The boundary value problem for wave equations with nonlinear dissipative and source terms

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Abstract: The boundary value problem for wave equations with nonlinear dissipative and source terms in an angular domain is considered. The questions of existence and non-existence of global and blow-up solutions of considering problem are investigated for some values of parameters in the nonlinear terms and the right-hand side of given equations. The uniqueness, local and global existence and blow-up of solutions of the problem mentioned are investigated.

Keywords: nonlinear wave equations; BVP; global and local solvability; blow-up; dissipative and source terms.

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1 Introduction

In the plane of independent variables x and t consider nonlinear wave equation of the following form [see, e.g., Lions, 1969, p.57]

$$Lu := u_{tt} - u_{xx} + \mu|u|^\rho u_t + \lambda|u|^\alpha u = f(x,t), \tag{1.1}$$

where μ, λ and $\rho, \alpha > 0$ are given constants; f is given and u is unknown real functions.

Denote by $D_T := \{(x,t) : 0 < x < t,\ 0 < t < T\}$ triangular domain, bounded by characteristic segment $\gamma_{1,T} : x = t,\ 0 \le t \le T$ and by segments $\gamma_{2,T} : x = 0$, $0 \le t \le T$, $\gamma_{3,T} : t = T,\ 0 \le x \le T$.

For equation (1.1) in angular domain D_T consider the boundary value problem on determination of solution $u(x,t)$ by conditions [see e.g., Bitsadze, 1981, p.228]

$$u\big|_{\gamma_{i,T}} = 0, \quad i = 1, 2. \tag{1.2}$$

Note that on questions of existence, uniqueness and blow-up of solutions of initial, mixed, nonlocal and other problems posed for nonlinear hyperbolic type equations there are devoted a number of papers (see, e.g., Lions, 1969; Bitsadze, 1981; John and Klainerman, 1984; Kato, 1980; Georgiev et al., 1977; Sideris, 1984; Hormander, 1997; Veron and Pohozaev, 2001; Mitidieri and Pohozaev, 2001; Todorova and Vitillaro, 2005; Jokhadze, 2008; Berikelashvili et al., 2008). In linear case, i.e., when $\mu = \lambda = 0$, problem (1.1), (1.2), as it is known, is well-posed and a global solvability takes place in corresponding function spaces (see, e.g., Bitsadze, 1981; Goursat, 1933; Kharibegashvili, 1995).

Definition 1.1: Let $f \in C(\overline{D}_T)$. Function u is called a strong generalised solution of problem (1.1), (1.2) of class C^1 in domain D_T, if $u \in C^1(\overline{D}_T)$ and there exists such the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, that $u_n \to u$ and $Lu_n \to f$ for $n \to \infty$ in spaces $C^1(\overline{D}_T)$ and $C(\overline{D}_T)$, respectively, where $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T) := \{v \in C^2(\overline{D}_T) : v\big|_{\Gamma_T} = 0\}$, $\Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Remark 1.1: It is clear that if $u \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a classical solution of problem (1.1), (1.2), then it is a strong generalised solution of this problem of class C^1 in domain D_T. In turn, if a strong generalised solution of problem (1.1), (1.2) of class C^1 in domain D_T belongs to space $C^2(\overline{D}_T)$, then it also is a classical solution of the problem.

Definition 1.2: Let $f \in C(\overline{D}_\infty)$. We say that problem (1.1), (1.2) is globally solvable in the class C^1, if for any finite $T > 0$ it has a strong generalised solution of the class C^1 in domain D_T.

The paper is organised as follows. In Section 2 it is obtained a priori estimates of the solution of problem (1.1), (1.2) in the spaces $C(\overline{D}_T)$ and $C^1(\overline{D}_T)$. In Section 3 problem (1.1), (1.2) is equivalently reduced to the system of nonlinear Volterra type integral equations and the local solvability of this problem is proved. In Section 4 it is shown the global solvability of the considered problem, and in Section 5 the uniqueness of the solution of the problem is proved. Finally, in Section 6 the nonexistence of global solvability of this problem is shown.

2 A priori estimates of the solution of problem (1.1), (1.2) in the spaces $C(\overline{D}_T)$ and $C^1(\overline{D}_T)$

Lemma 2.1: *Let $f \in C(\overline{D}_T)$ and*

$$\mu > 0, \quad \lambda > 0. \tag{2.1}$$

Then for a strong generalised solution of problem (1.1), (1.2) *of class C^1 in domain D_T it is valid the following a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \leq c_0 \|f\|_{C(\overline{D}_T)} \tag{2.2}$$

with positive constant $c_0 = c_0(T)$, not dependent on u and f.

Proof: Let u be a strong generalised solution of problem (1.1), (1.2) of class C^1 in domain D_T. Then due to Definition 1.1 there exists the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, such that

$$\lim_{n\to\infty} \|u_n - u\|_{C^1(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_T)} = 0, \tag{2.3}$$

and therefore

$$\lim_{n\to\infty} \||u_n|^\rho u_{nt} - |u|^\rho u_t\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \||u_n|^\alpha u_n - |u|^\alpha u\|_{C(\overline{D}_T)} = 0. \tag{2.4}$$

Consider function $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, as a solution of the following problem

$$Lu_n = f_n, \tag{2.5}$$

$$u_n\big|_{\Gamma_T} = 0. \tag{2.6}$$

Here

$$f_n := Lu_n. \tag{2.7}$$

Multiplying the both sides of equality (2.5) by $\frac{\partial u_n}{\partial t}$ and integrating the received in domain $D_\tau := \{(x,t) \in D_T : 0 < t < \tau\}$, $0 < \tau \le T$ we have

$$\frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_n}{\partial t}\right)^2 dxdt - \int_{D_\tau} \frac{\partial^2 u_n}{\partial x^2}\frac{\partial u_n}{\partial t} dxdt + \mu \int_{D_\tau} |u_n|^\rho \left(\frac{\partial u_n}{\partial t}\right)^2 dxdt \\ + \frac{\lambda}{\alpha+2}\int_{D_\tau} \frac{\partial}{\partial t}|u_n|^{\alpha+2} dxdt = \int_{D_\tau} f_n \frac{\partial u_n}{\partial t} dxdt.$$

Assume that $\Omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. Then by virtue of (2.6), integrating by parts the left side of the last equality, we obtain

$$\int_{D_\tau} f_n \frac{\partial u_n}{\partial t} dxdt = \int_{\gamma_{1,\tau}} \frac{1}{2\nu_t}\left[\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2 (\nu_t^2 - \nu_x^2)\right] ds \\ + \frac{1}{2}\int_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2\right] dx \\ + \mu \int_{D_\tau} |u_n|^\rho \left(\frac{\partial u_n}{\partial t}\right)^2 dxdt + \frac{\lambda}{\alpha+2}\int_{\Omega_\tau} |u_n|^{\alpha+2} dx, \tag{2.8}$$

where $\nu := (\nu_x, \nu_t)$ is unit vector of outer normal to ∂D_τ and $\gamma_{1,\tau} := \gamma_{1,T} \cap \{t \le \tau\}$.

Taking into account the fact that operator $\nu_t \frac{\partial}{\partial x} - \nu_x \frac{\partial}{\partial t}$ is an interior differential operator on $\gamma_{1,T}$, due to (2.6) we receive

$$\left.\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x\right)\right|_{\gamma_{1,\tau}} = 0. \tag{2.9}$$

Further, it is clear that

$$(\nu_t^2 - \nu_x^2)\big|_{\gamma_{1,\tau}} = 0. \tag{2.10}$$

Therefore, by virtue of (2.1) and (2.8)–(2.10) we get

$$w_n(\tau) := \int_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2\right] dx \le 2\int_{D_\tau} f_n \frac{\partial u_n}{\partial t} dxdt. \tag{2.11}$$

Taking into account inequality

$$2f_n \frac{\partial u_n}{\partial t} \le \left(\frac{\partial u_n}{\partial t}\right)^2 + f_n^2,$$

due to (2.11) we have

$$w_n(\tau) \le \int_{D_\tau} \left(\frac{\partial u_n}{\partial t}\right)^2 dxdt + \int_{D_\tau} f_n^2 dxdt.$$

According to the form of function $w_n(\tau)$ it follows that

$$w_n(\tau) \leq \int_0^\tau w_n(\sigma)d\sigma + \|f_n\|^2_{L_2(D_\tau)}.$$

Whence, having the fact that the value $\|f_n\|^2_{L_2(D_\tau)}$, as a function of τ is non-decreasing, by the Gronwall's lemma [see e.g., Henry, 1985, p.13] we receive

$$w_n(\tau) \leq \exp(\tau)||f_n||^2_{L_2(D_\tau)}. \tag{2.12}$$

If $(x,t) \in \overline{D}_T$, then by virtue of the condition (2.6) the following equality is valid

$$u_n(x,t) = u_n(x,t) - u_n(0,t) = \int_0^x \frac{\partial u_n(\sigma,t)}{\partial x} d\sigma.$$

Thus, taking into account obvious inequality

$$\|f_n\|^2_{L_2(D_T)} \leq \|f_n\|^2_{C(\overline{D}_T)} mes\ D_T,$$

due to (2.12) we have

$$\begin{aligned}
|u_n(x,t)|^2 &\leq \int_0^x d\sigma \int_0^x \left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \\
&\leq x \int_{\Omega_t} \left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \leq x w_n(t) \leq t w_n(t) \\
&\leq T \exp(T)\|f_n\|^2_{C(\overline{D}_T)} mes\ D_T = 2^{-1}T^3 \exp(T)\|f_n\|^2_{C(\overline{D}_T)}. \qquad (2.13)
\end{aligned}$$

From this it follows that

$$\|u_n\|_{C(\overline{D}_T)} \leq T\sqrt{2^{-1}T} \exp(2^{-1}T)\|f_n\|_{C(\overline{D}_T)}.$$

Passing in this inequality to limit for $n \to \infty$, and due to (2.3), (2.7), we have

$$\|u\|_{C(\overline{D}_T)} \leq T\sqrt{2^{-1}T} \exp(2^{-1}T)\|f\|_{C(\overline{D}_T)}. \tag{2.14}$$

This proves estimate (2.2).

Remark 2.1: From (2.14) it follows that constant c_0 in estimate (2.2) can be taken

$$c_0 := T\sqrt{2^{-1}T} \exp\left(2^{-1}T\right). \tag{2.15}$$

Below, taking into account estimate (2.2) and using the classical method of characteristics, we receive a priori estimate in the space $C^1(\overline{D}_T)$ for a strong generalised solution u of problem (1.1), (1.2) of class C^1 in domain D_T.

Lemma 2.2: *Under the conditions of Lemma* 2.1 *for a strong generalised solution* u *of problem* (1.1), (1.2) *of class* C^1 *in domain* D_T *it is valid the following a priori estimate*

$$\|u\|_{C^1(\overline{D}_T)} \le c_1, \tag{2.16}$$

with a positive constant $c_1 = c_1\big(T, c_0, \mu, \rho, \lambda, \alpha, \|f\|_{C(\overline{D}_T)}\big)$, *not dependent on* u, *where* $\|u\|_{C^1(\overline{D}_T)} := \max\big\{\|u\|_{C(\overline{D}_T)},\ \|u_x\|_{C(\overline{D}_T)},\ \|u_t\|_{C(\overline{D}_T)}\big\}$.

Proof: Let u be a strong generalised solution of problem (1.1), (1.2) of class C^1 in domain D_T. Then the limit equalities (2.3), (2.4) are valid, where u_n can be considered as a solution of problem (2.5), (2.6) with f_n given by (2.7). For fixed natural n let us introduce the following functions

$$u_{n1} := u_{nt} - u_{nx}, \quad u_{n2} := u_{nt} + u_{nx}, \quad u_{n3} := u_n, \tag{2.17}$$

which by taking into account (2.2) satisfy the following boundary conditions

$$u_{n1}(0,t) = -u_{n2}(0,t), \quad u_{n2}(t,t) = 0, \quad u_{n3}(t,t) = 0, \quad 0 \le t \le T. \tag{2.18}$$

With respect to unknown functions u_{n1}, u_{n2}, u_{n3} by virtue of (1.1) and (2.17) we have the following system of first order partial differential equations

$$\begin{cases} \dfrac{\partial u_{n1}}{\partial t} + \dfrac{\partial u_{n1}}{\partial x} = f_n(x,t) - 2^{-1}\mu|u_{n3}|^{\rho}(u_{n1}+u_{n2}) - \lambda|u_{n3}|^{\alpha}u_{n3}, \\ \dfrac{\partial u_{n2}}{\partial t} - \dfrac{\partial u_{n2}}{\partial x} = f_n(x,t) - 2^{-1}\mu|u_{n3}|^{\rho}(u_{n1}+u_{n2}) - \lambda|u_{n3}|^{\alpha}u_{n3}, \\ \dfrac{\partial u_{n3}}{\partial t} - \dfrac{\partial u_{n3}}{\partial x} = u_{n1}. \end{cases} \tag{2.19}$$

Integrating the received system along the corresponding characteristic curves, due to boundary conditions (2.18), we have

$$\begin{cases} u_{n1}(x,t) - u_{n1}(0,t) \\ \quad = \int_{t-x}^{t} \big(f_n - 2^{-1}\mu|u_{n3}|^{\rho}(u_{n1}+u_{n2}) - \lambda|u_{n3}|^{\alpha}u_{n3}\big)(P_\tau)\,d\tau, \\ u_{n2}(x,t) = \int_{2^{-1}(x+t)}^{t} \big(f_n - 2^{-1}\mu|u_{n3}|^{\rho}(u_{n1}+u_{n2}) - \lambda|u_{n3}|^{\alpha}u_{n3}\big)(Q_\tau)\,d\tau, \\ u_{n3}(x,t) = \int_{2^{-1}(x+t)}^{t} u_{n1}(Q_\tau)\,d\tau, \end{cases}$$

where $P_\tau := (x-t+\tau, \tau)$, $Q_\tau := (x+t-\tau, \tau)$.

From the second equation of the received system and the first equality (2.18), taking into account notation $P_{\tau_0} := (-t+\tau, \tau)$ this system can be rewritten as follows

$$\begin{cases} u_{n1}(x,t) = -\int_{t-x}^{t} \big(2^{-1}\mu|u_{n3}|^{\rho}(u_{n1}+u_{n2}) + \lambda|u_{n3}|^{\alpha}u_{n3}\big)(P_\tau)\,d\tau \\ \qquad\qquad + \int_{2^{-1}t}^{t} \big(2^{-1}\mu|u_{n3}|^{\rho}(u_{n1}+u_{n2}) + \lambda|u_{n3}|^{\alpha}u_{n3}\big)(P_{\tau_0})\,d\tau \\ \qquad\qquad + F_{n1}(x,t), \\ u_{n2}(x,t) = -\int_{2^{-1}(x+t)}^{t} \big(2^{-1}\mu|u_{n3}|^{\rho}(u_{n1}+u_{n2}) \\ \qquad\qquad + \lambda|u_{n3}|^{\alpha}u_{n3}\big)(Q_\tau)\,d\tau + F_{n2}(x,t), \\ u_{n3}(x,t) = \int_{2^{-1}(x+t)}^{t} u_{n1}(Q_\tau)\,d\tau. \end{cases} \tag{2.20}$$

Here

$$F_{n1}(x,t) := \int_{t-x}^{t} f_n(P_\tau)d\tau - \int_{2^{-1}t}^{t} f_n(t-\tau,\tau)d\tau,$$
$$F_{n2}(x,t) := \int_{2^{-1}(x+t)}^{t} f_n(Q_\tau)d\tau. \tag{2.21}$$

By passing in the equalities (2.20), (2.21) to limit as $n \to \infty$ in the space $C(\overline{D}_T)$ and taking into account (2.3), (2.4), (2.7) and (2.17) we have

$$\begin{cases} u_1(x,t) = -\int_{t-x}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2)+\lambda|u_3|^\alpha u_3\big)\big(P_\tau\big)d\tau \\ \qquad\qquad + \int_{2^{-1}t}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2)+\lambda|u_3|^\alpha u_3\big)\big(P_{\tau_0}\big)d\tau \\ \qquad\qquad +F_1(x,t), \\ u_2(x,t) = -\int_{2^{-1}(x+t)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2)+\lambda|u_3|^\alpha u_3\big)\big(Q_\tau\big)d\tau \\ \qquad\qquad +F_2(x,t), \\ u_3(x,t) = \int_{2^{-1}(x+t)}^{t} u_1(Q_\tau)d\tau, \end{cases} \tag{2.22}$$

where $u_i := \lim_{n\to\infty} u_{ni}$ (by the norm of space $C(\overline{D}_T)\big)$, $i=1,2,3$, and

$$F_1(x,t) := \int_{t-x}^{t} f(P_\tau)d\tau - \int_{2^{-1}t}^{t} f(t-\tau,\tau)d\tau,$$
$$F_2(x,t) := \int_{2^{-1}(x+t)}^{t} f(Q_\tau)d\tau. \tag{2.23}$$

It is clear that $u_3 = u$, which is a strong generalised solution of problem (1.1), (1.2) of class C^1 in domain D_T, besides

$$u_1 = u_t - u_x, \quad u_2 = u_t + u_x. \tag{2.24}$$

Let

$$K := c_0\|f\|_{C(\overline{D}_T)}, \tag{2.25}$$

where c_0 is defined by (2.15) and

$$v_i(t) := \sup_{(\xi,\tau)\in\overline{D}_t} |u_i(\xi,\tau)|, \quad i=1,2,3, \quad F(t) := \sup_{(\xi,\tau)\in\overline{D}_t} |f(\xi,\tau)|. \tag{2.26}$$

From (2.22), due (2.23) and (2.26) it follows that

$$|u_1(x,t)| \le \mu K^\rho \int_0^t \big(v_1(\tau)+v_2(\tau)\big)d\tau + 2\lambda T K^{\alpha+1} + 2TF(T),$$
$$|u_2(x,t)| \le 2^{-1}\mu K^\rho \int_0^t \big(v_1(\tau)+v_2(\tau)\big)d\tau + \lambda T K^{\alpha+1} + TF(T),$$
$$|u_3(x,t)| \le \int_0^t v_1(\tau)d\tau.$$

Whence for $(\xi,\tau)\in\overline{D}_t$ we receive

$$
\begin{aligned}
|u_1(\xi,\tau)| &\le \mu K^\rho \int_0^\tau \big(v_1(\tau_1)+v_2(\tau_1)\big)d\tau_1 + 2\lambda T K^{\alpha+1} + 2TF(T),\\
|u_2(\xi,\tau)| &\le 2^{-1}\mu K^\rho \int_0^\tau \big(v_1(\tau_1)+v_2(\tau_1)\big)d\tau_1 + \lambda T K^{\alpha+1} + TF(T),\\
|u_3(\xi,\tau)| &\le \int_0^\tau v_1(\tau_1)d\tau_1.
\end{aligned}
$$

Thus due to (2.26) it follows that

$$
\begin{aligned}
v_1(t) &\le \mu K^\rho \int_0^t \big(v_1(\tau)+v_2(\tau)\big)d\tau + 2\lambda T K^{\alpha+1} + 2TF(T),\\
v_2(t) &\le 2^{-1}\mu K^\rho \int_0^t \big(v_1(\tau)+v_2(\tau)\big)d\tau + \lambda T K^{\alpha+1} + TF(T),\\
v_3(t) &\le \int_0^t v_1(\tau)d\tau.
\end{aligned}
$$

Setting that $v(t):=\max_{1\le i\le 3} v_i(t)$, $0\le t\le T$, from the inequalities given above we have

$$
v(t)\le \big(2\mu K^\rho+1\big)\int_0^t v(\tau)d\tau + 2\lambda T K^{\alpha+1} + 2TF(T),\quad 0\le t\le T,
$$

whence by virtue of Gronwall's lemma we get

$$
\begin{aligned}
v(t) &\le 2T\big(\lambda K^{\alpha+1}+F(T)\big)\exp\big((2\mu K^\rho+1)t\big)\\
&\le 2T\big(\lambda c_0 K^\alpha+1\big)\exp\big((2\mu K^\rho+1)T\big)\|f\|_{C(\overline{D}_T)},\quad 0\le t\le T.
\end{aligned}
$$

Now from (2.24) it is easy to receive

$$
\|u\|_{C^1(\overline{D}_T)}\le \|v\|_{C[0,T]}\le 2T\big(\lambda c_0 K^\alpha+1\big)\exp\big((2\mu K^\rho+1)T\big)\|f\|_{C(\overline{D}_T)}.
$$

Lemma 2.2 is proved, besides

$$
c_1 := 2T\big(\lambda c_0 K^\alpha+1\big)\exp\big((2\mu K^\rho+1)T\big)\|f\|_{C(\overline{D}_T)},
$$

where K is defined by (2.25).

3 The equivalency of problem (1.1), (1.2) and the system of nonlinear volterra type integral equations (2.22) and the local solvability of problem (1.1), (1.2)

First of all let us show that problem (2.5), (2.6) is equivalent to problem (2.19), (2.18) in the classical sense. Indeed, if $u_n\in C^2$ is a solution of problem (2.5), (2.6), then the system of functions u_{n1}, u_{n2} and u_{n3} will, obviously, give the solution of problem (2.19), (2.18). Conversely, let $u_{n1}, u_{n2}, u_{n3} \in C^1$ be a solution of problem (2.19), (2.18). Let us show that $u_n := u_{n3} \in C^2$, is a solution of problem (2.5), (2.6) and satisfying the equalities (2.17). If we show that $u_{n2} = u_{nt} + u_{nx}$, then, obviously, the following equalities $u_{nt} = \frac{u_{n2}+u_{n1}}{2}$ and $u_{nx} = \frac{u_{n2}-u_{n1}}{2}$ hold, whence it follows that $u_n \in C^2$ represents a solution of problem (2.5), (2.6) in the classical sense.

Indeed, it follows from the first two equations of system (2.19) that

$$\frac{\partial u_{n1}}{\partial t} + \frac{\partial u_{n1}}{\partial x} = \frac{\partial u_{n2}}{\partial t} - \frac{\partial u_{n2}}{\partial x}. \tag{3.1}$$

Further, since $u_{n1} \in C^1$ then from the third equation of (2.19) it follows that $\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} \in C$ and $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} \in C$. Whence due to the commutative property of the first order differential operators with constant coefficients, we receive

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}u_{n3} \in C,$$
$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}u_{n3} \in C.$$

From these equalities, (3.1) and the third equation of system (2.19) we have

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)(u_{n2} - u_{nt} - u_{nx}) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n2} - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_n$$
$$-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_n = \frac{\partial u_{n2}}{\partial t} - \frac{\partial u_{n2}}{\partial x} - \frac{\partial u_{n1}}{\partial t} - \frac{\partial u_{n1}}{\partial x} = 0.$$

Whence according to the second and third equalities from (2.18) we conclude that $u_{n2} = u_{nt} + u_{nx}$. This proves the equivalency of problems (2.5), (2.6) and (2.19), (2.18) in the classical sense.

Above we have reduced problem (1.1), (1.2) to the system of Volterra type nonlinear integral equations (2.22). Before considering the question of local solvability of problem (1.1), (1.2), let us make a remark which follows from considerations given section 2.

Remark 3.1: Let u be a strong generalised solution of problem (1.1), (1.2) of class C^1 in domain D_T, then $u_1 := u_t - u_x$, $u_2 := u_t + u_x$, $u_3 := u$ is a continuous solution of the system of nonlinear Volterra type integral equations (2.22) and vice versa, if u_1, u_2, u_3 is a continuous solution of system (2.22), then $u := u_3$ is a strong generalised solution of problem (1.1), (1.2) of the class C^1 in domain D_T, besides the equalities $u_1 = u_t - u_x$, $u_2 = u_t + u_x$ are valid.

Now let us prove the local solvability of the system of Volterra type nonlinear integral equations (2.22).

Let

$$f \in C(\overline{D}_\infty), f_\infty := \sup_{(x,t)\in\overline{D}_\infty} |f(x,t)| < +\infty \tag{3.2}$$

and

$$\rho \geq 1. \tag{3.3}$$

Theorem 3.1: *Let the function f and the number ρ satisfy conditions* (3.2), (3.3), *respectively. Then there exists positive number $T_* := T_*(\mu, \rho, \lambda, \alpha, f)$, such that for $T \leq T_*$ the problem* (1.1), (1.2) *will have at least one strong generalised solution u of the class C^1 in domain D_T*

Proof: According to Remark 3.1 problem (1.1), (1.2) in the space $C^1(\overline{D}_T)$ is equivalent to the system of Volterra type nonlinear integral equations (2.22) in the space $C(\overline{D}_T)$. Below we will prove a unique solvability of system (2.22) by the principle of contraction mappings.

Let $U := (u_1, u_2, u_3)$. Consider vectorial operator $\Phi := (\Phi_1, \Phi_2, \Phi_3)$, acting by the formula

$$\begin{cases} (\Phi_1 U)(x,t) = -\int_{t-x}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2) + \lambda|u_3|^\alpha u_3\big)\big(P_\tau\big)d\tau \\ \qquad\qquad + \int_{2^{-1}t}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2) + \lambda|u_3|^\alpha u_3\big)\big(P_{\tau_0}\big)d\tau \\ \qquad\qquad + F_1(x,t), \\ (\Phi_2 U)(x,t) = -\int_{2^{-1}(x+t)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2) + \lambda|u_3|^\alpha u_3\big)\big(Q_\tau\big)d\tau \\ \qquad\qquad + F_2(x,t), \\ (\Phi_3 U)(x,t) = \int_{2^{-1}(x+t)}^{t} u_1(Q_\tau)d\tau. \end{cases} \tag{3.4}$$

Then the system (2.22) can be rewritten as follows

$$U = \Phi U. \tag{3.5}$$

Let $\|U\|_{X_T} := \max_{1\leq i\leq 3}\big\{\|u_i\|_{C(\overline{D}_T)}\big\}$, $U \in X_T := C(\overline{D}_T;\mathbb{R}^3)$, where $C(\overline{D}_T;\mathbb{R}^3)$ is the set of all continuous vector-functions $U : \overline{D}_T \to \mathbb{R}^3$. Let us denote by $B_R := \{U \in X_T : \|U\|_{X_T} \leq R\}$ closed ball of radius $R > 0$ in Banach space X_T with a centre in the null element.

Below we prove that: (i) Φ maps a ball B_R into itself; (ii) Φ is a contractive mapping on B_R.

Indeed, from (3.4) for $U : \|U\|_{X_T} \leq R$ we have

$$\begin{aligned} &|(\Phi_1 U)(x,t)| \leq 2T\big(|\mu|R^{\rho+1} + |\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_T)}\big), \\ &|(\Phi_2 U)(x,t)| \leq T\big(|\mu|R^{\rho+1} + |\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_T)}\big), \quad |(\Phi_3 U)(x,t)| \leq TR. \end{aligned}$$

From these estimates it follows that

$$\begin{aligned} \|\Phi U\|_{X_T} &\leq 2T\big(|\mu|R^{\rho+1} + |\lambda|R^{\alpha+1} + R + \|f\|_{C(\overline{D}_T)}\big) \\ &\leq 2T\big(|\mu|R^{\rho+1} + |\lambda|R^{\alpha+1} + R + f_\infty\big), \end{aligned}$$

where f_∞ is defined in (3.2).

Assume that the value of T with fixed $R > 0$ is too small that

$$2T\big(|\mu|R^{\rho+1} + |\lambda|R^{\alpha+1} + R + f_\infty\big) \leq R, \tag{3.6}$$

so $\Phi U \in B_R$ and therefore condition (i) is fulfilled.

Further, due to (3.3) and the first equality of (3.4) for $U^i : \|U^i\|_{X_T} \leq R$, $i = 1, 2$, we have

$$\begin{aligned}
&\big|(\Phi_1 U^2 - \Phi_1 U^1)(x,t)\big| \\
&\leq \int_{t-x}^{t} \big(2^{-1}|\mu|\big||u_3^2|^\rho - |u_3^1|^\rho\big||u_1^2 + u_2^2| + 2^{-1}|\mu||u_3^1|^\rho|u_1^2 - u_1^1 + u_2^2 - u_2^1| \\
&\quad + |\lambda|\big||u_3^2|^\alpha u_3^2 - |u_3^1|^\alpha u_3^1\big|\big)(P_\tau)\,d\tau + \int_{2^{-1}t}^{t} \big(2^{-1}|\mu|\big||u_3^2|^\rho - |u_3^1|^\rho\big||u_1^2 + u_2^2| \\
&\quad + 2^{-1}|\mu||u_3^1|^\rho|u_1^2 - u_1^1 + u_2^2 - u_2^1| + |\lambda|\big||u_3^2|^\alpha u_3^2 - |u_3^1|^\alpha u_3^1\big|\big)(P_{\tau_0})\,d\tau \\
&\leq 2T\big(|\mu|(\rho+1)R^\rho + |\lambda|(\alpha+1)R^\alpha\big)\|U_2 - U_1\|_{X_T}.
\end{aligned}$$

Analogously

$$|(\Phi_2 U^2 - \Phi_2 U^1)(x,t)| \leq T\big(|\mu|(\rho+1)R^\rho + |\lambda|(\alpha+1)R^\alpha\big)\|U_2 - U_1\|_{X_T}$$

and

$$|(\Phi_3 U^2 - \Phi_3 U^1)(x,t)| \leq T\|U_2 - U_1\|_{X_T}.$$

Assume that for fixed $R > 0$ number T is too small that

$$\max\big(T, 2T\big(|\mu|(\rho+1)R^\rho + |\lambda|(\alpha+1)R^\alpha\big)\big) \leq 2^{-1} < 1, \tag{3.7}$$

and therefore $\|\Phi U^2 - \Phi U^1\|_{X_T} \leq \frac{1}{2}\|U^2 - U^1\|_{X_T}$. Thus operator Φ is a contractive mapping on set B_R, i.e., condition (ii) is fulfilled.

From (3.6) and (3.7), in turn, follows that, if $T \leq T_*$, where

$$\begin{aligned}
T_* := \min\bigg\{&\frac{R}{2\big(|\mu|R^{\rho+1} + |\lambda|R^{\alpha+1} + R + f_\infty\big)}, \frac{1}{2}, \\
&\frac{1}{4\big(|\mu|(\rho+1)R^\rho + |\lambda|(\alpha+1)R^\alpha\big)}\bigg\},
\end{aligned} \tag{3.8}$$

then $\|\Phi U\|_{X_T} \leq R$ and $\|\Phi U^2 - \Phi U^1\|_{X_T} \leq \frac{1}{2}\|U^2 - U^1\|_{X_T}$ for U, U^1, $U^2 \in B_R$.

The application of the contraction mapping principle shows that there exists a unique solution U of (3.5) in $C(\overline{D}_T; \mathbb{R}^3)$ for $0 < T \leq T_*$. Theorem 3.1 is proved completely.

4 The case of global solvability of problem (1.1), (1.2)

Theorem 4.1: *Let the conditions* (2.1), (3.2) *and* (3.3) *are valid. Then for any* $T > 0$ *the problem* (1.1), (1.2) *has a strong generalised solution of the class* C^1 *in domain* D_T.

Proof: As it was noted in Remark 3.1 the problem (1.1), (1.2) in space $C^1(\overline{D}_T)$ is equivalent to the system of nonlinear integral equations (2.22) in space $C(\overline{D}_T)$.

In view of (3.2), (3.3) the truth of the Theorem 4.1 for sufficiently small T, namely for $T \le T_*$, where T_* is given by equality (3.8) follows from the Theorem 3.1. Let now $T > T_*$, and $U^{T_*} := (u_1^{T_*}, u_2^{T_*}, u_3^{T_*})$ is a solution of the system of nonlinear integral equations (2.22), or, the same, of vector equation (3.5) in domain D_{T_*} of space $C(\overline{D}_{T_*})$ according to Theorem 3.1. For $t > \Delta t_1 := T_*$ rewrite system (2.22) as follows

$$\begin{cases} u_1(x,t) = -\int_{\alpha_1(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2) + \lambda|u_3|^\alpha u_3\big)(P_\tau)d\tau \\ \qquad + \int_{\alpha_2(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2) + \lambda|u_3|^\alpha u_3\big)(P_{\tau_0})d\tau \\ \qquad + F_{1,\Delta t_1}(x,t), \\ u_2(x,t) = -\int_{\alpha_3(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2) + \lambda|u_3|^\alpha u_3\big)(Q_\tau)d\tau \\ \qquad + F_{2,\Delta t_1}(x,t), \\ u_3(x,t) = \int_{\alpha_3(x,t;\Delta t_1)}^{t} u_1(Q_\tau)d\tau + F_{3,\Delta t_1}(x,t), \end{cases} \tag{4.1}$$

where

$$\alpha_1(x,t;\Delta t_1) := \max(\Delta t_1, t-x), \alpha_2(x,t;\Delta t_1) := \max(\Delta t_1, 2^{-1}t),$$
$$\alpha_3(x,t;\Delta t_1) := \max(\Delta t_1, 2^{-1}(x+t));$$

$$\begin{cases} F_{1,\Delta t_1}(x,t) := -\int_{t-x}^{\alpha_1(x,t;\Delta t_1)} \big(2^{-1}\mu|u_3^{T_*}|^\rho(u_1^{T_*}+u_2^{T_*}) \\ \qquad + \lambda|u_3^{T_*}|^\alpha u_3^{T_*}\big)(P_\tau)d\tau \\ \qquad + \int_{2^{-1}t}^{\alpha_2(x,t;\Delta t_1)} \big(2^{-1}\mu|u_3^{T_*}|^\rho(u_1^{T_*}+u_2^{T_*}) \\ \qquad + \lambda|u_3^{T_*}|^\alpha u_3^{T_*}\big)(P_{\tau_0})d\tau + F_1(x,t), \\ F_{2,\Delta t_1}(x,t) := -\int_{2^{-1}(x+t)}^{\alpha_3(x,t;\Delta t_1)} \big(2^{-1}\mu|u_3^{T_*}|^\rho(u_1^{T_*}+u_2^{T_*}) \\ \qquad + \lambda|u_3^{T_*}|^\alpha u_3^{T_*}\big)(Q_\tau)d\tau + F_2(x,t), \\ F_{3,\Delta t_1}(x,t) := \int_{2^{-1}(x+t)}^{\alpha_3(x,t;\Delta t_1)} u_1^{T_*}(Q_\tau)d\tau. \end{cases} \tag{4.2}$$

Since the conditions of Lemma 2.2 are fulfilled, then for any positive $\tau \le T$ for a solution of vector equation (3.5) in domain D_τ of space X_τ due to the (2.16) it is valid a priori estimate

$$\|U\|_{X_\tau} \le R^T\big(\|f\|_{C(\overline{D}_\tau)}\big), \tag{4.3}$$

where $R^T = R^T(s)$ is a non-decreasing continuous function of its argument $s \ge 0$.

Let $R_* := R^T\big(\|f\|_{C(\overline{D}_T)}\big)$. As the second step Δt_2 with respect to t we take

$$\Delta t_2 := \frac{1}{4R_1\big(\mu\rho R_1^\rho + \lambda(\alpha+1)R_1^\alpha\big)}, \tag{4.4}$$

where

$$R_1 := 1 + 2T\big(\mu R_*^{\rho+1} + \lambda R_*^{\alpha+1}\big) + \|F\|_{X_T}, \quad F := (F_1, F_2, 0). \tag{4.5}$$

Rewrite the system of equations (4.1) for $t \in [T_*, T_* + \Delta t_2]$ in the form of one vector equation

$$U = \Psi U, \tag{4.6}$$

where the vectorial operator $\Psi := (\Psi_1, \Psi_2, \Psi_3)$ acts by the formula

$$\begin{cases} (\Psi_1 U)(x,t) = -\int_{\alpha_1(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2)+\lambda|u_3|^\alpha u_3\big)\big(P_\tau\big)d\tau \\ \qquad\qquad + \int_{\alpha_2(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2)+\lambda|u_3|^\alpha u_3\big)\big(P_{\tau_0}\big)d\tau \\ \qquad\qquad + F_{1,\Delta t_1}(x,t), \\ (\Psi_2 U)(x,t) = -\int_{\alpha_3(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu|u_3|^\rho(u_1+u_2)+\lambda|u_3|^\alpha u_3\big)\big(Q_\tau\big)d\tau \\ \qquad\qquad + F_{2,\Delta t_1}(x,t), \\ (\Psi_3 U)(x,t) = \int_{\alpha_3(x,t;\Delta t_1)}^{t} u_1(Q_\tau)d\tau + F_{3,\Delta t_1}(x,t). \end{cases} \tag{4.7}$$

Let, analogously as in Section 3, $\|U\|_{X_{[T_1,T_2]}} := \max_{1\le i\le 3}\big\{\|u_i\|_{C(\overline{D}_{[T_1,T_2]})}\big\}$, where $X_{[T_1,T_2]}$ is the set of all continuous vector-functions $U : \overline{D}_{[T_1,T_2]} \to \mathbb{R}^3$, $\overline{D}_{[T_1,T_2]} := \overline{D} \cap \{T_1 \le t \le T_2\}$.

First we show that the operator Ψ maps the ball $B([T_1,T_2];R_1) := \{U \in X_{[T_1,T_2]} : \|U\|_{X_{[T_1,T_2]}} \le R_1\}$ into itself, where $T_1 = T_*$ and $T_2 = T_* + \Delta t_2$.

Indeed, due to (4.2)–(4.5) and (4.7) we have

$$\begin{aligned} \|\Psi_1 U\|_{C(\overline{D}_{[T_1,T_2]})} &\le 2\big(\mu R_1^{\rho+1} + \lambda R_1^{\alpha+1}\big)\Delta t_2 + 2\big(\mu R_*^{\rho+1} + \lambda R_*^{\alpha+1}\big)\Delta t_1 \\ &\quad + \|F_1\|_{C(\overline{D}_{[T_1,T_2]})} \\ &\le 2^{-1} + 2T\big(\mu R_*^{\rho+1} + \lambda R_*^{\alpha+1}\big) + \|F\|_{X_T} \le R_1. \end{aligned}$$

Analogously: $\|\Psi_i U\|_{C(\overline{D}_{[T_1,T_2]})} \le R_1,\ i = 2,3$.

Now let us show that the operator Ψ is a contractive mapping in this ball. Indeed, for $(x,t) \in \overline{D}_{[T_1,T_2]}$ due to (4.4) and (4.7) we have

$$\begin{aligned} &|(\Psi_1 U^2 - \Psi_1 U^1)(x,t)| \\ &\le \int_{\alpha_1(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu\big||u_3^2|^\rho - |u_3^1|^\rho\big||u_1^2+u_2^2| \\ &\quad + 2^{-1}\mu|u_3^1|^\rho|u_1^2-u_1^1+u_2^2-u_2^1| + \lambda\big||u_3^2|^\alpha u_3^2 - |u_3^1|^\alpha u_3^1\big|\big)\big(P_\tau\big)d\tau \\ &\quad + \int_{\alpha_2(x,t;\Delta t_1)}^{t} \big(2^{-1}\mu\big||u_3^2|^\rho - |u_3^1|^\rho\big||u_1^2 \\ &\quad + u_2^2| + 2^{-1}\mu|u_3^1|^\rho|u_1^2-u_1^1+u_2^2-u_2^1| + \lambda\big||u_3^2|^\alpha u_3^2 - |u_3^1|^\alpha u_3^1\big|\big)\big(P_{\tau_0}\big)d\tau \\ &\le 2\big(\mu\rho R_1^\rho + \lambda(\alpha+1)R_1^\alpha\big)\Delta t_2\|u_3^2-u_3^1\|_{C(\overline{D}_{[T_1,T_2]})} \\ &\quad + 2\mu R_1^\rho \Delta t_2 \|U^2-U^1\|_{X_{[T_1,T_2]}} \\ &\le \big(2^{-1} + (2R_1)^{-1}\big)\|U^2-U^1\|_{X_{[T_1,T_2]}} = q_1\|U^2-U^1\|_{X_{[T_1,T_2]}}, \end{aligned}$$

where $q_1 := 2^{-1}\big(1 + R_1^{-1}\big) < 1$, since $R_1 > 1$ in view of (4.5).

Analogously we receive, that

$$|(\Psi_i U^2 - \Psi_i U^1)(x,t)| \le q_i\|U^2-U^1\|_{X_{[T_1,T_2]}}, \quad 0 < q_i := const < 1, \quad i = 2,3.$$

Thus, $\|\Psi U\|_{X_{[T_1,T_2]}} \le R_1$, $\|\Phi U^2 - \Phi U^1\|_{X_{[T_1,T_2]}} \le q_i\|U^2-U^1\|_{X_{[T_1,T_2]}}$, where $0 < q_i < 1$, $i = 1,2,3$ and due to the theorem about contraction mapping it follows the unique solvability of vector equation (4.6) in the space $X_{[T_1,T_2]}$.

Continuing this process step by step, and taking into account the fact that in view of global a priori estimate (4.3) the length of each step Δt_i does not depend on i we receive the global solvability of the system of equations (2.22), and therefore of problem (1.1), (1.2) in domain D_T for any $T > 0$.

5 Uniqueness of the solution of problem (1.1), (1.2)

Lemma 5.1: *Let the condition* (3.3) *is fulfilled. Then for any* $T > 0$ *problem* (1.1), (1.2) *can not have more than one strong generalised solution of the class* C^1 *in domain* D_T.

Proof: Indeed, suppose that problem (1.1), (1.2) has two different possible strong generalised solutions u^1 and u^2 of the class C^1 in domain D_T. According to Definition 1.1 there exists the sequence of functions $u_n^i \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, such that

$$\lim_{n\to\infty} \|u_n^i - u^i\|_{C^1(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|Lu_n^i - f\|_{C(\overline{D}_T)} = 0,$$
$$\lim_{n\to\infty} \||u_n^i|^\rho u_{nt}^i - |u^i|^\rho u_t^i\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \||u_n^i|^\alpha u_n^i - |u^i|^\alpha u^i\|_{C(\overline{D}_T)} = 0,$$
$$i = 1, 2. \tag{5.1}$$

Let us use known notation $\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$ and assume that $\omega_n := u_n^2 - u_n^1$. It is easy to see that function $\omega_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ and satisfies the following identities

$$\Box\omega_n + g_n = f_n, \tag{5.2}$$
$$\omega_n\big|_{\Gamma_T} = 0, \tag{5.3}$$

where

$$g_n := \mu\big(|u_n^2|^\rho u_{nt}^2 - |u_n^1|^\rho u_{nt}^1\big) + \lambda\big(|u_n^2|^\alpha u_n^2 - |u_n^1|^\alpha u_n^1\big), \quad f_n := Lu_n^2 - Lu_n^1. \tag{5.4}$$

Due to the first equality from (5.1) there exists the number $A := const > 0$, not dependent on indices i and n, such that

$$\|u_n^i\|_{C^1(\overline{D}_T)} \le A. \tag{5.5}$$

According to the second equalities from (5.1) and (5.4) it follows that

$$\lim_{n\to\infty} \|f_n\|_{C(\overline{D}_T)} = 0. \tag{5.6}$$

From (3.3), (5.5) and the first equality of (5.4) it is clear that

$$\begin{aligned} g_n^2 &\le 2\mu^2\big(|u_n^2|^\rho u_{nt}^2 - |u_n^1|^\rho u_{nt}^1\big)^2 + 2\lambda^2\big(|u_n^2|^\alpha u_n^2 - |u_n^1|^\alpha u_n^1\big)^2 \\ &= 2\mu^2\big(|u_n^2|^\rho \omega_{nt} + \big(|u_n^2|^\rho - |u_n^1|^\rho\big)u_{nt}^1\big)^2 + 2\lambda^2\big(|u_n^2|^\alpha u_n^2 - |u_n^1|^\alpha u_n^1\big)^2 \\ &\le 4\mu^2 A^{2\rho}\omega_{nt}^2 + \big(4\mu^2\rho^2 A^{2\rho} + 2\lambda^2(\alpha+1)^2 A^{2\alpha}\big)\omega_n^2. \end{aligned} \tag{5.7}$$

Multiplying the both sides of (5.2) by ω_{nt} and integrating the received equality in domain D_τ, due to boundary conditions (5.3), as it was in receiving of (2.11) from (2.5), (2.6) we shall have

$$w_n(\tau) := \int_{\Omega_\tau} \big(\omega_{nx}^2 + \omega_{nt}^2\big)dx = 2\int_{D_\tau} \big(f_n - g_n\big)\omega_{nt}dxdt. \tag{5.8}$$

Due to estimate (5.7) and the inequality of Cauchy we shall have

$$\begin{aligned} 2\int_{D_\tau} (f_n - g_n)\omega_{nt}dxdt &\le \int_{D_\tau} \big(f_n - g_n\big)^2 dxdt + \int_{D_\tau} \omega_{nt}^2 dxdt \\ &\le 2\int_{D_\tau} f_n^2 dxdt + 2\int_{D_\tau} g_n^2 dxdt + \int_{D_\tau} \omega_{nt}^2 dxdt \\ &\le 2\int_{D_\tau} f_n^2 dxdt + \big(1 + 8\mu^2 A^{2\rho}\big)\int_{D_\tau} \omega_{nt}^2 dxdt \\ &\quad + 4\big(2\mu^2\rho^2 A^{2\rho} + \lambda^2(\alpha+1)^2 A^{2\alpha}\big)\int_{D_\tau} \omega_n^2 dxdt. \end{aligned} \tag{5.9}$$

Further, from equality $\omega_n(x,t) = \int_x^t \omega_{nt}(x,\tau)d\tau$, $(x,t) \in \overline{D}_T$, which follows from (5.3), by use of standard considerations we receive inequality [see e.g. Ladyzhenskaya, 1973, p. 63]

$$\int_{D_\tau} \omega_n^2 dxdt \le \tau^2 \int_{D_\tau} \omega_{nt}^2 dxdt. \tag{5.10}$$

From (5.8)–(5.10) it follows that

$$\begin{aligned} w_n(\tau) &\le \big(1 + 8\mu^2 A^{2\rho} + 8\mu^2\tau^2\rho^2 A^{2\rho} + 4\lambda^2\tau^2(\alpha+1)^2 A^{2\alpha}\big)\int_{D_\tau} \omega_{nt}^2 dxdt \\ &\quad + 2\|f_n\|_{L_2(D_\tau)}^2 \\ &\le \big(1 + 8\mu^2 A^{2\rho} + 8\mu^2\tau^2\rho^2 A^{2\rho} + 4\lambda^2\tau^2(\alpha+1)^2 A^{2\alpha}\big)\int_0^\tau w_n(\sigma)d\sigma \\ &\quad + 2\|f_n\|_{L_2(D_T)}^2. \end{aligned}$$

Whence by the Gronwall's lemma we receive that

$$w_n(\tau) \le c_2\|f_n\|_{L_2(D_T)}^2, \quad 0 < \tau \le T, \tag{5.11}$$

where $c_2 := 2\exp(\big(1 + 8\mu^2 A^{2\rho} + 8\mu^2 T^2\rho^2 A^{2\rho} + 4\lambda^2 T^2(\alpha+1)^2 A^{2\alpha}\big)T$.

Conducting the same considerations, as those used for receiving of (2.13), and also due to (5.11), for $(x,t) \in \overline{D}_T$ we have

$$|\omega_n(x,t)|^2 \le tw_n(t) \le Tc_2 \; mes \; D_T\|f_n\|_{C(\overline{D}_T)}^2 = 2^{-1}c_2T^3\|f_n\|_{C(\overline{D}_T)}^2.$$

From this inequality it follows immediately that

$$\|\omega_n\|_{C(\overline{D}_T)} \le T\sqrt{2^{-1}c_2T}\|f_n\|_{C(\overline{D}_T)}. \tag{5.12}$$

Recalling the definition of function ω_n, according to the first equality from (5.1) we have

$$\lim_{n\to\infty} \|\omega_n\|_{C^1(\overline{D}_T)} = \|u^2 - u^1\|_{C^1(\overline{D}_T)}$$

and all the more

$$\lim_{n\to\infty} \|\omega_n\|_{C(\overline{D}_T)} = \|u^2 - u^1\|_{C(\overline{D}_T)}.$$

Due to this equality and (5.6), passing in (5.12) to limit for $n \to \infty$ we receive $\|u^2 - u^1\|_{C(\overline{D}_T)} = 0$, i.e., $u^1 = u^2$, which proves Lemma 5.1.

6 The case of nonexistence of a global solution of problem (1.1), (1.2)

Below we will show that violation of condition (2.1) may cause the absence of global solvability of problem (1.1), (1.2) in the sense of Definition 1.2. Indeed, in equation (1.1) we consider the case, when the parameter $\mu = 0$, while the parameter $\lambda < 0$.

Lemma 6.1: *Let u be a strong generalised solution of problem* (1.1), (1.2) *of the class C^1 in domain D_T in the sense of Definition 1.1. Then it is valid the following integral equality*

$$\int_{D_T} u\Box\varphi dxdt = -\lambda \int_{D_T} |u|^\alpha u\varphi dxdt + \int_{D_T} f\varphi dxdt \tag{6.1}$$

for any function φ, such that

$$\varphi \in C^2(\overline{D}_T), \quad \varphi\big|_{t=T} = 0, \quad \varphi_t\big|_{t=T} = 0, \quad \varphi\big|_{\gamma_{2,T}} = 0. \tag{6.2}$$

Proof: According to the definition of strong generalised solution u of problem (1.1), (1.2) of the class C^1 in domain D_T, function $u \in C^1(\overline{D}_T)$ and there exists the sequence of functions $u_n \in \overset{o}{C}{}^2(\overline{D}_T, \Gamma_T)$, such that the equalities (2.3), (2.4) are valid.

Suppose that $f_n := Lu_n$. Multiplying the both sides of equality $Lu_n = f_n$ by function φ let us integrate the received equality in domain D_T. As a result of integration by parts of the left side of this equality, due to (6.2) and the boundary conditions (1.2) we receive

$$\int_{D_T} u_n\Box\varphi dxdt = -\lambda \int_{D_T} |u_n|^\alpha u_n\varphi dxdt + \int_{D_T} f_n\varphi dxdt.$$

By passing to limit in this equality for $n \to \infty$, according to (2.3), (2.4) we receive equality (6.1). Thus Lemma 6.1 is proved.

Lemma 6.2: *Let $\lambda < 0$ and the function $u \in C^1(\overline{D}_T)$ be a strong generalised solution of the problem* (1.1), (1.2) *of the class C^1 in domain D_T. If $f \geq 0$ in domain D_T, then $u \geq 0$ in domain D_T.*

Proof: Let $P := P(x,t)$ be an arbitrary point in domain D_T. Denote by $G_{x,t}$ a quadrangle with vertices P and also P_1 and P_2, P_3, which lay on data supports $\gamma_{2,T}$ and $\gamma_{1,T}$, respectively, i.e., $P_1 := P_1(0, t-x), P_2 := P_2\big(\frac{t-x}{2}, \frac{t-x}{2}\big), P_3 := P_3\big(\frac{x+t}{2}, \frac{x+t}{2}\big)$.

Let $u \in C^2(\overline{D}_T)$ be a classical solution of problem (1.1), (1.2). By integration of equation (1.1) in domain $G_{x,t}$, using homogeneous boundary conditions (1.2) it is easy to see that function u satisfies the following Volterra type integral equation

$$u(x,t) = \int_{G_{x,t}} k(\xi,\eta)u(\xi,\eta)d\xi d\eta + F(x,t), \quad (x,t) \in \overline{D}_T, \tag{6.3}$$

where $k(x,t) := -\frac{\lambda}{2}|u(x,t)|^\alpha \in C(\overline{D}_T)$ and $F(x,t) := \frac{1}{2}\int_{G_{x,t}} f(\xi,\eta)d\xi d\eta, \quad (x,t) \in \overline{D}_T$. By virtue of suppositions made in Lemma 6.2 we have

$$k(x,t) \geq 0, \quad F(x,t) \geq 0, \quad \forall (x,t) \in \overline{D}_T. \tag{6.4}$$

Assuming that function $k(x,t)$ is given, let us consider Volterra type linear integral equation

$$v(x,t) = \int_{G_{x,t}} k(\xi,\eta)v(\xi,\eta)d\xi d\eta + F(x,t), \quad (x,t) \in \overline{D}_T \tag{6.5}$$

in the class $C(\overline{D}_T)$ with respect to unknown function v. As it is known [see e.g., Bitsadze, 1982, p.188], equation (6.5) in the class $C(\overline{D}_T)$ has unique continuous solution v, which can be obtained by use of the method of consecutive approximations:

$$v_0(x,t) = 0, \quad v_{n+1}(x,t) = \int_{G_{x,t}} k(\xi,\eta)v_n(\xi,\eta)d\xi d\eta + F(x,t),$$
$$n = 0,1,2,\dots, \quad (x,t) \in \overline{D}_T.$$

From these equalities according to (6.4) we have $v_n \geq 0$ in $\overline{D}_T$ for all $n = 0,1,\dots$ On the other hand, $v_n \to v$ in the class $C(\overline{D}_T)$ for $n \to \infty$. Therefore, limit function $v \geq 0$ in domain D_T. We have just note, that by virtue of equality (6.3) function u is also a solution of equation (6.5), and therefore due to the uniqueness of solution of this equation we finally receive $u = v \geq 0$ in domain D_T. Lemma 6.2 is proved.

For $\lambda < 0$, according to the last lemma, equality (6.1) can by rewritten in the form

$$\int_{D_T} |u|\Box\varphi dxdt = |\lambda| \int_{D_T} |u|^p \varphi dxdt + \int_{D_T} f\varphi dxdt, \quad p := \alpha + 1 > 1. \tag{6.6}$$

Let us introduce into consideration function [see e.g., Mitidieri and Pohozaev, 2001, pp.10–12] $\varphi^0 := \varphi^0(x,t)$ such that

$$\varphi^0 \in C^2(\overline{D}_\infty), \quad \varphi^0\big|_{D_{T=1}} > 0, \quad \varphi^0\big|_{\gamma_{2,\infty}} = 0, \quad \varphi^0\big|_{t\geq 1} = 0 \tag{6.7}$$

and

$$\kappa_0 := \int_{D_{T=1}} \frac{|\square\varphi^0|^{p'}}{|\varphi^0|^{p'-1}} dxdt < +\infty, \quad p' = 1 + \frac{1}{\alpha}. \tag{6.8}$$

It is easy to verify that in the role of function φ^0, satisfying conditions (6.7) and (6.8), one may use function

$$\varphi^0(x,t) := \begin{cases} x^n(1-t)^m, & (x,t) \in D_{T=1}, \\ 0, & t \geq 1, \end{cases}$$

for sufficiently large positive numbers n and m.

Suppose that $\varphi_T(x,t) := \varphi^0\left(\frac{x}{T}, \frac{t}{T}\right)$, $T > 0$. Due to (6.7) it is easy to see that

$$\varphi_T \in C^2(\overline{D}_T), \quad \varphi_T\big|_{D_T} > 0, \quad \varphi_T\big|_{\gamma_{2,T}} = 0, \quad \varphi_T\big|_{t=T} = 0, \quad \frac{\partial\varphi_T}{\partial t}\bigg|_{t=T} = 0. \tag{6.9}$$

Supposing that function f is fixed, let us introduce into consideration a function of one variable T

$$\zeta(T) := \int_{D_T} f\varphi_T dxdt, \quad T > 0. \tag{6.10}$$

The following theorem on the nonexistence of a global solution of problem (1.1), (1.2) is valid.

Theorem 6.1: *Let $\lambda < 0$, $\rho > 0$, $\alpha > 0$, $f \in C(\overline{D}_\infty)$ and $f \geq 0$ in domain D_∞. If*

$$\liminf_{T\to+\infty} \zeta(T) > 0, \tag{6.11}$$

then there exists positive number $T^* := T^*(f)$, such that for $T > T^*$ problem (1.1), (1.2) *cannot have strong generalised solution u of the class C^1 in domain D_T.*

Proof: Suppose, that in conditions of this theorem there exists strong generalised solution u of problem (1.1), (1.2) of the class C^1 in domain D_T. Then according to Lemmas 6.1 and 6.2 equality (6.6) holds, where due to (6.9) in the role of function φ can be taken function $\varphi = \varphi_T$, i.e.,

$$\int_{D_T} |u|\square\varphi_T dxdt = |\lambda| \int_{D_T} |u|^p\varphi_T dxdt + \int_{D_T} f\varphi_T dxdt.$$

Taking into account (6.10) this equality can be rewritten in the form

$$|\lambda| \int_{D_T} |u|^p\varphi_T dxdt = \int_{D_T} |u|\square\varphi_T dxdt - \zeta(T). \tag{6.12}$$

If in Young inequality with parameter $\varepsilon > 0$

$$ab \le \frac{\varepsilon}{p} a^p + \frac{1}{p' \varepsilon^{p'-1}} b^{p'}; \quad a, b \ge 0, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

we shall take $a = |u| \varphi_T^{\frac{1}{p}}, b = \frac{|\Box \varphi_T|}{\varphi_T^{\frac{1}{p}}}$, then since $\frac{p'}{p} = p' - 1$ we obtain

$$|u \Box \varphi_T| = |u| \varphi_T^{\frac{1}{p}} \frac{|\Box \varphi_T|}{\varphi_T^{\frac{1}{p}}} \le \frac{\varepsilon}{p} |u|^p \varphi_T + \frac{1}{p' \varepsilon^{p'-1}} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}}.$$

According last inequality from (6.12) we have

$$\left(|\lambda| - \frac{\varepsilon}{p} \right) \int_{D_T} |u|^p \varphi_T dxdt \le \frac{1}{p' \varepsilon^{p'-1}} \int_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} dxdt - \zeta(T),$$

whence for $\varepsilon < |\lambda| p$ we receive

$$\int_{D_T} |u|^p \varphi_T dxdt \le \frac{p}{(|\lambda| p - \varepsilon) p' \varepsilon^{p'-1}} \int_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} dxdt - \frac{p}{|\lambda| p - \varepsilon} \zeta(T).$$

Since $p' = \frac{p}{p-1}$, $p = \frac{p'}{p'-1}$ and $\min_{0<\varepsilon<|\lambda|p} \frac{p}{(|\lambda|p-\varepsilon)p'\varepsilon^{p'-1}} = \frac{1}{|\lambda|^{p'}}$, which is achieved for $\varepsilon = |\lambda|$, it follows, that

$$\int_{D_T} |u|^p \varphi_T dxdt \le \frac{1}{|\lambda|^{p'}} \int_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} dx\, dt - \frac{p'}{|\lambda|} \zeta(T). \tag{6.13}$$

Since $\varphi_T(x,t) := \varphi^0\big(\frac{x}{T}, \frac{t}{T}\big)$, then due to (6.7), (6.8), after changing variables $x = Tx'$, $t = Tt'$, it is easy to verify, that

$$\int_{D_T} \frac{|\Box \varphi_T|^{p'}}{\varphi_T^{p'-1}} dxdt = T^{-2(p'-1)} \int_{D_{T=1}} \frac{|\Box \varphi^0|^{p'}}{|\varphi^0|^{p'-1}} dx' dt' = T^{-2(p'-1)} \kappa_0.$$

According to (6.9) and the last inequality from (6.13) we receive

$$0 \le \int_{D_T} |u|^p \varphi_T dxdt \le \frac{1}{|\lambda|^{p'}} T^{-2(p'-1)} \kappa_0 - \frac{p'}{|\lambda|} \zeta(T). \tag{6.14}$$

Since $p' = \frac{p}{p-1} > 1$, then $-2(p'-1) < 0$ and due to (6.8) we have

$$\lim_{T \to +\infty} \frac{1}{|\lambda|^{p'}} T^{-2(p'-1)} \kappa_0 = 0.$$

Therefore, by virtue of (6.11) there exists positive number $T^* := T^*(f)$, such that for $T > T^*$ the right hand side of inequality (6.14) will be negative, whereas the left hand side of this inequality is non-negative. This means that if there exists strong generalised solution u of problem (1.1), (1.2) of the class C^1 in domain D_T, then necessarily $T \leq T^*$, which proves the Theorem 6.1.

Remark 6.1: It is easy to verify that if $f \in C(\overline{D}_\infty)$ and $f(x,t) \geq ct^{-m}$ for $t \geq 1$, where $c := const > 0, 0 \leq m := const \leq 2$, then condition (6.11) will be fulfilled, and so for $\lambda < 0, \rho > 0, \alpha > 0$ problem (1.1), (1.2) for sufficiently large T will not have strong generalised solution u of the class C^1 in domain D_T.

Indeed, let us introduce in (6.10) the transformation of independent variables x and t by formula $x = Tx_1,\ t = Tt_1$, after some estimates we have

$$\begin{aligned}\zeta(T) &= T^2 \int_{D_{T=1}} f(Tx_1, Tt_1)\varphi^0(x_1,t_1)dx_1dt_1 \\ &\geq cT^{2-m} \int_{D_{T=1}\cap\{t_1 \geq T^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)dx_1dt_1 \\ &\quad +T^2 \int_{D_{T=1}\cap\{t_1 < T^{-1}\}} f(Tx_1,Tt_1)\varphi^0(x_1,t_1)dx_1dt_1\end{aligned}$$

in supposition that $T > 1$. Further, let $T_1 > 1$ be any fixed number. Then from the last inequality for function ζ we have

$$\begin{aligned}\zeta(T) &\geq cT^{2-m} \int_{D_{T=1}\cap\{t_1 \geq T^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)dx_1dt_1 \\ &\geq c \int_{D_{T=1}\cap\{t_1 \geq T_1^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)dx_1dt_1,\end{aligned}$$

if $T \geq T_1 > 1$. From the latter inequality immediately follows the validity of (6.11).

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