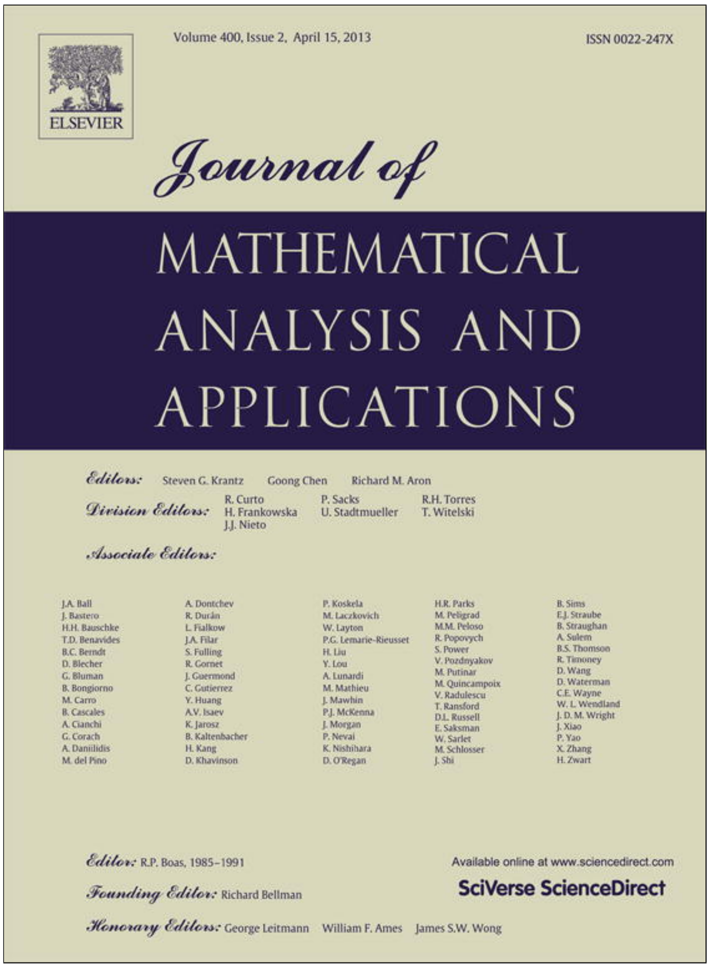
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# On the solvability of one boundary value problem for one class of semilinear second order hyperbolic systems

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ABSTRACT

For one class of semilinear second order hyperbolic systems it is considered the Sobolev problem in the conic domain of time type which represents a multidimensional version of the Darboux second problem. The questions on global and local solvability, uniqueness, and also nonexistence of a solution to this problem are studied.


## 1. Introduction

In the space $\mathbb{R}^{n+1}$ of the independent variables $x = (x_1, x_2, \ldots, x_n)$ and $t$ consider a second order semilinear hyperbolic system of the form

$$\Box u_i + f_i(u_1, \ldots, u_N) = F_i, \quad i = 1, \ldots, N, \tag{1.1}$$

where $f = (f_1, \ldots, f_N)$, $F = (F_1, \ldots, F_N)$ are given, and $u = (u_1, \ldots, u_N)$ is an unknown real vector-function, $n \geq 2$, $N \geq 2$, $\Box := \frac{\partial^2}{\partial t^2} - \Delta$, $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Let $D$ be a conic domain in the space $\mathbb{R}^{n+1}$, i.e. $D$ contains, along with the point $(x, t) \in D$, the whole ray $l : (\tau x, \tau t)$, $0 < \tau < \infty$. Denote by $S$ the conic surface $\partial D$. Suppose that $D$ is homeomorphic to the conic domain $\omega : t > |x|$, and $S \setminus O$ is a connected $n$-dimensional manifold of the class $C^\infty$, where $O = (0, \ldots, 0, 0)$ is the vertex of $S$. Suppose also that $D$ lies in the half-space $t > 0$ and $D_T := \{(x, t) \in D : t < T\}$, $S_T := \{(x, t) \in S : t \leq T\}$, $T > 0$. It is clear that if $T = \infty$, then $D_\infty = D$ and $S_\infty = S$.

For the system of Eq. (1.1) we consider the problem on finding of a solution $u(x, t)$ of this system in the domain $D_T$ by the boundary condition

$$u|_{S_T} = g, \tag{1.2}$$

where $g = (g_1, \ldots, g_n)$ is a given vector-function on $S_T$.

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In the linear case, when $f = 0$, $N = 1$ and the conic manifold $S = \partial D$ is time-oriented, i.e.

$$\left(\nu_0^2 - \sum_{i=1}^{n} \nu_i^2\right)\Bigg|_S < 0, \quad \nu_0|_S < 0, \tag{1.3}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ is the unit vector of the outer normal to $S \setminus O$, the problem (1.1), (1.2) was posed by S.L. Sobolev in the work [10], where the unique solvability of this problem in the corresponding functional spaces is proved. At the end of the work [10] the author suggested that the obtained results will be valid also for scalar nonlinear wave equation. In the work [5], for the scalar case ($N = 1$) and power nonlinearity $f(u) = \lambda|u|^p u$ ($\lambda = \text{const}$, $0 < p = \text{const} < \frac{2}{n-1}$), the global solvability of this problem for $\lambda > 0$ and the absence of a global solution for $\lambda < 0$ is shown, when the space dimension of the wave equation equals $n = 2$. In our work [6], also for the scalar case with more general nonlinearity, the classes of nonlinearity when in certain cases we have the global solvability of this problem are singled out, whereas in other cases not. Besides, here the restriction $n = 2$ is removed. It is noteworthy that this problem can be considered as a multidimensional version of the Darboux second problem, since the problem's data support $S$ represents a conic manifold of time type. In the case when one part of the boundary of the conic domain $D$ is of time type, while the other part represents a characteristic manifold, then the boundary value problem can be considered as a multidimensional version of the Darboux first problem. E.g., when $D : t > |x|, x_n > 0$ and boundary conditions have the form

$$u|_{\Gamma_0} = 0, \qquad u|_{\Gamma_1} = 0$$

or

$$\frac{\partial u}{\partial x_n}\bigg|_{\Gamma_0} = 0, \qquad u|_{\Gamma_1} = 0,$$

where $\Gamma_0 = \partial D \cap \{x_n = 0\}$ is a plane part of the boundary $\partial D$ of time type and $\Gamma_1 = \partial D \setminus \Gamma_0 : t = |x|, x_n > 0$ is a characteristic part of the boundary, then we have a multidimensional version of the Darboux first problem. Investigation of the multidimensional version of the Darboux second problem faces great difficulties as compared with the first problem. More detailed consideration of these problems in the linear case is given in A.V. Bitsadze's monograph [2].

The work is organized in the following way. In Section 2 it is given a notion of a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ and a definition of a global solution of this problem of the class $W_2^1$ in the domain $D_\infty$. In the Section 3 we consider the cases of local and global solvability of the problem (1.1), (1.2) in the class $W_2^1$. We suppose that the growth of nonlinearity in the Eq. (1.1) does not exceed power nonlinearity with exponent $\alpha = \text{const} \geq 0$. When $\alpha \leq 1$, then for the solution of boundary value problem the a priori estimate (Lemma 3.1) is valid and no restrictions are imposed on the structure of the vector-function $f = f(u)$. When $1 < \alpha < \frac{n+1}{n-1}$, as it turned out, the only constraint on the growth of nonlinearity of the vector-function $f = f(u)$ is not sufficient for the existence of the a priori estimate for the solution of the boundary value problem. Here we need structural constraints on the vector-function $f = f(u)$. E.g., when $f = \nabla G$, i.e. $f_i(u) = \frac{\partial}{\partial u_i}G(u)$, $u \in \mathbb{R}^N$, $i = 1, \dots, N$, where $G = G(u) \in C^1(\mathbb{R}^N)$ is a scalar function satisfying conditions $G(0) = 0$ and $G(u) \geq 0\, \forall u \in \mathbb{R}^N$, then the a priori estimate of the solution of the boundary value problem and, therefore, a global solvability of this problem (Theorem 3.3) are valid. If the vector-function $f$ cannot be represented in the form $f = \nabla G$, where the scalar function $G$ satisfies the conditions given above, then the boundary value problem may be globally insolvable. For example, when $N = n = 2$ and $f = (f_1, f_2)$, where $f_1 = u_1^2 - 2u_2^2$, $f_2 = -2u_1^2 + u_2^2$, the exponent of nonlinearity $\alpha = 2$ and $1 < \alpha < \frac{n+1}{n-1}$, and $f$ is not representable in the form $f = \nabla G$, then from the Theorem 5.1 we have that when $F_1 + F_2 \geq \frac{c}{t^\gamma}$, $t \geq 1$, where $c = \text{const} > 0$, $\gamma = \text{const} \leq 3$; $g = 0$, the problem under consideration is not globally solvable (see the Remark 5.1). In the Section 4 we give the conditions on the vector-function $f$ providing uniqueness and existence of a global solution of this problem of the class $W_2^1$. Finally, in the Section 5 for certain additional conditions on the vector-function $f$, $F$ and $g$ we prove nonexistence of a global solution of the problem (1.1), (1.2) of the class $W_2^1$ in $D_\infty$.

## 2. Definition of a generalized solution of the problem (1.1), (1.2) in $D_T$ and $D_\infty$

Let us rewrite the system (1.1) in the form of one vector equation

$$Lu := \Box u + f(u) = F. \tag{2.1}$$

Below we assume that the condition (1.3) is fulfilled and the nonlinear vector-function from (2.1) satisfies the following requirement

$$f \in C(\mathbb{R}^N), \quad |f(u)| \leq M_1 + M_2|u|^\alpha, \quad \alpha = \text{const} \geq 0,\ u \in \mathbb{R}^N, \tag{2.2}$$

where $|\cdot|$ is the norm in the space $\mathbb{R}^N$, $M_i = \text{const} \geq 0$, $i = 1, 2$.

Let $\overset{\circ}{C}{}^2(\overline{D}_T, S_T) := \{u \in C^2(\overline{D}_T) : u|_{S_T} = 0\}$. Denote by $W_2^k(\Omega)$ the Sobolev space consisting of elements $L_2(\Omega)$, having generalized derivatives up to $k$-th order inclusively from $L_2(\Omega)$. Let $\overset{\circ}{W}{}_2^1(D_T, S_T) := \{u \in W_2^1(D_T) : u|_{S_T} = 0\}$, where the equality $u|_{S_T} = 0$ must be understood in the sense of the trace theory [7]. Here and below we say that the vector $v = (v_1, \dots, v_N)$ belongs to the space $X$ if each component $v_i$, $1 \le i \le N$, of this vector belongs to the same $X$. In accordance with this, for simplicity of record, where this will not cause misunderstanding, instead of $v = (v_1, v_2, \dots, v_N) \in [X]^N$ we use the record $v \in X$.

**Remark 2.1.** The embedding operator $I : [W_2^1(D_T)]^N \to [L_q(D_T)]^N$ represents a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [7]. At the same time the Nemitski operator $K : [L_q(D_T)]^N \to [L_2(D_T)]^N$, acting by the formula $Ku = f(u)$, where $u = (u_1, \dots, u_N) \in [L_q(D_T)]^N$, and the vector-function $f = (f_1, \dots, f_N)$ satisfies the condition (2.2), is continuous and bounded for $q \ge 2\alpha$ [7]. Thus, if $\alpha < \frac{n+1}{n-1}$, i.e. $2\alpha < \frac{2(n+1)}{n-1}$, then there exists number $q$ such that $1 < q < \frac{2(n+1)}{n-1}$ and $q > 2\alpha$. Therefore, in this case the operator

$$K_0 = KI : \left[W_2^1(D_T)\right]^N \to \left[L_2(D_T)\right]^N \tag{2.3}$$

will be continuous and compact. It is clear that from $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u^m \to u$ in the space $W_2^1(D_T)$, then $f(u^m) \to f(u)$ in the space $L_2(D_T)$.

**Definition 2.1.** Let $f = (f_1, \dots, f_N)$ satisfy the condition (2.2), where $0 \le \alpha < \frac{n+1}{n-1}$, $F = (F_1, \dots, F_N) \in L_2(D_T)$ and $g = (g_1, \dots, g_N) \in W_2^1(S_T)$. We call a vector-function $u = (u_1, \dots, u_N) \in W_2^1(D_T)$ a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ if there exists a sequence of vector-functions $u_m \in C^2(\overline{D}_T)$ such that $u^m \to u$ in the space $W_2^1(D_T)$, $Lu^m \to F$ in the space $L_2(D_T)$ and $u^m|_{S_T} \to g$ in the space $W_2^1(S_T)$. The convergence of the sequence $\{f(u^m)\}$ to the function $f(u)$ in the space $L_2(D_T)$ when $u^m \to u$ in the space $W_2^1(D_T)$ follows from the Remark 2.1. When $g = 0$, i.e. in the case of homogeneous boundary conditions (1.2), we assume that $u^m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$. Then, it is clear that $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$.

It is obvious that a classical solution $u \in C^2(\overline{D}_T)$ of the problem (1.1), (1.2) represents a strong generalized solution of this problem of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.

**Definition 2.2.** Let $f$ satisfy the condition (2.2), where $0 \le \alpha < \frac{n+1}{n-1}$; $F \in L_{2,\mathrm{loc}}(D_\infty)$, $g \in W_{2,\mathrm{loc}}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (1.1), (1.2) is locally solvable in the class $W_2^1$ if there exists a number $T_0 = T_0(F, g) > 0$ such that for $T < T_0$ this problem has a strong generalized solution of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.

**Definition 2.3.** Let $f$ satisfy the condition (2.2), where $0 \le \alpha < \frac{n+1}{n-1}$; $F \in L_{2,\mathrm{loc}}(D_\infty)$, $g \in W_{2,\mathrm{loc}}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (1.1), (1.2) is globally solvable in the class $W_2^1$ if for any $T > 0$ this problem has a strong generalized solution of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.

**Definition 2.4.** Let $f$ satisfy the condition (2.2), where $0 \le \alpha < \frac{n+1}{n-1}$; $F \in L_{2,\mathrm{loc}}(D_\infty)$, $g \in W_{2,\mathrm{loc}}^1(S_\infty)$ and $F|_{D_T} \in L_2(D_T)$, $g|_{S_T} \in W_2^1(S_T)$ for any $T > 0$. A vector-function $u = (u_1, \dots, u_N) \in W_{2,\mathrm{loc}}^1(D_\infty)$ is called a global strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_\infty$ if for any $T > 0$ the vector-function $u|_{D_T}$ belongs to the space $W_2^1(D_T)$ and represents a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.

## 3. Some cases of global and local solvability of the problem (1.1), (1.2) in the class $W_2^1$

**Lemma 3.1.** *Let $f$ satisfy the condition (2.2), where $0 \le \alpha \le 1$; $F \in L_2(D_T)$ and $g \in W_2^1(S_T)$. Then for any strong generalized solution $u$ of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1 it is valid the following a priori estimate*

$$\|u\|_{W_2^1(D_T)} \le c_1 \|F\|_{L_2(D_T)} + c_2 \|g\|_{W_2^1(S_T)} + c_3 \tag{3.1}$$

*with nonnegative constants $c_i = c_i(S, f, T)$, $i = 1, 2, 3$, not depending on $u$, $g$ and $F$, with $c_j > 0$, $j = 1, 2$.*

**Proof.** Let $u \in W_2^1(D_T)$ be a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$. Then due to the Definition 2.1 there exists a sequence of vector-functions $u^m = (u_1^m, \dots, u_N^m) \in C^2(\overline{D}_T)$ such that

$$\lim_{m\to\infty} \|u^m - u\|_{W_2^1(D_T)} = 0, \qquad \lim_{m\to\infty} \|Lu^m - F\|_{L_2(D_T)} = 0, \tag{3.2}$$

$$\lim_{m\to\infty} \|u^m|_{S_T} - g\|_{W_2^1(S_T)} = 0. \tag{3.3}$$

Consider the vector-function $u^m \in C^2(\overline{D}_T)$ as a solution of the following problem

$$Lu^m = F^m, \tag{3.4}$$

$$u^m|_{S_T} = g^m. \tag{3.5}$$

Here

$$F^m := Lu^m, \qquad g^m := u^m|_{S_T}. \tag{3.6}$$

Scalarly multiplying the both sides of the vector equation (3.4) by $\frac{\partial u^m}{\partial t}$ and integrating in the domain $D_\tau$, $0 < \tau \le T$, we receive

$$\frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u^m}{\partial t}\right)^2 dxdt - \int_{D_\tau} \Delta u^m \frac{\partial u^m}{\partial t} dxdt + \int_{D_\tau} f(u^m)\frac{\partial u^m}{\partial t} dxdt = \int_{D_\tau} F^m \frac{\partial u^m}{\partial t} dxdt. \tag{3.7}$$

Let $\Omega_\tau := D \cap \{t = \tau\}$ and denote by $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ the unit vector of the outer normal to $S_T \setminus \{(0, \dots, 0, 0)\}$. Integrating by parts, by virtue of the equality (3.5) and $\nu|_{\Omega_\tau} = (0, \dots, 0, 1)$ we have

$$\begin{aligned}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u^m}{\partial t}\right)^2 dxdt &= \int_{\partial D_\tau} \left(\frac{\partial u^m}{\partial t}\right)^2 \nu_0 ds \\ &= \int_{\Omega_\tau} \left(\frac{\partial u^m}{\partial t}\right)^2 dx + \int_{S_\tau} \left(\frac{\partial u^m}{\partial t}\right)^2 \nu_0 ds,\end{aligned}$$

$$\begin{aligned}\int_{D_\tau} \frac{\partial^2 u^m}{\partial x_i^2}\frac{\partial u^m}{\partial t} dxdt &= \int_{\partial D_\tau} \frac{\partial u^m}{\partial x_i}\frac{\partial u^m}{\partial t}\nu_i ds - \frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u^m}{\partial x_i}\right)^2 dxdt \\ &= \int_{\partial D_\tau} \frac{\partial u^m}{\partial x_i}\frac{\partial u^m}{\partial t}\nu_i ds - \frac{1}{2}\int_{\partial D_\tau} \left(\frac{\partial u^m}{\partial x_i}\right)^2 \nu_0 ds = \int_{\partial D_\tau} \frac{\partial u^m}{\partial x_i}\frac{\partial u^m}{\partial t}\nu_i ds \\ &\quad - \frac{1}{2}\int_{S_\tau} \left(\frac{\partial u^m}{\partial x_i}\right)^2 \nu_0 ds - \frac{1}{2}\int_{\Omega_\tau} \left(\frac{\partial u^m}{\partial x_i}\right)^2 dx.\end{aligned}$$

Whence, in view of (3.7), it follows

$$\begin{aligned}\int_{D_\tau} F^m \frac{\partial u^m}{\partial t} dxdt &= \int_{S_\tau} \frac{1}{2\nu_0}\left[\sum_{i=1}^n \left(\frac{\partial u^m}{\partial x_i}\nu_0 - \frac{\partial u^m}{\partial t}\nu_i\right)^2 \right. \\ &\quad \left. + \left(\frac{\partial u^m}{\partial t}\right)^2\left(\nu_0^2 - \sum_{j=1}^n \nu_j^2\right)\right] ds + \frac{1}{2}\int_{\Omega_\tau}\left[\left(\frac{\partial u^m}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right] dx \\ &\quad + \int_{D_\tau} f(u^m)\frac{\partial u^m}{\partial t} dxdt.\end{aligned} \tag{3.8}$$

From (2.2), when $0 \le \alpha \le 1$, it follows that $|f(u)| \le M_1 + M_2 + M_2|u|$, $\forall u \in \mathbb{R}^N$, therefore,

$$\begin{aligned}\left|f(u^m)\frac{\partial u^m}{\partial t}\right| &\le \frac{1}{2}\left[f^2(u^m) + \left(\frac{\partial u^m}{\partial t}\right)^2\right] \le \frac{1}{2}\left[2(M_1 + M_2)^2 + 2M_2^2|u^m|^2 + \left(\frac{\partial u^m}{\partial t}\right)^2\right] \\ &= (M_1 + M_2)^2 + M_2^2|u^m|^2 + \frac{1}{2}\left(\frac{\partial u^m}{\partial t}\right)^2.\end{aligned} \tag{3.9}$$

Due to (1.3), (3.9) and $|F^m \frac{\partial u^m}{\partial t}| \le \frac{1}{2}\Big[\Big(\frac{\partial u^m}{\partial t}\Big)^2 + (F^m)^2\Big]$, from (3.8) we have

$$\frac{1}{2}\int_{\Omega_\tau}\left[\left(\frac{\partial u^m}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dx \le \int_{S_\tau}\frac{1}{2|\nu_0|}\left[\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\nu_0-\frac{\partial u^m}{\partial t}\nu_i\right)^2\right]ds$$
$$+(M_1+M_2)^2\operatorname{mes} D_\tau + M_2^2\int_{D_\tau}|u^m|^2dxdt$$
$$+\int_{D_\tau}\left(\frac{\partial u^m}{\partial t}\right)^2dxdt+\frac{1}{2}\int_{D_\tau}(F^m)^2dxdt. \quad (3.10)$$

Since $S$ is a conic surface then $\sup_{S\setminus O}|\nu_0|^{-1} = \sup_{S\cap\{t=1\}}|\nu_0|^{-1}$. At the same time $S\setminus O$ is a smooth manifold, $S\cap\{t=1\} = \partial\Omega_{\tau=1}$ is also a compact manifold. Therefore, noting that $\nu_0$ is a continuous function on $S\setminus O$ we have

$$M_0 := \sup_{S\setminus O}|\nu_0|^{-1} = \sup_{S\cap\{t=1\}}|\nu_0|^{-1} < +\infty, \quad |\nu_0| \le |\nu| = 1. \quad (3.11)$$

Taking into account that $\left(\nu_0\frac{\partial}{\partial x_i} - \nu_i\frac{\partial}{\partial t}\right)$, $i = 1,\dots,n$, is an inner differential operator on $S_T$, then due to (3.5) we have

$$\int_{S_\tau}\left[\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\nu_0-\frac{\partial u^m}{\partial t}\nu_i\right)^2\right]ds \le \|u^m|_{S_T}\|^2_{W_2^1(S_T)} = \|g^m\|^2_{W_2^1(S_T)}. \quad (3.12)$$

From (3.11) and (3.12) it follows that

$$\int_{S_\tau}\frac{1}{2|\nu_0|}\left[\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\nu_0-\frac{\partial u^m}{\partial t}\nu_i\right)^2\right]ds \le \frac{1}{2}M_0\|g^m\|^2_{W_2^1(S_T)}. \quad (3.13)$$

By virtue of (3.13) from (3.10) we have

$$\int_{\Omega_\tau}\left[\left(\frac{\partial u^m}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dx \le M_0\|g^m\|^2_{W_2^1(S_T)} + 2(M_1+M_2)^2\operatorname{mes} D_T$$
$$+2M_2^2\int_{D_\tau}|u^m|^2dxdt + 2\int_{D_\tau}\left(\frac{\partial u^m}{\partial t}\right)^2dxdt+\int_{D_T}(F^m)^2dxdt, \quad 0<\tau\le T. \quad (3.14)$$

If $t=\gamma(x)$ is the equation of the conic surface $S$, then due to (3.5) we have

$$u^m(x,\tau) = u^m(x,\gamma(x)) + \int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u^m(x,s)ds$$
$$= g^m(x) + \int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u^m(x,s)ds, \quad (x,\tau)\in\Omega_\tau.$$

Scalarly squaring the both parts of the obtained equality, integrating in the domain $\Omega_\tau$ and using the Schwartz inequality we have

$$\int_{\Omega_\tau}(u^m)^2dx \le 2\int_{\Omega_\tau}(g^m(x,\gamma(x)))^2dx + 2\int_{\Omega_\tau}\left(\int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u^m(x,s)ds\right)^2dx$$
$$\le 2\int_{S_\tau}(g^m)^2ds + 2\int_{\Omega_\tau}(\tau-\gamma(x))\left[\int_{\gamma(x)}^{\tau}\left(\frac{\partial u^m}{\partial t}\right)^2ds\right]dx \le 2\int_{S_\tau}(g^m)^2ds$$
$$+2T\int_{\Omega_\tau}\left[\int_{\gamma(x)}^{\tau}\left(\frac{\partial u^m}{\partial t}\right)^2ds\right]dx = 2\int_{S_\tau}(g^m)^2ds + 2T\int_{D_\tau}\left(\frac{\partial u^m}{\partial t}\right)^2dxdt. \quad (3.15)$$

From (3.14) and (3.15) it follows

$$\int_{\Omega_\tau}\left[(u^m)^2+\left(\frac{\partial u^m}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dx$$
$$\leq (M_0+2)\|g^m\|^2_{W_2^1(S_T)}+2(M_1+M_2)^2\operatorname{mes}D_T+2M_2^2\int_{D_\tau}|u^m|^2dxdt$$
$$+2(T+1)\int_{D_\tau}\left(\frac{\partial u^m}{\partial t}\right)^2dxdt+\|F^m\|^2_{L_2(D_T)}$$
$$\leq (2M_2^2+2(T+1))\int_{D_\tau}\left[(u^m)^2+\left(\frac{\partial u^m}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dxdt$$
$$+\left[\|F^m\|^2_{L_2(D_T)}+(M_0+2)\|g^m\|^2_{W_2^1(S_T)}+2(M_1+M_2)^2\operatorname{mes}D_T\right]. \tag{3.16}$$

Putting

$$w(\tau):=\int_{\Omega_\tau}\left[(u^m)^2+\left(\frac{\partial u^m}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dx, \tag{3.17}$$

from (3.16) we have

$$w(\tau)\leq (2M_2^2+2T+2)\int_0^\tau w(s)ds+\big[\|F^m\|^2_{L_2(D_T)}+(M_0+2)\|g^m\|^2_{W_2^1(S_T)}$$
$$+2(M_1+M_2)^2\operatorname{mes}D_T\big],\quad 0<\tau\leq T. \tag{3.18}$$

Whence by the Gronwall's lemma it follows that

$$w(\tau)\leq A_m\exp(2M_2^2+2T+2)\tau,\quad 0<\tau\leq T. \tag{3.19}$$

Here

$$A_m=\|F^m\|^2_{L_2(D_T)}+(M_0+2)\|g^m\|^2_{W_2^1(S_T)}+2(M_1+M_2)^2\operatorname{mes}D_T. \tag{3.20}$$

In view of (3.17) and (3.19) we find that

$$\|u^m\|^2_{W_2^1(D_T)}=\int_0^T w(\tau)d\tau\leq A_mT\exp(2M_2^2+2T+2)T. \tag{3.21}$$

Due to (3.2)–(3.5) and (3.20), passing to the limit in (3.21) when $m\to\infty$ we have

$$\|u\|^2_{W_2^1(D_T)}\leq A\,T\exp(2M_2^2+2T+2)T. \tag{3.22}$$

Here

$$A=\|F\|^2_{L_2(D_T)}+(M_0+2)\|g\|^2_{W_2^1(S_T)}+2(M_1+M_2)^2\operatorname{mes}D_T. \tag{3.23}$$

Taking a square root from the both parts of inequality (3.22) and using obvious inequality $\left(\sum_{i=1}^k a_i^2\right)^{1/2}\leq\sum_{i=1}^k|a_i|$, due to (3.23), finally we have

$$\|u\|_{W_2^1(D_T)}\leq c_1\|F\|_{L_2(D_T)}+c_2\|g\|_{W_2^1(S_T)}+c_3.$$

Here

$$\begin{cases} c_1=\sqrt{T}\exp(M_2^2+T+1)T,\\ c_2=\sqrt{T}(M_0+2)^{1/2}\exp(M_2^2+T+1)T,\\ c_3=\sqrt{2T}(M_1+M_2)(\operatorname{mes}D_T)^{1/2}\exp(M_2^2+T+1)T.\end{cases} \tag{3.24}$$

The Lemma 3.1 is completely proved. □

Before passing to the question of solvability of the problem (1.1), (1.2) let us consider the same question for the linear case of needed form, when in (1.1) the vector-function $f = 0$, i.e. for the problem

$$L_0 u := \Box u = F(x,t), \quad (x,t) \in D_T, \tag{3.25}$$

$$u|_{S_T} = g. \tag{3.26}$$

For the problem (3.25), (3.26), analogously to the Definition 2.1 for the problem (1.1), (1.2), we introduce a notion of a strong generalized solution $u = (u_1, \ldots, u_N) \in W_2^1(D_T)$ of the class $W_2^1$ in the domain $D_T$ with $F = (F_1, \ldots, F_N) \in L_2(D_T)$ and $g = (g_1, \ldots, g_N) \in W_2^1(S_T)$, for which there exists a sequence of vector-functions $u^m \in C^2(\overline{D}_T)$ such that

$$\lim_{m\to\infty} \|u^m - u\|_{W_2^1(D_T)} = 0, \qquad \lim_{m\to\infty} \|L_0 u^m - F\|_{L_2(D_T)} = 0, \tag{3.27}$$

$$\lim_{m\to\infty} \|u^m|_{S_T} - g\|_{W_2^1(S_T)} = 0. \tag{3.28}$$

Note that, as it is clear from the proof of Lemma 3.1, by virtue of (3.24) when $f = 0$, i.e. when $M_1 = M_2 = 0$ for the strong generalized solution $u \in W_2^1(D_T)$ of the problem (3.25), (3.26) of the class $W_2^1$ in the domain $D_T$ the following a priori estimate is valid

$$\|u\|_{W_2^1(D_T)} \le c\big(\|F\|_{L_2(D_T)} + \|g\|_{W_2^1(S_T)}\big), \tag{3.29}$$

where

$$c = \sqrt{T}(M_0 + 2)^{1/2} \exp(T+1)T. \tag{3.30}$$

Consider the Sobolev weight space $W_{2,\alpha}^k(D)$, $0 < \alpha < \infty$, $k = 1, 2, \ldots$, consisting of the functions belonging to the class $W_{2,\mathrm{loc}}^k(D)$, for which the following norm is finite [5]

$$\|w\|^2_{W^k_{2,\alpha}(D)} = \sum_{i=0}^{k} \int_D r^{-2\alpha-2(k-i)} \left| \frac{\partial^i w}{\partial x^{i'} \partial t^{i_0}} \right|^2 dxdt,$$

where

$$r = \left( \sum_{j=1}^n x_j^2 + t^2 \right)^{1/2}, \qquad \frac{\partial^i w}{\partial x^{i'} \partial t^{i_0}} := \frac{\partial^i w}{\partial x_1^{i_1} \ldots \partial x_n^{i_n} \partial t^{i_0}}, \quad i = i_1 + \cdots + i_n + i_0.$$

Analogously we introduce the space $W_{2,\alpha}^k(S)$, $S = \partial D$ [5].

Together with the problem (3.25), (3.26) consider an analogous problem in the infinite cone $D = D_\infty$:

$$L_0 u = F(x,t), \quad (x,t) \in D, \tag{3.31}$$

$$u|_S = g. \tag{3.32}$$

Due to (1.3), according to the result of the work [4], there exists a constant $\alpha_0 = \alpha_0(k) > 1$ such that for $\alpha \ge \alpha_0$ the problem (3.31), (3.32) has a unique solution $u = (u_1, \ldots, u_N) \in W_{2,\alpha}^k(D)$ for each $F = (F_1, \ldots, F_N) \in W_{2,\alpha-1}^{k-1}(D)$ and $g = (g_1, \ldots, g_N) \in W_{2,\alpha-1/2}^k(S)$, $k \ge 2$.

Since the space $C_0^\infty(D_T)$ of finite infinitely differentiable in $D_T$ functions is dense in $L_2(D_T)$, then for a given $F = (F_1, \ldots, F_N) \in L_2(D_T)$ there exists a sequence of vector-functions $F^m = (F_1^m, \ldots, F_N^m) \in C_0^\infty(D_T)$, such that $\lim_{m\to\infty} \|F^m - F\|_{L_2(D_T)} = 0$. For fixed $m$, extending the vector-function $F^m$ by zero beyond the domain $D_T$ and keeping the same notations, we have $F^m \in C_0^\infty(D)$. It is obvious that $F^m \in W_{2,\alpha-1}^{k-1}(D)$ for any $k \ge 2$ and $\alpha > 1$, and also for $\alpha \ge \alpha_0 = \alpha_0(k)$. If $g \in W_2^1(S_T)$, then there exists $\tilde{g} \in W_2^1(S)$ such that $g = \tilde{g}|_{S_T}$ and $\operatorname{diam} \operatorname{supp} \tilde{g} < +\infty$ [8]. Besides, the space $C_*^\infty(S) := \{g \in C^\infty(S) : \operatorname{diam} \operatorname{supp} g < +\infty,\ O \notin \operatorname{supp} g\}$ is dense in $W_2^1(S)$ [6]. Therefore, there exists a sequence $g^m \in C_*^\infty(S)$ such that $\lim_{m\to\infty} \|g^m - \tilde{g}\|_{W_2^1(S)} = 0$. It is easy to see that $g^m \in W_{2,\alpha-1/2}^k(S)$ for any $k \ge 2$ and $\alpha > 1$, and, therefore, for $\alpha \ge \alpha_0 = \alpha(k)$. According to what is mentioned above there exists a solution $\tilde{u}^m \in W_{2,\alpha}^k(D)$ of the problem (3.31), (3.32) for $F = F^m$ and $g = g^m$. Let $u^m = \tilde{u}^m|_{D_T}$. Since $u^m \in W_2^k(D_T)$, then, taking number $k$ sufficiently large, namely $k > \frac{n+1}{2} + 2$, we have $u^m \in C^2(\overline{D}_T)$. By virtue of estimate (3.29) we have

$$\|u^m - u^{m'}\|_{W_2^1(D_T)} \le c\big(\|F^m - F^{m'}\|_{L_2(D_T)} + \|g^m - g^{m'}\|_{W_2^1(S_T)}\big). \tag{3.33}$$

Since sequences $\{F^m\}$ and $\{g^m\}$ are fundamental in the spaces $L_2(D_T)$ and $W_2^1(S_T)$, respectively, then due to (3.33) the sequence $\{u^m\}$ will be fundamental in the space $W_2^1(D_T)$. Therefore, in view of the completeness of the space $W_2^1(D_T)$ there exists a vector-function $u \in W_2^1(D_T)$ such that $\lim_{m\to\infty} \|u^m - u\|_{W_2^1(D_T)} = 0$, and since $L_0 u^m = F^m \to F$ in the space $L_2(D_T)$ and $g^m = u^m|_{S_T} \to g$ in the space $W_2^1(S_T)$, i.e. the limit equalities (3.27) and (3.28) are fulfilled, then the vector-function $u$ is a strong generalized solution of the problem (3.25), (3.26) of the class $W_2^1$ in the domain $D_T$. The uniqueness of this solution of the problem (3.25), (3.26) of the class $W_2^1$ in the domain $D_T$ follows from a priori estimate (3.29). Therefore, for the solution $u$ of the problem (3.25), (3.26) we have $u = L_0^{-1}(F, g)$, where $L_0^{-1} : [L_2(D_T)]^N \times [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N$ is a linear continuous operator with a norm admitting in view of (3.29) the following estimate

$$\|L_0^{-1}\|_{[L_2(D_T)]^N \times [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N} \le c, \tag{3.34}$$

where constant $c$ is determined from (3.30).

Because of linearity of the operator $L_0^{-1} : [L_2(D_T)]^N \times [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N$ we have a representation

$$L_0^{-1}(F, g) = L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.35}$$

where $L_{01}^{-1} : [L_2(D_T)]^N \to [W_2^1(D_T)]^N$ and $L_{02}^{-1} : [W_2^1(S_T)]^N \to [W_2^1(D_T)]^N$ are linear continuous operators, and in view of (3.34) we have

$$\|L_{01}^{-1}\|_{[L_2(D_T)]^N \to [W_2^1(D_T)]^N} \le c, \qquad \|L_{02}^{-1}\|_{[W_2^1(S_T)]^N \to [W_2^1(D_T)]^N} \le c. \tag{3.36}$$

**Remark 3.1.** Note that for $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ and (2.2), where $0 \le \alpha < \frac{n+1}{n-1}$, in view of (3.34), (3.35), (3.36) and the Remark 2.1 the vector-function $u = (u_1, \ldots, u_N) \in W_2^1(D_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ if and only if $u$ is a solution of the following functional equation

$$u = L_{01}^{-1}(-f(u)) + L_{01}^{-1}(F) + L_{02}^{-1}(g) \tag{3.37}$$

in the space $W_2^1(D_T)$.

Rewrite the Eq. (3.37) in the form

$$u = A_0 u := -L_{01}^{-1}(K_0 u) + L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.38}$$

where the operator $K_0 : \left[W_2^1(D_T)\right]^N \to \left[L_2(D_T)\right]^N$ from (2.3) due to the Remark 2.1 is a continuous and compact operator. Therefore, according to (3.36), the operator $A_0 : \left[W_2^1(D_T)\right]^N \to \left[W_2^1(D_T)\right]^N$ is also continuous and compact. At the same time according to the Lemma 3.1 and the equalities (3.24) for any parameter $\tau \in [0, 1]$ and any solution $u$ of the equation $u = \tau A_0 u$ with parameter $\tau$ it is valid the same a priori estimate (3.1) with the constants $c_i$ from (3.24), not depending on $u$, $F$, $g$ and $\tau$. Therefore, due to Schaefer's fixed point theorem [3] the Eq. (3.38), and, therefore, according to the Remark 3.1, the problem (1.1), (1.2) has at least one solution $u \in W_2^1(D_T)$. Thus, we have proved the following theorem.

**Theorem 3.1.** *Let $f$ satisfy the condition (2.2), where $0 \le \alpha \le 1$. Then for any $F \in L_2(D_T)$ and $g \in W_2^1(S_T)$ the problem (1.1), (1.2) has at least one strong generalized solution $u$ of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.*

A global solvability of the problem (1.1), (1.2) in the class $W_2^1$ in the sense of the Definition 2.3 immediately follows from the Theorem 3.1 when the conditions of this theorem are fulfilled.

**Remark 3.2.** In the Theorem 3.1 a global solvability of the problem (1.1), (1.2) is proved for the case when $f$ satisfies the condition (2.2), where $0 \le \alpha \le 1$. In the case when $1 < \alpha < \frac{n+1}{n-1}$ the problem (1.1), (1.2), generally speaking, is not globally solvable, as it will be shown in the Section 5. At the same time below we prove that when $1 < \alpha < \frac{n+1}{n-1}$ the problem (1.1), (1.2) is locally solvable in the sense of the Definition 2.2.

**Theorem 3.2.** *Let $f$ satisfy the condition (2.2), where $1 < \alpha < \frac{n+1}{n-1}$; $g = 0$, $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for any $T > 0$. Then the problem (1.1), (1.2) is locally solvable in the class $W_2^1$, i.e. there exists number $T_0 = T_0(F) > 0$ such that for $T < T_0$ this problem has strong generalized solution of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.*

**Proof.** According to the Definition 2.1 and the Remark 3.1 the vector-function $u \in \mathring{W}_2^1(D_T, S_T) := \{v \in W_2^1(D_T) : v|_{S_T} = 0\}$ is a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ for $g = 0$ then and only then, when $u$ is a solution of the functional Eq. (3.38) for $g = 0$, i.e.

$$u = A_0 u := -L_{01}^{-1}(K_0 u) + L_{01}^{-1}(F) \tag{3.39}$$

in the space $\mathring{W}_2^1(D_T, S_T)$. Denote by $B(0, r_0) := \{u = (u_1, \ldots, u_N) \in \mathring{W}_2^1(D_T, S_T) : \|u\|_{\mathring{W}_2^1(D_T,S_T)} \le r_0\}$ a closed (convex) ball in the Hilbert space $\mathring{W}_2^1(D_T, S_T)$ with a center in null element and the radius $r_0 > 0$. Since the operator $A_0$ from (3.39), acting in the space $\mathring{W}_2^1(D_T, S_T)$ is a continuous compact operator, then, according to the Schauder theorem, for solvability of the Eq. (3.39) in the space $\mathring{W}_2^1(D_T, S_T)$ it suffices to prove that the operator $A_0$ maps the ball $B(0, r_0)$ into itself for certain $r_0 > 0$ [3]. Below we show that for any fixed $r_0 > 0$ there exists a number $T_0 = T_0(r_0, F) > 0$ such that for $T < T_0$ the operator $A_0$ from (3.39) maps the ball $B(0, r_0)$ into itself. With this purpose we evaluate $\|A_0 u\|_{\mathring{W}_2^1(D_T,S_T)}$ for $u \in \mathring{W}_2^1(D_T, S_T)$.

If $u=(u_1,\dots,u_N)\in \overset{\circ}{W}{}_2^1(D_T,S_T)$, then let us denote by $\tilde{u}$ the vector-function which represents an even extension of $u$ through plane $t=T$ in the domain $D_T^*$, symmetrical to the domain $D_T$ with respect to the same plane, i.e.

$$\tilde{u}=\begin{cases} u(x,t), & (x,t)\in D_T;\\ u(x,2T-t), & (x,t)\in D_T^* \end{cases}$$

and $\tilde{u}(x,t)=u(x,t)$ for $t=T$ in the sense of the trace theory. It is obvious that $\tilde{u}\in \overset{\circ}{W}{}_2^1(\tilde{D}_T):=\{v\in W_2^1(\tilde{D}_T): v|_{\partial\tilde{D}_T}=0\}$, where $\tilde{D}_T=D_T\cup\Omega_\tau\cup D_T^*$, $\Omega_\tau:=D\cap\{t=T\}$.

Using inequality

$$\int_\Omega |v|d\Omega\le(\text{mes }\Omega)^{1-1/p}\|v\|_{p,\Omega},\quad p\ge 1,$$

and taking into account equalities

$$\|\tilde{u}\|^p_{L_p(\tilde{D}_T)}=2\|u\|^p_{L_p(D_T)},\qquad \|\tilde{u}\|^2_{\overset{\circ}{W}{}_2^1(\tilde{D}_T)}=2\|u\|^2_{\overset{\circ}{W}{}_2^1(D_T,S_T)},$$

from known multiplicative inequality [8]

$$\|v\|_{p,\Omega}\le\beta\|\nabla_{x,t}v\|^{\tilde{\alpha}}_{m,\Omega}\|v\|^{1-\tilde{\alpha}}_{r,\Omega}\quad\forall v\in\overset{\circ}{W}{}_2^1(\Omega),\ \Omega\in\mathbb{R}^{n+1},$$

$$\nabla_{x,t}=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t}\right),\qquad \tilde{\alpha}=\left(\frac{1}{r}-\frac{1}{p}\right)\left(\frac{1}{r}-\frac{1}{\tilde{m}}\right)^{-1},\qquad \tilde{m}=\frac{(n+1)m}{n+1-m}$$

for $\Omega=\tilde{D}_T\subset\mathbb{R}^{n+1}$, $v=\tilde{u}$, $r=1$, $m=2$ and $1<p\le\frac{2(n+1)}{n+1-m}$, where $\beta=\text{const}>0$ does not depend on $v$ and $T$, it follows the following inequality

$$\|u\|_{L_p(D_T)}\le c_0(\text{mes }D_T)^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}}\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\quad\forall u\in\overset{\circ}{W}{}_2^1(D_T,S_T),\tag{3.40}$$

where $c_0=\text{const}>0$ does not depend on $u$ and $T$.

Since $\text{mes }D_T=\frac{\omega}{n+1}T^{n+1}$, where $\omega$ is the $n$-dimensional measure of section $\Omega_1:=D\cap\{t=1\}$, then for $p=2\alpha$ from (3.40) we have

$$\|u\|_{L_{2\alpha}(D_T)}\le C_T\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\quad\forall u\in\overset{\circ}{W}{}_2^1(D_T,S_T),\tag{3.41}$$

where

$$C_T=c_0\left(\frac{\omega}{n+1}\right)^{\alpha_1}T^{(n+1)\alpha_1},\qquad \alpha_1=\frac{1}{2\alpha}+\frac{1}{n+1}-\frac{1}{2}.\tag{3.42}$$

Since $\alpha<\frac{n+1}{n-1}$, then $\alpha_1=\frac{1}{2\alpha}+\frac{1}{n+1}-\frac{1}{2}>0$ and due to (3.41) and (3.42) for any $u\in\overset{\circ}{W}{}_2^1(D_T,S_T)$ we have

$$\|u\|_{L_{2\alpha}(D_T)}\le C_{T_1}\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}\quad\forall T\le T_1,\tag{3.43}$$

where $T_1$ is a fixed positive number.

For $\|K_0u\|_{L_2(D_T)}$, where $u\in\overset{\circ}{W}{}_2^1(D_T,S_T)$, $T\le T_1$, and operator $K_0$ acts according to the formula (2.3), due to (2.2) and (3.43) we have the following estimate

$$\begin{aligned}\|K_0u\|^2_{L_2(D_T)}&\le\int_{D_T}(M_1+M_2|u|^\alpha)^2dxdt\le 2M_1^2\text{ mes }D_T\\ &\quad+2M_2^2\int_{D_T}|u|^{2\alpha}dxdt=2M_1^2\text{ mes }D_T+2M_2^2\|u\|^{2\alpha}_{L_{2\alpha}(D_T)}\\ &\le 2M_1^2\text{ mes }D_{T_1}+2M_2^2C_{T_1}^{2\alpha}\|u\|^{2\alpha}_{\overset{\circ}{W}{}_2^1(D_T,S_T)},\end{aligned}$$

whence we have

$$\|K_0u\|_{L_2(D_T)}\le M_1(2\text{ mes }D_{T_1})^{\frac{1}{2}}+\sqrt{2}M_2C^\alpha_{T_1}\|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_T,S_T)}.\tag{3.44}$$

From (3.30), (3.36), (3.39) and (3.44) it follows that

$$\begin{aligned}\|A_0u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}&\le\|L_{01}^{-1}\|_{[L_2(D_T)]^N\to[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N}\|K_0u\|_{L_2(D_T)}\\ &\quad+\|L_{01}^{-1}\|_{[L_2(D_T)]^N\to[\overset{\circ}{W}{}_2^1(D_T,S_T)]^N}\|F\|_{L_2(D_T)}\end{aligned}$$

$$\leq c\big[\sqrt{2\,\mathrm{mes}\,D_{T_1}}M_1+\sqrt{2}M_2C^{\alpha}_{T_1}\|u\|^{\alpha}_{\mathring{W}^1_2(D_T,S_T)}+\|F\|_{L_2(D_{T_1})}\big]$$
$$\leq \sqrt{T}(M_0+2)^{1/2}\exp(T_1+1)T_1\big[\sqrt{2\,\mathrm{mes}\,D_{T_1}}M_1+\sqrt{2}M_2C^{\alpha}_{T_1}\|u\|^{\alpha}_{\mathring{W}^1_2(D_T,S_T)}$$
$$+\|F\|_{L_2(D_{T_1})}\big]\quad \forall T\leq T_1,\ \forall u\in \mathring{W}^1_2(D_T,S_T). \tag{3.45}$$

Since the right side of the inequality (3.45) contains $\sqrt{T}$ as a factor vanishing for $T\to 0$, then there exists positive number $T_0\leq T_1$ such that for $T<T_0$ and $\|u\|_{\mathring{W}^1_2(D_T,S_T)}\leq r_0$, due to (3.45) we have $\|A_0u\|_{\mathring{W}^1_2(D_T,S_T)}\leq r_0$, i.e. the operator $A_0:\mathring{W}^1_2(D_T,S_T)\to \mathring{W}^1_2(D_T,S_T)$ from (3.39) maps the ball $B(0,r_0)$ into itself. The Theorem 3.2 is proved. □

**Remark 3.3.** In the case when $f$ satisfies the condition (2.2), where $1<\alpha<\frac{n+1}{n-1}$, the Theorem 3.2 ensures a local solvability of the problem (1.1), (1.2), although in this case, with additional conditions imposed on $f$, as we show in the following theorem, this problem is globally solvable.

**Theorem 3.3.** *Let $f$ satisfy the condition (2.2), where $1<\alpha<\frac{n+1}{n-1}$, and $f=\nabla G$, i.e. $f_i(u)=\frac{\partial}{\partial u_i}G(u)$, $u\in\mathbb{R}^N$, $i=1,\dots,N$, where $G=G(u)\in C^1(\mathbb{R}^N)$ is a scalar function satisfying conditions $G(0)=0$ and $G(u)\geq 0\ \forall u\in\mathbb{R}^N$. Let $g=0$, $F\in L_{2,\mathrm{loc}}(D_\infty)$ and $F|_{D_T}\in L_2(D_T)$ for any $T>0$. Then the problem (1.1), (1.2) is globally solvable in the class $W^1_2$, i.e. for any $T>0$ this problem has a strong generalized solution of the class $W^1_2$ in the domain $D_T$ in the sense of the Definition 2.1.*

**Proof.** First let us show that for any fixed $T>0$, with the conditions of the Theorem 3.3, for a strong generalized solution $u$ of the problem (1.1), (1.2) of the class $W^1_2$ in the domain $D_T$ it is valid the following estimate

$$\|u\|_{\mathring{W}^1_2(D_T,S_T)}\leq c(T)\|F\|_{L_2(D_T)},\qquad c(T)=\sqrt{T}\exp\frac{1}{2}(T+T^2). \tag{3.46}$$

Indeed, according to the Definition 2.1 in the case when $g=0$ there exists a sequence of the vector-functions $u^m\in\mathring{C}^2(\overline{D}_T,S_T):=\{v\in C^2(\overline{D}_T):v|_{S_T}=0\}$ such that

$$\lim_{m\to\infty}\|u^m-u\|_{W^1_2(D_T)}=0,\qquad \lim_{m\to\infty}\|Lu^m-F\|_{L_2(D_T)}=0. \tag{3.47}$$

Putting

$$F^m:=Lu^m \tag{3.48}$$

and taking into account that $u^m|_{S_T}=0$ and the operator $\nu_0\frac{\partial}{\partial x_i}-\nu_i\frac{\partial}{\partial t}$ is an inner differential operator on $S_T$, and, therefore, $\big(\frac{\partial u^m}{\partial x_i}\nu_0-\frac{\partial u^m}{\partial t}\nu_i\big)|_{S_T}=0,\ i=1,\dots,n$, due to (1.3) from (3.8) we have

$$\int_{D_\tau}F^m\frac{\partial u^m}{\partial t}dxdt\geq\frac{1}{2}\int_{\Omega_\tau}\left[\left(\frac{\partial u^m}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dx+\int_{D_\tau}f(u^m)\frac{\partial u^m}{\partial t}dxdt. \tag{3.49}$$

Since $f=\nabla G$ then $f(u^m)\frac{\partial u^m}{\partial t}=\frac{\partial}{\partial t}G(u^m)$, and, taking into account that $u^m|_{S_T}=0$, $\nu_0|_{\Omega_\tau}=1$, $G(0)=0$, integrating by parts we receive

$$\int_{D_\tau}f(u^m)\frac{\partial u^m}{\partial t}dxdt=\int_{D_\tau}\frac{\partial}{\partial t}G(u^m)dxdt=\int_{\partial D_\tau}G(u^m)\nu_0ds$$
$$=\int_{S_\tau\cup\Omega_\tau}G(u^m)\nu_0ds=\int_{\Omega_\tau}G(u^m)dx. \tag{3.50}$$

Because $G(u)\geq 0\ \forall u\in\mathbb{R}^N$, then due to (3.50) from (3.49) we have

$$\int_{\Omega_\tau}\left[\left(\frac{\partial u^m}{\partial t}\right)^2+\sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dx\leq 2\int_{D_T}F^m\frac{\partial u^m}{\partial t}dxdt$$
$$\leq\int_{D_T}\left(\frac{\partial u^m}{\partial t}\right)^2dxdt+\int_{D_T}(F^m)^2dxdt,\quad 0<\tau\leq T. \tag{3.51}$$

Since $u^m|_{S_T}=0$, then $u(x,\tau)=\int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u^m(x,s)ds$, where $t=\gamma(x)$ is the equation of conic surface $S$. Therefore, as in receiving the inequality (3.15), we have

$$\int_{\Omega_\tau}(u^m)^2dx = \int_{\Omega_\tau}\left(\int_{\gamma(x)}^{\tau}\frac{\partial}{\partial t}u^m(x,s)ds\right)^2 dx$$
$$\leq \int_{\Omega_\tau}(\tau-|x|)\left[\int_{\gamma(x)}^{\tau}\left(\frac{\partial}{\partial t}u^m\right)^2 ds\right]dx$$
$$\leq T\int_{\Omega_\tau}\left[\int_{\gamma(x)}^{\tau}\left(\frac{\partial u^m}{\partial t}\right)^2 ds\right]dx = T\int_{D_\tau}\left(\frac{\partial u^m}{\partial t}\right)^2 dxdt. \tag{3.52}$$

Denoting $w(\tau) := \int_{\Omega_\tau}\left[(u^m)^2 + \left(\frac{\partial u^m}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dx$, in view of (3.51) and (3.52) we have

$$w(\tau) \leq (1+T)\int_{D_\tau}\left(\frac{\partial u^m}{\partial t}\right)^2 dxdt + \int_{D_\tau}(F^m)^2 dxdt$$
$$\leq (1+T)\int_{D_\tau}\left[(u^m)^2 + \left(\frac{\partial u^m}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dxdt + \|F^m\|^2_{L_2(D_\tau)}$$
$$= (1+T)\int_0^\tau w(s)ds + \|F^m\|^2_{L_2(D_\tau)}, \quad 0<\tau\leq T. \tag{3.53}$$

By virtue of the Gronwall's Lemma from (3.53) it follows that

$$w(\tau) \leq \|F\|^2_{L_2(D_\tau)}\exp(1+T)\tau \leq \|F\|^2_{L_2(D_T)}\exp(1+T)T, \quad 0<\tau\leq T. \tag{3.54}$$

According to (3.54) we have

$$\|u^m\|^2_{\mathring{W}_2^1(D_T,S_T)} = \int_{D_T}\left[(u^m)^2 + \left(\frac{\partial u^m}{\partial t}\right)^2 + \sum_{i=1}^n\left(\frac{\partial u^m}{\partial x_i}\right)^2\right]dxdt$$
$$= \int_0^T w(\tau)d\tau \leq T\|F^m\|_{L_2(D_T)}\exp(1+T)T,$$

whence, due to the limit equalities (3.47), it follows the estimate (3.46).

According to the Remark 3.1, when the conditions of the Theorem 3.3 are fulfilled, the vector-function $u \in \mathring{W}_2^1(D_T,S_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ if and only if $u$ is a solution of the following functional equation $u = A_0u$ from (3.39) in the space $\mathring{W}_2^1(D_T,S_T)$, where the operator $A_0$, acting in the space $\mathring{W}_2^1(D_T,S_T)$, is continuous and compact. At the same time, due to (3.46) for any $\mu \in [0,1]$ and for any solution of the equation $u = \mu A_0 u$ an a priori estimate is valid $\|u\|_{\mathring{W}_2^1(D_T,S_T)} \leq \mu c(T)\|F\|_{L_2(D_T)} \leq c(T)\|F\|_{L_2(D_T)}$ with positive constant $c(T)$, not depending on $u$, $\mu$ and $F$. Thus, according to Schaefer's fixed point theorem [3] the Eq. (3.46), and therefore the problem (1.1), (1.2), has at least one strong generalized solution of the class $W_2^1$ in the domain $D_T$ for any $T>0$. The Theorem 3.3 is proved. □

## 4. The uniqueness and existence of a global solution of the problem (1.1), (1.2) of the class $W_2^1$

Below we impose on the nonlinear vector-function $f = (f_1,\ldots,f_N)$ from (1.1) the additional requirements

$$f \in C^1(\mathbb{R}^N), \quad \left|\frac{\partial f_i(u)}{\partial u_j}\right| \leq M_3 + M_4|u|^\gamma \ \forall u \in \mathbb{R}^N,\ 1\leq i,j\leq N, \tag{4.1}$$

where $M_3, M_4, \gamma = \text{const} \geq 0$. For simplicity of reasoning we suppose that the vector-function $g = 0$ in the boundary condition (1.2).

**Remark 4.1.** It is obvious that from (4.1) it follows the condition (2.2) for $\alpha = \gamma+1$, and in the case $\gamma < \frac{2}{n-1}$ we have $\alpha < \frac{n+1}{n-1}$.

**Theorem 4.1.** *Let the condition (4.1) be fulfilled, where $0 \leq \gamma < \frac{2}{n-1}$; $F \in L_2(D_T)$ and $g = 0$. Then the problem (1.1), (1.2) cannot have more than one strong generalized solution of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.*

**Proof.** Let $F \in L_2(D_T)$, $g = 0$, and the problem (1.1), (1.2) have two strong generalized solutions $u^1$ and $u^2$ of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1, i.e. there exist two sequences of vector-functions $u^{im} \in \mathring{C}^2(\overline{D}_T,S_T) := \{u \in C^2(\overline{D}_T) : u|_{S_T} = 0\}$, $i = 1,2;\ m = 1,2,\ldots,$ such that

$$\lim_{m\to\infty} \|u^{im} - u^i\|_{\mathring{W}_2^1(D_T,S_T)} = 0, \qquad \lim_{m\to\infty} \|Lu^{im} - F\|_{L_2(D_T)} = 0, \quad i = 1, 2. \tag{4.2}$$

Let

$$w = u^2 - u^1, \qquad w^m = u^{2m} - u^{1m}, \qquad F^m = Lu^{2m} - Lu^{1m}. \tag{4.3}$$

In view of (4.2) and (4.3) we have

$$\lim_{m\to\infty} \|w^m - w\|_{\mathring{W}_2^1(D_T,S_T)} = 0, \qquad \lim_{m\to\infty} \|F^m\|_{L_2(D_T)} = 0. \tag{4.4}$$

In accordance with (4.3) consider the vector-function $w^m \in \mathring{C}^2(D_T, S_T)$ as a solution of the following problem

$$\square w^m = -[f(u^{2m}) - f(u^{1m})] + F^m, \tag{4.5}$$

$$w^m|_{S_T} = 0. \tag{4.6}$$

Analogously to how the inequality (3.49) was obtained from (4.5), (4.6) it follows

$$\int_{\Omega_\tau} \left[\left(\frac{\partial w^m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i}\right)^2\right] dx \le 2\int_{D_\tau} F^m \frac{\partial w^m}{\partial t} dxdt - 2\int_{D_\tau} [f(u^{2m}) - f(u^{1m})]\frac{\partial w^m}{\partial t} dxdt, \quad 0 < \tau \le T. \tag{4.7}$$

Taking into account the equality

$$f_i(u^{2m}) - f_i(u^{1m}) = \sum_{j=1}^N \int_0^1 \frac{\partial}{\partial u_j} f_i(u^{1m} + s(u^{2m} - u^{1m}))ds(u_j^{2m} - u_j^{1m}),$$

we receive

$$\left[f(u^{2m}) - f(u^{1m})\right]\frac{\partial w^m}{\partial t} = \sum_{i,j=1}^N \left[\int_0^1 \frac{\partial}{\partial u_j} f_i(u^{1m} + s(u^{2m} - u^{1m}))ds\right](u_j^{2m} - u_j^{1m})\frac{\partial w_i^m}{\partial t}. \tag{4.8}$$

By virtue of (4.1) and obvious inequality $|d_1 + d_2|^\gamma \le 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \le 2^\gamma(|d_1|^\gamma + |d_2|^\gamma)$ for $\gamma \ge 0$, $d_i \in \mathbb{R}$, we have

$$\left|\int_0^1 \frac{\partial}{\partial u_j} f_i(u^{1m} + s(u^{2m} - u^{1m}))ds\right| \le \int_0^1 [M_3 + M_4|(1-s)u^{1m} + su^{2m}|^\gamma]ds \le M_3 + 2^\gamma M_4\big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big). \tag{4.9}$$

From (4.8) and (4.9), taking into account (4.3), it follows

$$\begin{aligned}\left|\left[f(u^{2m}) - f(u^{1m})\right]\frac{\partial w^m}{\partial t}\right| &\le \sum_{i,j=1}^N \left[M_3 + 2^\gamma M_4\big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big)\right]|w_j^m|\left|\frac{\partial w_i^m}{\partial t}\right| \\ &\le N^2\left[M_3 + 2^\gamma M_4\big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big)\right]|w^m|\left|\frac{\partial w^m}{\partial t}\right| \\ &\le \frac{1}{2}N^2 M_3\left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t}\right)^2\right] \\ &\quad + 2^\gamma N^2 M_4\big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big)|w^m|\left|\frac{\partial w^m}{\partial t}\right|.\end{aligned} \tag{4.10}$$

Due to (4.7) and (4.10) we have

$$\begin{aligned}\int_{\Omega_\tau} \left[\left(\frac{\partial w^m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i}\right)^2\right] dx &\le \int_{D_\tau} \left[\left(\frac{\partial w^m}{\partial t}\right)^2 + (F^m)^2\right] dxdt \\ &\quad + N^2 M_3 \int_{D_\tau} \left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t}\right)^2\right] dxdt \\ &\quad + 2^{\gamma+1} N^2 M_4 \int_{D_T} \big(|u^{1m}|^\gamma + |u^{2m}|^\gamma\big)|w^m|\left|\frac{\partial w^m}{\partial t}\right| dxdt.\end{aligned} \tag{4.11}$$

The last integral in the right hand part of (4.11) can be estimated by Holder's inequality

$$\int_{D_T} \left(|u^{1m}|^\gamma + |u^{2m}|^\gamma\right)|w^m| \left|\frac{\partial w^m}{\partial t}\right| dxdt$$
$$\leq \left(\| \, |u^{1m}|^\gamma \|_{L_{n+1}(D_T)} + \| \, |u^{2m}|^\gamma \|_{L_{n+1}(D_T)}\right) \|w^m\|_{L_p(D_\tau)} \left\|\frac{\partial w^m}{\partial t}\right\|_{L_2(D_\tau)}. \tag{4.12}$$

Here $\frac{1}{n+1} + \frac{1}{p} + \frac{1}{2} = 1$, i.e. for

$$p = \frac{2(n+1)}{n-1}. \tag{4.13}$$

By virtue of (3.40) for $q \leq \frac{2(n+1)}{n-1}$ we have

$$\|v\|_{L_q(D_\tau)} \leq C_q(T)\|v\|_{\mathring{W}_2^1(D_T,S_T)} \quad \forall v \in \mathring{W}_2^1(D_\tau, S_\tau),\ 0 < \tau \leq T, \tag{4.14}$$

with positive constant $C_q(T)$, not depending on $v \in \mathring{W}_2^1(D_\tau, S_\tau)$ and $\tau \in (0, T]$.

According to the theorem $\gamma < \frac{2}{n-1}$ and, therefore, $\gamma(n+1) < \frac{2(n+1)}{n-1}$. Thus, from (4.13), (4.14) we receive

$$\| \, |u^{im}|^\gamma \|_{L_{n+1}(D_T)} = \|u^{im}\|^\gamma_{L_{\gamma(n+1)}(D_T)} \leq C^\gamma_{\gamma(n+1)}(T)\|u^{im}\|^\gamma_{\mathring{W}_2^1(D_T,S_T)}, \quad i = 1, 2;\ m \geq 1, \tag{4.15}$$

$$\|w^m\|_{L_p(D_\tau)} \leq C_p(T)\|w^m\|_{W_2^1(D_\tau)}, \quad m \geq m_0. \tag{4.16}$$

In view of the first limit equality from (4.2) there exists a natural number $m_0$ such that for $m \geq m_0$ we have

$$\|u^{im}\|^\gamma_{\mathring{W}_2^1(D_T,S_T)} \leq \|u^i\|^\gamma_{\mathring{W}_2^1(D_T,S_T)} + 1, \quad i = 1, 2;\ m \geq m_0.$$

In view of these inequalities from (4.12)–(4.16) it follows that

$$2^{\gamma+1}N^2M_4 \int_{D_T} \left(|u^{1m}|^\gamma + |u^{2m}|^\gamma\right)|w^m| \left|\frac{\partial w^m}{\partial t}\right| dxdt$$
$$\leq 2^{\gamma+1}N^2M_4C^\gamma_{\gamma(n+1)}(T)\left(\|u^1\|^\gamma_{\mathring{W}_2^1(D_T,S_T)} + \|u^2\|^\gamma_{\mathring{W}_2^1(D_T,S_T)} + 2\right)C_p(T)\|w^m\|_{W_2^1(D_\tau,S_\tau)} \left\|\frac{\partial w^m}{\partial t}\right\|_{L_2(D_\tau)}$$
$$\leq M_5\left(\|w^m\|^2_{W_2^1(D_\tau)} + \left\|\frac{\partial w^m}{\partial t}\right\|^2_{L_2(D_\tau)}\right) \leq 2M_5\|w^m\|^2_{W_2^1(D_\tau)}$$
$$= 2M_5 \int_{D_\tau}\left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i}\right)^2\right]dxdt, \tag{4.17}$$

where $M_5 = 2^\gamma N^2M_4C^\gamma_{\gamma(n+1)}(T)\left(\|u^1\|^\gamma_{\mathring{W}_2^1(D_T,S_T)} + \|u^2\|^\gamma_{\mathring{W}_2^1(D_T,S_T)} + 2\right)C_p(T)$.

Due to (4.17) from (4.11) we have

$$\int_{\Omega_\tau}\left[\left(\frac{\partial w^m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i}\right)^2\right]dx \leq M_6 \int_{D_\tau}\left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i}\right)^2\right]dxdt$$
$$+ \int_{D_T}(F^m)^2dxdt, \quad 0 < \tau \leq T, \tag{4.18}$$

where $M_6 = 1 + M_3N^2 + 2M_5$.

Note, that the inequality (3.52) is valid for $w^m$ too, and, therefore,

$$\int_{\Omega_\tau}(w^m)^2dx \leq T\int_{D_\tau}\left(\frac{\partial w^m}{\partial t}\right)^2 dxdt$$
$$\leq T\int_{D_T}\left[(w^m)^2 + \left(\frac{\partial w^m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i}\right)^2\right]dxdt. \tag{4.19}$$

Putting

$$\lambda_m(\tau) := \int_{\Omega_\tau} \left[ (w^m)^2 + \left(\frac{\partial w^m}{\partial t}\right)^2 + \sum_{i=1}^n \left(\frac{\partial w^m}{\partial x_i}\right)^2 \right] dx \tag{4.20}$$

and adding (4.18) to (4.19), we receive

$$\lambda_m(\tau) \le (M_6 + T) \int_0^\tau \lambda_m(s) ds + \|F^m\|^2_{L_2(D_T)}.$$

Whence, by the Gronwall's lemma, it follows that

$$\lambda_m(\tau) \le \|F^m\|^2_{L_2(D_T)} \exp(M_6 + T)\tau. \tag{4.21}$$

From (4.20) and (4.21) we have

$$\|w^m\|^2_{W_2^1(D_T)} = \int_0^T \lambda_m(\tau) d\tau \le T \|F^m\|^2_{L_2(D_T)} \exp(M_6 + T)T. \tag{4.22}$$

In view of (4.3), (4.4) from (4.22) it follows that

$$\begin{aligned} \|w\|_{W_2^1(D_T)} &= \lim_{m\to\infty} \|w - w^m + w^m\|_{W_2^1(D_T)} \\ &\le \lim_{m\to\infty} \|w - w^m\|_{W_2^1(D_T)} + \lim_{m\to\infty} \|w^m\|_{W_2^1(D_T)} \\ &= \lim_{m\to\infty} \|w - w^m\|_{W_2^1(D_T)} = \lim_{m\to\infty} \|w - w^m\|_{\mathring{W}_2^1(D_T,S_T)} = 0. \end{aligned}$$

Therefore $w = u_2 - u_1 = 0$, i.e. $u_2 = u_1$. The Theorem 4.1 is proved. □

From the Theorems 3.1, 3.3 and 4.1 and the Remark 4.1 it follows the following theorem of existence and uniqueness.

**Theorem 4.2.** *Let the vector-function $f$ satisfy the condition (4.1), where $0 \le \gamma < \frac{2}{n-1}$, and either $f$ satisfy the condition (2.2) for $\alpha \le 1$ or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \ge 0 \; \forall u \in \mathbb{R}^N$. Then for any $F \in L_2(D_T)$ and $g = 0$ the problem (1.1), (1.2) has unique strong generalized solution $u \in \mathring{W}_2^1(D_T, S_T)$ of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1.*

The following theorem on existence of global solution of this problem follows from the Theorem 4.2.

**Theorem 4.3.** *Let the vector-function $f$ satisfy the condition (4.1), where $0 \le \gamma < \frac{2}{n-1}$, and either $f$ satisfy the condition (2.2) for $\alpha \le 1$ or $f = \nabla G$, where $G \in C^1(\mathbb{R}^N)$, $G(0) = 0$ and $G(u) \ge 0 \; \forall u \in \mathbb{R}^N$. Let $g = 0$, $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ for each $T > 0$. Then the problem (1.1), (1.2) has unique global strong generalized solution $u \in W^1_{2,\mathrm{loc}}(D_\infty)$ of the class $W_2^1$ in the domain $D_\infty$ in the sense of the Definition 2.4.*

**Proof.** According to the Theorem 4.2 when the conditions of Theorem 4.3 are fulfilled for $T = k$, where $k$ is a natural number, there exists unique strong generalized solution $u^k \in \mathring{W}_2^1(D_T, S_T)$ of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_{T=k}$ in the sense of the Definition 2.1. Since $u^{k+1}|_{D_{T=k}}$ is also a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_{T=k}$, then in view of the Theorem 4.2 we have $u^k = u^{k+1}|_{D_{T=k}}$. Therefore one can construct unique global generalized solution $u \in \mathring{W}^1_{2,\mathrm{loc}}(D_\infty)$ of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_\infty$ in the sense of the Definition 2.4 in the following way:

$$u(x, t) = u^k(x, t), \quad (x, t) \in D_\infty, \; k = [t] + 1,$$

where $[t]$ is an integer part of the number $t$. The Theorem 4.3 is proved. □

## 5. The cases of nonexistence of a global solution of the problem (1.1), (1.2) of the class $W_2^1$

**Theorem 5.1.** *Let the vector-function $f = (f_1, \ldots, f_N)$ satisfy the condition (2.2), where $1 < \alpha < \frac{n+1}{n-1}$, and there exist numbers $l_1, \ldots, l_N$, $\sum_{i=1}^N |l_i| \ne 0$, such that*

$$\sum_{i=1}^N l_i f_i(u) \le c_0 - c_1 \left| \sum_{i=1}^N l_i u_i \right|^\beta \quad \forall u \in \mathbb{R}^N, \; 1 < \beta = \text{const} < \frac{n+1}{n-1}, \tag{5.1}$$

*where* $c_0, c_1 = \text{const}, c_1 > 0$. *Let* $F \in L_{2,\text{loc}}(D_\infty)$ *and* $F|_{D_T} \in L_2(D_T)$ *for any* $T > 0, g = 0$. *Let the scalar function* $F_0 = \sum_{i=1}^N l_i F_i - c_0$ *in the domain* $D_\infty$ *satisfy the following conditions*

$$F_0 \geq 0, \quad \lim_{t\to+\infty} \inf t^\gamma F_0(x,t) \geq c_2 = \text{const} > 0, \quad \gamma = \text{const} \leq n+1. \tag{5.2}$$

*Then there exists a finite positive number* $T_0 = T_0(F)$ *such that for* $T > T_0$ *the problem* (1.1), (1.2) *does not have a strong generalized solution of the class* $W_2^1$ *in the sense of the Definition* 2.1.

**Proof.** Let $u = (u_1, \dots, u_N)$ be a strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ in the sense of the Definition 2.1. It is easy to verify that

$$\int_{D_T} u\Box\varphi dxdt = -\int_{D_T} f(u)\varphi dxdt + \int_{D_T} F\varphi dxdt \tag{5.3}$$

for any test vector-function $\varphi = (\varphi_1, \dots, \varphi_N)$, such that

$$\varphi \in C^2(\overline{D}_T), \qquad \varphi|_{\partial D_T} = \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial D_T} = 0, \tag{5.4}$$

where $\nu$ is the unit vector of the outer normal to $\partial D_T$. Indeed, according to the definition of the strong generalized solution of the problem (1.1), (1.2) of the class $W_2^1$ in the domain $D_T$ there exists a sequence of the vector-functions $u^m \in \mathring{C}^2(\overline{D}_T, S_T)$, for which the limit equalities (3.47) are valid. Taking into account (3.48) and scalarly multiplying both parts of the equality $Lu^m = F^m$ by test vector-function $\varphi = (\varphi_1, \dots, \varphi_N)$, due to (5.4) after integrating by parts, we receive

$$\int_{D_T} u^m\Box\varphi dxdt = -\int_{D_T} f(u^m)\varphi dxdt + \int_{D_T} F^m\varphi dxdt. \tag{5.5}$$

By virtue of (3.47) and the Remark 2.1, passing in the equality (5.5) to the limit for $m \to \infty$ we receive (5.3).

Let us use the method of test functions [9]. Consider a scalar function $\varphi^0 = \varphi^0(x,t)$ such that

$$\varphi^0 \in C^2(\overline{D}_\infty), \quad \varphi^0|_{D_{T=1}} > 0, \qquad \varphi^0|_{t\geq 1} = 0, \qquad \varphi^0|_{\partial D_{T=1}} = \left.\frac{\partial\varphi^0}{\partial\nu}\right|_{\partial D_{T=1}} = 0 \tag{5.6}$$

and

$$æ_0 := \int_{D_{T=1}} \frac{|\Box\varphi^0|^{\beta'}}{|\varphi^0|^{\beta'-1}} dxdt < +\infty, \qquad \frac{1}{\beta} + \frac{1}{\beta'} = 1. \tag{5.7}$$

It is easy to see that in the capacity of the function $\varphi^0$, satisfying the conditions (5.6) and (5.7), we can choose the following function

$$\varphi^0(x,t) = \begin{cases} \omega^m\left(\dfrac{x}{t}\right)(1-t)^m t^k, & (x,t) \in D_{T=1}; \\ 0, & t \geq 1, \end{cases}$$

for sufficiently large positive $m$ and $k$, where the function $\omega \in C^\infty(\mathbb{R}^n)$ defines the equation of conic section $\partial\Omega_1 = S \cap \{t = 1\} : \omega(x) = 0,\ \nabla\omega|_{\partial\Omega_1} \neq 0$, and $\omega|_{\Omega_1} > 0$, $\Omega_1 := D \cap \{t = 1\}$.

Putting

$$\varphi_T(x,t) := \varphi^0\left(\frac{x}{T}, \frac{t}{T}\right), \quad T > 0, \tag{5.8}$$

due to (5.6) it is easy to see that

$$\varphi_T \in C^2(\overline{D}_T), \quad \varphi_T|_{D_T} > 0, \qquad \varphi_T|_{\partial D_T} = \left.\frac{\partial\varphi_T}{\partial\nu}\right|_{\partial D_T} = 0. \tag{5.9}$$

In the integral equality (5.3) for the test vector-function $\varphi$ we choose $\varphi = (l_1\varphi_T, l_2\varphi_T, \dots, l_N\varphi_T)$. For the chosen test vector-function $\varphi$, using notations

$$v = \sum_{i=1}^N l_i u_i, \qquad F_* = \sum_{i=1}^N l_i F_i, \qquad f_0 = \sum_{i=1}^N l_i f_i, \tag{5.10}$$

the integral equality (5.3) takes the following form

$$\int_{D_T} v\Box\varphi_T dxdt = -\int_{D_T} f_0(u)\varphi_T dxdt + \int_{D_T} F_*\varphi_T dxdt. \tag{5.11}$$

From (5.1), (5.9) and (5.11) it follows that

$$\int_{D_T} v\Box\varphi_T dxdt \geq \int_{D_T} [c_1|v|^\beta - c_0]\varphi_T dxdt + \int_{D_T} F_*\varphi_T dxdt$$
$$= c_1\int_{D_T} |v|^\beta\varphi_T dxdt + \chi(T), \tag{5.12}$$

where due to (5.2) and (5.9)

$$\chi(T) = \int_{D_T} (F_* - c_0)\varphi_T dxdt = \int_{D_T} F_0\varphi_T dxdt \geq 0. \tag{5.13}$$

In view of (5.2) there exists a number $T_1 = T_1(F) > 0$ such that

$$F_0(x,t) \geq \frac{c_2}{2}t^{-\gamma}, \quad t > T_1. \tag{5.14}$$

By virtue of (5.8) and (5.14), after substitution of variables $t = Tt'$, $x = Tx'$ in the integral (5.13), for $T > 2T_1$ we have

$$\chi(T) = T^{n+1}\int_{D_{T=1}} F_0(Tx', Tt')\varphi^0(x',t')dx'dt'$$
$$\geq T^{n+1}\int_{D_{T=1}\cap\left\{\frac{1}{2}<t'<1\right\}} F_0(Tx', Tt')\varphi^0(x',t')dx'dt'$$
$$\geq T^{n+1}\int_{D_{T=1}\cap\left\{\frac{1}{2}<t'<1\right\}} \frac{c_2}{2}(Tt')^{-\gamma}\varphi^0(x',t')dx'dt'$$
$$= \frac{c_2}{2}T^{n+1-\gamma}\int_{D_{T=1}\cap\left\{\frac{1}{2}<t'<1\right\}} (t')^{-\gamma}\varphi^0(x',t')dx'dt' = c_3T^{n+1-\gamma}, \quad T > 2T_1, \tag{5.15}$$

where due to $\varphi^0|_{D_{t=1}} > 0$

$$c_3 = \frac{c_2}{2}\int_{D_{T=1}\cap\left\{\frac{1}{2}<t'<1\right\}} (t')^{-\gamma}\varphi^0(x',t')dx'dt' = \text{const} > 0. \tag{5.16}$$

Since according to the conditions of the Theorem 5.1 the constant $\gamma \leq n+1$, then from (5.15) and (5.16) it follows

$$\lim_{T\to+\infty} \inf \chi(T) \geq c_3. \tag{5.17}$$

Further, in view of (5.13) rewrite the inequality (5.12) in the following form

$$c_1\int_{D_T} |v|^\beta\varphi_T dxdt \leq \int_{D_T} v\Box\varphi_T dxdt - \chi(T). \tag{5.18}$$

If in Young's inequality with the parameter $\varepsilon > 0 : ab \leq (\varepsilon/\beta)a^\beta + (\beta'\varepsilon^{\beta'-1})^{-1}b^{\beta}$, where $\beta' = \beta/(\beta-1)$, we take $a = |u|\varphi_T^{1/\beta}$, $b = |\Box\varphi_T|/\psi^{1/\beta}$, then taking into account equality $\beta'/\beta = \beta' - 1$, we have

$$|v\Box\varphi_T| = |v|\varphi_T^{\frac{1}{\beta}}\frac{|\Box\varphi_T|}{\varphi_T^{1/\beta}} \leq \frac{\varepsilon}{\beta}|v|^\beta\varphi_T + \frac{1}{\beta'\varepsilon^{\beta'-1}}\frac{|\Box\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}}. \tag{5.19}$$

In view of (5.19) from (5.18) we have

$$\left(c_1 - \frac{\varepsilon}{\beta}\right)\int_{D_T} |v|^\beta\varphi_T dxdt \leq \frac{1}{\beta'\varepsilon^{\beta'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}}dxdt - \chi(T),$$

whence for $\varepsilon < c_1\beta$ we receive

$$\int_{D_T} |v|^\beta\varphi_T dxdt \leq \frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}}dxdt - \frac{\beta}{c_1\beta-\varepsilon}\chi(T). \tag{5.20}$$

Taking into account equalities $\beta' = \beta/(\beta-1)$, $\beta = \beta'/(\beta'-1)$ and also equality

$$\min_{0<\varepsilon<c_1\beta}\frac{\beta}{(c_1\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}} = \frac{1}{c_1^{\beta'}},$$

which is reached for $\varepsilon = c_1$, then from (5.20) it follows that

$$\int_{D_T} |v|^\beta \varphi_T dxdt \le \frac{1}{c_1^{\beta'}} \int_{D_T} \frac{|\Box \varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}} dxdt - \frac{\beta'}{c_1} \chi(T). \tag{5.21}$$

By virtue of (5.6)–(5.8) after substitution of variables $x = Tx',\ t = Tt'$ one may easily verify that

$$\int_{D_T} \frac{|\Box \varphi_T|^{\beta'}}{\varphi_T^{\beta'-1}} dxdt = T^{n+1-2\beta'} \int_{D_{T=1}} \frac{|\Box \varphi^0|^{\beta'}}{(\varphi^0)^{\beta'-1}} dx'dt' = T^{n+1-2\beta'} æ_0 < +\infty.$$

Whence due to (5.9) from the inequality (5.21) we have

$$0 \le \int_{D_T} |v|^\beta \varphi_T dxdt \le \frac{1}{c_1^{\beta'}} T^{n+1-2\beta'} æ_0 - \frac{\beta'}{c_1} \chi(T). \tag{5.22}$$

Since by supposition $\beta < \frac{n+1}{n-1}$, then $n + 1 - 2\beta' < 0$ and, therefore,

$$\lim_{T \to +\infty} \frac{1}{c_1^{\beta'}} T^{n+1-2\beta'} æ_0 = 0. \tag{5.23}$$

From (5.16), (5.17) and (5.23) it follows that there exists a positive number $T_0 = T_0(F)$ such that for $T > T_0$ the right side of the inequality (5.22) will be a negative value, which is impossible. This means that if for the conditions of the Theorem 5.1 there exists a strong generalized solution of the problem (5.1), (5.2) of the class $W_2^1$ in the domain $D_T$, then $T \le T_0$ necessarily, which proves the Theorem 5.1. □

**Remark 5.1.** Let us consider one class of vector-functions $f$, satisfying the condition (5.1):

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^N a_{ij} |u_j|^{\beta_{ij}} + b_i, \quad i = 1, \dots, N, \tag{5.24}$$

where $a_{ij} = \text{const} > 0$, $b_i = \text{const}$, $1 < \beta_{ij} = \text{const} < \frac{n+1}{n-1}$; $i, j = 1, \dots, N$. In this case we can assume that $l_1 = l_2 = \cdots = l_N = -1$. Indeed, let us choose $\beta = \text{const}$ in such a way that $1 < \beta < \beta_{ij}$; $i, j = 1, \dots, N$. Then it is easy to verify that $|s|^{\beta_{ij}} \ge |s|^\beta - 1\ \forall s \in (-\infty, +\infty)$. Using the inequality [1]

$$\sum_{i=1}^N |y_i|^\beta \ge N^{1-\beta} \left| \sum_{i=1}^N y_i \right|^\beta \quad \forall y = (y_1, \dots, y_N) \in \mathbb{R}^N,\ \beta = \text{const} > 1,$$

we have

$$\begin{aligned} \sum_{i=1}^N f_i(u_1, \dots, u_N) &\ge a_0 \sum_{i,j=1}^N |u_j|^{\beta_{ij}} + \sum_{i=1}^N b_i \ge a_0 \sum_{i,j=1}^N (|u_j|^\beta - 1) + \sum_{i=1}^N b_i \\ &= a_0 N \sum_{j=1}^N |u_j|^\beta - a_0 N^2 + \sum_{i=1}^N b_i \ge a_0 N^{2-\beta} \left| \sum_{j=1}^N u_j \right|^\beta + \sum_{i=1}^N b_i - a_0 N^2, \end{aligned}$$

$$a_0 = \min_{i,j} a_{ij} > 0.$$

Whence the inequality (5.1) follows where

$$l_1 = l_2 = \cdots = l_N = -1, \qquad c_0 = a_0 N^2 - \sum_{i=1}^N b_i, \qquad c_1 = a_0 N^{2-\beta} > 0.$$

Note that the vector-function $f$ represented by the equalities (5.24) also satisfies the condition (5.1) for $l_1 = l_2 = \cdots = l_N = -1$ for less restrictive conditions, when $a_{ij} = \text{const} \ge 0$, but $a_{ik_i} > 0$, where $k_1, \dots, k_N$ represents an arbitrary fixed permutation of the numbers $1, 2, \dots, N$; $i, j = 1, \dots, N$.

When $N = n = 2$, $f_1 = a_{11}|u_1|^\gamma + a_{12}|u_2|^\beta$, $f_2 = a_{21}|u_1|^\gamma + a_{22}|u_2|^\beta$, $1 < \gamma, \beta < 3$, the restrictions $a_{ij} > 0$ can be removed and changed by the condition $\det(a_{ij}) \ne 0$. E.g., for $f_1 = u_1^2 - 2u_2^2$, $f_2 = -2u_1^2 + u_2^2$, the condition (5.1) for $l_1 = l_2 = 1$, $\beta = 2$, $c_0 = 0$ and $c_1 = \frac{1}{2}$ will be valid, since in this case $l_1 f_1(u) + l_2 f_2(u) = -(|u_1|^2 + |u_2|^2) \le -\frac{1}{2}|u_1 + u_2|^2$ and from the Theorem 5.1 we have that for $F_1 + F_2 \ge \frac{c}{t^\gamma}$, $t \ge 1$, where $c = \text{const} > 0$ and $\gamma = \text{const} \le 3$, $g = 0$ the considering boundary value problem is not globally solvable. More precisely, from (5.17) and (5.22) it follows that

$$0 \le \int_{D_T} |v|^\beta \varphi_T dxdt \le \frac{1}{c_1^{\beta'}} T^{n+1-2\beta'} æ_0 - \frac{\beta'}{c_1} c_3,$$

which right hand side becomes negative for $T > T_0 = \max([æ_0^{-1}\beta' c_1^{\beta'-1} c_3]^{\frac{1}{n+1-2\beta'}}, 1)$ and, therefore, for $T > T_0$ the problem (1.1), (1.2) does not have a solution. But for this concrete example, $n = 2$, $\beta = \beta' = 2$; $æ_0$ is determined from (5.7). The constants $c_1$, $c_2$ and $c_3$ are determined from (5.1), (5.2) and (5.16), respectively, and, therefore, in this case $c_1 = \frac{1}{2}$ and $T_0 = \frac{æ_0}{c_3}$. Further, due to the Theorem 3.2 on the local solvability and the Theorem 4.1 on the uniqueness of the solution of this problem there exist finite positive number $T_* = T_*(F)$ and unique vector-function $u = (u_1, u_2) \in W^1_{2,\mathrm{loc}}(D_{T_*})$, such that $u$ is a strong generalized solution of this problem of the class $W^1_2$ in the domain $D_T$ for $T < T_*$. From the aforesaid it follows that for the life-span $T_*$ of this solution we have the upper estimate $T_* \le T_0 = \max(\frac{æ_0}{c_3}, 1)$. The lower estimate for $T_*$ can be received from considerations given in the proof of the Theorem 3.2 on the local solvability.

**Remark 5.2.** From the Theorem 5.1 it follows that when its conditions are fulfilled, then the problem (1.1), (1.2) cannot have a global strong generalized solution of the class $W^1_2$ in the domain $D_\infty$ in the sense of the Definition 2.4.

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