Nonlinear Differ. Equ. Appl. 18 (2011), 117–138

1021-9722/11/020117-22
published online September 22, 2010
DOI 10.1007/s00030-010-0087-9

Nonlinear Differential Equations and Applications NoDEA

On the solvability of one boundary value problem for some semilinear wave equations with source terms

Sergo Kharibegashvili and Bidzina Midodashvili

Abstract. In a conic domain of time type for one class of semilinear wave equations with source terms we consider a Sobolev problem representing a multidimensional version of the Darboux second problem. The questions on global and local solvability, uniqueness and absence of solutions of this problem are investigated.

Mathematics Subject Classification (2010). 35L05, 35L70.

Keywords. Sobolev problem, Semilinear wave equations, Source terms, Global and local solvability, Nonexistence.

1. Introduction

Consider a semilinear wave equation of the form

$$L_\lambda u := \frac{\partial^2 u}{\partial t^2} - \triangle u + \lambda f(u) = F, \tag{1.1}$$

where $\lambda \neq 0$ is a given real constant, f and F are given real functions, besides, f is a nonlinear one and u is an unknown real function, $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

Let D be a conic domain in the space $\mathbb{R}^{n+1}$ of the variables $x = (x_1, x_2, \ldots, x_n)$ and t, i.e. D contains, along with the point $(x,t) \in D$, the whole ray $l : (\tau x, \tau t)$, $0 < \tau < \infty$. Denote by S the conic surface of ∂D. Suppose that D is homeomorphic to the conic domain $\omega : t > |x|$, and $S \backslash O$ is a connected n-dimensional manifold of the class C^∞, where $O = (0, \ldots, 0, 0)$ is the vertex of S. Suppose also that D lies in the half-space $t > 0$ and $D_T := \{(x,t) \in D : t < T\}$, $S_T := \{(x,t) \in S : t \leq T\}$, $T > 0$. It is clear that if $T = \infty$, then $D_\infty = D$ and $S_\infty = S$.

For the equation (1.1) we consider the problem on finding of a solution $u(x,t)$ of that equation in the domain D_T by the boundary condition

$$u\,|_{S_T} = g, \tag{1.2}$$

where g is a given real function on S_T.

In the linear case, when $\lambda = 0$ and the conic manifold $S = \partial D$ is time-oriented, i.e.

$$\left(\nu_0^2 - \sum_{i=1}^{n} \nu_i^2\right)|_S < 0, \quad \nu_0|_S < 0, \tag{1.3}$$

where $\nu = (\nu_1, \ldots, \nu_n, \nu_0)$ is the unit vector of the outer normal to $S \backslash O$, the problem (1.1), (1.2) was posed by S.L. Sobolev in the work [1], where the unique solvability of this problem in the corresponding functional spaces is given. In the same work [1] it is made a conjecture that the obtained results will be valid for nonlinear wave equations. The aim of the present paper is to reveal conditions for nonlinear source term in the equation (1.1) which will ensure a global solvability of the problem (1.1), (1.2) in some certain cases and be a cause of nonexistence of a global solution of this problem in other cases, although in all these cases the considering problem will be locally solvable. In the particular case of nonlinearity when $f(u) = |u|^p u$, $p = const > 0$ and the homogeneous boundary condition (1.2), i.e. for $g = 0$, the problem (1.1), (1.2) has been considered in [2], while for the nonhomogeneous boundary condition (1.2) this problem has been studied in [3]. Note that in the case (1.3) the problem (1.1), (1.2) can be considered as a multidimensional version of the Darboux second problem [4, pp. 228, 233] for the semilinear equation (1.1).

It should be noted also that for the nonlinear wave equations (1.1) other problems (Cauchy problem with initial conditions at $t = 0$, Cauchy characteristic problem in the light cone of future, mixed and periodical problems) are considered in numerous literature [see, e.g., [5–24]].

Below we require that the condition (1.3) is fulfilled and the nonlinear function f satisfies the following requirement

$$f \in C(\mathbb{R}), \quad |f(u)| \le M_1 + M_2|u|^\alpha, \quad \alpha = const > 0, \quad u \in \mathbb{R}, \tag{1.4}$$

besides, in the following sections we consider also the condition

$$\int_0^u f(s)ds \ge -M_3 - M_4 u^2, \quad u \in \mathbb{R}, \tag{1.5}$$

where $M_i = const \ge 0, i = 1, 2, 3, 4$.

Remark 1.1. In the case when $\alpha \le 1$ from the inequality (1.4) it follows the inequality (1.5). It is obvious also that the function $f(u) = |u|^\alpha sign u$ satisfies both conditions (1.4) and (1.5) for $M_1 = 0, M_2 = 1, M_3 = M_4 = 0$, since $\int_0^u |s|^\alpha sign s ds = \frac{1}{\alpha+1}|u|^{\alpha+1}$, $\alpha > 0$.

Denote by $W_2^k(\Omega)$ the Sobolev space consisting of the elements of $L_2(\Omega)$, having in $L_2(\Omega)$ all generalized derivatives up to the kth order, inclusively.

Remark 1.2. The embedding operator $I : W_2^1(D_T) \to L_q(D_T)$ is the linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [25, p. 86]. At the same time the Nemitsky operator $K : L_q(D_T) \to L_2(D_T)$, acting according to the formula $Ku = f(u)$, where the function f satisfies the condition (1.4), is continuous and bounded for $q \geq 2\alpha$ [26, p. 349], [27, pp. 66, 67]. Thus, if $\alpha < \frac{n+1}{n-1}$, i.e. $2\alpha < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \geq 2\alpha$. Therefore, in this case the operator

$$K_0 = KI : W_2^1(D_T) \to L_2(D_T) \tag{1.6}$$

is continuous and compact. Moreover, $f(u) \in L_2(D_T)$ follows from $u \in W_2^1(D_T)$, and if $u_m \to u$ in the space $W_2^1(D_T)$, then $f(u_m) \to f(u)$ in the space $L_2(D_T)$.

Definition 1.3. Let $F \in L_2(D_T), g \in W_2^1(S_T)$ and the condition (1.4) be fulfilled with $0 < \alpha < \frac{n+1}{n-1}$. We call a function $u \in W_2^1(D_T)$ a strong generalized solution of the nonlinear problem (1.1), (1.2) of the class W_2^1 in the domain D_T if there exists a sequence of functions $u_k \in C^2(\overline{D}_T)$ such that $u_k \to u$ in the space $W_2^1(D_T)$, $L_\lambda u_k \to F$ in the space $L_2(D_T)$ and $u_k|_{S_T} \to g$ in the space $W_2^1(S_T)$. The convergence of the sequence $\{\lambda f(u_k)\}$ to function $\lambda f(u)$ in the space $L_2(D_T)$ when $u_k \to u$ in the space $W_2^1(D_T)$ follows from Remark 1.2.

Definition 1.4. Let $0 < \alpha < \frac{n+1}{n-1}$, the condition (1.4) be fulfilled and $F \in L_{2,loc}(D)$, $g \in W_{2,loc}^1(S)$ and $F\big|_{D_T} \in L_2(D_T)$, $g\,|_{S_T} \in W_2^1(S_T)$ for every $T > 0$. We say that the problem (1.1), (1.2) is globally solvable in the class W_2^1 if for every $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T.

Definition 1.5. Let $0 < \alpha < \frac{n+1}{n-1}$, the condition (1.4) be fulfilled and $F \in L_{2,loc}(D)$, $g \in W_{2,loc}^1(S)$ and $F\,|_{D_T} \in L_2(D_T)$, $g\,|_{S_T} \in W_2^1(S_T)$ for every $T > 0$. We say that the problem (1.1), (1.2) is locally solvable in the class W_2^1 if there exists a number $T_0 = T_0(F, g) > 0$ such that for $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T.

The work is organized as follows. In Sect. 2 it is received a priori estimate of the solution of the problem (1.1), (1.2) of the class W_2^1, which is used in the proof of the global solvability of this problem in Sect. 3. In Sect. 4, under certain requirements on the nonlinear function f, it is proved the uniqueness of the solution of the problem (1.1), (1.2) of the class W_2^1 (this question has not been considered in [2,3]). In Sect. 5 we consider the cases of absence of global solvability and in the last section it is established the local solvability of the problem (1.1), (1.2).

2. A priori estimate of the solution of the problem (1.1), (1.2) of the class W_2^1

Lemma 2.1. *Let $\lambda > 0$, the conditions (1.3), (1.4), (1.5) be fulfilled, $0 < \alpha < \frac{n+1}{n-1}$, $F \in L_2(D_T)$ and $g \in W_2^1(S_T)$. Then for every strong generalized solution*

u of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T the a priori estimate

$$||u||_{W_2^1(D_T)} \le c_1||F||_{L_2(D_T)} + c_2||g||_{W_2^1(S_T)} + c_3||g||^{\frac{1}{2}}_{W_2^1(S_T)} + c_4 \tag{2.1}$$

is valid with nonnegative constants $c_i = c_i(\lambda, S, f, T), 1 \le i \le 4$, not depending on u, g and F, where $c_j > 0, j = 1, 2$.

Proof. Let $u \in W_2^1(D_T)$ be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T. Then, in view of Definition 1.3, there exists the sequence of functions $u_k \in C^2(\overline{D}_T)$ such that

$$\lim_{k\to\infty} ||u_k - u||_{W_2^1(D_T)} = 0, \quad \lim_{k\to\infty} ||L_\lambda u_k - F||_{L_2(D_T)} = 0, \tag{2.2}$$

$$\lim_{k\to\infty} ||u_k|_{S_T} - g||_{W_2^1(S_T)} = 0. \tag{2.3}$$

Consider the function $u_k \in C^2(\overline{D}_T)$ as a solution of the following problem

$$L_\lambda u_k = F_k, \tag{2.4}$$

$$u_k|_{S_T} = g_k. \tag{2.5}$$

Here

$$F_k := L_\lambda u_k, \quad g_k := u_k\, |_{S_T}. \tag{2.6}$$

Putting

$$\zeta(u) := \int_0^u f(s)ds \tag{2.7}$$

and multiplying the both sides of the equation (2.4) by $\frac{\partial u_k}{\partial t}$, after integration in the domain $D_\tau := \{(x,t) \in D : t < \tau\}$, $0 < \tau \le T$, we receive

$$\frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt - \int_{D_\tau} \triangle u_k \frac{\partial u_k}{\partial t} dx\,dt + \lambda\int_{D_\tau} \frac{\partial}{\partial t}\zeta(u_k)dx\,dt = \int_{D_\tau} F_k\frac{\partial u_k}{\partial t}dx\,dt. \tag{2.8}$$

Let $\Omega_\tau := D \cap \{t = \tau\}$ and denote by $\nu = (\nu_1, \ldots, \nu_n, \nu_0)$ the unit vector of the outer normal to $S_T \backslash \{(0, \ldots, 0, 0)\}$. Integrating by parts and taking into account the equalities (2.5) and $\nu\, |_{\Omega_\tau} = (0, \ldots, 0, 1)$ we receive

$$\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt = \int_{\partial D_\tau}\left(\frac{\partial u_k}{\partial t}\right)2\nu_0 ds = \int_{\Omega_\tau}\left(\frac{\partial u_k}{\partial t}\right)2dx + \int_{S_\tau}\left(\frac{\partial u_k}{\partial t}\right)2\nu_0 ds,$$

$$\lambda\int_{D_\tau}\frac{\partial}{\partial t}\zeta(u_k)dx\,dt = \lambda\int_{\partial D_\tau}\zeta(u_k)\nu_0 ds = \lambda\int_{S_\tau}\zeta(g_k)\nu_0 ds + \lambda\int_{\Omega_\tau}\zeta(u_k)dx,$$

$$\int_{D_\tau} \frac{\partial^2 u_k}{\partial x_i^2} \frac{\partial u_k}{\partial t} dx\, dt = \int_{\partial D_\tau} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t} \nu_i ds - \frac{1}{2} \int_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_k}{\partial x_i} \right)^2 dx\, dt$$
$$= \int_{\partial D_\tau} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t} \nu_i ds - \frac{1}{2} \int_{\partial D_\tau} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \nu_0 ds = \int_{\partial D_\tau} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial t} \nu_i ds$$
$$- \frac{1}{2} \int_{S_\tau} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \nu_0 ds - \frac{1}{2} \int_{\Omega_\tau} \left(\frac{\partial u_k}{\partial x_i} \right)^2 dx.$$

Whence due to (2.8) it follows that

$$\int_{D_\tau} F_k \frac{\partial u_k}{\partial t} dx\, dt = \int_{S_\tau} \frac{1}{2\nu_0} \left[\sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i} \nu_0 - \frac{\partial u_k}{\partial t} \nu_i \right)^2 \right.$$
$$\left. + \left(\frac{\partial u_k}{\partial t} \right)^2 \left(\nu_0^2 - \sum_{j=1}^n \nu_j^2 \right) \right] ds + \frac{1}{2} \int_{\Omega_\tau} \left[\left(\frac{\partial u_k}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i} \right)^2 \right] dx$$
$$+ \lambda \int_{S_\tau} \zeta(g_k) \nu_0 ds + \lambda \int_{\Omega_\tau} \zeta(u_k) dx. \tag{2.9}$$

Since $\lambda > 0$ and in view of (1.3), (1.5) and (2.7) from (2.9) it follows that

$$\frac{1}{2} \int_{\Omega_\tau} \left[\left(\frac{\partial u_k}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i} \right)^2 \right] dx \le \int_{S_\tau} \frac{1}{2|\nu_0|} \left[\sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i} \nu_0 - \frac{\partial u_k}{\partial t} \nu_i \right)^2 \right] ds$$
$$+ \lambda \int_{S_\tau} |\zeta(g_k)| |\nu_0| ds + \lambda \int_{\Omega_\tau} [M_3 + M_4 u_k^2] dx + \int_{D_\tau} F_k \frac{\partial u_k}{\partial t} dx\, dt. \tag{2.10}$$

Because S represents a conic manifold, then $\sup_{S \setminus O} |\nu_0|^{-1} = \sup_{S \cap \{t=1\}} |\nu_0|^{-1}$. At the same time $S \setminus O$ represents a smooth manifold and $S \cap \{t = 1\} = \partial\Omega_{\tau=1}$ is a compact manifold. Therefore, since ν_0 is the continuous function on $S \setminus O$, we have

$$M_0 := \sup_{S \setminus O} |\nu_0|^{-1} = \sup_{S \cap \{t=1\}} |\nu_0|^{-1} < +\infty, \quad |\nu_0| \le |\nu| = 1. \tag{2.11}$$

Noting that $(\nu_0 \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial t}), i = 1, \ldots, n$, is an inner differential operator on S_T and due to (2.5) we obtain

$$\int_{S_\tau} \left[\sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i} \nu_0 - \frac{\partial u_k}{\partial t} \nu_i \right)^2 \right] ds \le || \, u_k \, |_{S_T} ||^2_{W_2^1(S_T)} = ||g_k||^2_{W_2^1(S_T)}. \tag{2.12}$$

From (2.11), (2.12) it follows that

$$\int_{S_\tau} \frac{1}{2|\nu_0|} \left[\sum_{i=1}^n \left(\frac{\partial u_k}{\partial x_i} \nu_0 - \frac{\partial u_k}{\partial t} \nu_i \right)^2 \right] ds \le \frac{1}{2} M_0 ||g_k||^2_{W_2^1(S_T)}. \tag{2.13}$$

In view of (1.4) and (2.7) we have

$$|\zeta(u)| = \left| \int_0^u f(s) ds \right| \le M_1 |u| + \frac{M_2}{\alpha + 1} |u|^{\alpha+1}. \tag{2.14}$$

By virtue of the Cauchy inequality $2F_k\frac{\partial u_k}{\partial t} \le F_k^2 + (\frac{\partial u_k}{\partial t})^2$ and due to (2.13), (2.14) from (2.10) we have

$$\int_{\Omega_\tau}\left[\left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx \le M_0||g_k||^2_{W_2^1(S_T)} + 2\lambda M_1\int_{S_\tau}|g_k|ds$$
$$+\frac{2\lambda M_2}{\alpha+1}\int_{S_\tau}|g_k|^{\alpha+1}ds + 2\lambda M_3 mes\Omega_\tau + 2\lambda M_4\int_{\Omega_\tau}u_k^2dx$$
$$+\int_{D_T}\left(\frac{\partial u_k}{\partial t}\right)^2dx\,dt + \int_{D_T}F_k^2dx\,dt, 0<\tau\le T. \tag{2.15}$$

Further, according to the Schwartz inequality we have

$$\int_{S_\tau}|g_k|ds \le \left(\int_{S_\tau}1^2ds\right)^{\frac{1}{2}}\left(\int_{S_\tau}|g_k|^2ds\right)^{\frac{1}{2}}$$
$$\le (mesS_\tau)^{\frac{1}{2}}\ ||g_k\ ||_{W_2^1(S_\tau)} \le (mesS_T)^{\frac{1}{2}}\ ||g_k\ ||_{W_2^1(S_T)}. \tag{2.16}$$

Considerations from Remark 1.2 regarding to the space $W_2^1(S_T)$, taking into account that $dimS_T = dimD_T - 1 = n$, show that embedding operator $I : W_2^1(S_T) \to L_q(S_T)$ is a linear continuous compact operator for $1 < q < \frac{2n}{n-2}, n \ge 2$ [25, p. 86]. At the same time the Nemitsky operator $K_1 : L_q(S_T) \to L_2(S_T)$, acting according to the formula $K_1u = |u|^{\frac{\alpha+1}{2}}$ is continuous and bounded for $q \ge 2\frac{\alpha+1}{2} = \alpha+1$ [26, p. 349], [27, pp. 66, 67]. Thus, if $\alpha+1 < \frac{2n}{n-2}$, i.e. $\alpha < \frac{n+2}{n-2}$, then there exists number q, such that $1 < q < \frac{2n}{n-2}$ and $q \ge \alpha+1$. Therefore in this case the operator

$$K_2 = K_1I : W_2^1(S_T) \to L_2(S_T)$$

will be continuous and compact. Hence, according to the inequality $\frac{n+1}{n-1} < \frac{n+2}{n-2}$ and (2.3), (2.5), it follows that

$$\lim_{k\to\infty}\int_{S_\tau}|g_k|^{\alpha+1}ds = \int_{S_\tau}|g|^{\alpha+1}ds \tag{2.17}$$

and

$$\int_{S_T}|g|^{\alpha+1}ds \le C_1\ ||g\ ||^2_{W_2^1(S_T)} \tag{2.18}$$

with a positive constant $C_1 = C_1(T)$, not depending on $g \in W_2^1(S_T)$.

If $t = \gamma(x)$ represents an equation of the conic surface S, then due to (2.5) we have

$$u_k(x,\tau) = u_k(x,\gamma(x)) + \int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u_k(x,s)ds = g_k(x)$$
$$+\int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u_k(x,s)ds,\ (x,\tau) \in \Omega_\tau.$$

Squaring the both parts of the obtained equality, integrating it in the domain Ω_τ and using the Schwartz inequality, we receive

$$\int_{\Omega_\tau} u_k^2 dx \le 2\int_{\Omega_\tau} g_k^2(x,\gamma(x))dx + 2\int_{\Omega_\tau}\left(\int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u_k(x,s)ds\right)^2 dx$$
$$\le 2\int_{S_\tau} g_k^2 ds + 2\int_{\Omega_\tau}(\tau-\gamma(x))\left[\int_{\gamma(x)}^{\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 ds\right]dx \le 2\int_{S_\tau} g_k^2 ds$$
$$+2T\int_{\Omega_\tau}\left[\int_{\gamma(x)}^{\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 ds\right]dx = 2\int_{S_\tau} g_k^2 ds + 2T\int_{D_\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt. \tag{2.19}$$

In view of (2.16), (2.19) from (2.15) it follows that

$$\int_{\Omega_\tau}\left[u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx \le M_0\,||g_k\,||^2_{W_2^1(S_T)}$$
$$+2\lambda M_1(mesS_T)^{\frac{1}{2}}\,||g_k\,||_{W_2^1(S_T)} + \frac{2\lambda M_2}{\alpha+1}\int_{S_T}|g_k|^{\alpha+1}ds$$
$$+2\lambda M_3 mes\Omega_\tau + (2\lambda M_4+1)\left[2\int_{S_T} g_k^2 ds + 2T\int_{D_\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt\right]$$
$$+\int_{D_\tau}\left(\frac{\partial u_k}{\partial t}\right)^2 dx\,dt + \int_{D_\tau} F_k^2 dx\,dt$$
$$\le (4\lambda M_4 T + 2T + 1)\int_{D_\tau}\left[u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx$$
$$+\Big[M_0\,||g_k\,||^2_{W_2^1(S_T)} + 2\lambda M_1(mesS_T)^{\frac{1}{2}}\,||g_k\,||_{W_2^1(S_T)} + \frac{2\lambda M_2}{\alpha+1}\int_{S_T}|g_k|^{\alpha+1}ds$$
$$+2\lambda M_3 mes\Omega_\tau + (4\lambda M_4+2)\,||g_k\,||^2_{W_2^1(S_T)} + ||F_k\,||^2_{L_2(D_T)}\Big]. \tag{2.20}$$

Let

$$w(\tau) := \int_{\Omega_\tau}\left[u_k^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u_k}{\partial x_i}\right)^2\right]dx, \tag{2.21}$$

then from (2.20) we have

$$w(\tau) \le (4\lambda M_4 T + 2T + 1)\int_0^\tau w(s)ds + \Big[(M_0 + 4\lambda M_4 + 2)\,||g_k\,||^2_{W_2^1(S_T)}$$
$$+2\lambda M_1(mesS_T)^{\frac{1}{2}}\,||g_k\,||_{W_2^1(S_T)} + 2\lambda M_3 mes\Omega_T$$
$$+\frac{2\lambda M_2}{\alpha+1}\int_{S_T}|g_k|^{\alpha+1}ds + ||F_k\,||^2_{L_2(D_T)}\Big], \quad 0<\tau\le T.$$

Hence, due to the Gronwall's lemma [28, p. 13], it follows that

$$w(\tau) \le A_k exp(4\lambda M_4 T + 2T + 1)\tau, \quad 0<\tau\le T. \tag{2.22}$$

Here

$$A_k = (M_0 + 4\lambda M_4 + 2)\ ||g_k|\,|^2_{W_2^1(S_T)} + 2\lambda M_1 (mesS_T)^{\frac{1}{2}}\ ||g_k\ ||_{W_2^1(S_T)} \\ + 2\lambda M_3 mes\Omega_T + \frac{2\lambda M_2}{\alpha+1}\int_{S_T} |g_k|^{\alpha+1} ds + ||F_k\ ||^2_{L_2(D_T)}. \quad (2.23)$$

In view of (2.21) and (2.22) we have

$$||u_k||^2_{W_2^1(D_T)} = \int_0^T w(\tau)d\tau \le A_k T exp(4\lambda M_4 T + 2T + 1)T. \quad (2.24)$$

Further, by virtue of (2.2)–(2.5), (2.17), (2.18) and (2.23), passing to limit for $k \to \infty$ in (2.24), we obtain

$$||u||^2_{W_2^1(D_T)} \le AT \exp(4\lambda M_4 T + 2T + 1)T. \quad (2.25)$$

Here

$$A = (M_0 + 4\lambda M_4 + 2)||g||^2_{W_2^1(D_T)} + 2\lambda M_1 (mesS_T)^{\frac{1}{2}}||g||_{W_2^1(S_T)} \\ + 2\lambda M_3 mes\Omega_T + \frac{2\lambda M_2}{\alpha+1} C_1 ||g||^2_{W_2^1(S_T)} + ||F||^2_{L_2(D_T)}. \quad (2.26)$$

Taking square root from the both sides of the inequality (2.25) and using the obvious inequality $(\sum_{i=1}^m a_i^2)^{\frac{1}{2}} \le \sum_{i=1}^m |a_i|$, in view of (2.26) we finally obtain

$$||u||_{W_2^1(D_T)} \le c_1 ||F||_{L_2(D_T)} + c_2 ||g||_{W_2^1(S_T)} + c_3 ||g||^{\frac{1}{2}}_{W_2^1(S_T)} + c_4.$$

Here

$$\begin{cases} c_1 = \sqrt{T} exp(2\lambda M_4 T + T + \frac{1}{2})T; \\ c_2 = \sqrt{T}(M_0 + 4\lambda M_4 + \frac{2\lambda M_2}{\alpha+1} C_1 + 2)^{\frac{1}{2}} exp(2\lambda M_4 T + T + \frac{1}{2})T; \\ c_3 = \sqrt{T}\sqrt{2\lambda M_1}(mesS_T)^{\frac{1}{4}} exp(2\lambda M_4 T + T + \frac{1}{2})T; \\ c_4 = \sqrt{T}(2\lambda M_3 mes\Omega_T)^{\frac{1}{2}} exp(2\lambda M_4 T + T + \frac{1}{2})T. \end{cases} \quad (2.27)$$

This completely proves Lemma 2.1. □

3. Global solvability of the problem (1.1), (1.2) of the class W_2^1

First consider a solvability of a linear problem corresponding to the problem (1.1), (1.2) when $\lambda = 0$ in the equation (1.1), i.e. for the problem

$$L_0 u(x,t) = F(x,t), (x,t) \in D_T, \quad (3.1)$$

$$u(x,t) = g(x,t), (x,t) \in S_T. \quad (3.2)$$

In this case for $F \in L_2(D_T), g \in W_2^1(S_T)$ we introduce analogously a notion of strong generalized solution $u \in W_2^1(D_T)$ of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T, for which there exists a sequence of functions $u_k \in C^2(\overline{D}_T)$, such that $u_k \to u$ in the space $W_2^1(D_T)$, $L_0 u_k \to F$ in the space $L_2(D_T)$ and $u_k\ |_{S_T} \to g$ in the space $W_2^1(S_T)$. Note that, as it follows from Lemma 2.1, due to (2.27) when $\lambda = 0$ for the strong generalized solution $u \in W_2^1(D_T)$ of the problem (3.1), (3.2) the following estimate

$$||u||_{W_2^1(D_T)} \le c(||F||_{L_2(D_T)} + ||g||_{W_2^1(S_T)}) \quad (3.3)$$

is valid, where

$$c = \sqrt{T}(M_0 + 2)^{\frac{1}{2}} \exp\left(T + \frac{1}{2}\right) T. \tag{3.4}$$

Let us introduce into consideration the weighted Sobolev space $W^k_{2,\alpha}(D)$, $0 < \alpha < \infty, k = 1, 2, \ldots$, consisting of the functions from the class $W^k_{2,loc}(D)$ for which the norm [29]

$$||u||^2_{W^k_{2,\alpha}(D)} = \sum_{i=0}^{k} \int_D r^{-2\alpha - 2(k-i)} \Big| \frac{\partial^i u}{\partial x^{\hat{i}} \partial t^{i_0}} \Big|^2 dx\, dt,$$

is finite, where

$$r = \left(\sum_{j=1}^{n} x_j^2 + t^2 \right)^{\frac{1}{2}}, \quad \frac{\partial^i u}{\partial x^{\hat{i}} \partial t^{i_0}} = \frac{\partial^i u}{\partial x^{i_1} \ldots \partial x^{i_n} \partial t^{i_0}}, \quad i = i_1 + \cdots + i_n + i_0.$$

Analogously is introduced the space $W^k_{2,\alpha}(S)$, $S = \partial D$ [29].

Together with the problem (3.1), (3.2) consider an analogous problem in the infinite cone D with the following statement

$$L_0 u(x,t) = F(x,t), (x,t) \in D, \tag{3.5}$$
$$u(x,t) = g(x,t), (x,t) \in S. \tag{3.6}$$

According to (1.3) and the result of work [30, p. 114] there exists constant $\alpha_0 = \alpha_0(k) > 1$, such that for $\alpha \geq \alpha_0$ the problem (3.5), (3.6) has the unique solution $u \in W^k_{2,\alpha}(D)$ for every $F \in W^{k-1}_{2,\alpha-1}(D)$ and $g \in W^k_{2,\alpha-\frac{1}{2}}(S), k \geq 2$.

Since the space of finite and infinitely differentiable functions $C_0^\infty(D_T)$ is dense in $L_2(D_T)$, then for a given $F \in L_2(D_T)$ there exists a sequence of functions $F_l \in C_0^\infty(D_T)$, such that

$$\lim_{l \to \infty} ||F_l - F||_{L_2(D_T)} = 0.$$

For fixed l, extending the function F_l with zero beyond domain D_T and using the same notation for it, we will have $F_l \in C_0^\infty(D)$. It is obvious that $F_l \in W^{k-1}_{2,\alpha-1}(D)$ for every $k \geq 2$ and $\alpha > 1$, and, therefore, for $\alpha \geq \alpha_0 = \alpha_0(k)$ too. If $g \in W_2^1(S_T)$, then there exists a function $\tilde{g} \in W_2^1(S)$ such that $g = \tilde{g}\,|_{S_T}$ and $diam\, supp\, \tilde{g} < \infty$ [25, p. 56]. At the same time the space $C_*^\infty(S) := \{g \in C^\infty(S) : diam\, supp\, g < +\infty, 0 \neq supp\, g\}$ is dense in $W_2^1(S)$ [31]. Therefore there exists a sequence of functions $g_l \in C_*^\infty(S)$ such that $\lim_{l\to\infty} ||g_l - \tilde{g}\,||_{W_2^1(S)} = 0$. It is easy to see that $g_l \in W^k_{2,\alpha-\frac{1}{2}}(S)$ for every $k \geq 2$ and $\alpha > 1$, and therefore for $\alpha \geq \alpha_0 = \alpha_0(k)$ too. In view of the aforesaid there exists the solution $\tilde{u}_l \in W^k_{2,\alpha}(D)$ of the problem (3.5), (3.6) for $F = F_l$ and $g = g_l$. Let $u_l = \tilde{u}_l\,|_{D_T}$. Since $u_l \in W_2^k(D_T)$, then choosing number k sufficiently large, namely $k > \frac{n+1}{2} + 2$, according to the theorem of embedding [25, p. 84] we have $u_l \in C^2(\overline{D}_T)$. According to the estimate (3.3) we have

$$||u_l - u_{l'}||_{W_2^1(D_T)} \leq c(||F_l - F_{l'}||_{L_2(D_T)} + ||g_l - g_{l'}||_{W_2^1(S_T)}). \tag{3.7}$$

Since the sequences $\{F_l\}$ and $\{g_l\}$ are fundamental in the spaces $L_2(D_T)$ and $W_2^1(S_T)$, respectively, then due to (3.7) the sequence $\{u_l\}$ is fundamental in the space $W_2^1(D_T)$. Therefore, because of completeness of the space $W_2^1(D_T)$ there exists the function $u \in W_2^1(D_T)$ such that $\lim_{l\to\infty} ||u_l - u||_{W_2^1(D_T)} = 0$, and since $L_0 u_l = F_l \to F$ in the space $W_2^1(D_T)$ and $g_l = u_l|_{S_T} \to g$ in the space $W_2^1(S_T)$, then the function u will be a strong generalized solution of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T. The uniqueness of this solution of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T follows from a priori estimate (3.3). Therefore, for the solution u of the problem (3.1), (3.2) we have $u = L_0^{-1}(F, g)$, where $L_0^{-1} : L_2(D_T) \times W_2^1(S_T) \to W_2^1(D_T)$ is a linear continuous operator, whose norm, in view of (3.3), admits the following estimate

$$||L_0^{-1}||_{L_2(D_T)\times W_2^1(S_T)\to W_2^1(D_T)} \le c, \tag{3.8}$$

where constant c is determined from (3.4).

Because of linearity of the operator $L_0^{-1} : L_2(D_T) \times W_2^1(S_T) \to W_2^1(D_T)$ we have the following representation

$$L_0^{-1}(F, g) = L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.9}$$

where $L_{01}^{-1} : L_2(D_T) \to W_2^1(D_T)$ and $L_{02}^{-1} : W_2^1(S_T) \to W_2^1(D_T)$ are the linear continuous operators, which satisfy conditions

$$||L_{01}^{-1}||_{L_2(D_T)\to W_2^1(D_T)} \le c, \quad ||L_{02}^{-1}||_{W_2^1(S_T)\to W_2^1(D_T)} \le c \tag{3.10}$$

on the basis of (3.8).

Remark 3.1. Note that for $F \in L_2(D_T)$, $g \in W_2^1(S_T)$, $0 < \alpha < \frac{n+1}{n-1}$, by virtue of (3.8), (3.9), (3.10) and Remark 1.2, the function $u \in W_2^1(D_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T if and only if u satisfies the following functional equation

$$u = L_{01}^{-1}(-\lambda f(u)) + L_{01}^{-1}(F) + L_{02}^{-1}(g) \tag{3.11}$$

in the space $W_2^1(D_T)$.

Rewrite (3.11) in the form

$$u = A_0 u := -L_{01}^{-1}(K_0 u) + L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.12}$$

where the operator $K_0 : W_2^1(D_T) \to L_2(D_T)$ from (1.6), according to Remark 1.2, is a continuous and compact operator. Therefore, in view of (3.10) the operator $A_0 : W_2^1(D_T) \to W_2^1(D_T)$ is also continuous and compact. At the same time, because of Lemma 2.1 and (2.11), (2.27), for every parameter $\tau \in [0, 1]$ and every solution u of the equation $u = \tau A_0 u$ with parameter τ it is valid the same a priori estimate (2.1) with nonnegative constants c_i, not depending on u, F, g and τ. Thus, according to the Leray–Schauder theorem [32, p. 375], the equation (3.12), and therefore, according to Remark 3.1 the problem (1.1), (1.2) has at least one solution $u \in W_2^1(D_T)$. We proved the following theorem.

Theorem 3.2. *Let $\lambda > 0$ and the conditions (1.3), (1.4), (1.5) with $0 < \alpha < \frac{n+1}{n-1}$ be fulfilled. Then for every $F \in L_2(D_T)$ and $g \in W_2^1(S_T)$ the problem (1.1), (1.2) has at least one strong generalized solution u of the class W_2^1 in the domain D_T.*

From Theorem 3.2 it follows the global solvability of the problem (1.1), (1.2) of the class W_2^1 in the sense of Definition 1.4.

4. Uniqueness of the solution of the problem (1.1), (1.2) of the class W_2^1

Let us impose on the function f in the equation (1.1) additional requirement

$$f \in C^1(\mathbb{R}), \quad |f'(u)| \le a + b|u|^\gamma; \quad a, b, \gamma = const \ge 0. \tag{4.1}$$

It is obvious that the condition (1.4) with $\alpha = \gamma + 1$ follows from (4.1), and when $\gamma < \frac{2}{n-1}$ we have $\alpha = \gamma + 1 < \frac{n+1}{n-1}$.

Theorem 4.1. *Let $0 \le \gamma < \frac{2}{n-1}$ and the conditions (1.3), (4.1) be fulfilled. Then the problem (1.1), (1.2) cannot have more than one strong generalized solution of the class W_2^1 in the domain D_T in the sense of Definition 1.3.*

Proof. Suppose that $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ and the problem (1.1), (1.2) has two strong generalized solutions $u_1, u_2 \in W_2^1(D_T)$ in the sense of Definition 1.3, i.e. there exist two sequences of the functions $u_{ik} \in C^2(\overline{D}_T)$, $i = 1, 2$; $k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} ||u_{ik} - u_i||_{W_2^1(D_T)} = 0, \quad \lim_{k\to\infty} ||L_\lambda u_{ik} - F||_{L_2(D_T)} = 0, i = 1, 2, \tag{4.2}$$

$$\lim_{k\to\infty} ||g_{ik} - g||_{W_2^1(S_T)} = 0, \ g_{ik} = u_{ik}|_{S_T}, \ i = 1, 2. \tag{4.3}$$

Let

$$w = u_2 - u_1, \quad w_k = u_{2k} - u_{1k} \quad F_k = L_\lambda u_{2k} - L_\lambda u_{1k} \quad g_k^0 = w_k|_{S_T}. \tag{4.4}$$

Then in view of (4.2), (4.3) and (4.4) we have

$$\lim_{k\to\infty} ||w_k - w||_{W_2^1(D_T)} = 0, \quad \lim_{k\to\infty} ||F_k||_{L_2(D_T)} = 0, \tag{4.5}$$

$$\lim_{k\to\infty} ||g_k^0||_{W_2^1(S_T)} = 0. \tag{4.6}$$

In accordance with (4.4) we consider the function $w_k \in C^2(\overline{D}_T)$ as a solution of the following problem

$$\Box w_k := \frac{\partial^2 w_k}{\partial t^2} - \triangle w_k = -\lambda[f(u_{2k}) - f(u_{1k})] + F_k, \tag{4.7}$$

$$w_k|_{S_T} = g_k^0. \tag{4.8}$$

Multiplying the both sides of the equation (4.7) by the function $\frac{\partial w_k}{\partial t}$, after integration in the domain D_τ, in the same way as in derivation of the equality (2.9) we have

$$\int_{D_\tau} \{F_k - \lambda[f(u_{2k}) - f(u_{1k})]\} \frac{\partial w_k}{\partial t} dx\, dt$$
$$= \int_{S_\tau} \frac{1}{2\nu_0} \left[\sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i}\nu_0 - \frac{\partial w_k}{\partial t}\nu_i \right)^2 + \left(\frac{\partial w_k}{\partial t} \right)^2 \left(\nu_0^2 - \sum_{j=1}^n \nu_j^2 \right) \right] ds$$
$$+ \frac{1}{2} \int_{\Omega_\tau} \left[\left(\frac{\partial w_k}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i} \right)^2 \right] dx,\ 0 < \tau \le T. \tag{4.9}$$

Due to (1.3) and (2.12), (2.13) from (4.9), taking into account (4.8), we obtain

$$\int_{\Omega_\tau} \left[\left(\frac{\partial w_k}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i} \right)^2 \right] dx \le M_0 ||g_k^0||^2_{W_2^1(S_T)}$$
$$+ 2 \int_{D_\tau} F_k \frac{\partial w_k}{\partial t} dx\, dt + 2|\lambda| \int_{D_\tau} |f(u_{2k}) - f(u_{1k})| \left| \frac{\partial w_k}{\partial t} \right| dx\, dt. \tag{4.10}$$

Taking into account the inequality $|d_1 + d_2|^\gamma \le 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \le 2^\gamma(|d_1|^\gamma + |d_2|^\gamma)$ for $\gamma \ge 0$, due to (4.1) we have

$$|f(u_{2k}) - f(u_{1k})| = \left| (u_{2k} - u_{1k}) \int_0^1 f'(u_{1k} + t(u_{2k} - u_{1k}))dt \right|$$
$$\le |u_{2k} - u_{1k}| \int_0^1 (a + b|(1-t)u_{1k} + tu_{2k}|^\gamma)dt \le a|u_{2k} - u_{1k}|$$
$$+ 2^\gamma b|u_{2k} - u_{1k}|(|u_{1k}|^\gamma + |u_{2k}|^\gamma) = a|w_k| + 2^\gamma b|w_k|(|u_{1k}|^\gamma + |u_{2k}|^\gamma). \tag{4.11}$$

Applying the obvious inequality $2d_1d_2 \le d_1^2 + d_2^2$ and the inequality (4.11) to the right-side of (4.10) we obtain

$$\int_{\Omega_\tau} \left[\left(\frac{\partial w_k}{\partial t} \right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i} \right)^2 \right] dx \le M_0 ||g_k^0||^2_{W_2^1(S_T)} + \int_{D_T} F_k^2 dx\, dt$$
$$+ \int_{D_T} \left(\frac{\partial w_k}{\partial t} \right)^2 dx\, dt + a|\lambda| \int_{D_T} \left[w_k^2 + \left(\frac{\partial w_k}{\partial t} \right)^2 \right] dx\, dt$$
$$+ 2^{\gamma+1} b|\lambda| \int_{D_T} (|u_{1k}|^\gamma + |u_{2k}|^\gamma)|w_k| \left| \frac{\partial w_k}{\partial t} \right| dx\, dt. \tag{4.12}$$

The last integral in the right-side of (4.12) can be estimated by the Holder inequality

$$\int_{D_T} (|u_{1k}|^\gamma + |u_{2k}|^\gamma)|w_k| \left| \frac{\partial w_k}{\partial t} \right| dx\, dt \le (||\, |u_{1k}|^\gamma \,||_{L_{n+1}(D_T)}$$
$$+ ||\, |u_{2k}|^\gamma \,||_{L_{n+1}(D_T)}) ||w_k||_{L_p(D_T)} \left|\left| \frac{\partial w_k}{\partial t} \right|\right|_{L_2(D_T)}, \tag{4.13}$$

where $\frac{1}{n+1}+\frac{1}{p}+\frac{1}{2}=1$, i.e. for

$$p=\frac{2(n+1)}{n-1}. \tag{4.14}$$

Since $dimD_T = n+1$, then according to the embedding theorem of Sobolev [27, p. 111] for $1 \le q \le \frac{2(n+1)}{n-1}$ we have

$$||v||_{L_q(D_T)} \le C_q ||v||_{W_2^1(D_T)}\, \forall v \in W_2^1(D_T) \tag{4.15}$$

with positive constant C_q, not depending on $v \in W_2^1(D_T)$.

According to the conditions of the theorem $\gamma < \frac{2}{n-1}$ and, therefore, $\gamma(n+1) < \frac{2(n+1)}{n-1}$. Thus, in view of (4.14) and inequality (4.15) we have

$$\begin{aligned} || \, |u_{ik}|^\gamma \, ||_{L_{n+1}(D_T)} &= ||u_{ik}||_{L_{\gamma(n+1)}(D_T)} \\ &\le C^\gamma_{\gamma(n+1)} || \, u_{ik} \, ||^\gamma_{W_2^1(D_T)}, \quad i=1,2;\ k\ge 1, \end{aligned} \tag{4.16}$$

$$||w_k||_{L_p(D_T)} \le C_p ||w_k||_{W_2^1(D_T)}, \quad p=\frac{2(n+1)}{n-1}, k\ge 1. \tag{4.17}$$

According to the first equality of (4.2) there exists the natural number k_0 such that for $k \ge k_0$ we will have

$$||u_{ik}||^\gamma_{W_2^1(D_T)} \le ||u_i||^\gamma_{W_2^1(D_T)} + 1, \quad i=1,2; k \ge k_0.$$

Whence, taking into account (4.13), (4.16) and (4.17), it follows that

$$\begin{aligned} &2^{\gamma+1} b|\lambda| \int_{D_T} (|u_{1k}|^\gamma + |u_{2k}|^\gamma) |w_k| \left|\frac{\partial w_k}{\partial t}\right| dx\, dt \\ &\le 2^{\gamma+1} b|\lambda| C^\gamma_{\gamma(n+1)} (||u_1||^\gamma_{W_2^1(D_T)} \\ &\quad + ||u_2||^\gamma_{W_2^1(D_T)} + 2) C_p ||w_k||_{W_2^1(D_T)} \left|\left|\frac{\partial w_k}{\partial t}\right|\right|_{L_2(D_T)} \\ &\le M_1^\star \left(||w_k||^2_{W_2^1(D_T)} + \left|\left|\frac{\partial w_k}{\partial t}\right|\right|^2_{L_2(D_T)} \right) \le 2M_1^\star ||w_k||^2_{W_2^1(D_T)} \\ &= 2M_1^\star \int_{D_T} \left[w_k^2 + \left(\frac{\partial w_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i}\right)^2 \right] dx\, dt, \end{aligned} \tag{4.18}$$

where $2M_1^\star = 2^{\gamma+1} b|\lambda| C^\gamma_{\gamma(n+1)} (||u_1||^\gamma_{W_2^1(D_T)} + ||u_2||^\gamma_{W_2^1(D_T)} + 2) C_p$.

In view of (4.18) from (4.12) we have

$$\begin{aligned} \int_{\Omega_\tau} &\left[\left(\frac{\partial w_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i}\right)^2 \right] dx \le M_2^\star \\ &\times \int_{D_T} \left[w_k^2 + \left(\frac{\partial w_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i}\right)^2 \right] dx\, dt \\ &+ M_0 ||g_k^0||^2_{W_2^1(S_T)} + \int_{D_T} F_k^2 dx\, dt, \quad 0<\tau \le T, \end{aligned} \tag{4.19}$$

where $M_2^\star = 1 + a|\lambda| + 2M_1^\star$.

Note that the inequality (2.19) is still valid if we replace u_k and g_k with w_k and g_k^0, i.e.

$$\int_{\Omega_\tau} w_k^2 dx \le 2\int_{S_T} (g_k^0)^2 ds + 2T\int_{D_T}\left(\frac{\partial w_k}{\partial t}\right)^2 dx\,dt$$
$$\le 2||g_k^0||^2_{W_2^1(S_T)} + 2T\int_{D_T}\left[w_k^2 + \left(\frac{\partial w_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i}\right)^2\right] dx. \quad (4.20)$$

Putting

$$v_k(\tau) := \int_{\Omega_\tau}\left[w_k^2 + \left(\frac{\partial w_k}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w_k}{\partial x_i}\right)^2\right] dx, \quad (4.21)$$

and summing up the inequalities (4.19) and (4.20), we receive

$$v_k(\tau) \le (M_2^\star + 2T)\int_0^\tau v(s)ds + (M_0+2)||g_k^0||^2_{W_2^1(S_T)} + ||F_k||^2_{L_2(D_T)}.$$

Using the Gronwall's lemma [28, p. 13] we conclude that

$$v_k(\tau) \le [(M_0+2)||g_k^0||^2_{W_2^1(S_T)} + ||F_k||^2_{L_2(D_T)}]\exp(M_2^\star + 2T)\tau. \quad (4.22)$$

From (4.21) and (4.22) it follows that

$$||w_k||^2_{W_2^1(D_T)} = \int_0^T v_k(\tau)d\tau \le T[(M_0+2)||g_k^0||^2_{W_2^1(S_T)}$$
$$+ ||F_k||^2_{L_2(D_T)}]\exp(M_2^\star + 2T)T. \quad (4.23)$$

Due to (4.5) and (4.6) from (4.23) we have

$$\lim_{k\to\infty} ||w_k||_{W_2^1(D_T)} = 0.$$

Hence, in view of the first equality of (4.5), it follows that

$$||w||_{W_2^1(D_T)} = \lim_{k\to\infty} ||w - w_k + w_k||_{W_2^1(D_T)}$$
$$\le \lim_{k\to\infty} ||w - w_k||_{W_2^1(D_T)} + \lim_{k\to\infty} ||w_k||_{W_2^1(D_T)} = 0.$$

Therefore $w = u_2 - u_1 = 0$, i.e. $u_2 = u_1$, which proves Theorem 4.1. □

The theorem below immediately follows from Theorems 3.2 and 4.1.

Theorem 4.2. *Let $\lambda > 0$ and the conditions (1.3), (4.1), (1.5) with $0 \le \gamma < \frac{2}{n-1}$ be fulfilled. Then for every $F \in L_2(D_T)$ and $g \in W_2^1(S_T)$ the problem (1.1), (1.2) has the unique strong generalized solution u of the class W_2^1 in the domain D_T.*

5. Absence of the global solvability of the problem (1.1), (1.2)

Below we show that violation of the condition (1.5) may cause an absence of the global solvability of the problem (1.1), (1.2).

Lemma 5.1. *Let u be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T in the sense of Definition 1.3. Then the following integral equality*

$$\int_{D_T} u\Box\varphi dx\,dt = -\lambda\int_{D_T} f(u)\varphi dx\,dt + \int_{D_T} F\varphi dx\,dt \tag{5.1}$$

is valid for every function φ satisfying conditions

$$\varphi\in C^2(\overline{D}_T),\quad \varphi\Big|_{\partial D_T} = \frac{\partial\varphi}{\partial\nu}\Big|_{\partial D_T} = 0, \tag{5.2}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \triangle$, ν is the unit vector of the outer normal to ∂D_T.

Proof. According to the definition of a strong generalized solution u of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T there exists the sequence $u_k \in C^2(\overline{D}_T)$ such that the equalities (2.2)–(2.6) are valid. Let us multiply the both part of the equality (2.4) by the function φ and integrate the received equality in the domain D_T. Due to (5.2) integration by parts of the left-side of this equality yields

$$\int_{D_T} u_k\Box\varphi dx\,dt + \lambda\int_{D_T} f(u_k)\varphi dx\,dt = \int_{D_T} F\varphi dx\,dt. \tag{5.3}$$

Passing in the equality (5.3) to limit for $k\to\infty$ and taking into account (2.2) and Definition 1.3 we obtain the equality (5.1).

Let the nonlinear function f together with (1.4) satisfy the condition

$$f(u)\geq m_0|u|^\alpha,\quad m_0 = const > 0, \alpha > 1. \tag{5.4}$$

It is obvious that in this case the condition (1.5) will be violated since $\int_0^u f(s)ds \leq \frac{m_0}{\alpha+1}|u|^{\alpha+1} signu$ for $u<0$, and because $\alpha>1$, then for sufficiently large negative u the inequality (1.5) is not valid.

Let us use the method of test functions [14, pp. 10–12].

Let us introduce into consideration the function $\varphi^0 = \varphi^0(x,t)$ such that

$$\varphi^0\in C^2(\overline{D}_\infty), \varphi^0|_{D_{T=1}} > 0, \varphi^0|_{t\geq 1} = 0, \varphi^0|_{\partial D_{T=1}} = \frac{\partial\varphi^0}{\partial\nu}\Big|_{\partial D_{T=1}} = 0 \tag{5.5}$$

and

$$æ_0 := \int_{D_{T=1}} \frac{|\Box\varphi^0|^{\alpha'}}{|\varphi^0|^{\alpha'-1}} dx\,dt < +\infty, \frac{1}{\alpha} + \frac{1}{\alpha'} = 1. \tag{5.6}$$

It is easy to verify that for sufficiently large positive m and k for the function φ^0, satisfying conditions (5.5) and (5.6), may be chosen the function

$$\varphi^0(x,t) = \begin{cases} \omega^m(\frac{x}{t})(1-t)^m t^k, & (x,t)\in D_{T=1}, \\ 0, & t\geq 1, \end{cases}$$

where the function $\omega\in C^\infty(\mathbb{R}^n)$ defines an equation of the conic section $\partial\Omega_1 = S\cap\{t=1\} : \omega(x) = 0, \nabla\omega|_{\partial\Omega_1}\neq 0$, besides $\omega|_{\Omega_1} > 0, \Omega_1 : D\cap\{t=1\}$.

Setting that $\varphi_T(x,t) := \varphi^0(\frac{x}{T}, \frac{t}{T})$, $T > 0$, due to (5.5) it is easy to see that

$$\varphi_T \in C^2(\overline{D}_T), \quad \varphi_T|_{D_T} > 0, \quad \varphi_T|_{\partial D_T} = \frac{\partial \varphi_T}{\partial \nu}|_{\partial D_T} = 0. \tag{5.7}$$

Putting that the function F is fixed, let us introduce into consideration the function of one variable T

$$\zeta(T) := -\int_{D_T} F\varphi_T dx\,dt, \quad T > 0. \tag{5.8}$$

□

Theorem 5.2. *Let $\lambda > 0, 1 < \alpha < \frac{n+1}{n-1}$, the conditions (1.4), (5.4) be fulfilled, $F \in L_{2,loc}(D), F \leq 0$ and $F|_{D_T} \in L_2(D_T)$ for every $T > 0$, and $g = 0$. Then if*

$$\lim_{T\to+\infty} \inf \zeta(T) > 0, \tag{5.9}$$

then there exists the positive number $T_0 = T_0(F)$ such that for $T > T_0$ the problem (1.1), (1.2) cannot have a strong generalized solution of the class W_2^1 in the domain D_T.

Proof. In view of (5.7), assuming in the equality (5.1) $\varphi = \varphi_T$ and taking into account $\lambda > 0$, (5.4) and (5.8), we receive

$$\begin{aligned} \lambda_1 \int_D |u|^\alpha \varphi_T dx\,dt &\leq -\int_{D_T} u\Box\varphi_T dx\,dt - \zeta(T) \\ &\leq \int_{D_T} |u||\Box\varphi_T| dx\,dt - \zeta(T), \quad \lambda_1 = \lambda m_0 > 0. \end{aligned} \tag{5.10}$$

If in Young's inequality with the parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{\alpha}a^\alpha + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}b^{\alpha'}; \quad a, b \geq 0, \frac{1}{\alpha} + \frac{1}{\alpha'} = 1,$$

we take $a = |u|\varphi_T^{\frac{1}{\alpha}}, b = \frac{|\Box\varphi_T|}{\varphi_T^{\frac{1}{\alpha}}}$, then taking into account equality $\frac{\alpha\prime}{\alpha} = \alpha - 1$, we have

$$|u||\Box\varphi_T| = |u|\varphi_T^{\frac{1}{\alpha}}\frac{|\Box\varphi_T|}{\varphi_T^{\frac{1}{\alpha}}} \leq \frac{\varepsilon}{\alpha}|u|^\alpha\varphi_T + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}. \tag{5.11}$$

In view of (5.10) and (5.11) we have

$$\left(\lambda_1 - \frac{\varepsilon}{\alpha}\right)\int_{D_T} |u|^\alpha\varphi_T dx\,dt \leq \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}dx\,dt - \zeta(T),$$

whence for $\varepsilon < |\lambda_1|\alpha$ we obtain

$$\int_{D_T} |u|^\alpha\varphi_T dx\,dt \leq \frac{\alpha}{(\lambda_1\alpha - \varepsilon)\alpha'\varepsilon^{\alpha'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}}dx\,dt - \frac{\alpha}{\lambda_1\alpha - \varepsilon}\zeta(T). \tag{5.12}$$

Since $\alpha' = \frac{\alpha}{\alpha-1}, \alpha = \frac{\alpha'}{\alpha'-1}$ and $\min_{0<\varepsilon<\lambda_1\alpha} \frac{\alpha}{(\lambda_1\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{\lambda_1^{\alpha'}}$, which is reached for $\varepsilon = \lambda_1$, then from (5.12) it follows that

$$\int_{D_T} |u|^\alpha \varphi_T dx\, dt \le \frac{1}{\lambda_1^{\alpha'}} \int_{D_T} \frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}} dx\, dt - \frac{\alpha'}{\lambda_1}\zeta(T). \tag{5.13}$$

Since $\varphi_T := \varphi^0(\frac{x}{T}, \frac{t}{T})$ and due to (5.5), (5.6), after the change of variables $x = Tx', t = Tt'$, it is easy to verify that

$$\int_{D_T} \frac{|\Box\varphi_T|^{\alpha'}}{\varphi_T^{\alpha'-1}} dx\, dt = T^{n+1-2\alpha'} \int_{D_{T=1}} \frac{|\Box\varphi^0|^{\alpha'}}{(\varphi^0)^{\alpha'-1}} dx'dt' = T^{n+1-2\alpha'} æ_0 < +\infty.$$

Whence due to (5.7) from the inequality (5.13) we receive

$$0 \le \int_{D_T} |u|^\alpha \varphi_T dx\, dt \le \frac{1}{\lambda_1^{\alpha'}} T^{n+1-2\alpha'} æ_0 - \frac{\alpha'}{\lambda_1}\zeta(T). \tag{5.14}$$

Since $1 < \alpha < \frac{n+1}{n-1}$ and $\alpha' = \frac{\alpha}{\alpha-1} > 1$, then $n+1-2\alpha' < 0$ and according to (5.6) we have

$$\lim_{T\to\infty} \frac{1}{\lambda_1^{\alpha'}} T^{n+1-2\alpha'} æ_0 = 0.$$

Therefore, in view of (5.9) there exists the positive number $T_0 := T_0(F)$ such that for $T > T_0$ the right-side of the inequality (5.14) will be negative, whereas the left-side of this inequality is nonnegative. Thus, if in conditions of Theorem 5.2 there exists a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T, then necessarily $T \le T_0$. This proves Theorem 5.2. □

The theorem below immediately follows from the proof of Theorem 5.2.

Theorem 5.3. *Let $\lambda > 0, 1 < \alpha < \frac{n+1}{n-1}$, the conditions (1.4), (1.5) be fulfilled and $F^0 \in L_2(D_T), F^0 \le 0, ||F^0||_{L_2(D_T)} \ne 0$. Then if $F = \mu F^0, g = 0$, there exists the number $\mu_0 = \mu_0(F^0) > 0$ such that for $\mu > \mu_0$ the problem (1.1), (1.2) cannot have a strong generalized solution of the class W_2^1 in the domain D_T.*

6. The local solvability of the problem (1.1), (1.2)

Below we show that despite the absence of the global solvability of the problem (1.1), (1.2) in certain cases, for example as it was shown in the previous section with violation of the condition (1.5), the problem is locally solvable in the sense of Definition 1.5 with whatever sign of the parameter λ at the nonlinear term $f(u)$ in the equation (1.1). Below, for simplicity of consideration, we set $g = 0$ in the boundary condition (1.2).

Theorem 6.1. *Let $0 < \alpha < \frac{n+1}{n-1}$, the condition (1.3), (1.4) be fulfilled, $g = 0$, $F \in L_{2,loc}(D)$, and $F|_{D_T} \in L_2(D_T)$, for every $T > 0$. Then the problem (1.1), (1.2) is locally solvable in the class W_2^1, i.e. there exists the number $T_0 = T_0(F) > 0$ such that for $T < T_0$ the problem has a strong generalized solution of the class W_2^1 in the domain D_T.*

Proof. In the conditions of Theorem 6.1, according to Remark 3.1, the function $u \in \overset{0}{W}{}^1_2(D_T, S_T) := \{v \in W^1_2(D_T) : v|_{S_T} = 0\}$ is a strong generalized solution of the problem (1.1), (1.2) with $g = 0$ of the class W^1_2 in the domain D_T if and only if u is a solution of the functional equation (3.12), i.e.

$$u = A_0 u := -L^{-1}_{01}(K_0 u) + L^{-1}_{01}(F). \tag{6.1}$$

in the space $\overset{0}{W}{}^1_2(D_T)$. Here $L^{-1}_{01} : L_2(D_T) \to \overset{0}{W}{}^1_2(D_T, S_T)$ is the linear continuous operator with norm satisfying the first inequality (3.10), while the operator $K_0 : \overset{0}{W}{}^1_2(D_T, S_T) \to L_2(D_T)$, acting according to formula

$$K_0 u = -\lambda f(u), \tag{6.2}$$

is continuous and compact together with the operator $A_0 : \overset{0}{W}{}^1_2(D_T, S_T) \to \overset{0}{W}{}^1_2(D_T, S_T)$ from (6.1). Therefore, due to the Sobolev theorem [32, p. 270] for the solvability of the equation (6.1) in the space $\overset{0}{W}{}^1_2(D_T, S_T)$ it suffices to show that for any fixed number $R > 0$ and sufficiently small positive T the operator A_0 transforms the closed convex ball $B(0, R) := \{u \in \overset{0}{W}{}^1_2(D_T, S_T) : ||u||_{\overset{0}{W}{}^1_2(D_T,S_T)} \leq R\}$ with a center in null element and the radius R into itself in the Hilbert space $\overset{0}{W}{}^1_2(D_T, S_T)$. With this purpose let us estimate the value of $||A_0 u||_{\overset{0}{W}{}^1_2(D_T,S_T)}$ for $u \in \overset{0}{W}{}^1_2(D_T, S_T)$.

First let us put that $1 < \alpha < \frac{n+1}{n-1}$. If $u \in \overset{0}{W}{}^1_2(D_T, S_T)$, then we denote by $\tilde{u}$ the function which is the even continuation of the function u through the plane $t = T$ into the domain D^*_T, symmetrical to D_T with respect to the same plane $t = T$, i.e.

$$\tilde{u} = \begin{cases} u(x,t), & (x,t) \in D_T; \\ u(x, 2T - t), & (x,t) \in D^*_T, \end{cases}$$

and $\tilde{u}(x,t) = u(x,t)$ for $t = T$ in the sense of the trace theory. Obviously, $\tilde{u} \in \overset{0}{W}{}^1_2(\tilde{D}_T) := \{v \in W^1_2(\tilde{D}_T) : v|_{\partial \tilde{D}_T} = 0\}$, where $\tilde{D}_T = D_T \cup \Omega_T \cup D^*_T, \Omega_T := D \cap \{t = T\}$.

Using the inequality [33, p. 258]

$$\int_\Omega |v| d\Omega \leq (mes\Omega)^{1-\frac{1}{p}} ||v||_{p,\Omega}, \quad p \geq 1,$$

and taking into account the equalities

$$||\tilde{u}||^p_{L_p(\tilde{D}_T)} = 2||u||^p_{L_p(D_T)}, \quad ||\tilde{u}||^2_{\overset{0}{W}{}^1_2(\tilde{D}_T)} = 2||u||^2_{\overset{0}{W}{}^1_2(D_T,S_T)},$$

from the well-known multiplicative inequality [25, p. 78]

$$||v||_{p,\Omega} \leq \beta ||\nabla_{x,t} v||^{\tilde{\alpha}}_{m,\Omega} ||v||^{1-\tilde{\alpha}}_{r,\Omega} \quad \forall v \in \overset{0}{W}{}^1_2(\Omega), \Omega \subset \mathbb{R}^{n+1},$$

$$\nabla_{x,t} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right), \quad \tilde{\alpha} = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} - \frac{1}{\tilde{m}}\right)^{-1}, \quad \tilde{m} = \frac{(n+1)m}{n+1-m},$$

for $\Omega = \tilde{D}_T \in \mathbb{R}^{n+1}, v = \tilde{u}, r = 1, m = 2$ and $1 < p \le \frac{2(n+1)}{n-1}$, where $\beta = const > 0$ does not depend on v and T, we obtain the inequality

$$||u||_{L_p(D_T)} \le c_0 (mesD_T)^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}} ||u||_{\overset{0}{W}{}_2^1(D_T,S_T)} \quad \forall u \in \overset{0}{W}{}_2^1(D_T,S_T), \quad (6.3)$$

where $c_0 = const > 0$ is independent of u and T.

Taking into account that $messD_T = \frac{\omega_n}{n+1}T^{n+1}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, from (6.3) for $p = 2\alpha$ we obtain

$$||u||_{L_{2\alpha}(D_T)} \le C_T ||u||_{\overset{0}{W}{}_2^1(D_T,S_T)} \quad \forall u \in \overset{0}{W}{}_2^1(D_T,S_T), \quad (6.4)$$

where

$$C_T = c_0 \left(\frac{\omega_n}{n+1}\right)^{\alpha_1} T^{(n+1)\alpha_1}, \quad \alpha_1 = \frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}. \quad (6.5)$$

Since $\alpha < \frac{n+1}{n-1}$, then $\frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2} > 0$ and in view of (6.4) and (6.5), for every $u \in \overset{0}{W}{}_2^1(D_T,S_T)$, we have

$$||u||_{L_{2\alpha}(D_T)} \le C_{T_1} ||u||_{\overset{0}{W}{}_2^1(D_T,S_T)} \quad \forall T \le T_1, \quad (6.6)$$

where T_1 is a certain fixed positive number.

By virtue of (1.4), (6.2) and (6.6) for every $u \in \overset{0}{W}{}_2^1(D_T,S_T)$ and $T \le T_1$ we have

$$||K_0 u||^2_{L_2(D_T)} \le 2|\lambda|^2 \int_{D_T} (M_1^2 + M_2^2 |u|^{2\alpha}) dx\, dt \le 2|\lambda|^2 M_1^2 mesD_T$$
$$+ 2|\lambda|^2 M_2^2 ||u||^{2\alpha}_{L_{2\alpha}(D_T)} \le 2|\lambda|^2 M_1^2 mesD_T + 2|\lambda|^2 M_2^2 C_{T_1}^{2\alpha} ||u||^{2\alpha}_{\overset{0}{W}{}_2^1(D_T,S_T)}$$

and therefore

$$||K_0 u||_{L_2(D_T)} \le \sqrt{2}\,|\lambda| M_1 \sqrt{mesD_T} + \sqrt{2}\,|\lambda| M_2 C_{T_1}^{\alpha} ||u||^{\alpha}_{\overset{0}{W}{}_2^1(D_T,S_T)}. \quad (6.7)$$

Further, from (3.4), (3.10), (6.1) and (6.7) it follows that

$$||A_0 u||_{\overset{0}{W}{}_2^1(D_T,S_T)} \le ||L_{01}^{-1}||_{L_2(D_T)\to \overset{0}{W}{}_2^1(D_T,S_T)} ||K_0 u||_{L_2(D_T)}$$
$$+ ||L_{01}^{-1}||_{L_2(D_T)\to \overset{0}{W}{}_2^1(D_T,S_T)} ||F||_{L_2(D_T)}$$
$$\le c\Big[\sqrt{2}\,|\lambda| M_1 \sqrt{mesD_{T_1}} + \sqrt{2}\,|\lambda| M_2 C_{T_1}^{\alpha} ||u||^{\alpha}_{\overset{0}{W}{}_2^1(D_T,S_T)}$$
$$+ ||F||_{L_2(D_{T_1})}\Big]$$
$$\le \sqrt{T}\,(M_0+2)^{\frac{1}{2}} exp(T_1 + \frac{1}{2}) T_1 \Big[\sqrt{2}\,|\lambda| M_1 \sqrt{mesD_{T_1}}$$
$$+ \sqrt{2}\,|\lambda| M_2 C_{T_1}^{\alpha} ||u||^{\alpha}_{\overset{0}{W}{}_2^1(D_T,S_T)} + ||F||_{L_2(D_{T_1})}\Big],$$
$$\forall T \le T_1, \quad \forall u \in \overset{0}{W}{}_2^1(D_T,S_T). \quad (6.8)$$

Since the right-side of the inequality (6.8) contains $\sqrt{T}$ as a factor, which tends to zero for $T \to 0$, then there exists the positive number $T_0 \leq T_1$ such that for $T < T_0$ and $||u||_{\overset{0}{W}{}_2^1(D_T,S_T)} \leq R$, due to (6.8), we have $||A_0u||_{W_2^1(D_T)} \leq R$, i.e. the operator $A_0 : \overset{0}{W}{}_2^1(D_T,S_T) \to \overset{0}{W}{}_2^1(D_T,S_T)$ from (6.1) transforms the ball $B(0,R)$ into itself.

The case when $0 < \alpha \leq 1$ can be investigated in the same way if we use beforehand the known inequality [33, p. 258]

$$||v||_{L_\alpha(G)} \leq (mesG)^{\frac{\alpha_1-\alpha}{\alpha_1\alpha}} ||v||_{L_{\alpha_1}(G)},$$

where α_1 is a certain fixed number from $(1, \frac{n+1}{n-1})$. This proves Theorem 6.1. □

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S. Kharibegashvili
A. Razmadze Mathematical Institute,
1, M. Aleksidze St.,
0193 Tbilisi,
Georgia
e-mail: kharibegashvili@yahoo.com

S. Kharibegashvili
Department of Mathematics,
Georgian Technical University,
77, M. Kostava Str.,
0175 Tbilisi,
Georgia

B. Midodashvili
I. Javakhishvili Tbilisi State University,
2, University St.,
0143 Tbilisi,
Georgia
e-mail: bidmid@hotmail.com

Received: 17 February 2010.
Revised: 04 July 2010.
Accepted: 02 September 2010.