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THE CAUCHY MULTIDIMENSIONAL CHARACTERISTIC PROBLEM FOR ONE CLASS OF THE SECOND ORDER NONLINEAR HYPERBOLIC SYSTEMS

In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and t we consider a nonlinear hyperbolic second order system of the type

$$(Lu)_i := \frac{\partial^2 u_i}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u_i}{\partial x_i^2} + f_i(u_1, \dots, u_N) = F_i(x,t), \quad i = 1, \dots, N, \quad (1)$$

where $f = (f_1, \dots, f_N)$, $F = (F_1, \dots, F_N)$ are the given and $u = (u_1, \dots, u_N)$ is the unknown real vector functions, $n \geq 2$, $N \geq 2$.

For the system of equations (1) we consider the Cauchy characteristic problem of finding a solution $u(x,t)$ in a frustum of a cone of the future $D_T : |x| < t < T$, $T = \text{const} > 0$, under the boundary condition

$$u\big|_{S_T} = 0, \quad (2)$$

where $S_T : t = |x|$, $t \leq T$ is the conic surface, characteristic with respect to the system (1). For $T = \infty$, we assume that $D_\infty : t > |x|$ and $S_\infty = \partial D_\infty : t = |x|$.

Note that in the case of a scalar nonlinear wave equation this problem has been considered in the works [1]–[3].

Let $\overset{\circ}{C}{}^2(\overline{D}_T, S_T) := \{u \in C^2(\overline{D}_T) : u\big|_{S_T} = 0\}$ and $\overset{\circ}{W}{}_2^1(D_T, S_T) := \{u \in W_2^1(D_T) : u\big|_{S_T} = 0\}$, where $W_2^k(\Omega)$ is the Sobolev space consisting of elements $L_2(\Omega)$, whose generalized derivatives up to the k-th order, inclusive, belong to $L_2(\Omega)$, and the equality $u\big|_{S_T} = 0$ is understood in a sense of the trace theory.

Below, on the nonlinear vector function $f = (f_1, \dots, f_N)$ in (1) we impose the following requirement:

$$f \in C(\mathbb{R}^N), \ \ |f(u)| \leq M_1 + M_2 |u|^\alpha, \ \ \alpha = \text{const} \geq 0, \ \ u \in \mathbb{R}^N, \quad (3)$$

where $|\cdot|$ is the norm in the space $\mathbb{R}^N$, and $M_i = \text{const} \geq 0$, $i = 1, 2$.

Remark. The imbedding operator $I : W_2^1(D_T) \to L_q(D_T)$ is the linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$ and $n > 1$. At the same

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time, the Nemytskii operator $K : L_q(D_T) \to L_2(D_T)$ acting by the formula $Ku = f(u)$, where $u = (u_1, \dots, u_{_N}) \in L_q(D_T)$, and the vector function $f = (f_1, \dots, f_N)$ satisfies the condition (3), is continuous and bounded for $q \ge 2\alpha$. Therefore, if $\alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Thus, in this case the operator

$$K_0 = KI : \left[W_2^1(D_T)\right]^N \to \left[L_2(D_T)\right]^N$$

will be continuous and compact one. Hence from the fact that $u \in W_2^1(D_T)$, it follows that $f(u) \in L_2(D_T)$, and if $u^m \to u$ in the space $W_2^1(D_T)$, then $f(u^m) \to f(u)$ in the space $L_2(D_T)$.

Here and below, the belonging of the vector $v = (v_1, \dots, v_{_N})$ to some space X implies that every component v_i, $1 \le i \le N$, of that vector belongs to the space X.

Definition 1. Let $f = (f_1, \dots, f_N)$ satisfy the condition (3), where $0 \le \alpha < \frac{n+1}{n-1}$, $F = (F_1, \dots, F_N) \in L_2(D_T)$. The vector function $u = (u_1, \dots, u_{_N}) \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is said to be a strong generalized solution of the problem (1), (2) of the class W_2^1 in the space D_T if there exists a sequence of vector functions $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ such that $u^m \to u$ in the space $W_2^1(D_T)$ and $Lu^m \to F$ in the space $L_2(D_T)$. The convergence of the sequence $\{f(u^m)\}$ to $f(u)$ in the space $L_2(D_T)$, as $u^m \to u$ in the space $W_2^1(D_T)$, follows from the above Remark.

Obviously, a classical solution $u \in C^2(\overline{D}_T)$ of the problem (1), (2) is likewise a strong generalized solution of that problem of the class W_2^1 in the domain D_T in a sense of Definition 1.

Definition 2. Let f satisfy the condition (3), where $0 \le \alpha < \frac{n+1}{n-1}$, $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. We say that the problem (1), (2), is locally solvable in the class W_2^1 if there exists a number $T_0 = T_0(F) > 0$ such that for any $T < T_0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in a sense of Definition 1.

Definition 3. Let f satisfy the condition (3), where $0 \le \alpha < \frac{n+1}{n-1}$; $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. We say that the problem (1), (2) is globally solvable in the class W_2^1 if for any $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T in a sense of Definition 1.

Theorem 1. *Let f satisfy the condition* (3), *where* $1 < \alpha < \frac{n+1}{n-1}$; $F \in L_{2,\mathrm{loc}}(D_\infty)$ *and* $F|_{D_T} \in L_2(D_T)$ *for any* $T > 0$. *Then the problem* (1), (2) *is locally solvable in the class* W_2^1 *in a sense of Definition* 2.

Theorem 2. *Let f satisfy the condition* (3), *where* $0 \le \alpha \le 1$; $F \in L_{2,\mathrm{loc}}(D_\infty)$ *and* $F|_{D_T} \in L_2(D_T)$ *for any* $T > 0$. *Then the problem* (1), (2) *is globally solvable in the class* W_2^1 *in a sense of Definition* 3.

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