v \geq 0$ in the domain D_T. The proof of Lemma 7 is complete.

If $\lambda < 0$, then, by virtue of Lemma 7, Eq. (53) can be represented in the form

$$\int\limits_{D_T} |u| \Box\varphi \, dx\, dt = |\lambda| \int\limits_{D_T} |u|^{\alpha+1} \varphi \, dx\, dt + \int\limits_{D_T} f\varphi \, dx\, dt. \tag{61}$$

Let us introduce a function $\varphi^0 := \varphi^0(x,t)$ such that

$$\varphi^0 \in C^2\left(\bar{D}_\infty\right), \qquad \varphi^0\Big|_{D_{T=1}} > 0, \qquad \varphi^0\Big|_{\gamma_{2,\infty}} = 0, \qquad \varphi^0\Big|_{t \geq 1} = 0, \tag{62}$$

$$\varkappa_0 := \int\limits_{D_{T=1}} \frac{\left|\Box\varphi^0\right|^{p'}}{\left|\varphi^0\right|^{p'-1}} dx\, dt < +\infty, \quad p' = 1 + \frac{1}{\alpha}. \tag{63}$$

One can readily see that a function φ^0 satisfying conditions (62) and (63) can be chosen in the form

$$\varphi^0(x,t) = \begin{cases} (x+kt)^n (1-t)^m & \text{if} \quad (x,t) \in D_{T=1} \\ 0 & \text{if} \quad t \geq 1 \end{cases}$$

for sufficiently large positive constants n and m.

By setting $\varphi_T(x,t) := \varphi^0\left(\frac{x}{T}, \frac{t}{T}\right)$, $T > 0$, and by using (62), we obtain

$$\varphi_T \in C^2\left(\bar{D}_T\right), \qquad \varphi_T|_{D_T} > 0, \qquad \varphi_T|_{\gamma_{2,T}} = 0, \qquad \varphi_T|_{t=T} = 0, \qquad \left.\frac{\partial \varphi_T}{\partial t}\right|_{t=T} = 0. \tag{64}$$

We assume that f is a fixed function and introduce the following function of a single variable T:

$$\zeta(T) := \int\limits_{D_T} f\varphi_T dx\,dt, \qquad T > 0. \tag{65}$$

We have the following assertion on the absence of global solvability of problem (1), (2).

Theorem 3. *Let $\lambda < 0$, $\alpha > 0$, $f \in C\left(\bar{D}_\infty\right)$, and $f \ge 0$ in the domain D_∞. If*

$$\liminf_{T\to+\infty} \zeta(T) > 0, \tag{66}$$

then there exists a positive number $T_0 := T_0(f)$ such that for $T > T_0$ problem (1), (2) *has no strong generalized solution u of the class C in the domain D_T.*

Proof. Suppose that, under the assumptions of this theorem, there exists a strong generalized solution u of problem (1), (2) of the class C in the domain D_T. Then, by virtue of Lemmas 6 and 7, we have relation (61), where, by virtue of (64), the function φ can be chosen in the form $\varphi = \varphi_T$, i.e.,

$$\int\limits_{D_T} |u|\Box\varphi_T dx\,dt = |\lambda| \int\limits_{D_T} |u|^p \varphi_T dx\,dt + \int\limits_{D_T} f\varphi_T\,dx\,dt, \qquad p := \alpha + 1. \tag{67}$$

By using (65), we rewrite relation (67) in the form

$$|\lambda| \int\limits_{D_T} |u|^p \varphi_T dx\,dt = \int\limits_{D_T} |u|\Box\varphi_T dx\,dt - \zeta(T). \tag{68}$$

If we set $a = |u|\varphi_T^{1/p}$, $b = |\Box\varphi_T|/\varphi_T^{1/p}$ in the Young inequality

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{p'\varepsilon^{p'-1}}b^{p'}, \quad a, b \ge 0, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \qquad p := \alpha + 1 > 1,$$

with the parameter $\varepsilon > 0$, then, by using the relation $p'/p = p' - 1$, we obtain

$$|u\Box\varphi_T| = |u|\varphi_T^{1/p}\frac{|\Box\varphi_T|}{\varphi_T^{1/p}} \le \frac{\varepsilon}{p}|u|^p\varphi_T + \frac{1}{p'\varepsilon^{p'-1}}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}. \tag{69}$$

By virtue of (68) and (69), we have

$$\left(|\lambda| - \frac{\varepsilon}{p}\right) \int\limits_{D_T} |u|^p\varphi_T dx\,dt \le \frac{1}{p'\varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}} dx\,dt - \zeta(T),$$

which, for $\varepsilon < |\lambda|p$, implies the inequality

$$\int\limits_{D_T} |u|^p\varphi_T dx\,dt \le \frac{p}{(|\lambda|p - \varepsilon)p'\varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}} dx\,dt - \frac{p}{|\lambda|p - \varepsilon}\zeta(T). \tag{70}$$

By using the relations

$$p' = \frac{p}{p-1}, \qquad p = \frac{p'}{p'-1}, \qquad \min_{0<\varepsilon<|\lambda|p} \frac{p}{(|\lambda|p-\varepsilon)p'\varepsilon^{p'-1}} = \frac{1}{|\lambda|^{p'}}$$

(the minimum is attained for $\varepsilon = |\lambda|$), from (70), we obtain

$$\int\limits_{D_T} |u|^p \varphi_T dx\, dt \le \frac{1}{|\lambda|^{p'}} \int\limits_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} dx\, dt - \frac{p'}{|\lambda|}\zeta(T). \tag{71}$$

Since $\varphi_T(x,t) := \varphi^0\left(\frac{x}{T}, \frac{t}{T}\right)$, it follows from (62) and (63) that

$$\int\limits_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} dx\, dt = T^{-2(p'-1)} \int\limits_{D_{T=1}} \frac{|\Box \varphi^0|^{p'}}{|\varphi^0|^{p'-1}} dx'\, dt' = T^{-2(p'-1)} \varkappa_0 < +\infty \tag{72}$$

after the change of variables $x = Tx'$, $t = Tt'$. By virtue of (64) and (72), from (71), we have

$$0 \le \int\limits_{D_T} |u|^p \varphi_T dx\, dt \le \frac{1}{|\lambda|^{p'}} T^{-2(p'-1)} \varkappa_0 - \frac{p'}{|\lambda|}\zeta(T). \tag{73}$$

Since $p' = p/(p-1) > 1$, we have $-2\,(p'-1) < 0$, and it follows from (63) that

$$\lim_{T\to\infty} \frac{1}{|\lambda|^{p'}} T^{-2(p'-1)} \varkappa_0 = 0.$$

Therefore, by virtue of inequality (66), there exists a positive number $T_0 := T_0(f)$ such that if $T > T_0$, then the right-hand side of inequality (73) is negative, while the left-hand side of this inequality is nonnegative. It follows that if there exists a strong generalized solution u of problem (1), (2) of the class C in the domain D_T, then necessarily $T \le T_0$, which completes the proof of Theorem 3.

Remark 4. One can readily see that if $f \in C\left(\bar{D}_\infty\right)$, $f \ge 0$, and $f(x,t) \ge ct^{-m}$ for $t \ge 1$, where $c :=$ const > 0 and $0 \le m :=$ const ≤ 2, then condition (66) is satisfied, and hence problem (1), (2) with sufficiently large T has no strong generalized solution u of the class C in the domain D_T for $\lambda < 0$ and $\alpha > 0$.

Indeed, by introducing the transformation of the independent variables x and t by the formulas $x = Tx'$ and $t = Tt'$ in (65) and by performing simple estimates, we obtain

$$\begin{aligned}\zeta(T) &= T^2 \int\limits_{D_1} f\left(Tx', Tt'\right) \varphi^0\left(x', t'\right) dx'\, dt' \\ &\ge cT^{2-m} \int\limits_{D_1\cap\{t'\ge T^{-1}\}} t'^{-m} \varphi^0\left(x', t'\right) dx'\, dt' \\ &\quad + T^2 \int\limits_{D_1\cap\{t'<T^{-1}\}} f\left(Tx', Tt'\right) \varphi^0\left(x', t'\right) dx'\, dt'\end{aligned}$$

under the assumption that $T > 1$. Further, let $T_1 > 1$ be an arbitrary fixed number. Then from the last inequality for the function ζ, we have

$$\zeta(T) \ge cT^{2-m} \int\limits_{D_1\cap\{t'\ge T^{-1}\}} t'^{-m} \varphi^0\left(x', t'\right) dx'\, dt' \ge cT^{2-m} \int\limits_{D_1\cap\{t'\ge T_1^{-1}\}} t'^{-m} \varphi^0\left(x', t'\right) dx'\, dt'$$

if $T \ge T_1 > 1$. The last inequality readily implies relation (66).

7. LOCAL SOLVABILITY OF PROBLEM (1), (2) FOR THE CASE IN WHICH $\lambda < 0$ AND $\alpha > 0$

Theorem 4. *Suppose that* $\lambda < 0$, $\alpha > 0$, $f \in C\left(\bar{D}_\infty\right)$, *and* $f \not\equiv 0$. *Then there exists a positive number* $T_* := T_*(f)$ *such that for* $T \le T_*$ *problem* (1), (2) *has at least one strong generalized solution* u *of the class* C *in the domain* D_T.

Proof. In Section 4, we have equivalently reduced problem (1), (2) in the space $C\left(\bar{D}_T\right)$ to the functional equation (37), where $A : C\left(\bar{D}_T\right) \to C\left(\bar{D}_T\right)$ is a continuous compact operator. Therefore, by the Schauder theorem, to justify the solvability of Eq. (37), it suffices to show that the operator A maps some ball $B_R := \{v \in C\left(\bar{D}_T\right) : \|v\|_{C(\bar{D}_T)} \le R\}$ of radius $R > 0$, which is a closed convex set in the Banach space $C\left(\bar{D}_T\right)$, into itself. Let us show that this is the case for sufficiently small T.

Indeed, by virtue of (31) and (37), we have

$$\begin{aligned}\|Au\|_{C(\bar{D}_T)} &\le \left\|L_0^{-1}\right\|_{C(\bar{D}_T)\to C(\bar{D}_T)} \left[|\lambda|\,\|u\|^{\alpha+1}_{C(\bar{D}_T)} + \|f\|_{C(\bar{D}_T)}\right] \\ &\le \frac{1}{2} \sup_{(x,t)\in\bar{D}_T} \operatorname{mes} G_{x,t} \left[|\lambda|\,\|u\|^{\alpha+1}_{C(\bar{D}_T)} + \|f\|_{C(\bar{D}_T)}\right] \\ &\le \frac{1}{2} \operatorname{mes} D_T \left[|\lambda|\,\|u\|^{\alpha+1}_{C(\bar{D}_T)} + \|f\|_{C(\bar{D}_T)}\right] \\ &= \frac{1+k}{4} T^2 \left[|\lambda|\,\|u\|^{\alpha+1}_{C(\bar{D}_T)} + \|f\|_{C(\bar{D}_T)}\right] \le \frac{1+k}{4} T^2 \left[|\lambda| R^{\alpha+1} + \|f\|_{C(\bar{D}_T)}\right] \qquad (74)\end{aligned}$$

for $\|u\|_{C(\bar{D}_T)} \le R$.

We fix an arbitrary positive number T_2. Then, by virtue of the estimate (74), we have

$$\|Au\|_{C(\bar{D}_T)} \le \frac{1+k}{4} T^2 \left[|\lambda| R^{\alpha+1} + \|f\|_{C(\bar{D}_{T_2})}\right]$$

for $0 < T \le T_2$, which, in turn, implies that if

$$T_*^2 := \min\left\{ T_2^2, \frac{4R}{(1+k)\left(|\lambda| R^{\alpha+1} + \|f\|_{C(\bar{D}_{T_2})}\right)} \right\},$$

then $\|Au\|_{C(\bar{D}_T)} \le R$ for $\|u\|_{C(\bar{D}_T)} \le R$, $0 < T \le T_*$. The proof of Theorem 4 is complete.

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REFERENCES

1. Bitsadze, A.V., *Nekotorye klassy uravnenii v chastnykh proizvodnykh* (Some Classes of Partial Differential Equations), Moscow: Nauka, 1981.
2. John, F., *Manuscripta Math.*, 1979, vol. 28, no. 1–3, pp. 235–268.
3. John, F., *Comm. Pure Appl. Math.*, 1981, vol. 34, no. 1, pp. 29–51.
4. John, F. and Klainerman, S., *Comm. Pure Appl. Math.*, 1984, vol. 37, no. 4, pp. 443–455.
5. Kato, T., *Comm. Pure Appl. Math.*, 1980, vol. 33, no. 4, pp. 501–505.
6. Georgiev, V., Lindblad, H., and Sogge, C., *Amer. J. Math.*, 1977, vol. 119, no. 6, pp. 1291–1319.
7. Sideris, T.G., *J. Differ. Equat.*, 1984, vol. 52, no. 3, pp. 378–406.
8. Hörmander, L., *Lectures on Nonlinear Hyperbolic Differential Equations*, Berlin, 1997 (Math. and Appl., vol. 26).

9. Veron, L. and Pohozaev, S.I., *Ann. Sc. Norm. Super. Pisa Cl. Sci. Ser. 4*, 2001, vol. 29, no. 2, pp. 393–420.
10. Mitidieri, E. and Pokhozhaev, S.I., A Priori Estimates and the Absence of Solutions of Nonlinear Partial Differential Equations and Inequalities, *Tr. Mat. Inst. Steklova*, 2001, vol. 34, pp. 1–384.
11. Todorova, G. and Vitillaro, E., *J. Math. Anal. Appl.*, 2005, vol. 303, no. 1, pp. 242–257.
12. Goursat, E., *Kurs matematicheskogo analiza* (Course of Mathematical Analysis), Moscow: GTTI, 1933, vol. 3, part 1.
13. Moiseev, E.I., *Differ. Uravn.*, 1984, vol. 20, no. 1, pp. 73–87.
14. Moiseev, E.I., *Uravneniya smeshannogo tipa so spektral'nym parametrom* (Equations of Mixed Type with a Spectral Parameter), Moscow: Moskov. Gos. Univ., 1988.
15. Kharibegashvili, S., *Mem. Differential Equations Math. Phys.*, 1995, vol. 4, pp. 1–127.
16. Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Heidelberg: Springer-Verlag, 1981. Translated under the title *Geometricheskaya teoriya polulineinykh parabolicheskikh uravnenii*, Moscow, 1985.
17. Ladyzhenskaya, O.A., *Kraevye zadachi matematicheskoi fiziki* (Boundary Value Problems of Mathematical Physics), Moscow: Nauka, 1973.
18. Narasimhan, R., *Analysis on Real and Complex Manifolds*, Amsterdam: North-Holland, 1968. Translated under the title *Analiz na deistvitel'nykh i kompleksnykh mnogoobraziyakh*, Moscow: Mir, 1971.
19. Gilbarg, D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, Berlin: Springer-Verlag, 1983. Translated under the title *Ellipticheskie uravneniya s chastnymi proizvodnymi vtorogo poryadka*, Moscow: Nauka, 1989.
20. Trenogin, V.A., *Funktsional'nyi analiz* (Functional Analysis), Moscow: Nauka, 1993.