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PARTIAL DIFFERENTIAL EQUATIONS

Cauchy Problem for a Generalized Nonlinear Liouville Equation

S. S. Kharibegashvili and O. M. Dzhokhadze

Georgia Technical University, Tbilisi, Georgia
Tbilisi State University, Tbilisi, Georgia

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Abstract—We consider the Cauchy problem for a generalized Liouville equation. We study the existence, uniqueness, and absence of a global solution of this problem. We also discuss the local solvability of the problem.

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1. STATEMENT OF THE PROBLEM

For the nonlinear equation

$$Lu := u_{tt} - u_{xx} + a(x,t)e^u = f(x,t) \tag{1.1}$$

in the half-plane $\Omega := \{(x,t):\ x \in \mathbb{R},\ t > 0\}$, we consider the Cauchy problem with the initial conditions

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \mathbb{R} := (-\infty, +\infty), \tag{1.2}$$

where a, f, φ, and ψ are given real functions and u is the unknown real function.

Note that if $a = \text{const} \neq 0$ and $f = 0$, then Eq. (1.1) is the classical Liouville equation, for which the Cauchy problem has been completely studied (e.g., see [1, 2]). At the same time, methods suggested in these papers for the analysis of the Cauchy problem cannot be used for a broad class of functions a and f.

Let $P_0 := P_0(x_0, t_0)$ be an arbitrary point of the domain Ω, and let $D_{P_0} := \{(x,t):\ t + x_0 - t_0 < x < -t + x_0 + t_0,\ t > 0\}$ be the triangular domain bounded by the characteristic segments γ_{1,P_0}: $x = t + x_0 - t_0$ and γ_{2,P_0}: $x = -t + x_0 + t_0$, $0 \le t \le t_0$, of Eq. (1.1) and by the segment $\gamma_{P_0}:\ t = 0$, $x_0 - t_0 \le x \le x_0 + t_0$.

First, the Cauchy problem for Eq. (1.1) is posed in the finite domain D_{P_0}: find a solution $u = u(x,t)$, $(x,t) \in D_{P_0}$, of Eq. (1.1) satisfying the initial conditions

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \gamma_{P_0}, \tag{1.3}$$

where φ and ψ are given real functions on γ_{P_0}.

Definition 1.1. Let $a, f \in C(\overline{D}_{P_0})$, $\varphi \in C^1(\gamma_{P_0})$, and $\psi \in C(\gamma_{P_0})$. A function u is called a *generalized solution of problem* (1.1), (1.3) *in the class C in the domain D_{P_0}* if $u \in C(\overline{D}_{P_0})$ and there exists a sequence of functions $u_n \in C^2(\overline{D}_{P_0})$ such that $u_n \to u$ and $Lu_n \to f$ in the space $C(\overline{D}_{P_0})$ and $u_n(\cdot,0) \to \varphi$ and $u_{nt}(\cdot,0) \to \psi$ in the spaces $C^1(\gamma_{P_0})$ and $C(\gamma_{P_0})$, respectively, as $n \to \infty$.

Remark 1.1. Obviously, a classical solution of problem (1.1), (1.3) in the class $C^2(\overline{D}_{P_0})$ is a *strong generalized solution* of this problem in the class C in the domain D_{P_0}. In turn, if a strong generalized solution of problem (1.1), (1.3) in the class C in the domain D_{P_0} belongs to the space $C^2(\overline{D}_{P_0})$, then it is also a *classical solution* of that problem.

Definition 1.2. Now let $a, f \in C(\overline{\Omega})$, $\varphi \in C^1(\mathbb{R})$, and $\psi \in C(\mathbb{R})$. We say that problem (1.1), (1.3) is *globally solvable in the class* C if, for any point $P_0 \in \Omega$, this problem has a strong generalized solution in the class C in the domain D_{P_0}.

Definition 1.3. Let $a, f \in C(\overline{\Omega})$, $\varphi \in C^1(\mathbb{R})$, and $\psi \in C(\mathbb{R})$. A function $u \in C(\overline{\Omega})$ is called a *global strong generalized solution of problem* (1.1), (1.2) *in the class* C if, for any point $P_0 \in \Omega$, it is a strong generalized solution of problem (1.1), (1.3) in the class C in the domain D_{P_0} in the sense of Definition 1.1.

Remark 1.2. Note that in the case where the existence and uniqueness theorem holds for a strong generalized solution of problem (1.1), (1.3) in the class C in the domain D_{P_0} for any $P_0 \in \Omega$, we obtain the existence of a unique global strong generalized solution of problem (1.1), (1.2) in the class C in the sense of Definition 1.3.

The present paper is organized as follows. In Section 2, under some constraints for the coefficient a of Eq. (1.1), we derive an a priori estimate for a strong generalized solution of problem (1.1), (1.3) in the class C in the domain D_{P_0} in the sense of Definition 1.1. In Section 3, we prove the existence of that solution. In Section 4, we perform an equivalent reduction of the posed problem to a nonlinear integral equation of Volterra type in the class of continuous functions. In Section 5, we prove the existence of a strong generalized solution of this problem in the sense of Definition 1.1, and in Section 6 we provide a proof of the existence of a unique global classical solution of problem (1.1), (1.2). In Section 7, we consider the problem on the local solvability of problem (1.1), (1.3) and (1.1), (1.2) with no constraint imposed on the continuous function a, and in Section 8, we analyze the absence of a strong generalized solution of problem (1.1), (1.3) in the class C in the domain D_{P_0} in the sense of Definition 1.1.

2. A PRIORI ESTIMATE FOR A STRONG GENERALIZED SOLUTION OF PROBLEM (1.1), (1.3) IN THE CLASS C IN THE DOMAIN D_{P_0}

Consider the conditions

$$a, a_t \in C(\overline{\Omega}); \quad a \ge 0, \quad a_t \le 0 \quad \text{everywhere in} \quad \overline{\Omega} \tag{2.1}$$

and

$$\varphi \in C^1(\mathbb{R}), \qquad \psi \in C(\mathbb{R}). \tag{2.2}$$

Lemma 2.1. *Let conditions* (2.1) *and* (2.2) *be satisfied, and let* P_0 *be an arbitrary point of the domain* Ω. *If* u *is a strong generalized solution of problem* (1.1), (1.3) *in the class* C *in the domain* D_{P_0}, *then one has the a priori estimate*

$$\|u\|_{C(\overline{D}_{P_0})} \le c_1(\|f\|_{C(\overline{D}_{P_0})} + \|\varphi\|_{C^1(\gamma_{P_0})} + \|a(\cdot,0)e^{\varphi}\|^{1/2}_{C(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})}) \tag{2.3}$$

with a positive constant $c_1 = c_1(t_0)$ *independent of* u, a, f, φ, *and* ψ.

Proof. Let u be a strong generalized solution of problem (1.1), (1.3) in the class C in the domain D_{P_0}, and let $P_0 \in \Omega$. Then, by virtue of Definition 1.1, there exists a sequence of functions $u_n \in C^2(\overline{D}_{P_0})$ such that

$$\begin{gathered}\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_{P_0})} = 0, \qquad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_{P_0})} = 0, \\ \lim_{n\to\infty} \|u_n(\cdot,0) - \varphi\|_{C^1(\gamma_{P_0})} = 0, \qquad \lim_{n\to\infty} \|u_{nt}(\cdot,0) - \psi\|_{C(\gamma_{P_0})} = 0,\end{gathered} \tag{2.4}$$

and consequently, by virtue of the continuity of the function a,

$$\lim_{n\to\infty} \|a(e^{u_n} - e^u)\|_{C(\overline{D}_{P_0})} = 0. \tag{2.5}$$

Consider the function $u_n \in C^2(\overline{D}_{P_0})$ as a solution of the Cauchy problem

$$Lu_n = f_n, \tag{2.6}$$

$$u_n(x,0) = \varphi_n(x), \qquad u_{nt}(x,0) = \psi_n(x), \qquad x \in \gamma_{P_0}. \tag{2.7}$$

Here

$$f_n := Lu_n, \qquad \varphi_n := u_n(\cdot,0), \qquad \psi_n := u_{nt}(\cdot,0). \tag{2.8}$$

Let $P_0' := P_0'(x_0', t_0') \in D_{P_0}$. Then, obviously, $D_{P_0'} \subset D_{P_0}$ and $\gamma_{P_0'} \subset \gamma_{P_0}$. By multiplying both sides of relation (2.6) by u_{nt} and by integrating the resulting relation over the domain

$$D_{P_0',\tau} := \{(x,t) \in D_{P_0'} :\ 0 < t < \tau\}, \qquad 0 < \tau < t_0',$$

we obtain

$$\int\limits_{D_{P_0',\tau}} (u_{nt}^2)_t \, dx\, dt - 2 \int\limits_{D_{P_0',\tau}} (u_{nx}u_{nt})_x \, dx\, dt + \int\limits_{D_{P_0',\tau}} (u_{nx}^2)_t \, dx\, dt$$
$$+ 2 \int\limits_{D_{P_0',\tau}} (ae^{u_n})_t \, dx\, dt - 2 \int\limits_{D_{P_0',\tau}} a_t e^{u_n} \, dx\, dt = 2 \int\limits_{D_{P_0',\tau}} f_n u_{nt} \, dx\, dt.$$

Set $\Omega_{P_0',\tau} := \overline{D}_{P_0'} \cap \{t = \tau\},\ 0 < \tau < t_0'$. Then, by taking into account (2.7) and by integrating the left-hand side in the last relation by parts, we reduce it to the form

$$2 \int\limits_{D_{P_0',\tau}} f_n u_{nt} \, dx\, dt + 2 \int\limits_{D_{P_0',\tau}} a_t e^{u_n} \, dx\, dt$$
$$= \sum_{i=1}^{2} \int\limits_{\gamma_{i,P_0',\tau}} \nu_t^{-1} [(\nu_t u_{nx} - \nu_x u_{nt})^2 + u_{nt}^2(\nu_t^2 - \nu_x^2) + 2ae^{u_n}\nu_t^2] \, ds$$
$$- \int\limits_{x_0'-t_0'}^{x_0'+t_0'} [\varphi_n'^2(x) + \psi_n^2(x) + 2a(x,0)e^{\varphi_n}] \, dx + \int\limits_{\Omega_{P_0',\tau}} [u_{nx}^2 + u_{nt}^2 + 2a(x,\tau)e^{u_n}] \, dx, \tag{2.9}$$

where $\nu := (\nu_x, \nu_t)$ is the unit outward normal on $\partial D_{P_0',\tau}$ and $\gamma_{i,P_0',\tau} := \gamma_{i,P_0'} \cap \{t \le \tau\}$, $i = 1, 2$.

Since the relations

$$\nu_t|_{\gamma_{i,P_0'}} > 0, \qquad (\nu_t^2 - \nu_x^2)|_{\gamma_{i,P_0'}} = 0, \qquad i = 1, 2,$$

are satisfied everywhere on $\gamma_{i,P_0'}$, $i = 1, 2$, it follows from (2.1) and (2.9) that

$$w_n(\tau) \le 2 \int\limits_{D_{P_0',\tau}} f_n u_{nt} \, dx\, dt + \alpha_n. \tag{2.10}$$

Here

$$w_n(\tau) := \int\limits_{\Omega_{P_0',\tau}} (u_{nx}^2 + u_{nt}^2) \, dx + \sum_{i=1}^{2} \int\limits_{\gamma_{i,P_0',\tau}} \nu_t^{-1} (\nu_t u_{nx} - \nu_x u_{nt})^2 \, ds, \tag{2.11}$$

$$\alpha := \int\limits_{x_0'-t_0'}^{x_0'+t_0'} [\varphi_n'^2(x) + \psi_n^2(x) + 2a(x,0)e^{\varphi_n}] \, dx. \tag{2.12}$$

By using the relation

$$2\int\limits_{D_{P_0',\tau}} f_n u_{nt}\,dx\,dt \le \int\limits_{D_{P_0',\tau}} u_{nt}^2\,dx\,dt + \|f_n\|^2_{L_2(D_{P_0,\tau})},$$

we rewrite inequality (2.10) in the form

$$w_n(\tau) \le \int\limits_{D_{P_0',\tau}} u_{nt}^2\,dx\,dt + \|f_n\|^2_{L_2(D_{P_0,\tau})} + \alpha_n.$$

This, together with (2.11), implies that

$$w_n(\tau) \le \int\limits_0^\tau w_n(\sigma)\,d\sigma + \|f_n\|^2_{L_2(D_{P_0,\tau})} + \alpha_n, \qquad 0<\tau<t_0'.$$

Since the quantity $\|f_n\|^2_{L_2(D_{P_0,\tau})}$ is a nondecreasing function of τ, it follows from the last inequality and the Gronwall lemma (e.g., see [3, p. 13]) that

$$w_n(\tau) \le e^\tau(\|f_n\|^2_{L_2(D_{P_0,\tau})} + \alpha_n). \tag{2.13}$$

One can readily see that $\nu_t\dfrac{\partial}{\partial x} - \nu_x\dfrac{\partial}{\partial t}$ is an operator of interior differentiation along the unit tangent of $\gamma_{1,P_0'}$. Therefore, the integration along the segment $\gamma_{1,P_0'}$ leads to the relation

$$u_n(x_0',t_0') = \varphi_n(x_0'-t_0') + \int\limits_{\gamma_{1,P_0'}} (\nu_t u_{nx} - \nu_x u_{nt})\,ds.$$

Then, by squaring both sides of this relation and by using the Cauchy and Schwarz inequalities, we obtain

$$\begin{aligned}|u_n(x_0',t_0')|^2 &\le 2\varphi_n^2(x_0'-t_0') + 2\int\limits_{\gamma_{1,P_0'}} ds \int\limits_{\gamma_{1,P_0'}} (\nu_t u_{nx} - \nu_x u_{nt})^2\,ds \\ &\le 2\varphi_n^2(x_0'-t_0') + 2\sqrt{2}\,t_0 \int\limits_{\gamma_{1,P_0'}} (\nu_t u_{nx} - \nu_x u_{nt})^2\,ds.\end{aligned}$$

This, together with (2.11)–(2.13), implies that

$$\begin{aligned}|u_n(x_0',t_0')|^2 &\le 2\varphi_n^2(x_0'-t_0') + 4t_0e^{t_0}(\|f_n\|^2_{L_2(D_{P_0})} + \alpha_n) \le 2\varphi_n^2(x_0'-t_0') \\ &\quad + 4t_0e^{t_0}(\|f_n\|^2_{C(\overline{D}_{P_0})}\operatorname{mes} D_{P_0} + 2t_0\|\varphi_n'\|^2_{C(\gamma_{P_0})} + 2t_0\|\psi_n\|^2_{C(\gamma_{P_0})} + 4t_0\|a(\cdot,0)e^{\varphi_n}\|_{C(\gamma_{P_0})}) \\ &= 2\varphi_n^2(x_0'-t_0') + 4t_0^2e^{t_0}(t_0\|f_n\|^2_{C(\overline{D}_{P_0})} + 2\|\varphi_n'\|^2_{C(\gamma_{P_0})} + 2\|\psi_n\|^2_{C(\gamma_{P_0})} + 4\|a(\cdot,0)e^{\varphi_n}\|_{C(\gamma_{P_0})}).\end{aligned}$$

Here we have used the obvious inequalities

$$\|\cdot\|^2_{L_2(D_{P_0})} \le \|\cdot\|^2_{C(\overline{D}_{P_0})}\operatorname{mes} D_{P_0} = t_0^2\|\cdot\|^2_{C(\overline{D}_{P_0})}; \qquad \|\cdot\|^2_{L_2(\gamma_{P_0})} \le 2t_0\|\cdot\|^2_{C(\gamma_{P_0})}.$$

Consequently,

$$|u_n(x_0',t_0')| \le c_1(\|f_n\|_{C(\overline{D}_{P_0})} + \|\varphi_n\|_{C^1(\gamma_{P_0})} + \|a(\cdot,0)e^{\varphi_n}\|^{1/2}_{C(\gamma_{P_0})} + \|\psi_n\|_{C(\gamma_{P_0})}),$$

where

$$c_1^2 := \max\{4t_0^3 e^{t_0}, 2 + 8t_0^2 e^{t_0}, 16t_0^2 e^{t_0}\}. \tag{2.14}$$

By passing in this inequality to the limit as $n \to \infty$ and by taking into account (2.4) and (2.8), we obtain the estimate

$$|u(x_0', t_0')| \le c_1(\|f\|_{C(\overline{D}_{P_0})} + \|\varphi\|_{C^1(\gamma_{P_0})} + \|a(\cdot, 0)e^{\varphi}\|^{1/2}_{C(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})}). \tag{2.15}$$

Since $P_0' := P_0'(x_0', t_0') \in D_{P_0}$ is an arbitrary point of the domain D_{P_0}, we have the estimate (2.3).

3. UNIQUENESS OF A STRONG GENERALIZED SOLUTION OF PROBLEM (1.1), (1.3) IN THE CLASS C IN THE DOMAIN D_{P_0}

Theorem 3.1. *Let $a, f \in C(\overline{\Omega})$, and let condition* (2.2) *be satisfied. Then, for any given point $P_0 \in \Omega$, problem* (1.1), (1.3) *has at most one strong generalized solution of the class C in the domain D_{P_0}.*

Proof. Indeed, suppose that problem (1.1), (1.3) has two distinct strong generalized solutions u^1 and u^2 of the class C in the domain D_{P_0}. Then, by Definition 1.1, there exists a sequence of functions $u_n^i \in C^2(\overline{D}_{P_0})$ such that

$$\begin{aligned} &\lim_{n\to\infty} \|u_n^i - u^i\|_{C(\overline{D}_{P_0})} = 0, \qquad \lim_{n\to\infty} \|Lu_n^i - f\|_{C(\overline{D}_{P_0})} = 0, \\ &\lim_{n\to\infty} \|u_n^i(\cdot, 0) - \varphi\|_{C^1(\gamma_{P_0})} = 0, \qquad \lim_{n\to\infty} \|u_{nt}^i(\cdot, 0) - \psi\|_{C(\gamma_{P_0})} = 0 \end{aligned} \tag{3.1}$$

and

$$\lim_{n\to\infty} \|a(e^{u_n^i} - e^{u^i})\|_{C(\overline{D}_{P_0})} = 0, \qquad i = 1, 2. \tag{3.2}$$

Set $\omega_n := u_n^2 - u_n^1$ and $\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$. One can readily see that the function $\omega_n \in C^2(\overline{D}_{P_0})$ satisfies the relations

$$\Box\omega_n + g_n = f_n, \tag{3.3}$$

$$\omega_n|_{\gamma_{P_0}} = \tau_n, \qquad \omega_{nt}|_{\gamma_{P_0}} = \nu_n, \tag{3.4}$$

where

$$g_n := a(e^{u_n^2} - e^{u_n^1}), \quad f_n := Lu_n^2 - Lu_n^1, \quad \tau_n := (u_n^2 - u_n^1)|_{\gamma_{P_0}}, \quad \nu_n := (u_n^2 - u_n^1)_t|_{\gamma_{P_0}}. \tag{3.5}$$

By virtue of the first relation in (3.1), there exists a number $K := \mathrm{const} > 0$ independent of the indices i and n and such that

$$\|u_n^i\|_{C(\overline{D}_{P_0})} \le K. \tag{3.6}$$

By virtue of relations (3.1) and (3.5), we have

$$\lim_{n\to\infty} \|\tau_n\|_{C^1(\gamma_{P_0})} = 0, \qquad \lim_{n\to\infty} \|\nu_n\|_{C(\gamma_{P_0})} = 0, \qquad \lim_{n\to\infty} \|f_n\|_{C(\overline{D}_{P_0})} = 0. \tag{3.7}$$

By virtue of the estimate (3.6) and the first relation in (3.5), one can readily see that

$$|g_n| \le a_0 e^K |\omega_n|, \tag{3.8}$$

where $a_0 := \|a\|_{C(\overline{D}_{P_0})}$.

By multiplying both sides of relation (3.3) by ω_{nt} and by integrating the resulting relation over the domain $D_{P_0',\tau}$, where $P_0' := P_0'(x_0', t_0') \in D_{P_0}$, by using (3.4), and by following the lines of the derivation of relation (2.9) from (2.6) and (2.7), we obtain

$$v_n(\tau) := \int\limits_{\Omega_{P_0',\tau}} (\omega_{nx}^2 + \omega_{nt}^2)\,dx + \sum_{i=1}^{2} \int\limits_{\gamma_{i,P_0',\tau}} \nu_t^{-1}(\nu_t\omega_{nx} - \nu_x\omega_{nt})^2\,ds$$
$$= 2\int\limits_{D_{P_0',\tau}} (f_n - g_n)\omega_{nt}\,dx\,dt + \|\tau_n'\|^2_{L_2(\gamma_{P_0'})} + \|\nu_n\|^2_{L_2(\gamma_{P_0'})}. \tag{3.9}$$

By virtue of the estimate (3.8) and the Cauchy inequality, we obtain the relation

$$2\int\limits_{D_{P_0',\tau}} (f_n - g_n)\omega_{nt}\,dx\,dt \le 2\int\limits_{D_{P_0',\tau}} \omega_{nt}^2\,dx\,dt + \int\limits_{D_{P_0,\tau}} f_n^2\,dx\,dt + \int\limits_{D_{P_0',\tau}} g_n^2\,dx\,dt$$
$$\le 2\int\limits_{D_{P_0',\tau}} \omega_{nt}^2\,dx\,dt + \|f_n\|^2_{L_2(D_{P_0,\tau})} + a_0^2e^{2K}\int\limits_{D_{P_0',\tau}} \omega_n^2\,dx\,dt. \tag{3.10}$$

Next, by virtue of the first relation in (3.4), one can readily see that

$$\omega_n(x,t) = \tau_n(x) + \int\limits_0^t \omega_{nt}(x,\sigma)d\sigma, \qquad (x,t) \in \overline{D}_{P_0',\tau}.$$

By squaring both sides of this relation and by using the Cauchy and Schwarz inequalities, we obtain

$$|\omega_n(x,t)|^2 \le 2\tau_n^2(x) + 2t\int\limits_0^t \omega_{nt}^2(x,\sigma)\,d\sigma, \qquad (x,t) \in \overline{D}_{P_0',\tau}.$$

By setting

$$v(x,t) = \begin{cases} \omega_{nt}(x,t) & \text{for} \quad (x,t) \in \overline{D}_{P_0',\tau} \\ 0 & \text{for} \quad (x,t) \notin \overline{D}_{P_0',\tau}, \end{cases}$$

and by using the inequality $t \le \tau$ satisfied for $(x,t) \in \overline{D}_{P_0',\tau}$, we arrive at the inequality

$$\int\limits_{D_{P_0',\tau}} \omega_n^2\,dx\,dt \le 2\tau\|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau\int\limits_{x_0'-t_0'}^{x_0'+t_0'} dx\int\limits_0^\tau \left(\int\limits_0^\tau v^2(x,\sigma)\,d\sigma\right)dt$$
$$= 2\tau\|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau^2\int\limits_{x_0'-t_0'}^{x_0'+t_0'} dx\int\limits_0^\tau v^2(x,t)\,dt = 2\tau\|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau^2\int\limits_{D_{P_0',\tau}} \omega_{nt}^2\,dx\,dt. \tag{3.11}$$

It follows from relations (3.9)–(3.11) that

$$v_n(\tau) \le 2(\tau^2a_0^2e^{2K} + 1)\int\limits_{D_{P_0',\tau}} \omega_{nt}^2\,dx\,dt$$
$$+ \|f_n\|^2_{L_2(D_{P_0,\tau})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 2\tau a_0^2e^{2K}\|\tau_n\|^2_{L_2(\gamma_{P_0})}$$
$$\le 2(t_0^2a_0^2e^{2K} + 1)\int\limits_0^\tau v_n(\sigma)d\sigma + \|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 2\tau a_0^2e^{2K}\|\tau_n\|^2_{L_2(\gamma_{P_0})}.$$

Consequently, by the Gronwall lemma, for $0 < \tau \le t_0'$, we obtain the estimate

$$v_n(\tau) \le c_2(\|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 2t_0 a_0^2 e^{2K} \|\tau_n\|^2_{L_2(\gamma_{P_0})}),$$

where $c_2 := \exp\{2t_0(t_0^2 a_0^2 e^{2K} + 1)\}$. This, together with (3.9), implies that

$$\int\limits_{\gamma_{1,P_0'}} (\nu_t \omega_{nx} - \nu_x \omega_{nt})^2 \, ds$$
$$\le \sqrt{2} c_2(\|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 2t_0 a_0^2 e^{2K} \|\tau_n\|^2_{L_2(\gamma_{P_0})}), \qquad 0 < \tau \le t_0'.$$

Next, by reproducing the argument used in the derivation of the estimate (2.15), we obtain

$$|\omega_n(x_0', t_0')|^2 \le 2\tau_n^2(x_0' - t_0') + 4t_0^2 c_2 (t_0 \|f_n\|^2_{C(\overline{D}_{P_0})} + 2\|\tau_n'\|^2_{C(\gamma_{P_0})}$$
$$+ 2\|\nu_n\|^2_{C(\gamma_{P_0})} + 4t_0 a_0^2 e^{2K} \|\tau_n\|^2_{C(\gamma_{P_0})}).$$

Consequently, by virtue of (3.7), we have $\lim_{n\to\infty} |\omega_n(x_0', t_0')| = 0$; i.e., $u^2(x_0', t_0') = u^1(x_0', t_0')$ for any $(x_0', t_0') \in \overline{D}_{P_0}$. This completes the proof of Theorem 3.1.

4. EQUIVALENT REDUCTION OF PROBLEM (1.1), (1.3) TO A NONLINEAR INTEGRAL EQUATION OF THE VOLTERRA TYPE IN THE CLASS OF CONTINUOUS FUNCTIONS

Let $u \in C^2(\overline{\Omega})$ be a classical solution of problem (1.1), (1.3). Set

$$D_{x,t} := \{(x_1, t_1) : \; t_1 + x - t < x_1 < -t_1 + x + t, \; t_1 > 0\}, \qquad (x,t) \in \Omega.$$

Note that $D_{P_0} = D_{x,t}$ for $x = x_0$ and $t = t_0$. By using the initial conditions (1.3) for $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$ and by integrating Eq. (1.1) over the domain $D_{x,t}$, we obtain the relation

$$u(x,t) + (\Box^{-1}(ae^u))(x,t) = F(x,t), \qquad (x,t) \in \overline{D}_{P_0}. \tag{4.1}$$

Here

$$F(x,t) := (\Box^{-1} f)(x,t) + (l_1\varphi)(x,t) + (l_2\psi)(x,t), \tag{4.2}$$

and the continuous operators

$$\begin{aligned} l_1 :\; & C^k(\gamma_{P_0}) \to C^k(\overline{D}_{P_0}), \qquad k = 0,1,2, \qquad l_2 : C^k(\gamma_{P_0}) \to C^{k+1}(\overline{D}_{P_0}), \qquad k = 0,1, \\ \Box^{-1} :\; & C^k(\overline{D}_{P_0}) \to C^{k+1}(\overline{D}_{P_0}), \qquad k = 0,1, \end{aligned} \tag{4.3}$$

act by the formulas (e.g., see [4, p. 173])

$$\begin{aligned} (l_1\varphi)(x,t) &:= \frac{1}{2}[\varphi(x+t) + \varphi(x-t)], \qquad (l_2\psi)(x,t) := \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(\xi)\, d\xi, \\ (\Box^{-1} f)(x,t) &:= \frac{1}{2}\int\limits_{D_{x,t}} f(\xi,\tau)\, d\xi\, d\tau. \end{aligned} \tag{4.4}$$

Remark 4.1. Relation (4.1) can be treated as a nonlinear integral equation of Volterra type.

Lemma 4.1. *A function $u \in C(\overline{D}_{P_0})$ is a strong generalized solution of problem* (1.1), (1.3) *of the class C in the domain D_{P_0} if and only if it is a continuous solution of the nonlinear integral equation* (4.1).

Proof. Indeed, let $u \in C(\overline{D}_{P_0})$ be a solution of Eq. (4.1). Since $u(f) \in C(\overline{D}_{P_0})$ and the space $C^2(\overline{D}_{P_0})$ is dense in $C(\overline{D}_{P_0})$ (e.g., see [5, p. 37]), it follows that there exists a sequence of functions $w_n(f_n) \in C^2(\overline{D}_{P_0})$ such that $w_n(f_n) \to u(f)$ in the space $C(\overline{D}_{P_0})$ as $n \to \infty$.

Likewise, since $\varphi \in C^1(\gamma_{P_0})$ [respectively, $\psi \in C(\gamma_{P_0})$], it follows that there exists a sequence of functions $\varphi_n \in C^2(\gamma_{P_0})$ [respectively, $\psi_n \in C^1(\gamma_{P_0})$] such that $\varphi_n \to \varphi$ (respectively, $\psi_n \to \psi$) in the space $C^1(\gamma_{P_0})$ [respectively, $C(\gamma_{P_0})$] as $n \to \infty$.

Set $u_n := -\Box^{-1}(ae^{w_n}) + \Box^{-1} f_n + l_1\varphi_n + l_2\psi_n$, $n = 1, 2, \ldots$ One can readily see that $u_n \in C^2(\overline{D}_{P_0})$, and since, by virtue of (4.3) and (4.4), l_1, l_2, and $\Box^{-1}$ are linear continuous operators in the corresponding spaces and moreover,

$$\lim_{n\to\infty} \|w_n - u\|_{C(\overline{D}_{P_0})} = 0, \qquad \lim_{n\to\infty} \|f_n - f\|_{C(\overline{D}_{P_0})} = 0,$$
$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{C^1(\gamma_{P_0})} = 0, \qquad \lim_{n\to\infty} \|\psi_n - \psi\|_{C(\gamma_{P_0})} = 0,$$

it follows from (2.5) that $u_n \to -\Box^{-1}(ae^u) + \Box^{-1} f + l_1\varphi + l_2\psi$, $u_n(\cdot, 0) \to \varphi$, and $u_{nt}(\cdot, 0) \to \psi$ in the spaces $C(\overline{D}_{P_0})$, $C^1(\gamma_{P_0})$, and $C(\gamma_{P_0})$, respectively, as $n \to \infty$. But from relations (4.1) and (4.2), we obtain $-\Box^{-1}(ae^u) + \Box^{-1} f + l_1\varphi + l_2\psi = u$. Therefore, we have

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_{P_0})} = 0.$$

On the other hand, $\Box u_n = -ae^{w_n} + f_n$, which, together with the relations

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_{P_0})} = 0, \qquad \lim_{n\to\infty} \|w_n - u\|_{C(\overline{D}_{P_0})} = 0, \qquad \lim_{n\to\infty} \|f_n - f\|_{C(\overline{D}_{P_0})} = 0,$$

implies that

$$Lu_n = \Box u_n + ae^{u_n} = -ae^{w_n} + f_n + ae^{u_n} = a(e^{u_n} - e^u) - a(e^{w_n} - e^u) + f_n \to f$$

in the space $C(\overline{D}_{P_0})$ as $n \to \infty$. The converse is obvious.

5. EXISTENCE OF A STRONG GENERALIZED SOLUTION OF PROBLEM (1.1), (1.3) OF THE CLASS C IN THE DOMAIN D_{P_0} AND THE GLOBAL SOLVABILITY IN THE SENSE OF DEFINITION 1.2

As was mentioned above, the operator $\Box^{-1}$ occurring in (4.4) is a linear continuous operator acting, by virtue of (4.3), from the space $C(\overline{D}_{P_0})$ to the space of continuously differentiable functions $C^1(\overline{D}_{P_0})$. Next, since the space $C^1(\overline{D}_{P_0})$ is compactly embedded in the space $C(\overline{D}_{P_0})$ (e.g., see [6, p. 135 of the Russian translation]), we readily obtain the following assertion.

Lemma 5.1. *The operator* $\Box^{-1}: \; C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$ *occurring in* (4.4) *is a linear compact operator.*

We rewrite Eq. (4.1) in the form

$$u = Au := -\Box^{-1}(ae^u) + F, \tag{5.1}$$

where the operator $A: \; C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$ is continuous and compact, because the nonlinear operator $N: \; C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$ acting by the formula $Nu := ae^u$ is bounded and continuous and the linear operator $\Box^{-1}: \; C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$ is compact by virtue of Lemma 5.1. At the same time, by Lemma 2.1 and relation (2.14), for any value of the parameter $\tau \in [0, 1]$ and for any solution $u \in C(\overline{D}_{P_0})$ of the equation $u = \tau Au$, we have the a priori estimate (2.3) with the same positive constant c_1, which occurs in the estimate (2.3) and is independent of u, a, f, φ, ψ, and τ. Therefore, by the Leray–Schauder theorem (e.g., see [7, p. 375]), under the assumptions of Lemma 2.1, Eq. (5.1) has at least one solution $u \in C(\overline{D}_{P_0})$. Therefore, by Lemma 4.1, the following assertion holds.

Theorem 5.1. *If conditions* (2.1) *and* (2.2) *are satisfied, then problem* (1.1), (1.3) *is globally solvable in the class* C *in the sense of Definition* 1.2; *i.e., for each point* $P_0 \in \Omega$, *this problem has a strong generalized solution of the class* C *in the domain* D_{P_0}.

6. EXISTENCE OF A CLASSICAL SOLUTION OF PROBLEM (1.1), (1.2) IN THE HALF-PLANE Ω

Lemma 4.1 and relations (5.1), (4.2), and (4.3) readily imply the following assertion.

Lemma 6.1. *Let* $a, f \in C^1(\overline{\Omega})$, $\varphi \in C^2(\mathbb{R})$, *and* $\psi \in C^1(\mathbb{R})$. *Then any strong generalized solution* u *of problem* (1.1), (1.3) *of the class* C *in the domain* D_{P_0} *in the sense of Definition* 1.1 *is classical, i.e., belongs to the class* $C^2(\overline{D}_{P_0})$.

Lemma 6.2. *Let* $a \in C^1(\overline{\Omega})$, *and let condition* (2.1) *be satisfied. Then, for arbitrary* $f \in C^1(\overline{\Omega})$, $\varphi \in C^2(\mathbb{R})$, *and* $\psi \in C^1(\mathbb{R})$, *problem* (1.1), (1.2) *has a unique global classical solution* $u \in C^2(\overline{\Omega})$ *in the half-plane* Ω.

Proof. By Theorem 3.1, Lemma 6.1, and the argument carried out in Section 5, in the domain D_{x_0,t_0} for $t_0 = n \in \mathbb{N}$, there exists a unique classical solution $u_n \in C^2(\overline{D}_{x_0,n})$ of problem (1.1), (1.3). Since u_{n+1} is a classical solution of problem (1.1), (1.3) in the domain $D_{x_0,n}$ as well, it follows from Theorem 3.1 that $u_{n+1}|_{D_{x_0,n}} = u_n$. Therefore, the function u constructed in the domain Ω by the rule $u(x,t) = u_n(x,t)$ for $n = [t]+1$, where $(x,t) \in \Omega$ and $[t]$ is the integer part of the number t, is the unique classical solution of problem (1.1), (1.2) of the class $C^2(\overline{\Omega})$ in the half-plane Ω. This completes the proof of Lemma 6.2.

7. LOCAL SOLVABILITY OF PROBLEMS (1.1), (1.3) AND (1.1), (1.2)

In what follows, we assume that $a, f \in C(\overline{\Omega})$, $\varphi \in C^1(\mathbb{R})$, and $\psi \in C(\mathbb{R})$.

Theorem 7.1. *For any given* $x_0 \in \mathbb{R}$, *there exists a positive number* $T := T(x_0; a, f, \varphi, \psi) > 0$ *such that, for* $t_0 \in (0, T)$, *problem* (1.1), (1.3) *in the domain* D_{P_0} *has at least one strong generalized solution* u *of the class* C *in the domain* D_{P_0}.

Proof. In Section 4, problem (1.1), (1.3) in the space $C(\overline{D}_{P_0})$ has been equivalently reduced to the integral equation (4.1) of the Volterra type or, which is the same, (5.1). Since $A: \ C(\overline{D}_{P_0}) \to C(\overline{D}_{P_0})$ is a continuous and compact operator, it follows from the Schauder theorem that, for the solvability of Eq. (5.1), it suffices to show that the operator A brings some ball

$$B_R := \{v \in C(\overline{D}_{P_0}) : \ \|v\|_{C(\overline{D}_{P_0})} \le R\}$$

of radius $R > 0$, which is a closed and convex set in the Banach space $C(\overline{D}_{P_0})$, into itself. Let us show that this is the case for sufficiently small t_0.

Take a number $T_0 > 0$ and set

$$\varphi_0 := \|\varphi\|_{C[x_0-T_0,x_0+T_0]}, \qquad \psi_0 := \|\psi\|_{C[x_0-T_0,x_0+T_0]}, \qquad a_1 := \|a\|_{C(\overline{D}_{P_1})}, \qquad f_1 := \|f\|_{C(\overline{D}_{P_1})},$$

where $P_1 := P_1(x_0, T_0)$. This, together with (4.2) and (4.4) for $t_0 < T_0$, implies that

$$\|Au\|_{C(\overline{D}_{P_0})} \le 2^{-1} t_0^2 a_1 \|e^u\|_{C(\overline{D}_{P_0})} + \|F\|_{C(\overline{D}_{P_0})} \le \varphi_0 + [2^{-1} T_0 (a_1 e^R + f_1) + \psi_0] t_0. \tag{7.1}$$

Now, by setting $R := 2\varphi_0$ and $T := \min\{T_0, d^{-1}\varphi_0\}$, where $d := 2^{-1}T_0(a_1 e^R + f_1) + \psi_0$, for $t_0 < T$ from (7.1), we obtain the estimate

$$\|Au\|_{C(\overline{D}_{P_0})} \le \varphi_0 + \varphi_0 = 2\varphi_0 = R.$$

The proof of the theorem is complete.

Now consider the following conditions:

$$\widetilde{a}_1 := \sup_{(x,t)\in\overline{\Omega}} |a(x,t)| < +\infty, \qquad \widetilde{f}_1 := \sup_{(x,t)\in\overline{\Omega}} |f(x,t)| < +\infty,$$
$$\widetilde{\varphi}_0 := \sup_{x\in\mathbb{R}} |\varphi(x)| < +\infty, \qquad \widetilde{\psi}_1 := \sup_{x\in\mathbb{R}} |\psi(x)| < +\infty. \tag{7.2}$$

Theorem 7.2. *Let $a, f \in C^1(\overline{\Omega})$, $\varphi \in C^2(\mathbb{R})$, and $\psi \in C^1(\mathbb{R})$, and let condition* (7.2) *be satisfied. Then there exists a number $T_* := T_*(a, f, \varphi, \psi) > 0$ such that problem* (1.1), (1.2) *in the strip $\Omega_1 := \mathbb{R} \times (0, T_*)$ has a unique classical solution $u \in C^2(\overline{\Omega}_1)$.*

The proof of this theorem follows directly from the uniqueness theorem (Theorem 3.1), Theorem 7.1, and the argument carried out in Lemma 6.2.

8. THE CASE OF ABSENCE OF A GLOBAL SOLUTION OF PROBLEM (1.1), (1.3)

Remark 8.1. Below we show that, under certain conditions imposed on the functions $a, f \in C(\overline{\Omega})$, $\varphi \in C^1(\mathbb{R})$, and $\psi \in C(\mathbb{R})$, for any fixed $x_0 \in \mathbb{R}$, there exists a number

$$T^* := T^*(x_0; a, f, \varphi, \psi) > 0$$

such that, for $t_0 \in (0, T^*)$, problem (1.1), (1.3) has a strong generalized solution of the class C in the domain D_{P_0}, and for $t_0 > T^*$ it does not have such a solution in this domain.

Lemma 8.1. *Let u be a strong generalized solution of problem* (1.1), (1.3) *of the class C in the domain D_{P_0}. Then one has the integral relation*

$$\int\limits_{D_{P_0}} ae^u \chi \, dx\, dt = \int\limits_{x_0-t_0}^{x_0+t_0} [\psi(x)\chi(x,0) - \varphi(x)\chi_t(x,0)]\, dx - \int\limits_{D_{P_0}} u\Box\chi \, dx\, dt + \int\limits_{D_{P_0}} f\chi \, dx\, dt \tag{8.1}$$

for any function χ such that

$$\chi \in C^2(\overline{D}_{P_0}), \qquad \chi|_{\gamma_{i,P_0}} = 0, \qquad i = 1, 2. \tag{8.2}$$

Proof. By the definition of a strong generalized solution u of problem (1.1), (1.3) of the class C in the domain D_{P_0}, we have $u \in C(\overline{D}_{P_0})$, and there exists a sequence of functions $u_n \in C^2(\overline{D}_{P_0})$ satisfying relations (2.4) and (2.5).

Set $f_n := Lu_n$. We multiply both sides of the relation $Lu_n = f_n$ by the function χ and integrate the resulting relation over the domain D_{P_0}. As a result of integration by parts on the left-hand side in this relation with regard of (8.2), we obtain

$$\int\limits_{D_{P_0}} ae^{u_n} \chi \, dx\, dt = \int\limits_{x_0-t_0}^{x_0+t_0} [\psi_n(x)\chi(x,0) - \varphi_n(x)\chi_t(x,0)]\, dx$$
$$- \int\limits_{D_{P_0}} u_n\Box\chi \, dx\, dt + \int\limits_{D_{P_0}} f_n\chi \, dx\, dt. \tag{8.3}$$

Here

$$\varphi_n := u_n(\cdot, 0), \qquad \psi_n := u_{nt}(\cdot, 0). \tag{8.4}$$

By passing in relation (8.3) to the limit as $n \to \infty$ and by taking into account (2.4), (2.5), and (8.4), we obtain the desired relation (8.1).

Lemma 8.2. *Let a function $u \in C(\overline{D}_{P_0})$ be a strong generalized solution of problem* (1.1), (1.3) *of the class C in the domain D_{P_0}. If*

$$a(x,t) \le -c_3 := \text{const} < 0, \quad f(x,t) \ge 0, \quad (x,t) \in \overline{\Omega}, \quad \varphi(x) \ge 0, \quad \psi(x) \ge 0, \quad x \in \mathbb{R}, \tag{8.5}$$

then $u \ge 0$ in the domain D_{P_0}.

Proof. By virtue of (4.2), (4.4) and the property of the operator $\Box^{-1}$, the nonnegativity of the function u readily follows from the representation (5.1).

Under the assumptions of the lemma, by taking into account the inequality $2e^u \ge u^2$ for $u \ge 0$, from (8.1), we obtain

$$c_3 \int\limits_{D_{P_0}} u^2 \chi \, dx \, dt$$
$$\le 2 \int\limits_{x_0-t_0}^{x_0+t_0} [\varphi(x)\chi_t(x,0) - \psi(x)\chi(x,0)] \, dx + 2 \int\limits_{D_{P_0}} u \Box \chi \, dx \, dt - 2 \int\limits_{D_{P_0}} f\chi \, dx \, dt. \tag{8.6}$$

We use the method of test functions (e.g., see [8, pp. 10–12]). Let us introduce a function $\chi^0 := \chi^0(x,t)$ such that

$$\chi^0 \in C^2(\overline{D}_{(0,1)}), \qquad \chi^0|_{D_{(0,1)}} > 0, \qquad \chi^0|_{\gamma_{i,(0,1)}} = 0, \qquad i = 1, 2, \tag{8.7}$$

and

$$\varkappa_0 := \int\limits_{D_{(0,1)}} \frac{|\Box \chi^0|^2}{\chi^0} \, dx \, dt < +\infty. \tag{8.8}$$

One can readily see that, for a function χ^0 satisfying conditions (8.7) and (8.8), one can take, e.g., the function

$$\chi^0 = \chi^*(x,t) := [(1-t)^2 - x^2]^n, \qquad (x,t) \in \overline{D}_{(0,1)}, \tag{8.9}$$

for a sufficiently large positive integer n.

Now, by assuming that $\chi_{P_0}(x,t) = \chi^0((x-x_0)/t_0, t/t_0)$ and by taking into account (8.7), one can readily see that

$$\chi_{P_0} \in C^2(\overline{D}_{P_0}), \qquad \chi_{P_0}|_{D_{P_0}} > 0, \qquad \chi_{P_0}|_{\gamma_{i,P_0}} = 0, \qquad i = 1, 2. \tag{8.10}$$

We assume that the functions f, φ, and ψ and the number x_0 are fixed and introduce a function of the single variable t_0,

$$\zeta(t_0) := \int\limits_{x_0-t_0}^{x_0+t_0} \left[\psi(x)\chi_{P_0}(x,0) - \varphi(x) \frac{\partial \chi_{P_0}(x,0)}{\partial t} \right] dx + \int\limits_{D_{P_0}} f \chi_{P_0} \, dx \, dt. \tag{8.11}$$

We have the following assertion on the absence of the global solvability of problem (1.1), (1.3).

Theorem 8.1. *Let condition* (8.5) *be satisfied, and let a function $u \in C(\overline{D}_{P_0})$ be a strong generalized solution of problem* (1.1), (1.3) *of the class C in the domain D_{P_0}. If*

$$\liminf_{t_0 \to +\infty} \zeta(t_0) > 0, \tag{8.12}$$

then there exists a positive number $T^0 := T^0(x_0; a, f, \varphi, \psi) > 0$ such that, for $t_0 > T^0$, problem (1.1), (1.3) *does not have a strong generalized solution of the class C in the domain D_{P_0}.*

Proof. Suppose that, under assumptions of this lemma, there exists a strong generalized solution u of problem (1.1), (1.3) of the class C in the domain D_{P_0}. Then, by Lemmas 8.1 and 8.2, we have inequality (8.6), where, by virtue of (8.10), for the function χ, one can take the function $\chi = \chi_{P_0}$; i.e.,

$$c_3 \int\limits_{D_{P_0}} u^2 \chi_{P_0}\, dx\, dt \le 2 \int\limits_{D_{P_0}} u \Box \chi_{P_0}\, dx\, dt - 2\zeta(t_0). \tag{8.13}$$

By using the ε-inequality, we obtain

$$2u\Box\chi_{P_0} \le 2|u\Box\chi_{P_0}| = 2|u|\sqrt{\chi_{P_0}}\,\frac{|\Box\chi_{P_0}|}{\sqrt{\chi_{P_0}}} \le \varepsilon u^2\chi_{P_0} + \frac{|\Box\chi_{P_0}|^2}{\varepsilon\chi_{P_0}}.$$

From the last inequality and from (8.13), we obtain the inequality

$$(c_3 - \varepsilon) \int\limits_{D_{P_0}} u^2 \chi_{P_0}\, dx\, dt \le \frac{1}{\varepsilon} \int\limits_{D_{P_0}} \frac{|\Box\chi_{P_0}|^2}{\chi_{P_0}}\, dx\, dt - 2\zeta(t_0),$$

which, for $\varepsilon < c_3$, implies that

$$\int\limits_{D_{P_0}} u^2 \chi_{P_0}\, dx\, dt \le \frac{1}{\varepsilon(c_3 - \varepsilon)} \int\limits_{D_{P_0}} \frac{|\Box\chi_{P_0}|^2}{\chi_{P_0}}\, dx\, dt - \frac{2}{c_3 - \varepsilon}\,\zeta(t_0). \tag{8.14}$$

Since $\min_{0<\varepsilon<c_3} \dfrac{1}{\varepsilon(c_3 - \varepsilon)} = \dfrac{4}{c_3^2}$ is attained for $\varepsilon = \dfrac{c_3}{2}$, we find that inequality (8.14) acquires the form

$$\int\limits_{D_{P_0}} u^2 \chi_{P_0}\, dx\, dt \le \frac{4}{c_3^2} \int\limits_{D_{P_0}} \frac{|\Box\chi_{P_0}|^2}{\chi_{P_0}}\, dx\, dt - \frac{4}{c_3}\zeta(t_0). \tag{8.15}$$

Since $\chi_{P_0}(x,t) = \chi^0((x - x_0)/t_0, t/t_0)$, it follows from conditions (8.7) and (8.8) that, after the substitution $x = x_0 + t_0 x'$, $t = t_0 t'$, we obtain

$$\int\limits_{D_{P_0}} \frac{|\Box\chi_{P_0}|^2}{\chi_{P_0}}\, dx\, dt = \frac{1}{t_0^2} \int\limits_{D_{(0,1)}} \frac{|\Box\chi^0|^2}{\chi^0}\, dx'\, dt' = \frac{\varkappa_0}{t_0^2} < +\infty.$$

Then, by virtue of condition (8.10), from inequality (8.15), we derive the inequality

$$0 \le \int\limits_{D_{P_0}} u^2 \chi_{P_0}\, dx\, dt \le \left(\frac{2}{c_3 t_0}\right)^2 \varkappa_0 - \frac{4}{c_3}\,\zeta(t_0). \tag{8.16}$$

By virtue of the definition (8.8), we have $\lim_{t_0\to+\infty}(2/(c_3t_0))^2\varkappa_0 = 0$. Therefore, by (8.12), there exists a positive number $T^0 := T^0(x_0; a, f, \varphi, \psi) > 0$ such that for $t_0 > T^0$ the right-hand side of inequality (8.16) is negative, while the left-hand side in this inequality is nonnegative. This implies that if there exists a strong generalized solution u of problem (1.1), (1.3) of the class C in the domain D_{P_0}, then we necessarily have $t_0 \le T^0$, which completes the proof of Theorem 8.1.

Remark 8.2. By Remark 8.1, by $T^* := T^*(x_0; a, f, \varphi, \psi)$ we denote the least upper bound of $t_0 > 0$ for which problem (1.1), (1.3) is solvable in the domain D_{P_0}. It follows from Theorems 7.1 and 8.1 that $0 < T^* \le T^0$; moreover, problem (1.1), (1.3) is solvable in the domain D_{P_0} for $t_0 < T^*$ and has no solution for $t_0 > T^*$.

Remark 8.3. One can readily see that if, in addition to (8.5), we additionally require that one of the following inequalities is satisfied:

$$\text{(i) } f(x,t) \ge c, \quad (x,t) \in \overline{\Omega}; \qquad \text{(ii) } \varphi(x) \ge c; \qquad \text{(iii) } \psi(x) \ge c, \quad x \in \mathbb{R}, \tag{8.17}$$

where $c := \mathrm{const} > 0$, and for the function χ_{P_0}, we take

$$\chi_{P_0}(x,t) = \chi^*((x-x_0)/t_0, t/t_0),$$

where χ^* is given by (8.9), then condition (8.12) is satisfied, and, therefore, in this case, problem (1.1), (1.3) with sufficiently large t_0 does not have a strong generalized solution u of the class C in the domain D_{P_0}.

Indeed, by performing the transformation of the independent variable x by the formula $x = x_0 + t_0\tau$ in the first integral in (8.11) in the case in which, for example, the third condition (8.17) is satisfied, after simple manipulations, we obtain

$$\begin{aligned}\zeta(t_0) &\geq \int\limits_{x_0-t_0}^{x_0+t_0} \psi(x)\chi_{P_0}(x,0)\,dx = t_0 \int\limits_{-1}^{1} \psi(x_0+t_0\tau)\chi^*(\tau,0)\,d\tau \geq ct_0 \int\limits_{-1}^{1} (1-\tau^2)^n\,d\tau \\ &= 2ct_0 \int\limits_{0}^{1} (1-\tau^2)^n\,d\tau = ct_0 B(2^{-1}, n+1) > 0,\end{aligned} \tag{8.18}$$

where $B(a,b)$ is a well-known Euler integral of the first kind (e.g., see [9, p. 750]). It readily follows from (8.18) that inequality (8.12) holds. The remaining cases in (8.17) can be considered in a similar way. The proof of Theorem 8.1 is complete.

Remark 8.4. If condition (8.17) fails, then, in general, problem (1.1), (1.2) can have a global solution. Indeed, the function

$$u(x,t) = \ln \frac{8}{(e^x + e^{-x})^2}$$

is a global solution of problem (1.1), (1.2) for

$$f = 0, \qquad \psi = 0, \qquad \varphi(x) = \ln \frac{8}{(e^x + e^{-x})^2}, \qquad x \in \mathbb{R}.$$

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