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ON ONE BOUNDARY VALUE PROBLEM FOR A NONLINEAR EQUATION WITH THE ITERATED WAVE OPERATOR IN THE PRINCIPAL PART

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Dedicated to the memory of Professor J.-L. Lions

Abstract. One boundary value problem for a hyperbolic equation with power nonlinearity and the iterated wave operator in the principal part is considered in a conical domain. Depending on the index of nonlinearity and spatial dimensionality of the equation the question on the existence and uniqueness of a solution of a boundary value problem is investigated. The question as to the absence of a solution of this problem is also considered.

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1. Statement of the Problem

In the Euclidean space R^{n+1} of independent variables $x_1, x_2, \dots, x_n, t$ consider a nonlinear equation of the form

$$L_\lambda u := \Box^2 u = \lambda f(u) + F, \tag{1.1}$$

where λ is a given real constant, $f : R \to R$ is a given continuous nonlinear function, $f(0) = 0$, F is a given, and u is an unknown real function, $\Box = \frac{\partial^2}{\partial t^2} - \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

Let $D_T : |x| < t < T$ be the domain which is the intersection of the light cone of future $K_O^+ : t > |x|$ with apex at the origin $O(0,0,\dots,0)$ and a half-space $H_T : t < T$, $T = const > 0$. Assume $S_T = \partial D_T \cap \partial K_O^+$, $S_T^0 = \partial D_T \cap \partial H_T$. It is obvious that $S_T : t = |x|,\ 0 \leq t \leq T$, is a characteristic conical manifold for equation (1.1), $S_T^0 : |x| \leq T$, $t = T$, and $\partial D_T = S_T \cup S_T^0$.

For equation (1.1) consider the boundary value problem on determination of its solution $u(x_1, \dots, x_n, t)$ in the domain D_T by the boundary conditions

$$u\big|_{\partial D_T} = 0, \tag{1.2}$$

$$\left.\frac{\partial u}{\partial t}\right|_{S_T^0} = 0. \tag{1.3}$$

It should be noted that for nonlinear hyperbolic equations the question of the local or global solvability of the Cauchy problem with the initial conditions for $t = 0$ is considered in vast literature [see, e.g., 1–20].

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As is known, for a nonlinear wave equation of the form $\square u = \lambda f(u) + F$ the characteristic problem in the light cone of future $K_O^+ : t > |x|$ with a boundary condition of the form $u|_{\partial K_O^+} = g$ in the linear case, i.e., with $\lambda = 0$ is well-posed and the global solvability takes place in appropriate function spaces [21–25], while in the nonlinear case, when the function $f(u)$ has exponential nature and $\lambda \neq 0$, this problem was considered in [26–28].

Assume $\overset{\circ}{C}{}^k(\overline{D}_T, \partial D_T, S_T^0) = \Big\{u \in C^k(\overline{D}_T) : u|_{\partial D_T} = 0,\ \frac{\partial u}{\partial t}\big|_{S_T^0} = 0\Big\}$, $k \geq 1$.

Let $u \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T, S_T^0)$ be a classical solution of problem (1.1), (1.2), (1.3). Multiplying both parts of equation (1.1) by an arbitrary function $\phi \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T, S_T^0)$ and integrating the resulting equation by parts in the domain D_T we obtain

$$\int\limits_{D_T} \square u \square \phi \, dx\, dt = \lambda \int\limits_{D_T} f(u)\phi \, dx\, dt + \int\limits_{D_T} F\phi \, dx\, dt. \tag{1.4}$$

Here we have used the equality

$$\int\limits_{D_T} \square u \square \phi \, dx\, dt = \int\limits_{\partial D_T} \frac{\partial \phi}{\partial N} \square u \, ds - \int\limits_{\partial D_T} \phi \frac{\partial}{\partial N} \square u \, ds + \int\limits_{D_T} \phi \square^2 u \, dx\, dt$$

and the fact that since $S_T = \partial D_T \cap \partial K_O^+$ is a characteristic manifold, the derivative on the conormal $\frac{\partial}{\partial N} = \gamma_{n+1}\frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i}$, where $\gamma = (\gamma_1, \dots, \gamma_n, \gamma_{n+1})$ is the unit vector of the outer normal to ∂D_T, is an inner differential operator on the characteristic manifold S_T and thus if $v \in C^1(\overline{D}_T)$ and $v|_{S_T} = 0$, then $\frac{\partial v}{\partial N}\big|_{S_T} = 0$.

Let us introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\square}(D_T)$ as the completion with respect to the norm

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\square}(D_T)} = \int\limits_{D_T} \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 + (\square u)^2 \right] dx\, dt \tag{1.5}$$

of the classical space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T, S_T^0)$. It follows from (1.5) that if $u \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$, then $u \in \overset{\circ}{W}{}^1_2(D_T)$ and $\square u \in L_2(D_T)$. Here $W_2^1(D_T)$ is the known Sobolev space [29, p. 56] consisting of elements from $L_2(D_T)$, which have first order generalized derivatives in $L_2(D_T)$, and $\overset{\circ}{W}{}^1_2(D_T) = \{u \in W_2^1(D_T) : u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ should be understood in the sense of the trace theory [29, p. 70]. Moreover, since from (1.5) it follows that $u \in \overset{\circ}{W}{}^1_2(D_T)$ and $\square u \in L_2(D_T)$, then an arbitrary function u from the space $\overset{\circ}{W}{}^1_{2,\square}(D_T)$ satisfies the homogeneous condition (1.3) in the sense of the trace theory [29, p. 130].

Let us assume that equality (1.4) underlies the determination of a generalized solution of problem (1.1), (1.2), (1.3).

Definition 1. Let $F \in L_2(D_T)$. We call a function $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ a weak generalized solution of problem (1.1), (1.2), (1.3) if $f(u) \in L_2(D_T)$ and for any function $\phi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ the integral equality (1.4) is valid, i.e.

$$\int\limits_{D_T} \Box u \Box \phi \, dx \, dt = \lambda \int\limits_{D_T} f(u)\phi \, dx \, dt + \int\limits_{D_T} F\phi \, dx \, dt \quad \forall \phi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{1.6}$$

It is easy to verify that if a solution u of problem (1.1), (1.2), (1.3) in the sense of Definition 1 belongs to the class $C^4(\overline{D}_T)$, then it will be a classical solution of this problem.

2. Solvability of Problem (1.1), (1.2), (1.3) in the Case of a Nonlinearity of the Form $f(u) = |u|^\alpha \operatorname{sgn} u$

Assume that in equation (1.1) the nonlinear function f has the form

$$f(u) = |u|^\alpha \operatorname{sgn} u, \quad \alpha = const > 0, \quad \alpha \neq 1. \tag{2.1}$$

Then in accordance with (2.1) equation (1.1) and the integral equality (1.6) take the form

$$L_\lambda u := \Box^2 u = \lambda |u|^\alpha \operatorname{sgn} u + F \tag{2.2}$$

and

$$\int\limits_{D_T} \Box u \Box \phi \, dx \, dt = \lambda \int\limits_{D_T} \phi |u|^\alpha \operatorname{sgn} u \, dx \, dt + \int\limits_{D_T} F\phi \, dx \, dt \quad \forall \phi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{2.3}$$

Lemma 1. *The inequality*

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq c \|\Box u\|_{L_2(D_T)} \quad \forall u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T) \tag{2.4}$$

is valid, where the norm of the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ *is given by equality* (1.5) *and the positive constant* c *does not depend on* u.

Proof. Let $\Omega_\tau := \overline{D}_T \cap \{t = \tau\}$, $D_\tau = D_T \cap \{t < \tau\}$, $S_\tau = \{(x,t) \in \partial D_\tau : t = |x|\}$, $0 < \tau \leq T$ and $\gamma = (\gamma_1, \dots, \gamma_n, \gamma_{n+1})$ be the unit vector of the outer normal to ∂D_τ.

Since the space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T, S_T^0)$ is a dense subspace of the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ it is sufficient to prove inequality (2.4) for functions from the space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T, S_T^0)$. For $u \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T, S_T^0)$, taking into account the equalities $u|_{S_\tau} = 0$, $\Omega_\tau = \partial D_\tau \cap \{t = \tau\}$ and $\gamma|_{\Omega_\tau} = (0, \dots, 0, 1)$, it is easy to obtain by integration by parts

$$\int\limits_{D_\tau} \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\,dx\,dt = \frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt = \frac{1}{2}\int\limits_{\partial D_\tau}\left(\frac{\partial u}{\partial t}\right)^2 \gamma_{n+1} ds$$

$$= \frac{1}{2}\int\limits_{\Omega_\tau}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\int\limits_{S_\tau}\left(\frac{\partial u}{\partial t}\right)^2 \gamma_{n+1} ds, \quad \tau \le T, \qquad (2.5)$$

$$\int\limits_{D_\tau} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial u}{\partial t}\,dx\,dt = \int\limits_{\partial D_\tau} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t}\gamma_i ds - \frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial x_i}\right)^2 dx\,dt$$

$$= \int\limits_{\partial D_\tau} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t}\gamma_i ds - \frac{1}{2}\int\limits_{\partial D_\tau}\left(\frac{\partial u}{\partial x_i}\right)^2 \gamma_{n+1} ds$$

$$= \int\limits_{\partial D_\tau} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t}\gamma_i ds - -\frac{1}{2}\int\limits_{S_\tau}\left(\frac{\partial u}{\partial x_i}\right)^2 \gamma_{n+1} ds$$

$$- \frac{1}{2}\int\limits_{\Omega_\tau}\left(\frac{\partial u}{\partial x_i}\right)^2 ds, \quad \tau \le T. \qquad (2.6)$$

It follows from (2.5) and (2.6) that

$$\int\limits_{D_\tau} \Box u \frac{\partial u}{\partial t}\,dx\,dt = \int\limits_{S_\tau} \frac{1}{2\gamma_{n+1}}\Bigg[\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\gamma_{n+1} - \frac{\partial u}{\partial t}\gamma_i\right)^2$$

$$+ \left(\frac{\partial u}{\partial t}\right)^2\left(\gamma_{n+1}^2 - \sum_{j=1}^{n}\gamma_j^2\right)\Bigg] ds$$

$$+ \frac{1}{2}\int\limits_{\Omega_\tau}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right] dx, \quad \tau \le T. \qquad (2.7)$$

Since $u|_{S_\tau} = 0$ and the operator $(\gamma_{n+1}\frac{\partial}{\partial x_i} - \gamma_i \frac{\partial}{\partial t})$, $1 \le i \le n$, is an internal differential operator on S_τ, we have the equalities

$$\left(\frac{\partial u}{\partial x_i}\gamma_{n+1} - \frac{\partial u}{\partial t}\gamma_i\right)\Bigg|_{S_\tau} = 0, \quad i = 1, \dots, n. \qquad (2.8)$$

Therefore, taking into account that $\gamma_{n+1}^2 - \sum\limits_{j=1}^{n}\gamma_j^2 = 0$ on the characteristic manifold S_τ, in view of (2.7) and (2.8) we have

$$\int\limits_{\Omega_\tau}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right] dx = 2\int\limits_{D_\tau} \Box u \frac{\partial u}{\partial t}\,dx\,dt, \quad \tau \le T. \qquad (2.9)$$

Assuming $w(\delta)=\int\limits_{\Omega_\delta}\left[\left(\frac{\partial u}{\partial t}\right)^2+\sum\limits_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2\right]dx$, and using inequality $2\,\Box u\frac{\partial u}{\partial t}\leq\varepsilon\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{\varepsilon}|\Box u|^2$, which is valid for any $\varepsilon=const>0$, from (2.9) we obtain

$$w(\delta)\leq\varepsilon\int\limits_0^\delta w(\sigma)d\sigma+\frac{1}{\varepsilon}\|\Box\|^2_{L_2(D_\delta)},\quad 0<\delta\leq T. \tag{2.10}$$

From (2.10), taking into account that the value $\|\Box\|^2_{L_2(D_\delta)}$ as a function of δ is non-decreasing, in view of Gronwall's lemma [30, p. 13] it follows that

$$w(\delta)\leq\frac{1}{\varepsilon}\|\Box\|^2_{L_2(D_\delta)}\exp\delta\varepsilon.$$

Hence, taking into account the fact that $\inf\limits_{\varepsilon>0}\frac{1}{\varepsilon}\exp\delta\varepsilon=e\delta$ and it is reached at $\varepsilon=\frac{1}{\delta}$, we obtain

$$w(\delta)\leq e\delta\|\Box\|^2_{L_2(D_\delta)},\quad 0<\delta\leq T. \tag{2.11}$$

From (2.11), in turn, it follows that

$$\int\limits_{D_T}\left[\left(\frac{\partial u}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2\right]dx\,dt=\int\limits_0^T w(\delta)d\delta\leq\frac{e}{2}T^2\|\Box u\|^2_{L_2(D_T)}. \tag{2.12}$$

Using the equalities $u|_{S_T}=0$ and $u(x,t)=\int\limits_{|x|}^t\frac{\partial u(x,t)}{\partial t}d\tau$, $(x,t)\in\overline{D}_T$, which are valid for any function $u\in\overset{\circ}{C}{}^2(\overline{D}_T,\partial D_T,S_T^0)$, by a standard reasoning [29, p. 63] we easily obtain the inequality

$$\int\limits_{D_T}u^2(x,t)\,dx\,dt\leq T^2\int\limits_{D_T}\left(\frac{\partial u}{\partial t}\right)^2dx\,dt. \tag{2.13}$$

By virtue of (2.11) and (2.13) we have

$$\begin{aligned}\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)}&=\int\limits_{D_T}\left[u^2+\left(\frac{\partial u}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\right)^2+(\Box u)^2\right]dx\,dt\\&\leq\left(1+\frac{e}{2}T^2+\frac{e}{2}T^4\right)\|\Box\|^2_{L_2(D_T)},\end{aligned}$$

whence inequality (2.4) with the constant $c^2=1+\frac{e}{2}T^2+\frac{e}{2}T^4$ follows. □

Lemma 2. *Assume $F\in L_2(D_T)$, $0<\alpha<1$, and in the case $\alpha>1$ additionally require that $\lambda<0$. Then in the case with a nonlinearity of form* (2.1) *for a weak generalized solution $u\in\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ of problem* (1.1), (1.2), (1.3) *, i.e., problem* (2.2), (1.2), (1.3) *in the sense of the integral equality* (2.3) *with $|u|^\alpha\in L_2(D_T)$, we have an a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)}\leq c_1\|F\|_{L_2(D_T)}+c_2 \tag{2.14}$$

with non-negative constants $c_i\,(T,\alpha,\lambda)$, $i=1,2$, *which do not depend on* u, F *and* $c_1>0$.

Proof. First, let $\alpha>1$ and $\lambda<0$. Assuming in equality (2.3) that $\phi=u\in \overset{\circ}{W}{}^{1}_{2,\Box}(D_T)$ and taking into account (1.5), for any $\varepsilon>0$ we have

$$\begin{aligned}\|\Box u\|^2_{L_2(D_T)}&=\int\limits_{D_T}(\Box u)^2\,dx\,dt=\lambda\int\limits_{D_T}|u|^{\alpha+1}\,dx\,dt+\int\limits_{D_T}Fu\,dx\,dt\\&\le\int\limits_{D_T}Fu\,dx\,dt\le\frac{1}{4\varepsilon}\int\limits_{D_T}F^2\,dx\,dt+\varepsilon\|u\|^2_{L_2(D_T)}\\&\le\frac{1}{4\varepsilon}\|F\|^2_{L_2(D_T)}+\varepsilon\|u\|^2_{\overset{\circ}{W}{}^{1}_{2,\Box}(D_T)}.\end{aligned}\tag{2.15}$$

Due to (2.4) and (2.15) we have

$$\|u\|^2_{\overset{\circ}{W}{}^{1}_{2,\Box}(D_T)}\le c^2\|\Box u\|^2_{L_2(D_T)}\le\frac{c^2}{4\varepsilon}\|F\|^2_{L_2(D_T)}+c^2\varepsilon\|u\|^2_{\overset{\circ}{W}{}^{1}_{2,\Box}(D_T)},$$

from which for $\varepsilon=\frac{1}{2c^2}<\frac{1}{c^2}$ we obtain

$$\|u\|^2_{\overset{\circ}{W}{}^{1}_{2,\Box}(D_T)}\le\frac{c^2}{4\varepsilon\,(1-\varepsilon c^2)}\|F\|^2_{L_2(D_T)}=c^4\|F\|^2_{L_2(D_T)}.\tag{2.16}$$

From (2.16) in the case $\alpha>1$ and $\lambda<0$ follows inequality (2.14) with $c_1=c^2$ and $c_2=0$.

Let now $0<\alpha<1$. Using the known inequality

$$ab\le\frac{\varepsilon a^p}{p}+\frac{b^q}{q\varepsilon^{q-1}}$$

with a parameter $\varepsilon>0$ for $a=|u|^{\alpha+1}$, $b=1$, $p=\frac{2}{\alpha+1}>1$, $q=\frac{2}{1-\alpha}$, $\frac{1}{p}+\frac{1}{q}=1$, in the same way as for inequality (2.15) we have

$$\begin{aligned}\|\Box u\|^2_{L_2(D_T)}&=\int\limits_{D_T}(\Box u)^2\,dx\,dt=\lambda\int\limits_{D_T}|u|^{\alpha+1}\,dx\,dt+\int\limits_{D_T}Fu\,dx\,dt\\&\le|\lambda|\int\limits_{D_T}\left[\varepsilon\frac{1+\alpha}{2}|u|^2+\frac{1-\alpha}{2\varepsilon^{q-1}}\right]dx\,dt+\frac{1}{4\varepsilon}\int\limits_{D_T}F^2\,dx\,dt+\varepsilon\int\limits_{D_T}u^2\,dx\,dt\\&=\frac{1}{4\varepsilon}\|F\|^2_{L_2(D_T)}+\varepsilon\left(|\lambda|\frac{1+\alpha}{2}+1\right)\|u\|^2_{L_2(D_T)}+|\lambda|\frac{1-\alpha}{2\varepsilon^{q-1}}\operatorname{mes}D_T.\end{aligned}\tag{2.17}$$

In view of (1.5) and (2.4) it follows from (2.17) that

$$\begin{aligned}\|u\|^2_{\overset{\circ}{W}{}^{1}_{2,\Box}(D_T)}&\le c^2\|\Box u\|^2_{L_2(D_T)}\le\frac{c^2}{4\varepsilon}\|F\|^2_{L_2(D_T)}\\&+\varepsilon c^2\left(|\lambda|\frac{1+\alpha}{2}+1\right)\|u\|^2_{\overset{\circ}{W}{}^{1}_{2,\Box}(D_T)}+c^2|\lambda|\frac{1-\alpha}{2\varepsilon^{q-1}}\operatorname{mes}D_T,\quad q=\frac{2}{1-\alpha},\end{aligned}$$

whence for $\varepsilon = \frac{1}{2}c^{-2}\left(|\lambda|\frac{1+\alpha}{2}+1\right)^{-1}$ we obtain

$$\begin{aligned}\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} &\le \left[1-\varepsilon c^2\left(|\lambda|\frac{1+\alpha}{2}+1\right)\right]^{-1} \\ &\qquad\times\left(\frac{c^2}{4\varepsilon}\|F\|^2_{L_2(D_T)} + c^2|\lambda|\frac{1-\alpha}{2\varepsilon^{q-1}}\operatorname{mes} D_T\right) \\ &= c^4\left(|\lambda|\frac{1+\alpha}{2}+1\right)\|F\|^2_{L_2(D_T)} + 2c^2|\lambda|\frac{1-\alpha}{2\varepsilon^{q-1}}\operatorname{mes} D_T. \quad (2.18)\end{aligned}$$

From (2.18), in the case $0<\alpha<1$, follows inequality (2.14) with $c_1 = c^2\left(|\lambda|\frac{1+\alpha}{2}+1\right)^{\frac{1}{2}}$ and $c_2 = c\left(2|\lambda|\frac{1-\alpha}{2\varepsilon^{q-1}}\operatorname{mes} D_T\right)^{\frac{1}{2}}$, where $q=\frac{1}{1-\alpha}$. Lemma 2 is completely proved. □

Remark 1. From the proof of Lemma 2 it follows that in estimate (2.14) the constants c_1 and c_2 are equal:

$$1)\ \ \alpha>1,\quad \lambda<0:\ \ c_1=c^2,\ \ c_2=0; \qquad (2.19)$$

$$2)\ \ 0<\alpha<1,\quad -\infty<\lambda<+\infty:$$

$$c_1 = c^2\left(|\lambda|\frac{1+\alpha}{2}+1\right)^{\frac{1}{2}},\quad c_2 = c\left(2|\lambda|\frac{1-\alpha}{2\varepsilon^{q-1}}\operatorname{mes} D_T\right)^{\frac{1}{2}}, \qquad (2.20)$$

where the constant $c = \left(1+\frac{e}{2}T^2+\frac{e}{2}T^4\right)^{\frac{1}{2}}$ is taken from estimate (2.4) and $q=\frac{2}{1-\alpha}$.

Remark 2. Below we will consider the linear problem corresponding to (1.1), (1.2), (1.3), when $\lambda = 0$. In that case, for $F\in L_2(D_T)$ we analogously to the above introduce the concept of a weak generalized solution $u\in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$, when the integral equality

$$(u,\phi)_\Box := \int_{D_T} \Box u \Box\phi\,dx\,dt = \int_{D_T} F\phi\,dx\,dt\ \ \forall\phi\in \overset{\circ}{W}{}^1_{2,\Box}(D_T) \qquad (2.21)$$

holds.

Remark 3. In view of (1.5) and (2.4), taking into account that

$$\begin{aligned}\left|(\Box u,\Box\phi)_{L_2(D_T)}\right| &= \left|\int_{D_T}\Box u\Box\phi\,dx\,dt\right| \le \|\Box u\|_{L_2(D_T)}\|\Box\phi\|_{L_2(D_T)} \\ &\le \|\Box u\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)}\|\Box\phi\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)},\end{aligned}$$

the bilinear form

$$(u,\phi)_\Box := \int_{D_T}\Box u\Box\phi\,dx\,dt$$

of (2.21) can be considered as a scalar product in the Hilbert space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$. Therefore, since for $F \in L_2(D_T)$

$$\left| \int\limits_{D_T} F\phi \, dx \, dt \right| \leq \|F\|_{L_2(D_T)} \|\phi\|_{L_2(D_T)} \leq \|F\|_{L_2(D_T)} \|\phi\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)},$$

due to the theorem of Riesz [31, p. 83] there is a unique function u in the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ that satisfies equality (2.21) for any $\phi \in \overset{\circ}{W}{}_{2,\Box}(D_T)$ and for whose norm an estimate

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq \|F\|_{L_2(D_T)} \tag{2.22}$$

is valid. Thus, being introduced the notation $u = L_0^{-1}F$, we find that to the linear problem corresponding to (1.1), (1.2), (1.3), when $\lambda = 0$, there corresponds a linear bounded operator

$$L_0^{-1} : L_2(D_T) \to \overset{\circ}{W}{}^1_{2,\Box}(D_T),$$

for whose norm, by virtue of (2.22), an estimate

$$\left\|L_0^{-1}\right\|_{L_2(D_T) \to \overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq \|F\|_{L_2(D_T)} \tag{2.23}$$

is true.

Taking into account Definition 1 and Remark 3, equality (2.3), which is equivalent to problem (2.2), (1.2), (1.3), can be rewritten in the form of an equivalent equation

$$u = L_0^{-1}\left[\lambda |u|^\alpha \operatorname{sgn} u + F\right] \tag{2.24}$$

in the Hilbert space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.

Remark 4. The embedding operator $I : \overset{\circ}{W}{}^1_2(D_T) \to L_q(D_T)$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$ when $n \geq 2$ [29, p. 81]. At the same time, the operator of Nemytskii $N : L_q(D_T) \to L_2(D_T)$, which acts according to the formula $Nu = \lambda |u|^\alpha \operatorname{sgn} u$, $\alpha > 1$, is continuous and bounded for $q \geq 2\alpha$ [32, p. 349], [33, pp. 66, 67]. Thus, if $1 < \alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < 2\alpha \leq q < \frac{2(n+1)}{n-1}$ and hence the operator

$$N_1 = NI : \overset{\circ}{W}{}^1_2(D_T) \to L_2(D_T) \tag{2.25}$$

is a continuous and compact operator. In that case, since $u \in \overset{\circ}{W}{}^1_2(D_T)$, it is clear that $f(u) = |u|^\alpha \operatorname{sgn} u \in L_2(D_T)$. Further, since in view of (1.5) the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ is continuously embedded into the space $\overset{\circ}{W}{}^1_2(D_T)$, taking into account (2.25) the operator

$$N_2 = NII_1 : \overset{\circ}{W}{}^1_{2,\Box}(D_T) \to L_2(D_T), \tag{2.26}$$

where $I_1 : \overset{\circ}{W}{}^1_{2,\Box}(D_T) \to \overset{\circ}{W}{}^1_2(D_T)$ is the embedding operator, is continuous and compact for $1 < \alpha < \frac{n+1}{n-1}$. For $0 < \alpha < 1$, operator (2.26) is also continuous and compact, since according to the theorem of Rellich [29, p. 64] the space $\overset{\circ}{W}{}_2^1(D_T)$ is continuously and compactly embedded into $L_2(D_T)$, and the space $L_2(D_T)$, in turn, is continuously embedded into $L_p(D_T)$ for $0 < p < 2$.

Let us rewrite equation (2.24) in the form

$$u = Au := L_0^{-1}(N_2 u + F), \tag{2.27}$$

where the operator $N_2 : \overset{\circ}{W}{}_{2,\Box}^1(D_T) \to L_2(D_T)$, for $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \neq 1$, is continuous and compact in view of Remark 4. Then taking into account (2.23) the operator $A : \overset{\circ}{W}{}_{2,\Box}^1(D_T) \to \overset{\circ}{W}{}_{2,\Box}^1(D_T)$ in (2.27) is also continuous and compact. At the same, time according to the a priori estimate (2.14) of Lemma 2, in which the constants c_1 and c_2 are given by equalities (2.19) and (2.20), for any parameter $\tau \in [0,1]$ and for any solution $u \in \overset{\circ}{W}{}_{2,\Box}^1(D_T)$ of the equation $u = \tau Au$ with this parameter we have the a priori estimation (2.14) with constants $c_1 > 0$ and $c_2 \geq 0$ not depending on u, τ and F. Therefore, according to the theorem of Lere–Schauder [34, p. 375], equation (2.27) and, consequently, problem (2.2), (1.2), (1.3) have at least one weak generalized solution u in the space $\overset{\circ}{W}{}_{2,\Box}^1(D_T)$.

Thus the following statement is valid.

Theorem 1. *Let $0 < \alpha < \frac{n+1}{n-1}, \alpha \neq 1, \lambda \neq 0$ and in the case $\alpha > 1$ additionally require that $\lambda < 0$. Then for any $F \in L_2(D_T)$ problem* (2.2), (1.2), (1.3) *has at least one weak generalized solution $u \in \overset{\circ}{W}{}_{2,\Box}^1(D_T)$.*

3. The Uniqueness of a Solution of Problem (1.1), (1.2), (1.3) in the Case of a Nonlinearity of the Form $f(u) = |u|^\alpha \operatorname{sgn} u$

Let $F \in L_2(D_T)$, and u_1, u_2 be two weak generalized solutions of problem (2.2), (1.2), (1.3) in the space $\overset{\circ}{W}{}_{2,\Box}^1(D_T)$, i.e., according to (2.3) the following equalities

$$\int_{D_T} \Box u_i \Box \phi \, dx \, dt = \lambda \int_{D_T} \phi |u_i|^\alpha \operatorname{sgn} u_i \, dx \, dt + \int_{D_T} F\phi \, dx \, dt \quad \forall \phi \in \overset{\circ}{W}{}_{2,\Box}^1(D_T) \tag{3.1}$$

are fulfilled and $|u_i|^\alpha \in L_2(D_T)$, $i = 1, 2$.

For the difference $v = u_2 - u_1$, from (3.1) it follows that

$$\int_{D_T} \Box v \Box \phi \, dx \, dt = \lambda \int_{D_T} \phi \left(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1 \right) dx \, dt \quad \forall \phi \in \overset{\circ}{W}{}_{2,\Box}^1(D_T). \tag{3.2}$$

Assuming $\phi = v \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ in equality (3.2), we obtain

$$\int\limits_{D_T} (\Box v)^2 \, dx\, dt = \lambda \int\limits_{D_T} \left(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1\right)(u_2 - u_1)\, dx\, dt. \tag{3.3}$$

Note that for the finite values of u_1 and u_2 with $\alpha > 0$ the inequality

$$\left(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1\right)(u_2 - u_1) \geq 0 \tag{3.4}$$

holds.

From (3.3) and inequality (3.4), which is true for almost all points $(x,t) \in D_T$ with $u_i \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$, $i = 1,2$, when $\alpha > 0$ and $\lambda < 0$, it follows that

$$\int\limits_{D_T} (\Box v)^2 \, dx\, dt \leq 0,$$

whence, due to (2.4), we obtain $v = 0$, i.e. $u_1 = u_2$.

Thus the following statement is valid.

Theorem 2. *Let $\alpha > 0$, $\alpha \neq 1$ and $\lambda < 0$. Then for any $F \in L_2(D_T)$, problem* (2.2), (1.2), (1.3) *cannot have more than one generalized solution in the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

In turn, Theorems 1 and 2 give rise to

Theorem 3. *Let $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \neq 1$ and $\lambda < 0$. Then for any $F \in L_2(D_T)$, problem* (2.2), (1.2), (1.3) *has a unique weak generalized solution $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

4. The Absence of a Solution of Problem (1.1), (1.2), (1.3) in the Case of a Nonlinearity of the Form $f(u) = |u|^\alpha$

Assume now in equation (1.1) and therefore in the integral equality (1.3) that $f(u) = |u|^\alpha$, $\alpha > 1$.

Theorem 4. *Let $F^0 \in L_2(D_T)$, $\|F^0\|_{L_2(D_T)} \neq 0$, $F^0 \geq 0$, and $F = \mu F^0$, $\mu = const > 0$. Then in the case $f(u) = |u|^\alpha$, $\alpha > 1$, with $\lambda > 0$ there exists a number $\mu_0 = \mu_0(F^0, \lambda, \alpha) > 0$ such that for $\mu > \mu_0$ problem* (1.1), (1.2), (1.3) *cannot have a weak generalized solution in the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

Proof. Let us assume that when the conditions of the theorem are satisfied the solution $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ of problem (1.1), (1.2), (1.3) exists for any fixed $\mu > 0$. Then equality (1.6) takes the form

$$\int\limits_{D_T} \Box u \Box \phi \, dx\, dt = \lambda \int\limits_{D_T} |u|^\alpha \phi \, dx\, dt + \mu \int\limits_{D_T} F^0 \phi \, dx\, dt \quad \forall \phi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{4.1}$$

It is easy to verify that

$$\int\limits_{D_T} \Box u \Box \phi \, dx\, dt = \int\limits_{D_T} u \Box^2 \phi \, dx\, dt \quad \forall \phi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T, S_T^0), \tag{4.2}$$

where $\overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T, S_T^0) = \left\{ u \in C^4\left(\overline{D}_T\right) : u|_{\partial D_T} = 0, \ \frac{\partial u}{\partial t}\big|_{S_T^0} = 0 \right\} \subset \overset{\circ}{W}{}^1_{2,\Box}(D_T)$. Indeed, since $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ and the space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T, S_T^0)$ is dense in $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$, there exists a sequence $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T, S_T^0)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} = 0. \tag{4.3}$$

Taking into account that

$$\int\limits_{D_T} \Box u_k \Box \phi \, dx\, dt = \int\limits_{\partial D_T} \frac{\partial u_k}{\partial N} \Box \phi ds - \int\limits_{\partial D_T} u_k \frac{\partial}{\partial N} \Box \phi ds + \int\limits_{D_T} u_k \Box^2 \phi \, dx\, dt, \tag{4.4}$$

where the derivative on the conormal $\frac{\partial}{\partial N} = \gamma_{n+1} \frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i}$ is an inner differential operator on the characteristic manifold S_T and therefore $\frac{\partial u_k}{\partial N}\big|_{S_T} = 0$ since $u_k|_{S_T} = 0$, from (4.4), due to the fact that $u_k|_{S_T^0} = \frac{\partial u_k}{\partial t}\big|_{S_T^0} = 0$, and $\partial D_T = S_T \cup S_T^0$, we obtain

$$\int\limits_{D_T} \Box u_k \Box \phi \, dx\, dt = \int\limits_{D_T} u_k \Box^2 \phi \, dx\, dt. \tag{4.5}$$

Passing in (4.5) to the limit as $k \to \infty$, in view of (1.5) and (4.3) we obtain (4.2).

Taking into account (4.2), we rewrite equality (4.1) as

$$\lambda \int\limits_{D_T} |u|^\alpha \phi \, dx\, dt = \int\limits_{D_T} u \Box^2 \phi \, dx\, dt - \mu \int\limits_{D_T} F^0 \phi \, dx\, dt \quad \forall \phi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T, S_T^0). \tag{4.6}$$

Below we use the method of test functions [12, p. 10–12]. Let us select a test function $\phi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T, S_T^0)$ such that $\phi|_{D_T} > 0$. If in Young's inequality with a parameter $\varepsilon > 0$

$$ab \le \frac{\varepsilon}{\alpha} a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} b^{\alpha'}; \quad a, b \ge 0, \quad \alpha' = \frac{\alpha}{\alpha - 1}$$

we take $a = |u| \phi^{\frac{1}{\alpha}}$, $b = \frac{|\Box^2 \phi|}{\phi^{\frac{1}{\alpha}}}$, then due to the fact that $\frac{\alpha'}{\alpha} = \alpha' - 1$ we have

$$|u \Box^2 \phi| = |u| \phi^{\frac{1}{\alpha}} \frac{|\Box^2 \phi|}{\phi^{\frac{1}{\alpha}}} \le \frac{\varepsilon}{\alpha} |u|^\alpha \phi + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \frac{|\Box^2 \phi|^{\alpha'}}{\phi^{\alpha'-1}}. \tag{4.7}$$

By virtue of (4.7) and (4.6) we obtain the inequality

$$\Big(\lambda-\frac{\varepsilon}{\alpha}\Big)\int\limits_{D_T}|u|^\alpha\phi\,dx\,dt\ \le\frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int\limits_{D_T}\frac{|\Box^2\phi|^{\alpha'}}{\phi^{\alpha'-1}}\,dx\,dt-\mu\int\limits_{D_T}F^0\phi\,dx\,dt,$$

which, for $\varepsilon<\lambda\alpha$, implies

$$\int\limits_{D_T}|u|^\alpha\phi\,dx\,dt\ \le\frac{\alpha}{(\lambda\alpha-\varepsilon)\,\alpha'\varepsilon^{\alpha'-1}}\int\limits_{D_T}\frac{|\Box^2\phi|^{\alpha'}}{\phi^{\alpha'-1}}\,dx\,dt-\frac{\alpha\mu}{\lambda\alpha-\varepsilon}\int\limits_{D_T}F^0\phi\,dx\,dt. \quad (4.8)$$

Taking into account the equalities $\alpha'=\frac{\alpha}{\alpha-1}$, $\alpha=\frac{\alpha'}{\alpha'-1}$ and $\min\limits_{0<\varepsilon<\lambda\alpha}\frac{\alpha}{(\lambda\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}}=\frac{1}{\lambda^{\alpha'}}$ which is obtained at $\varepsilon=\lambda$, it follows from (4.8) that

$$\int\limits_{D_T}|u|^\alpha\phi\,dx\,dt\ \le\frac{1}{\lambda^{\alpha'}}\int\limits_{D_T}\frac{|\Box^2\phi|^{\alpha'}}{\phi^{\alpha'-1}}\,dx\,dt-\frac{\alpha'\mu}{\lambda}\int\limits_{D_T}F^0\phi\,dx\,dt. \quad (4.9)$$

Note that the existence of a test function ϕ such that

$$\phi\in\overset{\circ}{C}{}^4(\overline{D}_T,\partial D_T,S_T^0),\ \ \phi|_{D_T}>0,\ \ \kappa=\int\limits_{D_T}\frac{|\Box^2\phi|^{\alpha'}}{\phi^{\alpha'-1}}\,dx\,dt<+\infty \quad (4.10)$$

is not difficult to show. Indeed, it is easy to verify that the function

$$\phi(x,t)=\big[\big(t^2-|x|^2\big)\big((T-t)^2-|x|^2\big)\big]^m$$

satisfies conditions (4.10) for a sufficiently large positive m.

Since, by assumption, $F^0\in L_2(D_T)$, $\|F^0\|_{L_2(D_T)}\ne0$, $F^0\ge0$, and $\operatorname{mes}D_T<+\infty$, due to the fact that $\phi|_{D_T}>0$ we have

$$0<\kappa_1=\int\limits_{D_T}F^0\phi\,dx\,dt<+\infty. \quad (4.11)$$

Let us denote by $g(\mu)$ the right side of inequality (4.9) which is a linear function with respect to μ, then in view of (4.10) and (4.11)

$$g(\mu)<0\ \text{ for }\ \mu>\mu_0\ \text{ and }\ g(\mu)>0\ \text{ for }\ \mu<\mu_0, \quad (4.12)$$

where

$$g(\mu)=\frac{\kappa_0}{\lambda^{\alpha'}}-\frac{\alpha'\mu}{\lambda}\kappa_1,\ \ \mu_0=\frac{\lambda}{\alpha'\lambda^{\alpha'}}\frac{\kappa_0}{\kappa_1}>0.$$

According to (4.12) with $\mu>\mu_0$, the right side of inequality (4.9) is negative, while the left side of this inequality is non-negative. The obtained contradiction proves Theorem 4. □

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