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PARTIAL DIFFERENTIAL EQUATIONS

Some Properties and Applications of the Riemann and Green–Hadamard Functions for Linear Second-Order Hyperbolic Equations

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Abstract— We study some properties of the Riemann and Green–Hadamard functions for linear second-order hyperbolic equations of general form. We consider their sign definiteness and symmetry in a certain sense and prove a comparison type theorem. Some applications are presented.

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1. GREEN–HADAMARD FUNCTION OF THE FIRST DARBOUX PROBLEM FOR A GENERAL SECOND-ORDER HYPERBOLIC EQUATION WITH TWO INDEPENDENT VARIABLES

In the plane of independent variables x and y, consider the general linear second-order hyperbolic partial differential equation

$$Lu := u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \tag{1}$$

where a, b, c, and f are given real functions and u is the unknown real function. In what follows, we assume that the functions a, a_x, b, b_y, c, and f are continuous in their domain.

For Eq. (1), consider the first Darboux problem in the following setting: in the domain D_T : $0 < y < x < T$, find a regular solution $u(x,y)$ of Eq. (1) with the boundary conditions (e.g., see [1, p. 228; 2, p. 107; 3; 4, p. 7; 5, p. 12])

$$u(x,0) = 0, \qquad u(y,y) = 0, \qquad 0 \le x, y \le T. \tag{2}$$

It is well known that the Green–Hadamard function plays an important role in the study of problem (1), (2); this function was defined and constructed in [3, 4, 6–11] for some special cases of Eq. (1) with singular coefficients. In this section, we present an in a sense modified (as compared with the approaches used in the above-mentioned papers) approach to defining the Green–Hadamard function of problem (1), (2) for Eq. (1) with regular coefficients a, b, and c.

Along with (1), consider the equation

$$L^*v := v_{xy} - (a(x,y)v)_x - (b(x,y)v)_y + c(x,y)v = 0, \tag{3}$$

where L^* is the Lagrange adjoint of L. One can readily see that

$$vLu - uL^*v = (vu_x)_y - (v_yu)_x + (auv)_x + (buv)_y. \tag{4}$$

Take a point $P := P(\xi,\eta) \in D_T$ and consider the corresponding quadrangle

$$\Omega_P := \{(x,y) : \; y < x < \xi, \; 0 < y < \eta\} \subset D_T$$

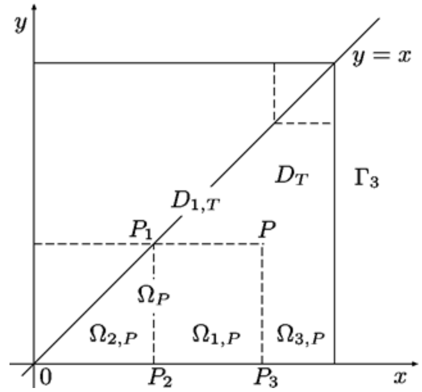

Figure.

with vertices $O(0,0)$, $P_1(\eta,\eta)$, $P(\xi,\eta)$, and $P_3(\xi,0)$, rectangle

$$\Omega_{1,P} := \{(x,y):\ \eta < x < \xi,\ 0 < y < \eta\}$$

with vertices P, P_1, $P_2(\eta,0)$, and P_3, and triangle $\Omega_{2,P} := \{(x,y):\ y < x < \eta,\ 0 < y < \eta\}$ with vertices O, P_1, and P_2 (see the figure).

By using relation (4), where u is a regular solution of Eq. (1) in the domain D_T and v is a regular solution of Eq. (3) in the domains $\Omega_{1,P}$ and $\Omega_{2,P}$, and by applying the Green formula to the above-mentioned subdomains $\Omega_{1,P}$ and $\Omega_{2,P}$ of the domain Ω_P, one can readily justify the relations

$$\int\limits_{\Omega_{i,P}} fv\,dx\,dy = \int\limits_{\partial\Omega_{i,P}} (vu_x\nu_y - v_y u\nu_x + auv\nu_x + buv\nu_y)\,ds =: I_i, \qquad i = 1,2, \tag{5}$$

where $\nu := (\nu_x,\nu_y)$ is the unit outward normal on $\partial\Omega_{i,P}$, $i = 1,2$.

In what follows, we assume that the function v satisfies the conditions

$$(v_y - av)|_{x=\xi} = 0, \qquad (v_x - bv)|_{y=\eta} = 0, \qquad v(\xi,\eta) = 1; \tag{6}$$

i.e., it is the Riemann function $R(x,y;\xi,\eta)$ of the operator L in the domain $\Omega_{1,P}$ (e.g., see [12, p. 449]). Thus, below we use the notation

$$v|_{\Omega_{1,P}} = R(x,y;\xi,\eta). \tag{7}$$

By virtue of (5), (6), and the boundary conditions (2), we have

$$\begin{aligned} I_1 &= -\int\limits_0^\eta v_y u|_{x=\xi}\,dy + \int\limits_0^\eta auv|_{x=\xi}\,dy + \int\limits_\eta^\xi vu_x|_{y=\eta}\,dx + \int\limits_\eta^\xi buv|_{y=\eta}\,dx \\ &\quad + \int\limits_0^\eta v_y u|_{x=\eta+}\,dy - \int\limits_0^\eta auv|_{x=\eta+}\,dy \\ &= (vu)|_{y=\eta}\Big|_{x=\eta}^{x=\xi} - \int\limits_0^\eta u(v_y - av)|_{x=\xi}\,dy - \int\limits_\eta^\xi u(v_x - bv)|_{y=\eta}\,dx + \int\limits_0^\eta u(v_y - av)|_{x=\eta+}\,dy \\ &= u(\xi,\eta) + \int\limits_0^\eta u(v_y - av)|_{x=\eta+}\,dy, \end{aligned} \tag{8}$$

$$I_2 = \int\limits_{OP_1} vu_x\nu_y\,ds - \int\limits_0^{\eta} u(v_y - av)|_{x=\eta-}\,dy. \tag{9}$$

By adding relations (8) and (9) and by taking into account (5), we obtain

$$\int\limits_{\Omega_P} fv\,dx\,dy = u(\xi,\eta) + \int\limits_0^{\eta} u([v]_y - a[v])\,dy + \int\limits_{OP_1} vu_x\nu_y\,ds, \tag{10}$$

where $[v] := v|_{x=\eta+} - v|_{x=\eta-}$.

Now, assuming that

$$v|_{OP_1} = 0 \tag{11}$$

and

$$[v]_y - a[v] = 0, \tag{12}$$

from (10), we directly obtain

$$u(\xi,\eta) = \int\limits_{\Omega_P} fv\,dx\,dy, \qquad P := P(\xi,\eta) \in D_T. \tag{13}$$

Remark 1. Equation (12) with respect to the variable y for the function $v^-(y) := v(x,y)|_{x=\eta-}$ can be treated as the ordinary differential equation

$$v_y^-(y) - a(\eta,y)v^-(y) = d(y),$$

where

$$d(y) := R_y(x,y)|_{x=\eta+} - a(\eta,y)R(x,y)|_{x=\eta+}.$$

In view of the Cauchy condition $v^-(\eta) = 0$, we have

$$v^-(y) = \int\limits_{\eta}^{y} d(y_1)\exp\left\{\int\limits_{y_1}^{y} a(\eta,y_2)\,dy_2\right\}dy_1. \tag{14}$$

The function v satisfying Eq. (3) and conditions (11) and (14) is uniquely determined in the domain $\Omega_{2,P}$ as the solution of the first Darboux problem for hyperbolic equations (e.g., see [2, p. 107; 11, p. 204]).

It follows from the above argument that the function v depends not only on the variables x and y but also on the coordinates ξ and η of the point P; i.e., $v =: G(x,y;\xi,\eta)$, where $(x,y) \in \Omega_P$ with $P := P(\xi,\eta) \in D_T$. In the literature, the function G is known as the Green–Hadamard (or Riemann–Hadamard) function [3, 4, 6–11]; by (7), for $P := P(\xi,\eta) \in D_T$, it has the form

$$G(x,y;\xi,\eta) = \begin{cases} R(x,y;\xi,\eta) & \text{if} \quad x > \eta \text{ and } (x,y) \in \Omega_{1,P} \\ H(x,y;\xi,\eta) & \text{if} \quad x < \eta \text{ and } (x,y) \in \Omega_{2,P}, \end{cases} \tag{15}$$

where

$$H(x,y;\xi,\eta) := v|_{\Omega_{2,P}}.$$

Thus, in view of (15), formula (13) acquires the form

$$u(\xi,\eta) = \int\limits_{\Omega_P} G(x,y;\xi,\eta)f(x,y)\,dx\,dy, \qquad P := P(\xi,\eta) \in D_T. \tag{16}$$

2. REDUCTION OF THE DARBOUX PROBLEM TO THE GOURSAT PROBLEM AND THE CASE IN WHICH THE GREEN–HADAMARD FUNCTION OF THE FIRST DARBOUX PROBLEM (1), (2) CAN BE EXPRESSED VIA THE RIEMANN FUNCTION OF EQ. (1)

Let u be a regular solution of problem (1), (2) in the domain D_T. Consider the function $\widetilde{u}$ defined in the square $D_{1,T} := \{(x, y) :\ 0 < x, y < T\}$ by the formula

$$\widetilde{u}(x, y) = \begin{cases} u(x, y) & \text{if} \quad y < x \\ -u(y, x) & \text{if} \quad y > x. \end{cases} \tag{17}$$

Obviously, it satisfies the equation

$$\widetilde{L}w := w_{xy} + \widetilde{a}(x, y)w_x + \widetilde{b}(x, y)w_y + \widetilde{c}(x, y)w = \widetilde{f}(x, y) \tag{18}$$

in $D_{1,T}\backslash\{(x, y) : y = x\}$ and the homogeneous Goursat conditions

$$w(x, 0) = 0, \qquad w(0, y) = 0, \qquad 0 < x, y < T, \tag{19}$$

where

$$\begin{gathered} \widetilde{a}(x, y) := \begin{cases} a(x, y) & \text{if} \ \ y < x \\ b(y, x) & \text{if} \ \ y > x, \end{cases} \qquad \widetilde{b}(x, y) := \begin{cases} b(x, y) & \text{if} \ \ y < x \\ a(y, x) & \text{if} \ \ y > x, \end{cases} \\ \widetilde{c}(x, y) := \begin{cases} c(x, y) & \text{if} \ \ y < x \\ c(y, x) & \text{if} \ \ y > x, \end{cases} \qquad \widetilde{f}(x, y) := \begin{cases} f(x, y) & \text{if} \ \ y < x \\ -f(y, x) & \text{if} \ \ y > x. \end{cases} \end{gathered} \tag{20}$$

For the Riemann function $\widetilde{R}(x, y; \xi, \eta)$ for Eq. (18) in the domain $D_{1,T}$ to be well defined, we require that $\widetilde{a}$, $\widetilde{a}_x$, $\widetilde{b}$, $\widetilde{b}_y$, and $\widetilde{c}$ be continuous functions in $\overline{D}_{1,T}$, which is equivalent to the relations

$$a(x, x) = b(x, x), \qquad a_x(x, x) = b_y(x, x), \qquad 0 \le x \le T. \tag{21}$$

In what follows, we require that $\widetilde{f} \in C(\overline{D}_{1,T})$, which is equivalent to the conditions

$$f \in C(\overline{D}_T) \quad \text{and} \quad f(x, x) = 0, \quad 0 \le x \le T. \tag{22}$$

If conditions (21) and (22) hold, then one can show that the function $\widetilde{u}$ given by (17) is a regular solution of the Goursat problem (18), (19) in $D_{1,T}$. In this case, the solution of problem (18), (19) at the point $P := P(\xi, \eta)$ of the domain $D_{1,T}$ can be represented in the form (e.g., see [12, p. 450; 13])

$$w(\xi, \eta) = \int_0^\eta dy \int_0^\xi \widetilde{R}(x, y; \xi, \eta)\widetilde{f}(x, y)\, dx. \tag{23}$$

By taking into account relations (20), one can readily show that the function $-w(y, x)$, as well as $w(x, y)$, is a solution of the Goursat problem (18), (19). By virtue of the uniqueness of the solution of this problem, we obtain $w(x, y) = -w(y, x)$, $(x, y) \in D_{1,T}$. This, in turn, implies that, in addition to the first condition in (19), the function w satisfies the condition $w(x, x) = 0$, $0 \le x \le T$; i.e., w is a solution of problem (1), (2) in D_T. Now, by virtue of the uniqueness of the solution of the first Darboux problem (1), (2), we obtain $u = w$ in the domain D_T. Therefore, it follows from the representation (23) of the solution u of problem (1), (2) at the point $P := P(\xi, \eta)$ of the domain D_T and from (20) that

$$u(\xi, \eta) = -\int_0^\eta dy \int_0^y \widetilde{R}(x, y; \xi, \eta) f(y, x)\, dx + \int_0^\eta dy \int_y^\xi \widetilde{R}(x, y; \xi, \eta) f(x, y)\, dx$$

$$= -\int_0^\eta dx \int_0^x \widetilde{R}(y,x;\xi,\eta) f(x,y)\,dy + \int_0^\eta dx \int_0^x \widetilde{R}(x,y;\xi,\eta) f(x,y)\,dy$$
$$+ \int_\eta^\xi dx \int_0^\eta \widetilde{R}(x,y;\xi,\eta) f(x,y)\,dy$$
$$= \int_0^\eta dx \int_0^x [\widetilde{R}(x,y;\xi,\eta) - \widetilde{R}(y,x;\xi,\eta)] f(x,y)\,dy + \int_\eta^\xi dx \int_0^\eta \widetilde{R}(x,y;\xi,\eta) f(x,y)\,dy$$
$$= \int_{\Omega_{2,P}} [\widetilde{R}(x,y;\xi,\eta) - \widetilde{R}(y,x;\xi,\eta)] f(x,y)\,dx\,dy + \int_{\Omega_{1,P}} \widetilde{R}(x,y;\xi,\eta) f(x,y)\,dx\,dy. \tag{24}$$

By virtue of the representations (16) and (24), we find that, for $P := P(\xi,\eta) \in D_T$, the function $H(x,y;\xi,\eta)$ in (15) can be expressed via the Riemann function $\widetilde{R}(x,y;\xi,\eta)$ of Eq. (18) by the formula
$$H(x,y;\xi,\eta) = \widetilde{R}(x,y;\xi,\eta) - \widetilde{R}(y,x;\xi,\eta), \qquad (x,y) \in \Omega_{2,P}; \tag{25}$$
moreover, $\widetilde{R}(x,y;\xi,\eta) = R(x,y;\xi,\eta)$, $(x,y) \in \Omega_{1,P}$.

Now let us additionally show that
$$\widetilde{R}(x,y;\xi,\eta) = R(y,x;\eta,\xi), \qquad (y,x) \in \Omega_{3,P}, \tag{26}$$
for $\widetilde{P} := \widetilde{P}(\eta,\xi) \in D_T$, where $\Omega_{3,P} := \{(x,y) :\ \eta < x < T,\ 0 < y < \xi,\ y < x\}$.

To prove relation (26), note the following.

1. The Riemann function $R(x,y;\xi,\eta)$ satisfies the equation
$$L^*R := R_{xy}(x,y;\xi,\eta) - (a(x,y)R(x,y;\xi,\eta))_x - (b(x,y)R(x,y;\xi,\eta))_y + c(x,y)R(x,y;\xi,\eta) = 0 \tag{27}$$
for $(x,y) \in \Omega_{3,P}$ and the Goursat conditions
$$\begin{aligned} R|_{x=\xi} &:= R(\xi,y;\xi,\eta) = \exp\left\{\int_\eta^y a(\xi,\eta_1)\,d\eta_1\right\}, \\ R|_{y=\eta} &:= R(x,\eta;\xi,\eta) = \exp\left\{\int_\xi^x b(\xi_1,\eta)\,d\xi_1\right\}. \end{aligned} \tag{28}$$

2. By virtue of (20), (27), and (28), the Riemann function $\widetilde{R}(x,y;\xi,\eta)$ satisfies the equation
$$\widetilde{L}^*\widetilde{R} := \widetilde{R}_{xy} - (b(y,x)\widetilde{R})_x - (a(y,x)\widetilde{R})_y + c(y,x)\widetilde{R} = 0 \tag{29}$$
for $(y,x) \in \Omega_{3,P}$ and the Goursat conditions
$$\begin{aligned} \widetilde{R}|_{x=\xi} &:= \widetilde{R}(\xi,y;\xi,\eta) = \exp\left\{\int_\eta^y b(\eta_1,\xi)\,d\eta_1\right\}, \\ \widetilde{R}|_{y=\eta} &:= \widetilde{R}(x,\eta;\xi,\eta) = \exp\left\{\int_\xi^x a(\eta,\xi_1)\,d\xi_1\right\}. \end{aligned} \tag{30}$$

Let us show that the function $R_1(x,y;\xi,\eta)$ given by the formula

$$R_1(x,y;\xi,\eta)=R(y,x;\eta,\xi) \quad \text{for} \quad (y,x)\in\Omega_{3,P}, \tag{31}$$

satisfies problem (29), (30).

Indeed, we have

$$\widetilde{L}^*R_1=R_{xy}(y,x;\eta,\xi)-(a(y,x)R(y,x;\eta,\xi))_y-(b(y,x)R(y,x;\eta,\xi))_x+c(y,x)R(y,x;\eta,\xi). \tag{32}$$

By exchanging the variables x and y and the variables ξ and η in relation (27) and by taking into account (32), we obtain $\widetilde{L}^*R_1=0$; i.e., the function R_1 satisfies Eq. (29). Next, by virtue of relations (28) and (31), we have

$$R_1|_{x=\xi}:=R_1(\xi,y;\xi,\eta)=R(y,\xi;\eta,\xi)=\exp\left\{\int\limits_{\eta}^{y}b(\eta_1,\xi)\,d\eta_1\right\}$$

and

$$R_1|_{y=\eta}:=R_1(x,\eta;\xi,\eta)=R(\eta,x;\eta,\xi)=\exp\left\{\int\limits_{\xi}^{x}a(\eta,\xi_1)\,d\xi_1\right\}.$$

Hence it readily follows that the function R_1, as well as the function $\widetilde{R}$, satisfies problem (29), (30); therefore, by virtue of the uniqueness of the solution of the Goursat problem, we obtain $\widetilde{R}(x,y;\xi,\eta)=R_1(x,y;\xi,\eta)$ for $(y,x)\in\Omega_{3,P}$. This completes the proof of relation (26).

3. NONNEGATIVITY OF THE GREEN–HADAMARD FUNCTION OF THE FIRST DARBOUX PROBLEM (1), (2)

First, let us consider a sufficient condition on the coefficients a, b, and c of Eq. (1) that guarantees the nonnegativity of the Green–Hadamard function of problem (1), (2), that is, the property

$$G(x,y;\xi,\eta)\ge 0, \qquad (x,y)\in\Omega_P, \qquad P:=P(\xi,\eta)\in D_T; \tag{33}$$

at the end of this section, we present its application to the problem on the absence of a global solution of the first Darboux problem for hyperbolic equations with a power-law nonlinearity.

Theorem 1. *Let*

$$k\ge 0 \tag{34}$$

in the domain D_T, where $k:=b_y+ab-c$ is the Laplace invariant of Eq. (1). Then condition (33) is satisfied.

Proof. The operator L in Eq. (1) can be represented in the form

$$Lu=lu-ku, \tag{35}$$

where

$$lu:=\left(\frac{\partial}{\partial y}+a\right)\left(\frac{\partial}{\partial x}+b\right)u. \tag{36}$$

By a straightforward integration, one can readily show that the solution of the problem

$$lU=F; \qquad U(x,0)=0, \qquad U(y,y)=0, \qquad 0\le x,y\le T, \tag{37}$$

can be represented at the point $P:=P(\xi,\eta)$ in the form

$$U(\xi,\eta)=\int\limits_{\Omega_{1,P}}R(x,y;\xi,\eta)F(x,y)\,dx\,dy. \tag{38}$$

Here

$$R(x,y;\xi,\eta) := \exp\left\{\int\limits_{\eta}^{y} a(x,\eta_1)\,d\eta_1 + \int\limits_{\xi}^{x} b(\xi_1,\eta)\,d\xi_1\right\} > 0 \tag{39}$$

is the Riemann function of the operator l given by (36) (e.g., see [14, p. 16]).

In view of relations (35) and (38), the solution of problem (1), (2) satisfies the following integral equation at the point $P := P(\xi,\eta) \in \overline{D}_T$:

$$u(\xi,\eta) = \int\limits_{\Omega_{1,P}} R(x,y;\xi,\eta)k(x,y)u(x,y)\,dx\,dy + \int\limits_{\Omega_{1,P}} R(x,y;\xi,\eta)f(x,y)\,dx\,dy, \quad (\xi,\eta) \in \overline{D}_T. \tag{40}$$

It is known that the Volterra integral equation (40) can be solved by the successive approximation method (e.g., see [5, p. 48])

$$\begin{aligned} u_0 = 0, \qquad u_n(\xi,\eta) \ &= \int\limits_{\Omega_{1,P}} R(x,y;\xi,\eta)k(x,y)u_{n-1}(x,y)\,dx\,dy \\ &+ \int\limits_{\Omega_{1,P}} R(x,y;\xi,\eta)f(x,y)\,dx\,dy, \qquad n \ge 1, \qquad (x,y) \in \overline{D}_T. \end{aligned} \tag{41}$$

By virtue of inequalities (34) and (39) for $f \ge 0$, it follows from the recursion relations (41) that

$$u_n(\xi,\eta) \ge 0, \qquad n = 0,1,\ldots, \qquad (\xi,\eta) \in \overline{D}_T. \tag{42}$$

Since $\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0$, from (42), we have

$$u(\xi,\eta) \ge 0 \quad \text{if} \quad f(\xi,\eta) \ge 0, \quad (\xi,\eta) \in \overline{D}_T. \tag{43}$$

Therefore, by virtue of inequality (43), the right-hand side of relation (16) is nonnegative for any nonnegative function $f \in C(\overline{D}_T)$. This readily implies that condition (33) is satisfied. The proof of Theorem 1 is complete.

Remark 2. Note that if the coefficients of the operator L are constant, then the sufficient condition (34) for the nonnegativity of the Green–Hadamard function of problem (1), (2) is also necessary.

Indeed, suppose that the coefficients a, b, and c in Eq. (1) are constant and

$$k := ab - c < 0. \tag{44}$$

We write out problem (1), (2) for the new unknown function $\widetilde{u} = u\exp(ay + bx)$ in the form

$$\widetilde{u}_{xy} + (c - ab)\widetilde{u} = f\exp(ay + bx), \qquad (x,y) \in \overline{D}_T, \tag{45}$$

$$\widetilde{u}(x,0) = 0, \qquad \widetilde{u}(y,y) = 0, \qquad 0 \le x,y \le T. \tag{46}$$

The solution of this problem at the point $P := P(\xi,\eta) \in \overline{D}_T$ can be represented as follows:

$$\widetilde{u}(\xi,\eta) = \int\limits_{\Omega_P} \widetilde{G}(x,y;\xi,\eta)f(x,y)\exp(ay + bx)\,dx\,dy, \qquad (\xi,\eta) \in \overline{D}_T, \tag{47}$$

where $\widetilde{G}(x,y;\xi,\eta)$ is the Green–Hadamard function of problem (45), (46).

By comparing the representations (16) and (47), one can readily show that

$$G(x,y;\xi,\eta) = \widetilde{G}(x,y;\xi,\eta)\exp\{a(y-\eta) + b(x-\xi)\}.$$

In accordance with (15), the Green–Hadamard function $\widetilde{G}(x,y;\xi,\eta)$ of problem (45), (46) coincides in the characteristic rectangle $\Omega_{1,P}$ with the Riemann function of Eq. (45), which can be represented via the Bessel function J_0 by the formula (e.g., see [12, p. 455])

$$\widetilde{G}(x,y;\xi,\eta) = J_0(2\sqrt{k(x-\xi)(y-\eta)}\,), \qquad (x,y)\in\Omega_{1,P}.$$

It remains to note that the Bessel function J_0 is sign alternating and has infinitely many zeros.

Similar results on the sign definiteness of the Riemann function were obtained in [15].

By way of application of Theorem 1, let us study the global solvability of the first Darboux problem for the nonlinear equation

$$L_0 u := u_{xy} + c(x,y)u = \lambda|u|^\alpha + f(x,y) \tag{48}$$

in the domain D_T with the boundary conditions (2), where $\lambda > 0$ and $\alpha > 1$ are constants and c and f are given continuous functions in the closed domain $\overline{D}_\infty$.

Let u be a classical solution of problem (48), (2) in the domain D_T. Then, by a straightforward integration by parts, one can readily justify the integral relation

$$\int\limits_{D_T} uL_0\varphi\,dx\,dy = \lambda\int\limits_{D_T}|u|^\alpha\varphi\,dx\,dy + \int\limits_{D_T} f\varphi\,dx\,dy \tag{49}$$

for any function φ such that

$$\varphi\in C^2(\overline{D}_T), \qquad \varphi(x,x)=0, \qquad \varphi(T,y)=0, \qquad 0\le x,y\le T. \tag{50}$$

Lemma 1. *Let a function u be a classical solution of problem* (48), (2) *in the domain D_T. If the conditions $c\le 0$ and $f\ge 0$ are satisfied, then $u\ge 0$ in D_T.*

Proof. Obviously, by virtue of (16), the function u is a solution of the nonlinear integral Volterra equation

$$u(\xi,\eta) = \lambda\int\limits_{\Omega_P} G(x,y;\xi,\eta)|u(x,y)|^\alpha\,dx\,dy + \int\limits_{\Omega_P} G(x,y;\xi,\eta)f(x,y)\,dx\,dy, \qquad P := P(\xi,\eta)\in D_T.$$

Next, by virtue of the condition $c\le 0$, we have inequality (34); consequently, inequality (33) holds by Theorem 1. Now, from the last relation, we readily obtain $u\ge 0$ in the domain D_T. Under the assumptions of Lemma 1, relation (49) can be rewritten in the form

$$\int\limits_{D_T}|u|L_0\varphi\,dx\,dy = \lambda\int\limits_{D_T}|u|^\alpha\varphi\,dx\,dy + \int\limits_{D_T} f\varphi\,dx\,dy. \tag{51}$$

In what follows, we use the method of test functions [16, pp. 10–12]. Let us introduce a function $\varphi^0 := \varphi^0(x,y)$ such that

$$\varphi^0\in C^2(\overline{D}_\infty), \quad \varphi^0|_{D_{T=1}} > 0, \quad \varphi^0(x,x)=0, \quad 0\le x\le 1, \quad \varphi^0|_{x\ge 1}=0 \tag{52}$$

and

$$\varkappa := \int\limits_{D_{T=1}} \frac{|\varphi^0_{xy}|^{\alpha'}}{|\varphi^0|^{\alpha'-1}}\,dx\,dy < +\infty, \qquad \alpha' = 1+\frac{1}{\alpha}. \tag{53}$$

One can readily see that the function

$$\varphi^0(x,y) = \begin{cases} (1-x)^n(x-y)^m & \text{for } (x,y)\in D_{T=1}\\ 0 & \text{for } x\ge 1,\end{cases}$$

with sufficiently large positive constants n and m satisfies conditions (52) and (53).

Set $\varphi_T(x,y) := \varphi^0(x/T, y/T)$, $T > 0$. In view of (52), it is easily seen that

$$\varphi_T \in C^2(\overline{D}_T), \quad \varphi_T|_{D_T} > 0, \quad \varphi_T(x,x) = 0, \quad \varphi_T(T,y) = 0, \quad 0 \le x, y \le T. \tag{54}$$

We assume that f is fixed and introduce a function of one variable T by setting

$$\zeta(T) := \int\limits_{D_T} f\varphi_T\,dx\,dy, \qquad T > 0. \tag{55}$$

Proposition [on the absence of global solvability of problem (48), (2)]. *Let $c, f \in C(\overline{D}_\infty)$, and let the conditions $c \le 0$ and $f \ge 0$ be satisfied in the closed domain $\overline{D}_\infty$. If*

$$\liminf_{T\to+\infty} \zeta(T) > 0, \tag{56}$$

then there exists a positive number $T_0 := T_0(f)$ such that for $T > T_0$ problem (48), (2) *has no classical solution in the domain D_T.*

Proof. Suppose that, under the assumptions of this proposition, there exists a classical solution u of problem (48), (2) in the domain D_T. Then, by Lemma 1, relation (51) holds, where, by virtue of (54), the function φ can be chosen in the form $\varphi = \varphi_T$; i.e.,

$$\int\limits_{D_T} |u| L_0 \varphi_T\,dx\,dy = \lambda \int\limits_{D_T} |u|^\alpha \varphi_T\,dx\,dy + \int\limits_{D_T} f\varphi_T\,dx\,dy.$$

This, together with relations (54) and (55) and the condition $c \le 0$, implies the inequality

$$\lambda \int\limits_{D_T} |u|^\alpha \varphi_T\,dx\,dy \le \int\limits_{D_T} |u|\varphi_{Txy}\,dx\,dy - \zeta(T). \tag{57}$$

If we set $a = |u|\varphi_T^{1/p}$ and $b = |\varphi_{Txy}|/\varphi_T^{1/p}$ in the Young inequality with parameter $\varepsilon > 0$,

$$ab \le \frac{\varepsilon}{p} a^p + \frac{1}{p'\varepsilon^{p'-1}} b^{p'}; \qquad a, b \ge 0, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \qquad p := \alpha > 1,$$

then, by using the relation $p'/p = p' - 1$, we obtain

$$|u\varphi_{Txy}| = |u|\varphi_T^{1/p} \frac{|\varphi_{Txy}|}{\varphi_T^{1/p}} \le \frac{\varepsilon}{p} |u|^p \varphi_T + \frac{1}{p'\varepsilon^{p'-1}} \frac{|\varphi_{Txy}|^{p'}}{\varphi_T^{p'-1}}.$$

By taking into account (57), from the last inequality, we obtain

$$\left(\lambda - \frac{\varepsilon}{p}\right) \int\limits_{D_T} |u|^p \varphi_T\,dx\,dy \le \frac{1}{p'\varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\varphi_{Txy}|^{p'}}{\varphi_T^{p'-1}}\,dx\,dy - \zeta(T),$$

which, for $\varepsilon < \lambda p$, implies that

$$\int\limits_{D_T} |u|^p \varphi_T\,dx\,dy \le \frac{p}{(\lambda p - \varepsilon)p'\varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\varphi_{Txy}|^{p'}}{\varphi_T^{p'-1}}\,dx\,dy - \frac{p}{\lambda p - \varepsilon}\,\zeta(T).$$

By using the notation $p' = p/(p-1)$ and $p = p'/(p'-1)$ and the relation

$$\min_{0<\varepsilon<\lambda p} \frac{p}{(\lambda p - \varepsilon)p'\varepsilon^{p'-1}} = \frac{1}{\lambda^{p'}},$$

where the minimum is attained for $\varepsilon = \lambda$, we obtain the inequality

$$\int\limits_{D_T} |u|^p \varphi_T \, dx \, dy \le \frac{1}{\lambda^{p'}} \int\limits_{D_T} \frac{|\varphi_{Txy}|^{p'}}{\varphi_T^{p'-1}} \, dx \, dy - \frac{p'}{\lambda} \, \zeta(T). \tag{58}$$

Since $\varphi_T(x, y) := \varphi^0(x/T, y/T)$, it follows from (52) and (53) that, after the substitution $x = Tx_1$ and $y = Ty_1$, one obtains the easy-to-verify relations

$$\int\limits_{D_T} \frac{|\varphi_{Txy}|^{p'}}{\varphi_T^{p'-1}} \, dx \, dy = T^{-2(p'-1)} \int\limits_{D_{T=1}} \frac{|\varphi^0_{xy}|^{p'}}{|\varphi^0|^{p'-1}} \, dx_1 \, dy_1 = T^{-2(p'-1)} \varkappa < +\infty.$$

This, together with condition (54) and inequality (58), implies that

$$0 \le \int\limits_{D_T} |u|^p \varphi_T \, dx \, dy \le \frac{1}{\lambda^{p'}} T^{-2(p'-1)} \varkappa - \frac{p'}{\lambda} \, \zeta(T). \tag{59}$$

Since $p' = p/(p - 1) > 1$, it follows that $-2(p' - 1) < 0$, and by (53), we have the relation

$$\lim_{T \to \infty} \frac{1}{\lambda^{p'}} T^{-2(p'-1)} \varkappa = 0.$$

Therefore, by virtue of (56), there exists a positive number $T_0 := T_0(f)$ such that if $T > T_0$, then the right-hand side of inequality (59) is negative, while the left-hand side in this inequality is nonnegative. It follows that if there exists a classical solution u of problem (48), (2) in the domain D_T, then we necessarily have $T \le T_0$, which proves the desired assertion.

Remark 3. One can readily see that if $f \in C(\overline{D}_\infty)$, $f \ge 0$ in $\overline{D}_\infty$, and $f(x, y) \ge cx^{-m}$ for $x \ge 1$, where $c :=$ const > 0 and $0 \le m :=$ const ≤ 2, then condition (56) is satisfied, and therefore, for $\lambda > 0$, $\alpha > 1$, and $c \le 0$ in $\overline{D}_\infty$, problem (48), (2) with sufficiently large T has no classical solution u in the domain D_T.

Indeed, introducing the transformation of the independent variables x and y in (55) by the formula $x = Tx_1$, $y = Ty_1$, after simple calculations, we obtain

$$\begin{aligned} \zeta(T) &= T^2 \int\limits_{D_{T=1}} f(Tx_1, Ty_1) \varphi^0(x_1, y_1) \, dx_1 \, dy_1 \\ &\ge cT^{2-m} \int\limits_{D_{T=1} \cap \{x_1 \ge T^{-1}\}} x_1^{-m} \varphi^0(x_1, y_1) \, dx_1 \, dy_1 + T^2 \int\limits_{D_{T=1} \cap \{x_1 < T^{-1}\}} f(Tx_1, Ty_1) \varphi^0(x_1, y_1) \, dx_1 \, dy_1 \end{aligned}$$

under the assumption that $T > 1$. Next, let $T_1 > 1$ be an arbitrary given number. Then from the last inequality for the function ζ, we have

$$\zeta(T) \ge cT^{2-m} \int\limits_{D_{T=1} \cap \{x_1 \ge T^{-1}\}} x_1^{-m} \varphi^0(x_1, y_1) \, dx_1 \, dy_1 \ge c \int\limits_{D_{T=1} \cap \{x_1 \ge T_1^{-1}\}} x_1^{-m} \varphi^0(x_1, y_1) \, dx_1 \, dy_1$$

provided that $T \ge T_1 > 1$ and $m \le 2$, which, together with (54), readily implies inequality (56).

Remark 4. It is known (e.g., see [10, p. 38; 11, p. 230]) that the Green–Hadamard function coincides in the characteristic rectangle $\Omega_{1,P}$ with the Riemann function of the operator L in (1). At the same time, if the Laplace invariant k of Eq. (1) is zero, i.e., by (35), the operator $L = l$ admits the representation (36), then the Green–Hadamard function of problem (37) is zero in the triangular part $\Omega_{2,P}$ of the domain Ω_P. It turns out that the vanishing of the Laplace invariant k of Eq. (1) is a necessary condition for the converse assertion. Indeed, the following theorem holds.

Theorem 2. *The Green–Hadamard function* $G(x,y;\xi,\eta)$ *is zero in the triangular domain* $\Omega_{2,P}$ *for any* $P := P(\xi,\eta) \in \overline{D}_T$ *if and only if the Laplace invariant* k *is the identically zero*,

$$k \equiv 0. \tag{60}$$

Proof. The sufficiency of condition (60) was proved above in this section (see Remark 4). The necessity is a consequence of the following argument. Let $G(x,y;\xi,\eta) = 0$ in the domain $\Omega_{2,P}$ for any $P := P(\xi,\eta) \in \overline{D}_T$. Let us show that $k \equiv 0$. Indeed, $(H_y - aH)|_{x=\eta} = 0$ by virtue of (15). This, together with (12), implies that

$$(R_y - aR)|_{x=\eta} = 0. \tag{61}$$

A simple verification shows that, by virtue of (27), the following relation holds:

$$(\beta R_y)_x - (\beta a R)_x - \beta k R = 0, \tag{62}$$

where $\beta(x,y;\xi) := \exp\{\int_x^\xi b(\xi_1,y)\,d\xi_1\} \neq 0$.

We replace the variable x in relation (62) by ξ_1 and integrate the resulting relation with respect to ξ_1 from x to ξ. As a result, by taking into account the first relation in (6), we obtain [15]

$$R_y(x,y;\xi,\eta) = a(x,y)R(x,y;\xi,\eta) - \beta^{-1}(x,y;\xi)\int_x^\xi \beta(\xi_1,y;\xi)k(\xi_1,y)R(\xi_1,y;\xi,\eta)\,d\xi_1. \tag{63}$$

By (61), relation (63) with $x = \eta$ acquires the form

$$\int_\eta^\xi \beta(\xi_1,y;\xi)k(\xi_1,y)R(\xi_1,y;\xi,\eta)\,d\xi_1 = 0.$$

By differentiating the last relation with respect to η, we obtain

$$\beta(\eta,y;\xi)R(\eta,y;\xi,\eta)k(\eta,y) - \int_\eta^\xi \beta(\xi_1,y;\xi)R_\eta(\xi_1,y;\xi,\eta)k(\xi_1,y)\,d\xi_1 = 0. \tag{64}$$

Now let us represent the expression $R(\eta,y;\xi,\eta)$ in closed form. By the definition of the Riemann function, we have

$$R(\eta,y;\xi,\eta)|_{y=\eta} = \exp\left\{\int_\xi^\eta b(\xi_1,\eta)\,d\xi_1\right\}. \tag{65}$$

Obviously, the solution of problem (61), (65) is given by the formula

$$R(\eta,y;\xi,\eta) = \exp\left\{\int_\eta^y a(\eta,\eta_1)\,d\eta_1 + \int_\xi^\eta b(\xi_1,\eta)\,d\xi_1\right\} \neq 0.$$

In addition, note that $\beta(\eta,y;\xi) \neq 0$.

Therefore, after the division by the nonzero function $\beta(\eta,y;\xi)R(\eta,y;\xi,\eta)$, Eq. (64) is a homogeneous Volterra equation of the second kind for the function k. Hence it readily follows that $k \equiv 0$.

4. COMPARISON THEOREM FOR THE GREEN–HADAMARD FUNCTION OF THE FIRST DARBOUX PROBLEM (1), (2)

Let a function u^i be a classical solution of the problem

$$L_i u^i := u^i_{xy} + a_i(x,y)u^i_x + b_i(x,y)u^i_y + c_i(x,y)u^i = f_i(x,y), \quad (x,y) \in D_T, \quad i = 1,2, \tag{66}$$

$$u^i(x,0) = 0, \qquad u^i(y,y) = 0, \qquad 0 \le x, y \le T, \qquad i = 1,2, \tag{67}$$

in the domain D_T. By (16), the solution u^i of problem (66), (67) can be represented at the point $P := P(\xi,\eta) \in \overline{D}_T$ in the form

$$u^i(\xi,\eta) = \int\limits_{\Omega_P} G_i(x,y;\xi,\eta) f_i(x,y)\, dx\, dy, \tag{68}$$

where G_i is the Green–Hadamard function of problem (66), (67), $i = 1,2$.

By analogy with (35), we rewrite the operator L_i in (66) in the form

$$L_i u^i = l_i u^i - k_i u^i, \tag{69}$$

where

$$l_i u^i := \left(\frac{\partial}{\partial y} + a_i\right)\left(\frac{\partial}{\partial x} + b_i\right) u^i, \tag{70}$$

and $k_i := b_{iy} + a_i b_i - c_i$ is the Laplace invariant of the operator L_i, $i = 1,2$. By a straightforward integration, one can readily show that the solution of the problem

$$l_i U^i = F_i; \qquad U^i(x,0) = 0, \qquad U^i(y,y) = 0, \qquad 0 \le x, y \le T,$$

can be represented at the point $P := P(\xi,\eta) \in \overline{D}_T$ in the form

$$U^i(\xi,\eta) = \int\limits_{\Omega_{1,P}} R_i(x,y;\xi,\eta) F_i(x,y)\, dx\, dy, \qquad i = 1,2. \tag{71}$$

Here

$$R_i(x,y;\xi,\eta) := \exp\left\{\int\limits_{\eta}^{y} a_i(x,\eta_1)\, d\eta_1 + \int\limits_{\xi}^{x} b_i(\xi_1,\eta)\, d\xi_1\right\} > 0, \qquad i = 1,2, \tag{72}$$

is the Riemann function of the operator l_i acting by formula (70).

By virtue of relations (66), (69), and (71), the solution u^i of problem (66), (67) satisfies the following integral equation at the point $P := P(\xi,\eta) \in \overline{D}_T$:

$$u^i(\xi,\eta) = \int\limits_{\Omega_{1,P}} R_i(x,y;\xi,\eta) k_i(x,y) u^i(x,y)\, dx\, dy + \int\limits_{\Omega_{1,P}} R_i(x,y;\xi,\eta) f_i(x,y)\, dx\, dy, \qquad i = 1,2, \qquad (\xi,\eta) \in \overline{D}_T. \tag{73}$$

It is known that the integral Volterra equation (73) can be solved by the successive approximation method

$$u^i_0 = 0, \qquad u^i_n(\xi,\eta) = \int\limits_{\Omega_{1,P}} R_i(x,y;\xi,\eta) k_i(x,y) u^i_{n-1}(x,y)\, dx\, dy + \int\limits_{\Omega_{1,P}} R_i(x,y;\xi,\eta) f_i(x,y)\, dx\, dy, \quad i = 1,2, \quad n \ge 1, \quad (\xi,\eta) \in \overline{D}_T. \tag{74}$$

Let $f_1 \equiv f_2 =: f \geq 0$, and let the following conditions be satisfied:

$$R_2 \geq R_1 \geq 0, \tag{75}$$
$$k_2 \geq k_1 \geq 0. \tag{76}$$

It follows from (74)–(76) that $u_n^2 \geq u_n^1$, $n = 0, 1, \ldots$, and since

$$\lim_{n\to\infty} \|u_n^i - u^i\|_{C(\overline{D}_T)} = 0, \quad i = 1, 2,$$

we have $u^2 \geq u^1$ in the domain $\overline{D}_T$. In turn, this, together with the representation (68), implies that

$$\int\limits_{\Omega_P} [G_2(x, y; \xi, \eta) - G_1(x, y; \xi, \eta)] f(x, y)\, dx\, dy \geq 0, \qquad P := P(\xi, \eta) \in \overline{D}_T,$$

and consequently, by virtue of the arbitrary choice of the function $f \geq 0$, we have

$$G_2(x, y; \xi, \eta) \geq G_1(x, y; \xi, \eta), \qquad (\xi, \eta) \in \overline{D}_T, \qquad (x, y) \in \overline{\Omega}_P. \tag{77}$$

Note that, by virtue of inequalities (72), condition (75) is equivalent to the condition

$$\int\limits_{\eta}^{y} a_2(x, \eta_1)\, d\eta_1 + \int\limits_{\xi}^{x} b_2(\xi_1, \eta)\, d\xi_1 \geq \int\limits_{\eta}^{y} a_1(x, \eta_1)\, d\eta_1 + \int\limits_{\xi}^{x} b_1(\xi_1, \eta)\, d\xi_1,$$

which, in turn, is equivalent to the simultaneous inequalities

$$\int\limits_{\eta}^{y} a_2(x, \eta_1)\, d\eta_1 \geq \int\limits_{\eta}^{y} a_1(x, \eta_1)\, d\eta_1 \quad \text{and} \quad \int\limits_{\xi}^{x} b_2(\xi_1, \eta)\, d\xi_1 \geq \int\limits_{\xi}^{x} b_1(\xi_1, \eta)\, d\xi_1. \tag{78}$$

We have thereby proved the following assertion.

Theorem 3. *Let conditions* (76) *and* (78) *be satisfied. Then inequality* (77) *holds.*

5. SOME SYMMETRY PROPERTIES OF THE RIEMANN FUNCTION OF EQUATION (1)

As was shown in Section 2, if conditions (21) and (22) are satisfied, then the Green–Hadamard function $G(x, y; \xi, \eta)$ can be represented by relation (25) in the domain $\Omega_{2,P}$ and, by Theorem 2, is zero only under condition (60). At the same time, the vanishing of the right-hand side of relation (25) expresses, in a sense, the symmetry of the Riemann function.

Below, under the assumption that Eq. (1) is defined on the entire plane of the variables x and y, we obtain some symmetry properties of the Riemann function depending on the coefficients of Eq. (1), which are assumed to be smooth.

Theorem 4. *The relation*

$$R(x, y; \xi, \eta) = R(y, x; \xi, \eta) \tag{79}$$

holds if and only if

$$a(x, y) = b(y, x), \qquad c(x, y) = c(y, x), \qquad k(x, y) = 0. \tag{80}$$

Proof. Necessity. Let relation (79) hold. First, we prove the first relation in (80). Indeed, by substituting $\eta = \xi$ and $y = x$ into condition (28), we obtain

$$\int\limits_{\xi}^{x} b(\xi_1, \xi)\, d\xi_1 = \int\limits_{\xi}^{x} a(\xi, \eta_1)\, d\eta_1.$$

By differentiating the resulting relation with respect to x, we obtain $a(x, y) = b(y, x)$.

Now prove the third relation in (80). It follows from (79) that

$$R_y(\eta, y; \xi, \eta) = R_x(y, \eta; \xi, \eta). \tag{81}$$

Next, the second relation in (28) implies that

$$R_x(x, \eta; \xi, \eta) = b(x, \eta) R(x, \eta; \xi, \eta). \tag{82}$$

By virtue of relations (79), (81), and (82) and the above-proved first relation in (80), we obtain

$$R_y(\eta, y; \xi, \eta) = a(\eta, y) R(\eta, y; \xi, \eta),$$

which, together with (63) for $x = \eta$, implies that

$$\int\limits_{\xi}^{\eta} \beta_1^{-1}(t, y; \xi) k(t, y) R(t, y; \xi, \eta)\, dt = 0.$$

Consequently, just as in the proof of Theorem 2, we obtain $k(x, y) = 0$.

Finally, let us prove the second relation in (80). Indeed, by virtue of the already proved third relation in (80), the Riemann function of Eq. (1) can be represented in the form (39); therefore, relation (79) acquires the form

$$\int\limits_{\eta}^{y} a(x, \eta_1)\, d\eta_1 + \int\limits_{\xi}^{x} b(\xi_1, \eta)\, d\xi_1 = \int\limits_{\eta}^{x} a(y, \eta_1)\, d\eta_1 + \int\limits_{\xi}^{y} b(\xi_1, \eta)\, d\xi_1. \tag{83}$$

By differentiating relation (83) first with respect to x and then with respect to y, we obtain

$$a_x(x, y) = a_x(y, x). \tag{84}$$

Next, by taking into account relation (84), the first and third relations in (80), and the definition of the invariant k, one can readily justify the second relation in (80).

Sufficiency. Let us show that relation (79) follows from (80). Indeed, by virtue of the third relation in (80), the Riemann function of Eq. (1) can be represented in the form (39). Therefore, to justify relation (79), it suffices to prove relation (83), which, by virtue of the first relation in (80), acquires the form

$$\int\limits_{\eta}^{y} a(x, \eta_1)\, d\eta_1 + \int\limits_{\xi}^{x} a(\eta, \xi_1)\, d\xi_1 = \int\limits_{\eta}^{x} a(y, \eta_1)\, d\eta_1 + \int\limits_{\xi}^{y} a(\eta, \xi_1)\, d\xi_1. \tag{85}$$

To prove relation (85), consider the function

$$F(x, y; \xi, \eta) := \int\limits_{\eta}^{y} a(x, \eta_1)\, d\eta_1 + \int\limits_{\xi}^{x} a(\eta, \xi_1)\, d\xi_1 - \int\limits_{\eta}^{x} a(y, \eta_1)\, d\eta_1 - \int\limits_{\xi}^{y} a(\eta, \xi_1)\, d\xi_1. \tag{86}$$

One can readily see that relation (84) follows from (80), which, together with (86), implies that

$$F_x = \int_{\eta}^{y} a_x(x,\eta_1)\,d\eta_1 + a(\eta,x) - a(y,x) = 0$$

and

$$F_y = a(x,y) - \int_{\eta}^{x} a_x(y,\eta_1)\,d\eta_1 - a(\eta,y) = 0.$$

By virtue of the relation $F(\xi,\eta;\xi,\eta)=0$, hence we have $F \equiv 0$; i.e., relations (85) and hence (79) are satisfied.

The following assertion can be proved in a similar way.

Theorem 5. *The relation* $R(x,y;\xi,\eta) = R(x,y;\eta,\xi)$ *holds if and only if*

$$a(x,y) = b(y,x), \qquad c(x,y) = c(y,x), \qquad h(x,y) = 0,$$

where $h := a_x + ab - c$ *is the Laplace invariant of Eq.* (1).

Theorem 6. *The relation*

$$R(x,y;\xi,\eta) = R(y,x;\eta,\xi) \tag{87}$$

holds if and only if

$$a(x,y) = b(y,x), \qquad c(x,y) = c(y,x). \tag{88}$$

Proof. By the definition of Riemann functions (e.g., see [17, p. 178]), they satisfy the integral equation

$$R(x,y;\xi,\eta) - \int_{\xi}^{x} b(t,y)R(t,y;\xi,\eta)\,dt - \int_{\eta}^{y} a(x,\tau)R(x,\tau;\xi,\eta)\,d\tau + \int_{\xi}^{x} dt \int_{\eta}^{y} c(t,\tau)R(t,\tau;\xi,\eta)\,d\tau = 1. \tag{89}$$

By replacing the variables x and ξ in this equation by y and η, respectively, we obtain

$$R(y,x;\eta,\xi) - \int_{\eta}^{y} b(\tau,x)R(\tau,x;\eta,\xi)\,d\tau - \int_{\xi}^{x} a(y,t)R(y,t;\eta,\xi)dt + \int_{\eta}^{y} d\tau \int_{\xi}^{x} c(\tau,t)R(\tau,t;\eta,\xi)\,dt = 1. \tag{90}$$

Necessity. Let us assume that relation (87) holds and show that relations (88) hold. Obviously, it follows from (89) and (90) that

$$\int_{\xi}^{x} [a(y,t) - b(t,y)]R(t,y;\xi,\eta)\,dt + \int_{\eta}^{y} [b(\tau,x) - a(x,\tau)]R(x,\tau;\xi,\eta)\,d\tau + \int_{\xi}^{x} dt \int_{\eta}^{y} [c(t,\tau) - c(\tau,t)]R(t,\tau;\xi,\eta)\,d\tau = 0. \tag{91}$$

For $\xi = x$, relation (91) acquires the form

$$\int\limits_{\eta}^{y} [b(\tau, x) - a(x, \tau)] R(x, \tau; x, \eta)\, d\tau = 0.$$

By differentiating the resulting equation with respect to y and by taking into account the first equation in (28), we readily obtain the first equation in (88). To prove the second equation in (88), we rewrite (91) in the form

$$\int\limits_{x}^{\xi} dt \int\limits_{y}^{\eta} \omega(t, \tau) R(t, \tau; \xi, \eta)\, d\tau = 0, \tag{92}$$

where $\omega(t, \tau) := c(t, \tau) - c(\tau, t)$.

By successively differentiating (92) with respect to the variables ξ and η, we obtain

$$R(\xi, \eta; \xi, \eta)\omega(\xi, \eta) + \int\limits_{x}^{\xi} \omega(t, \eta) R_{\xi}(t, \eta; \xi, \eta)\, dt + \int\limits_{y}^{\eta} \omega(\xi, \tau) R_{\eta}(\xi, \tau; \xi, \eta)\, d\tau + \int\limits_{x}^{\xi} dt \int\limits_{y}^{\eta} \omega(t, \tau) R_{\xi\eta}(t, \tau; \xi, \eta)\, d\tau = 0.$$

This, together with the third relation in (6), implies that $\omega = 0$, which, in turn, justifies the second relation in (88).

The sufficiency of relations (88) for the validity of (87) is obvious by virtue of the uniqueness theorem for the solution of a Volterra integral equation of the second kind and relations (89) and (90).

Remark 5. It is well known [12, p. 452] that the symmetry property $R(x, y; \xi, \eta) = R(\xi, \eta; x, y)$ of the Riemann function holds if L is a self-adjoint operator, i.e., if $a(x, y) = b(x, y) = 0$. On the other hand, as follows from the argument similar to the one carried out in proof of Theorem 6, the last condition is necessary as well.

REFERENCES

1. Bitsadze, A.V., *Nekotorye klassy uravnenii v chastnykh proizvodnykh* (Some Classes of Partial Differential Equations), Moscow: Nauka, 1981.
2. Goursat, E., *Kurs matematicheskogo analiza* (Course of Mathematical Analysis), Vol. 3, Moscow, 1933, part I.
3. Moiseev, E.I., Approximation of the Classical Solution of the Darboux Problem by Smooth Solutions, *Differ. Uravn.*, 1984, vol. 20, no. 1, pp. 73–87.
4. Moiseev, E.I., *Uravneniya smeshannogo tipa so spektral'nym parametrom* (Equations of Mixed Type with a Spectral Parameter), Moscow: Moskov. Gos. Univ., 1988.
5. Kharibegashvili, S., Goursat and Darboux Type Problems for Linear Hyperbolic Partial Differential Equations and Systems, *Mem. Differential Equations Math. Phys.*, 1995, vol. 4, pp. 1–127.
6. Gellerstedt, S., Sur un problème aux limits pour une équation linéaire aux dérivées partielles du second ordre de type mixte, *Thésis*, Uppsala, 1935.
7. Copson, E.T., On the Riemann–Green Function, *J. Ratl. Mech. Anal.*, 1958, vol. 1, pp. 324–348.
8. Smirnov, M.M., *Vyrozhdayushchiesya ellipticheskie i giperbolicheskie uravneniya* (Degenerating Elliptic and Hyperbolic Equations), Moscow: Nauka, 1966.
9. Smirnov, M.M., *Vyrozhdayushchiesya giperbolicheskie uravneniya* (Degenerating Hyperbolic Equations), Minsk, 1967.
10. Smirnov, M.M., *Uravneniya smeshannogo tipa* (Equations of Mixed Type), Moscow: Nauka, 1970.

11. Nakhushev, A.M., *Uravneniya matematicheskoi biologii* (Equations of Mathematical Biology), Moscow, 1995.
12. Courant, R., *Partial Differential Equations*, New York, 1962. Translated under the title *Uravneniya s chastnymi proizvodnymi*, Moscow: Mir, 1964.
13. Dzhokhadze, O.M., The Riemann Function for Higher-Order Hyperbolic Equations and Systems with Dominated Lower Terms, *Differ. Uravn.*, 2003, vol. 39, no. 10, pp. 1366–1378.
14. Zhegalov, V.I. and Mironov, A.N., *Differentsial'nye uravneniya so starshimi proizvodnymi* (Differential Equations with Higher-Order Derivatives), Kazan, 2001.
15. Lerner, M.E., Qualitative Properties of the Riemann Function, *Differ. Uravn.*, 1991, vol. 27, no. 12, pp. 2106–2120.
16. Mitidieri, E. and Pokhozhaev, S.I., A Priori Estimates and the Absence of Solutions of Nonlinear Partial Differential Equations and Inequalities, *Tr. Mat. Inst. Steklova*, 2001, vol. 234, pp. 1–384.
17. Bitsadze, A.V., *Uravneniya matematicheskoi fiziki* (Equations of Mathematical Physics), Moscow: Nauka, 1976.