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The Cauchy–Goursat Problem for Wave Equations with Nonlinear Dissipative Term

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Abstract—The Cauchy–Goursat problem for wave equations with nonlinear dissipative term is studied. The existence, uniqueness, and blow-up of global solutions of this problem are considered. The local solvability of this problem is also discussed.

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1. STATEMENT OF THE PROBLEM

In the plane of independent variables x and t, let us consider the following wave equation with nonlinear dissipative term [1, p. 57 (Russian transl.)], [2]

$$Lu := u_{tt} - u_{xx} + g(x,t,u)u_t = f(x,t), \tag{1.1}$$

where f and g are given functions, and u is an unknown real function.

Let $D_T := \{(x,t) : 0 < x < t,\ 0 < t < T\}$ denote the triangular domain bounded by the characteristic closed interval $\gamma_{1,T}$: $x = t$, $0 \le t \le T$, as well as by the closed intervals $\gamma_{2,T}$: $x = 0$, $0 \le t \le T$ and $\gamma_{3,T}$: $t = T$, $0 \le x \le T$. For Eq. (1.1), consider the Cauchy–Goursat problem of finding the solution $u(x,t)$ in the domain D_T, subject to the conditions [3, p. 284]

$$u_x|_{\gamma_{2,T}} = 0, \qquad u|_{\gamma_{1,T}} = 0. \tag{1.2}$$

Note that, for hyperbolic-type nonlinear equations, the questions of the existence, uniqueness, and blow-up of global solutions of initial, mixed, nonlocal, and other problems were studied in numerous papers (see, for example, [4]–[18]). It is well known that, in the linear case, i.e., for $g(x,t,u) = g(x,t)$, problem (1.1), (1.2), is well posed and its global solvability was established in the corresponding function spaces (see, for example, [1], [19]–[23]).

In what follows, it will be shown that, under certain conditions on the nonlinear function $g(x,t,u)$, problem (1.1), (1.2) is locally solvable; we shall also obtain conditions for the global solvability whose violation, in general, may lead to the blow-up of the solution in finite time.

Definition 1.1. Let $f \in C(\overline{D}_T)$, and let $g \in C(\overline{D}_T \times \mathbb{R})$, where $\mathbb{R} := (-\infty, +\infty)$. A function u is called a *strong generalized solution of Problem* (1.1), (1.2) of class C^1 in the domain D_T if $u \in C^1(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that $u_n \to u$ and $Lu_n \to f$ as $n \to \infty$ in the spaces $C^1(\overline{D}_T)$ and $C(\overline{D}_T)$, respectively, where

$$\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T) := \{v \in C^2(\overline{D}_T) : v_x|_{\gamma_{2,T}} = 0,\ v|_{\gamma_{1,T}} = 0\}, \qquad \Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}.$$

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Remark 1.1. Obviously, the classical solution of problem (1.1), (1.2) from the space $u \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a strong generalized solution of this problem of class C^1 in the domain D_T. In turn, if the strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T belongs to the space $C^2(\overline{D}_T)$, then it will also be the classical solution of this problem.

Definition 1.2. Let $f \in C(\overline{D}_\infty)$, and let $g \in C(\overline{D}_\infty \times \mathbb{R})$. We say that problem (1.1), (1.2) is *globally solvable for the class* C^1 if, for any finite $T > 0$, this problem has a strong generalized solution of class C^1 in the domain D_T.

2. A PRIORI ESTIMATES OF THE SOLUTION OF PROBLEM (1.1), (1.2) FOR THE CLASSES $C(\overline{D}_T)$, $C^1(\overline{D}_T)$

Lemma 2.1. *Let $f \in C(\overline{D}_T)$, let $g \in C(\overline{D}_T \times \mathbb{R})$, and let*

$$g(x,t,s) \geq -M_T, \qquad (x,t,s) \in \overline{D}_T \times \mathbb{R}, \quad M_T := \mathrm{const} > 0. \tag{2.1}$$

Then, in the domain D_T, the strong generalized solution of problem (1.1), (1.2) *of class C^1 satisfies the a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \leq c_0 \|f\|_{C(\overline{D}_T)}, \tag{2.2}$$

where $c_0 = c_0(T, M_T)$ is a positive constant independent of u and f.

Proof. Let u be a strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T. Then, in view of Definition 1.1, there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C^1(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_T)} = 0, \tag{2.3}$$

and, therefore, also

$$\lim_{n\to\infty} \|g(x,t,u_n)u_{nt} - g(x,t,u)u_t\|_{C(\overline{D}_T)} = 0. \tag{2.4}$$

Consider the function $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ as the solution of the following problem:

$$Lu_n = f_n, \tag{2.5}$$

$$\left.\frac{\partial u_n}{\partial x}\right|_{\gamma_{2,T}} = 0, \qquad u_n|_{\gamma_{1,T}} = 0. \tag{2.6}$$

Here

$$f_n := Lu_n. \tag{2.7}$$

Multiplying both sides of relation (2.5) by $\partial u_n/\partial t$ and integrating the resulting equality over the domain $D_\tau := \{(x,t) \in D_T : 0 < t < \tau\}$, $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt - \int_{D_\tau} \frac{\partial^2 u_n}{\partial x^2}\frac{\partial u_n}{\partial t}\,dx\,dt + \int_{D_\tau} g(x,t,u_n)\left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt$$
$$= \int_{D_\tau} f_n \frac{\partial u_n}{\partial t}\,dx\,dt.$$

Set $\Omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \leq T$. Then, in view of (2.6), applying Green's formula to the left-hand side of the last equality, we can write

$$\int_{D_\tau} f_n \frac{\partial u_n}{\partial t}\,dx\,dt = \int_{\gamma_{1,\tau}} \frac{1}{2\nu_t}\left[\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2(\nu_t^2 - \nu_x^2)\right] ds$$
$$+ \frac{1}{2}\int_{\Omega_\tau}\left[\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2\right] dx + \int_{D_\tau} g(x,t,u_n)\left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt, \tag{2.8}$$

where $\nu := (\nu_x, \nu_t)$ is the unit vector of the outer/exterior normal to ∂D_τ and $\gamma_{1,\tau} := \gamma_{1,T} \cap \{t \le \tau\}$. Since the operator $\nu_t\,(\partial/\partial x) - \nu_x\,(\partial/\partial t)$ is the inner differential operator on $\gamma_{1,T}$, in view of the second condition from (2.6), it follows that

$$\left(\frac{\partial u_n}{\partial x}\,\nu_t - \frac{\partial u_n}{\partial t}\,\nu_x\right)\bigg|_{\gamma_{1,\tau}} = 0. \tag{2.9}$$

Further, it is easy to see that

$$(\nu_t^2 - \nu_x^2)|_{\gamma_{1,\tau}} = 0. \tag{2.10}$$

Therefore, from (2.8)–(2.10) we obtain

$$w_n(\tau) := \int_{\Omega_\tau}\left[\left(\frac{\partial u_n}{\partial x}\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2\right] dx \le 2\int_{D_\tau} f_n \frac{\partial u_n}{\partial t}\,dx\,dt + 2M_T\int_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt. \tag{2.11}$$

Taking into account the inequality

$$2f_n\,\frac{\partial u_n}{\partial t} \le \left(\frac{\partial u_n}{\partial t}\right)^2 + f_n^2$$

and using (2.11), we obtain

$$w_n(\tau) \le (1 + 2M_T)\int_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt + \int_{D_\tau} f_n^2\,dx\,dt.$$

Hence, using the expression for the function $w_n(\tau)$, we can write

$$w_n(\tau) \le m_T\int_0^\tau w_n(\sigma)\,d\sigma + \|f_n\|^2_{L_2(D_\tau)},$$

where $m_T := \max(1 + 2M_T, 1)$. Hence, since $\|f_n\|^2_{L_2(D_\tau)}$ is a nondecreasing function of τ, by Gronwall's lemma [24, p. 13 (Russian transl.)], we have

$$w_n(\tau) \le \exp(m_T\tau)\|f_n\|^2_{L_2(D_\tau)}. \tag{2.12}$$

If $(x,t) \in \overline{D}_T$, then, in view of the second condition from (2.6), the following equality holds:

$$u_n(x,t) = u_n(x,t) - u_n(t,t) = \int_t^x \frac{\partial u_n(\sigma,t)}{\partial x}\,d\sigma;$$

hence, in view of (2.12) we have

$$\begin{aligned} |u_n(x,t)|^2 &\le \int_x^t d\sigma\int_x^t\left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \le (t-x)\int_{\Omega_t}\left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \\ &\le (t-x)w_n(t) \le tw_n(t) \le T\exp(m_T T)\|f_n\|^2_{C(\overline{D}_T)}\operatorname{mes} D_T \\ &= 2^{-1}T^3\exp(m_T T)\|f_n\|^2_{C(\overline{D}_T)}. \end{aligned} \tag{2.13}$$

Using (2.13), we obtain

$$\|u_n\|_{C(\overline{D}_T)} \le T\sqrt{\frac{T}{2}}\exp\left(\frac{m_T T}{2}\right)\|f_n\|_{C(\overline{D}_T)}.$$

Passing to the limit as $n \to \infty$ in this inequality and taking into account (2.3), (2.7), we can write

$$\|u\|_{C(\overline{D}_T)} \le T\sqrt{\frac{T}{2}}\exp\left(\frac{m_T T}{2}\right)\|f\|_{C(\overline{D}_T)}. \tag{2.14}$$

Estimate (2.2) is proved. □

Remark 2.1. It follows from (2.14) that the constant c_0 in estimate (2.2) can be taken to be

$$c_0 := T\sqrt{\frac{T}{2}}\exp\left(\frac{m_T T}{2}\right). \tag{2.15}$$

Below, using the classical method of characteristics and taking into account estimate (2.2), we obtain an a priori estimate in the space $C^1(\overline{D}_T)$ for the strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T.

The following statement is valid.

Lemma 2.2. *Under the assumptions of Lemma* 2.1*, the strong generalized solution of problem* (1.1)*,* (1.2) *of class C^1 satisfies the following a priori estimate in the domain D_T:*

$$\|u\|_{C^1(\overline{D}_T)} \le c_1 \tag{2.16}$$

with positive constant $c_1 = c_1(T, c_0, \|f\|_{C(\overline{D}_T)})$, where

$$\|u\|_{C^1(\overline{D}_T)} := \max\{\|u\|_{C(\overline{D}_T)}, \|u_x\|_{C(\overline{D}_T)}, \|u_t\|_{C(\overline{D}_T)}\}.$$

Proof. Let u be a strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T. Then we have the limit equalities (2.3), (2.4), where u_n can be regarded as the solution of problem (2.5), (2.6) with the right-hand side of f_n from (2.7). For a fixed natural number n, we introduce the following functions:

$$u_{n1} := u_{nt} - u_{nx}, \qquad u_{n2} := u_{nt} + u_{nx}, \qquad u_{n3} := u_n, \tag{2.17}$$

which, in view of (2.2), for $0 \le t \le T$, satisfy the boundary conditions

$$u_{n1}(0,t) = u_{n2}(0,t), \qquad u_{n2}(t,t) = 0, \qquad u_{n3}(t,t) = 0. \tag{2.18}$$

In view of (1.1) and (2.17), the unknown functions u_{n1}, u_{n2}, u_{n3} satisfy the following system of first-order partial differential equations:

$$\begin{cases} \dfrac{\partial u_{n1}}{\partial t} + \dfrac{\partial u_{n1}}{\partial x} = f_n(x,t) - \dfrac{1}{2}g(x,t,u_{n3})(u_{n1}+u_{n2}), \\ \dfrac{\partial u_{n2}}{\partial t} - \dfrac{\partial u_{n2}}{\partial x} = f_n(x,t) - \dfrac{1}{2}g(x,t,u_{n3})(u_{n1}+u_{n2}), \\ \dfrac{\partial u_{n3}}{\partial t} - \dfrac{\partial u_{n3}}{\partial x} = u_{n1}. \end{cases} \tag{2.19}$$

Integrating the equations of system (2.19) along the corresponding characteristic curves and taking into account the boundary conditions (2.18), we obtain

$$\begin{cases} u_{n1}(x,t) - u_{n1}(0,t) = \displaystyle\int_{t-x}^{t}\left[f_n(P_\tau) - \frac{1}{2}g(P_\tau, u_{n3}(P_\tau))(u_{n1}(P_\tau) + u_{n2}(P_\tau))\right]d\tau, \\ u_{n2}(x,t) = \displaystyle\int_{(x+t)/2}^{t}\left[f_n(Q_\tau) - \frac{1}{2}g(Q_\tau, u_{n3}(Q_\tau))(u_{n1}(Q_\tau) + u_{n2}(Q_\tau))\right]d\tau, \\ u_{n3}(x,t) = \displaystyle\int_{(x+t)/2}^{t} u_{n1}(Q_\tau)\,d\tau, \end{cases}$$

where $P_\tau := (x - t + \tau, \tau)$ and $Q_\tau := (x + t - \tau, \tau)$. Using the second equation of this system and the first equality from (2.18), as well as the notation $P_{\tau_0} := (t-\tau, \tau)$, we can rewrite this system as

$$\begin{cases} u_{n1}(x,t) = -\dfrac{1}{2}\displaystyle\int_{t-x}^{t} [g(P_\tau, u_{n3}(P_\tau))(u_{n1}(P_\tau) + u_{n2}(P_\tau))]\,d\tau \\ \qquad\qquad\qquad - \dfrac{1}{2}\displaystyle\int_{t/2}^{t} [g(P_{\tau_0}, u_{n3}(P_{\tau_0}))(u_{n1}(P_{\tau_0}) + u_{n2}(P_{\tau_0}))]\,d\tau + F_{n1}(x,t), \\ u_{n2}(x,t) = -\dfrac{1}{2}\displaystyle\int_{(x+t)/2}^{t} g(Q_\tau, u_{n3}(Q_\tau))(u_{n1}(Q_\tau) + u_{n2}(Q_\tau))\,d\tau + F_{n2}(x,t), \\ u_{n3}(x,t) = \displaystyle\int_{(x+t)/2}^{t} u_{n1}(Q_\tau)\,d\tau. \end{cases} \tag{2.20}$$

Here

$$F_{n1}(x,t) := \int_{t-x}^{t} f_n(P_\tau)\,d\tau + \int_{t/2}^{t} f_n(P_{\tau_0})\,d\tau, \qquad F_{n2}(x,t) := \int_{(x+t)/2}^{t} f_n(Q_\tau)\,d\tau. \tag{2.21}$$

Passing in relations (2.20), (2.21) to the limit as $n \to \infty$ in the space $C(\overline{D}_T)$ and taking into account relations (2.3), (2.4), (2.7), and (2.17), we obtain

$$\begin{cases} u_{1}(x,t) = -\dfrac{1}{2}\displaystyle\int_{t-x}^{t} [g(P_\tau, u_{3}(P_\tau))(u_{1}(P_\tau) + u_{2}(P_\tau))]\,d\tau \\ \qquad\qquad\qquad - \dfrac{1}{2}\displaystyle\int_{t/2}^{t} [g(P_{\tau_0}, u_{3}(P_{\tau_0}))(u_{1}(P_{\tau_0}) + u_{2}(P_{\tau_0}))]\,d\tau + F_{1}(x,t), \\ u_{2}(x,t) = -\dfrac{1}{2}\displaystyle\int_{(x+t)/2}^{t} g(Q_\tau, u_{3}(Q_\tau))(u_{1}(Q_\tau) + u_{2}(Q_\tau))\,d\tau + F_{2}(x,t), \\ u_{3}(x,t) = \displaystyle\int_{(x+t)/2}^{t} u_{1}(Q_\tau)\,d\tau, \end{cases} \tag{2.22}$$

where $u_i := \lim_{n\to\infty} u_{ni}$ (in the norm of the space $C(\overline{D}_T)$) $i = 1, 2, 3$, and

$$F_{1}(x,t) := \int_{t-x}^{t} f(P_\tau)\,d\tau + \int_{t/2}^{t} f(P_{\tau_0})\,d\tau, \qquad F_{2}(x,t) := \int_{(x+t)/2}^{t} f(Q_\tau)\,d\tau. \tag{2.23}$$

Obviously, $u_3 = u$ is a strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T. Further,

$$u_1 := u_t - u_x, \qquad u_2 := u_t + u_x. \tag{2.24}$$

Let $G_T := \{(x,t,s) \in \mathbb{R}^3 : (x,t) \in \overline{D}_T,\ |s| \le c_0 \|f\|_{C(\overline{D}_T)}\}$, and let

$$K := \sup_{(x,t,s)\in G_T} |g(x,t,s)| < +\infty, \tag{2.25}$$

where $K = K(T, c_0, \|f\|_{C(\overline{D}_T)})$. Then in view of the a priori estimate (2.2), the strong generalized solution $u_3 = u$ of problem (1.1), (1.2) of class C^1 in the domain D_T satisfies the estimate

$$|g(x,t,u_3(x,t))| \le K, \qquad (x,t) \in \overline{D}_T. \tag{2.26}$$

Let

$$v_i(t) := \sup_{(\xi,\tau)\in\overline{D}_t} |u_i(\xi,\tau)|, \quad i = 1, 2, 3, \qquad F(t) := \sup_{(\xi,\tau)\in\overline{D}_t} |f(\xi,\tau)|. \tag{2.27}$$

In view of (2.23), (2.26), and (2.27), relations (2.22) imply

$$|u_1(x,t)| \le K \int_0^t (v_1(\tau) + v_2(\tau))\,d\tau + 2tF(t),$$

$$|u_2(x,t)| \le \frac{K}{2}\int_0^t (v_1(\tau)+v_2(\tau))\,d\tau + tF(t),$$
$$|u_3(x,t)| \le \int_0^t v_1(\tau)\,d\tau.$$

Hence, for $(\xi,\tau)\in\overline{D}_t$, we have

$$|u_1(\xi,\tau)| \le K\int_0^\tau (v_1(\tau_1)+v_2(\tau_1))\,d\tau_1 + 2\tau F(\tau),$$
$$|u_2(\xi,\tau)| \le \frac{K}{2}\int_0^\tau (v_1(\tau_1)+v_2(\tau_1))\,d\tau_1 + \tau F(\tau),$$
$$|u_3(\xi,\tau)| \le \int_0^\tau v_1(\tau_1)\,d\tau_1$$

and, therefore, in view of (2.27) and the fact that $tF(t)$ is a nondecreasing function, we can write

$$v_1(t) \le K\int_0^t (v_1(\tau)+v_2(\tau))\,d\tau + 2tF(t),$$
$$v_2(t) \le \frac{K}{2}\int_0^t (v_1(\tau)+v_2(\tau))\,d\tau + tF(t),$$
$$v_3(t) \le \int_0^t v_1(\tau)\,d\tau.$$

Setting $v(t) := \max_{1\le i\le 3} v_i(t)$, from the above inequalities we obtain

$$v(t) \le 2K\int_0^t v(\tau)\,d\tau + 2tF(t),$$

whence, applying Gronwall's lemma, we have

$$v(t) \le 2tF(t)\exp(2tK) \le 2T\exp(2TK)\|f\|_{C(\overline{D}_T)}, \qquad 0\le t\le T.$$

Now it readily follows from (2.24) that

$$\|u\|_{C^1(\overline{D}_T)} \le \|v\|_{C[0,T]} \le 2T\exp(2TK)\|f\|_{C(\overline{D}_T)}.$$

Lemma 2.2 is proved and

$$c_1 := 2T\exp(2TK)\|f\|_{C(\overline{D}_T)}, \tag{2.28}$$

where K is given by (2.25). □

3. EQUIVALENCE OF PROBLEM (1.1), (1.2) TO A SYSTEM OF VOLTERRA-TYPE NONLINEAR INTEGRAL EQUATIONS AND ITS LOCAL SOLVABILITY

First, let us show that problem (2.5), (2.6) is equivalent to problem (2.19), (2.18) in the classical sense. Indeed, if $u_n\in C^2$ is a solution of problem (2.5), (2.6), then the system of functions u_{n1}, u_{n2}, and u_{n3} will, obviously, be a solution of problem (2.19), (2.18). Conversely, let $u_{n1}, u_{n2}, u_{n3}\in C^1$ be solutions of problem (2.19), (2.18). Let us show that $u_n := u_{n3}\in C^2$ is a solution of problem (2.5), (2.6) and satisfies relations (2.17). If we show that $u_{n2} = u_{nt}+u_{nx}$, then, obviously, we have the equalities

$$u_{nt} = \frac{u_{n2}+u_{n1}}{2} \qquad \text{and} \qquad u_{nx} = \frac{u_{n2}-u_{n1}}{2},$$

whence it immediately follows that $u_n\in C^2$ is a solution of problem (2.5), (2.6) in the classical sense.

Indeed, it follows from the first and second equations of system (2.19) that

$$\frac{\partial u_{n1}}{\partial t} + \frac{\partial u_{n1}}{\partial x} = \frac{\partial u_{n2}}{\partial t} - \frac{\partial u_{n2}}{\partial x}. \tag{3.1}$$

Further, because $u_{n1} \in C^1$, it follows from the third equation of system (2.19) that

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} \in C \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} \in C.$$

Hence, taking into account the fact that the first-order differential operators with constant coefficients are interchangeable, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}u_{n3} \in C,$$
$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n3} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}u_{n3} \in C.$$

In view of these equalities, (3.1), and the third equality of system (2.19), we can write

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)&(u_{n2} - u_{nt} - u_{nx}) \\ &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_{n2} - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_n - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u_n \\ &= \frac{\partial u_{n2}}{\partial t} - \frac{\partial u_{n2}}{\partial x} - \frac{\partial u_{n1}}{\partial t} - \frac{\partial u_{n1}}{\partial x} = 0.\end{aligned}$$

Therefore, in view of the second and the third equality from (2.18), we find that $u_{n2} = u_{nt} + u_{nx}$. This establishes the equivalence of problems (2.5), (2.6) and (2.19), (2.18) in the classical sense.

Above we have reduced problem (1.1), (1.2) to the system of Volterra-type nonlinear integral equations (2.22). Before considering the local solvability of problem (1.1), (1.2), let us make the following remark immediately following from the arguments given in Sec. 2.

Remark 3.1. Let u be a strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T; then $u_1 := u_t - u_x$, $u_2 := u_t + u_x$, $u_3 := u$ is a continuous solution of the system of Volterra-type nonlinear integral equations (2.22). Conversely, if u_1, u_2, u_3 is a continuous solution of system (2.22), then $u := u_3$ is a strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T, and the relations $u_1 := u_t - u_x$, $u_2 := u_t + u_x$ hold.

Now let us turn our attention to the proof of the local solvability of the system of Volterra-type nonlinear integral equations (2.22).

Let

$$f \in C(\overline{D}_\infty), \qquad f_\infty := \sup_{(x,t)\in\overline{D}_\infty} |f(x,t)| < +\infty, \qquad g \in C(\overline{D}_\infty \times \mathbb{R}) \tag{3.2}$$

and, for $(x,t) \in \overline{D}_\infty$ and s, s_1, s_2 such that $|s|, |s_1|, |s_2| \le R$, we have

$$|g(x,t,s)| \le M(R), \qquad |g(x,t,s_2) - g(x,t,s_1)| \le c(R)|s_2 - s_1|, \tag{3.3}$$

where $M(R)$ and $c(R)$ are nonnegative continuous functions of the argument $R \ge 0$.

Theorem 3.1. *Let the functions f and g satisfy conditions* (3.2), (3.3). *Then there exists a positive number $T_* := T_*(f,g)$ such that, for $T \le T_*$, problem* (1.1), (1.2) *has at least one strong generalized solution u of class C^1 in the domain D_T.*

Proof. By Remark 3.1, problem (1.1), (1.2) in the space $C^1(\overline{D}_T)$ is equivalent to the system of Volterra-type nonlinear integral equations (2.22) for the class $C(\overline{D}_T)$. To prove the unique solvability of system (2.22), we use the contraction mapping principle [26, p. 390].

Set $U := (u_1, u_2, u_3)$. Let us introduce the vector operator $\Phi := (\Phi_1, \Phi_2, \Phi_3)$, by the formula

$$\begin{cases} (\Phi_1 U)(x,t) = -\dfrac{1}{2}\displaystyle\int_{t-x}^{t} g(P_\tau, u_3(P_\tau))(u_1(P_\tau) + u_2(P_\tau))\,d\tau \\ \qquad\qquad\qquad + \dfrac{1}{2}\displaystyle\int_{t/2}^{t} g(P_{\tau 0}, u_3(P_{\tau 0}))(u_1(P_{\tau 0}) + u_2(P_{\tau 0}))\,d\tau + F_1(x,t), \\ (\Phi_2 U)(x,t) = -\dfrac{1}{2}\displaystyle\int_{(x+t)/2}^{t} g(Q_\tau, u_3(Q_\tau))(u_1(Q_\tau) + u_2(Q_\tau))\,d\tau + F_2(x,t), \\ (\Phi_3 U)(x,t) = \displaystyle\int_{(x+t)/2}^{t} u_1(Q_\tau)\,d\tau. \end{cases} \tag{3.4}$$

Then system (2.22) can be rewritten in vector form:

$$U = \Phi U. \tag{3.5}$$

Let

$$\|U\|_{X_T} := \max_{1\le i\le 3}\{\|u_i\|_{C(\overline{D}_T)}\}, \qquad U \in X_T := C(\overline{D}_T; \mathbb{R}^3),$$

where $C(\overline{D}_T; \mathbb{R}^3)$ is the set of continuous vector functions $U\colon \overline{D}_T \to \mathbb{R}^3$. Let

$$B_R := \{U \in X_T : \|U\|_{X_T} \le R\}$$

denote the closed ball of radius $R > 0$ in the Banach space X_T centered at the zero element.

Below we shall prove that

(1) Φ maps the ball B_R into itself;

(2) Φ is a contraction mapping on B_R.

Indeed, in view of the first inequality (3.3) from (3.4), for U such that $\|U\|_{X_T} \le R$, we have

$$|(\Phi_1 U)(x,t)| \le 2T(RM(R) + \|f\|_{C(\overline{D}_T)}),$$
$$|(\Phi_2 U)(x,t)| \le T(RM(R) + \|f\|_{C(\overline{D}_T)}), \qquad |(\Phi_3 U)(x,t)| \le TR.$$

It follows from these estimates that

$$\|\Phi U\|_{X_T} \le 2T(RM(R) + R + \|f\|_{C(\overline{D}_T)}) \le 2T(RM(R) + R + f_\infty),$$

where f_∞ is defined in (3.2).

For a fixed $R > 0$, let the number T be small so that

$$2T(RM(R) + R + f_\infty) \le R, \tag{3.6}$$

i.e., $\Phi U \in B_R$, and thus condition (1) holds.

Further, in view of (3.3) from (3.4), for U^i such that $\|U^i\|_{X_T} \le R$, $i = 1, 2$, we have

$$\begin{aligned} &|(\Phi_1 U^2 - \Phi_1 U^1)(x,t)| \\ &\quad\le \frac{1}{2}\int_{t-x}^{t} \big(|g(P_\tau, u_3^2(P_\tau)) - g(P_\tau, u_3^1(P_\tau))||u_1^2(P_\tau) + u_2^2(P_\tau)| \\ &\qquad\qquad + |g(P_\tau, u_3^1(P_\tau))||u_1^2(P_\tau) - u_1^1(P_\tau) + u_2^2(P_\tau) - u_2^1(P_\tau)|\big)\,d\tau \\ &\qquad + \frac{1}{2}\int_{t/2}^{t} \big(|g(P_{\tau 0}, u_3^2(P_{\tau 0})) - g(P_{\tau 0}, u_3^1(P_{\tau 0}))||u_1^2(P_{\tau 0}) + u_2^2(P_{\tau 0})| \\ &\qquad\qquad + |g(P_{\tau 0}, u_3^1(P_{\tau 0}))||u_1^2(P_{\tau 0}) - u_1^1(P_{\tau 0}) + u_2^2(P_{\tau 0}) - u_2^1(P_{\tau 0})|\big)\,d\tau \\ &\quad\le 2T[Rc(R) + M(R)]\|U^2 - U^1\|_{X_T}. \end{aligned}$$

Similarly,

$$
\begin{aligned}
&|(\Phi_2 U^2 - \Phi_2 U^1)(x,t)| \\
&\qquad \le \frac{1}{2}\int_{(x+t)/2}^{t} \big(|g(Q_\tau, u_3^2(Q_\tau)) - g(Q_\tau, u_3^1(Q_\tau))||u_1^2(Q_\tau) + u_2^2(Q_\tau)| \\
&\qquad\qquad\qquad + |g(Q_\tau, u_3^1(Q_\tau))||u_1^2(Q_\tau) - u_1^1(Q_\tau) + u_2^2(Q_\tau) - u_2^1(Q_\tau)|\big)\, d\tau \\
&\qquad \le T[Rc(R) + M(R)]\|U^2 - U^1\|_{X_T},
\end{aligned}
$$

$$
|(\Phi_3 U^2 - \Phi_3 U^1)(x,t)| \le \int_{(x+t)/2}^{t} |u_1^2(Q_\tau) - u_1^1(Q_\tau)|\, d\tau \le T\|U^2 - U^1\|_{X_T}.
$$

For a fixed $R > 0$, let the number T be small so that

$$
\max\{T, 2T(Rc(R) + M(R))\} \le \frac{1}{2} < 1, \tag{3.7}
$$

and thus

$$
\|\Phi U^2 - \Phi U^1\|_{X_T} \le \frac{1}{2}\|U^2 - U^1\|_{X_T}.
$$

Thus, the operator Φ is a contraction mapping on the set B_R, i.e., condition (2) holds.

In turn, it follows from (3.6) and (3.7) that if $0 < T \le T_*$, where

$$
T_* := \min\left\{\frac{R}{2(RM(R) + R + f_\infty)}, \frac{1}{2}, \frac{1}{4(Rc(R) + M(R))}\right\}, \tag{3.8}
$$

then

$$
\|\Phi U\|_{X_T} \le R \qquad \text{and} \qquad \|\Phi U^2 - \Phi U^1\|_{X_T} \le \frac{1}{2}\|U^2 - U^1\|_{X_T} \quad \text{for } U, U^1, U^2 \in B_R.
$$

Therefore, in view of the contraction mapping principle, there exists a solution U of Eq. (3.5) in the space $C(\overline{D}_T; \mathbb{R}^3)$. Theorem 3.1 is proved. $\square$

4. THE CASE OF THE GLOBAL SOLVABILITY OF PROBLEM (1.1), (1.2)

The following statement is valid.

Theorem 4.1. *Let conditions* (2.1), (3.2), *and* (3.3) *hold. Then, for any* $T > 0$, *problem* (1.1), (1.2) *has a strong generalized solution of class* C^1 *in the domain* D_T.

Proof. As was noted in Remark 3.1, problem (1.1), (1.2) for the class $C^1(\overline{D}_T)$ is equivalent to the system of nonlinear integral equations (2.22) for the class $C(\overline{D}_T)$. In view of (3.2), (3.3), the validity of this theorem for sufficiently small T, namely, for $T \le T_*$, where T_* is given by equality (3.8), follows from Theorem 3.1. Now suppose that $T > T_*$ and $U^{T_*} := (u_1^{T_*}, u_2^{T_*}, u_3^{T_*})$ is a solution of the system of nonlinear integral equations (2.22) or, equivalently, of the vector equation (3.5) in the domain D_{T_*} of class $C(\overline{D}_{T_*})$ by Theorem 3.1. For $t > \Delta t_1 := T_*$, system (2.22) can be rewritten as

$$
\begin{cases}
u_1(x,t) = -\dfrac{1}{2}\displaystyle\int_{\alpha_1(x,t,\Delta t_1)}^{t} g(P_\tau, u_3(P_\tau))(u_1(P_\tau) + u_2(P_\tau))\, d\tau \\
\qquad\qquad + \dfrac{1}{2}\displaystyle\int_{\alpha_2(x,t,\Delta t_1)}^{t} g(P_{\tau_0}, u_3(P_{\tau_0}))(u_1(P_{\tau_0}) + u_2(P_{\tau_0}))\, d\tau + F_{1,\Delta t_1}(x,t), \\
u_2(x,t) = -\dfrac{1}{2}\displaystyle\int_{\alpha_3(x,t,\Delta t_1)}^{t} g(Q_\tau, u_3(Q_\tau))(u_1(Q_\tau) + u_2(Q_\tau))\, d\tau + F_{2,\Delta t_1}(x,t), \\
u_3(x,t) = \displaystyle\int_{\alpha_3(x,t,\Delta t_1)}^{t} u_1(Q_\tau)\, d\tau + F_{3,\Delta t_1}(x,t),
\end{cases} \tag{4.1}
$$

where

$$\alpha_1(x,t,\Delta t_1) := \max(\Delta t_1, t-x), \qquad \alpha_2(x,t,\Delta t_1) := \max\left(\Delta t_1, \frac{t}{2}\right),$$
$$\alpha_3(x,t,\Delta t_1) := \max\left(\Delta t_1, \frac{x+t}{2}\right);$$

$$\begin{cases} F_{1,\Delta t_1}(x,t) := -\dfrac{1}{2}\displaystyle\int_{t-x}^{\alpha_1(x,t,\Delta t_1)} g(P_\tau, u_3^{T_*}(P_\tau))(u_1^{T_*}(P_\tau) + u_2^{T_*}(P_\tau))\,d\tau \\ \qquad\qquad + \dfrac{1}{2}\displaystyle\int_{t/2}^{\alpha_2(x,t,\Delta t_1)} g(P_{\tau 0}, u_3^{T_*}(P_{\tau 0}))(u_1^{T_*}(P_{\tau 0}) + u_2^{T_*}(P_{\tau 0}))\,d\tau + F_1(x,t), \\ F_{2,\Delta t_1}(x,t) := -\dfrac{1}{2}\displaystyle\int_{(x+t)/2}^{\alpha_3(x,t,\Delta t_1)} g(Q_\tau, u_3^{T_*}(Q_\tau))(u_1^{T_*}(Q_\tau) + u_2^{T_*}(Q_\tau))\,d\tau + F_2(x,t), \\ F_{3,\Delta t_1}(x,t) := \displaystyle\int_{(x+t)/2}^{\alpha_3(x,t,\Delta t_1)} u_1^{T_*}(Q_\tau)\,d\tau. \end{cases} \tag{4.2}$$

Since the assumptions of Lemma 2.2 hold, it follows that, for any positive $\tau \le T$, the solution of the vector equation (3.5) in the domain D_τ of class $C(\overline{D}_\tau)$ satisfies the a priori estimate

$$\|U\|_{C(\overline{D}_\tau)} \le R^T(\|f\|_{C(\overline{D}_\tau)}), \tag{4.3}$$

where $R^T = R^T(s)$ is a nondecreasing continuous function of the argument $s \ge 0$. Set

$$R_* := R^T(\|f\|_{C(\overline{D}_T)}).$$

In the second step with respect to t, for Δt_2 we take

$$\Delta t_2 := \min\left\{\frac{1}{4M(R_1)R_1}, \frac{1}{4c(R_1)R_1}\right\}, \tag{4.4}$$

where

$$R_1 := 1 + 2TM(R_*)R_* + \|F\|_{C(\overline{D}_T)}, \qquad F := (F_1, F_2, F_3). \tag{4.5}$$

For $t \in [T_*, T_* + \Delta t_2]$, the system of equations (4.1) can be rewritten as the single vector equation

$$U = \Psi U, \tag{4.6}$$

where the operator $\Psi := (\Psi_1, \Psi_2, \Psi_3)$ acts by the formula

$$\begin{cases} (\Psi_1 U)(x,t) = -\dfrac{1}{2}\displaystyle\int_{\alpha_1(x,t,\Delta t_1)}^{t} g(P_\tau, u_3(P_\tau))(u_1(P_\tau) + u_2(P_\tau))\,d\tau \\ \qquad\qquad + \dfrac{1}{2}\displaystyle\int_{\alpha_2(x,t,\Delta t_1)}^{t} g(P_{\tau 0}, u_3(P_{\tau 0}))(u_1(P_{\tau 0})) + u_2(P_{\tau 0}))\,d\tau + F_{1,\Delta t_1}(x,t), \\ (\Psi_2 U)(x,t) = -\dfrac{1}{2}\displaystyle\int_{\alpha_3(x,t,\Delta t_1)}^{t} g(Q_\tau, u_3(Q_\tau))(u_1(Q_\tau) + u_2(Q_\tau))\,d\tau + F_{2,\Delta t_1}(x,t), \\ (\Psi_3 U)(x,t) = \displaystyle\int_{\alpha_3(x,t,\Delta t_1)}^{t} u_1(Q_\tau)d\tau + F_{3,\Delta t_1}(x,t). \end{cases} \tag{4.7}$$

First, let us show that the operator Ψ takes the ball

$$B([T_1,T_2];R_1) := \{U \in C(\overline{D}_{T_1,T_2}) : \|U\|_{C(\overline{D}_{T_1,T_2})} \le R_1\}$$

into itself, where

$$T_1 = T_*, \qquad T_2 = T_* + \Delta t_2, \qquad \overline{D}_{T_1,T_2} := \overline{D} \cap \{T_1 \le t \le T_2\}.$$

Indeed, in view of (3.3), (4.2)–(4.5), and (4.7) we can write

$$\begin{aligned}\|\Psi_1 U\|_{C(\overline{D}_{T_1,T_2})} &\le 2M(R_1)R_1\Delta t_2 + 2M(R_*)R_*\Delta t_1 + \|F_1\|_{C(\overline{D}_T)}\\ &\le 2^{-1} + 2TM(R_*)R_* + \|F\|_{C(\overline{D}_T)} \le R_1.\end{aligned}$$

Similarly,

$$\|\Psi_i U\|_{C(\overline{D}_{T_1,T_2})} \le R_1, \qquad i=2,3,$$

and thus, finally, we have

$$\|\Psi U\|_{C(\overline{D}_{T_1,T_2})} \le R_1.$$

Let us now show that the operator Ψ is a contraction mapping in this ball. Indeed, for $(x,t)\in\overline{D}_{T_1,T_2}$, using (3.3), (4.4), and (4.7), we obtain

$$\begin{aligned}&|(\Psi_1U^2-\Psi_1U^1)(x,t)|\\ &\quad\le \frac12\int_{\alpha_1(x,t,\Delta t_1)}^{t}\big(|g(P_\tau,u_3^2(P_\tau))-g(P_\tau,u_3^1(P_\tau))||u_1^2(P_\tau)+u_2^2(P_\tau)|\\ &\qquad\qquad + |g(P_\tau,u_3^1(P_\tau))||u_1^2(P_\tau)-u_1^1(P_\tau)+u_2^2(P_\tau)-u_2^1(P_\tau)|\big)\,d\tau\\ &\qquad+\frac12\int_{\alpha_1(x,t,\Delta t_1)}^{t}\big(|g(P_{\tau_0},u_3^2(P_{\tau_0}))-g(P_{\tau_0},u_3^1(P_{\tau_0}))||u_1^2(P_{\tau_0})+u_2^2(P_{\tau_0})|\\ &\qquad\qquad + |g(P_{\tau_0},u_3^1(P_{\tau_0}))||u_1^2(P_{\tau_0})-u_1^1(P_{\tau_0})+u_2^2(P_{\tau_0})-u_2^1(P_{\tau_0})|\big)\,d\tau\\ &\quad\le 2c(R_1)R_1\Delta t_2\|u_3^2-u_3^1\|_{C(\overline{D}_{T_1,T_2})} + 2M(R_1)\Delta t_2\|U^2-U^1\|_{C(\overline{D}_{T_1,T_2})}\\ &\quad\le \frac12\|u_3^2-u_3^1\|_{C(\overline{D}_{T_1,T_2})} + \frac{1}{2R_1}\|U^2-U^1\|_{C(\overline{D}_{T_1,T_2})}\\ &\quad\le\left(\frac12+\frac{1}{2R_1}\right)\|U^2-U^1\|_{C(\overline{D}_{T_1,T_2})} = q_1\|U^2-U^1\|_{C(\overline{D}_{T_1,T_2})},\end{aligned}$$

where $q_1 := (1/2)(1+1/R_1) < 1$, because $R_1>1$ in view of (4.5). Similarly, we find that

$$|(\Psi_iU^2-\Psi_iU^1)(x,t)| \le q_i\|U^2-U^1\|_{C(\overline{D}_{T_1,T_2})}, \quad 0<q_i:=\mathrm{const}<1, \qquad i=2,3.$$

Hence, in view of the contraction mapping theorem, we establish the solvability of system (4.6) for the class $C(\overline{D}_{T_1,T_2})$. Continuing this process step by step and taking into account the fact that, in view of the global a priori estimate (4.3), the length of each step Δt_i is independent of its number i, we establish the global solvability of system (3.5), and hence also that of problem (1.1), (1.2) itself in the domain D_T for any $T>0$. □

5. UNIQUENESS OF THE SOLUTION OF PROBLEM (1.1), (1.2)

Lemma 5.1. *Let conditions* (3.2), (3.3) *hold. Then, for any $T>0$, problem* (1.1), (1.2) *cannot have more than one strong generalized solution of class C^1 in the domain D_T.*

Proof. Indeed, suppose that problem (1.1), (1.2) has two possible different strong generalized solutions u^1 and u^2 of class C^1 in the domain D_T. By definition 1.1, there exists a sequence of functions $u_n^i\in\mathring{C}^2(\overline{D}_T,\Gamma_T)$ such that

$$\lim_{n\to\infty}\|u_n^i-u^i\|_{C^1(\overline{D}_T)}=0, \quad \lim_{n\to\infty}\|Lu_n^i-f\|_{C(\overline{D}_T)}=0, \qquad i=1,2, \tag{5.1}$$

$$\lim_{n\to\infty}\|g(x,t,u_n^i)u_{nt}^i-g(x,t,u^i)u_t^i\|_{C(\overline{D}_T)}=0, \qquad i=1,2. \tag{5.2}$$

Let us use the well-known notation $\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$ and set $\omega_{nm} := u_n^2 - u_m^1$. It is easy to see that the function $\omega_{nm} \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ satisfies the following equalities:

$$\Box \omega_{nm} + g_{nm} = f_{nm}, \tag{5.3}$$

$$\frac{\partial \omega_{nm}}{\partial x}\bigg|_{\gamma_{2,T}} = 0, \qquad \omega_{nm}|_{\gamma_{1,T}} = 0, \tag{5.4}$$

where

$$g_{nm} := g(x,t,u_n^2)u_{nt}^2 - g(x,t,u_m^1)u_{mt}^1, \qquad f_{nm} := Lu_n^2 - Lu_m^1. \tag{5.5}$$

By the first equality from (5.1), there exists a number $A := \text{const} > 0$ independent of the indices i and n such that

$$\|u_n^i\|_{C^1(\overline{D}_T)} \le A. \tag{5.6}$$

By the second equalities from (5.1) and (5.5), we have

$$\lim_{n,m\to\infty} \|f_{nm}\|_{C(\overline{D}_T)} = 0. \tag{5.7}$$

In view of (3.2), (3.3), (5.6) and the first equality from (5.5), it is easy to see that

$$\begin{aligned} g_{nm}^2 &= \biggl(g(x,t,u_n^2)\frac{\partial \omega_{nm}}{\partial t} + (g(x,t,u_n^2) - g(x,t,u_m^1))u_{mt}^1\biggr)^2 \\ &\le 2M^2(A)\biggl(\frac{\partial \omega_{nm}}{\partial t}\biggr)^2 + 2A^2c^2(A)\omega_{nm}^2. \end{aligned} \tag{5.8}$$

Multiplying both sides of relation (5.3) by $\partial\omega_{nm}/\partial t$, integrating the resulting equality over the domain D_τ, and using (5.4), just as in the derivation of inequality (2.11), from (2.5), (2.6), we obtain

$$w_{nm}(\tau) := \int_{\Omega_\tau} \biggl[\biggl(\frac{\partial \omega_{nm}}{\partial x}\biggr)^2 + \biggl(\frac{\partial \omega_{nm}}{\partial t}\biggr)^2\biggr]\,dx = 2\int_{D_\tau} (f_{nm} - g_{nm})\frac{\partial \omega_{nm}}{\partial t}\,dx\,dt. \tag{5.9}$$

By estimate (5.8) and Cauchy's inequality, we have

$$\begin{aligned} 2\int_{D_\tau} &(f_{nm} - g_{nm})\frac{\partial \omega_{nm}}{\partial t}\,dx\,dt \\ &\le \int_{D_\tau} (f_{nm} - g_{nm})^2\,dx\,dt + \int_{D_\tau}\biggl(\frac{\partial \omega_{nm}}{\partial t}\biggr)^2\,dx\,dt \\ &\le 2\int_{D_\tau} f_{nm}^2\,dx\,dt + 2\int_{D_\tau} g_{nm}^2\,dx\,dt + \int_{D_\tau}\biggl(\frac{\partial \omega_{nm}}{\partial t}\biggr)^2\,dx\,dt \\ &\le 2\int_{D_\tau} f_{nm}^2\,dx\,dt + 4A^2c^2(A)\int_{D_\tau} \omega_{nm}^2\,dx\,dt + (1 + 4M^2(A))\int_{D_\tau}\biggl(\frac{\partial \omega_{nm}}{\partial t}\biggr)^2\,dx\,dt. \end{aligned} \tag{5.10}$$

Further, in view of the equality

$$\omega_{nm}(x,t) = \int_x^t \frac{\partial \omega_{nm}(x,\tau)}{\partial t}\,d\tau, \qquad (x,t) \in \overline{D}_T,$$

which follows from the second equality in (5.4), using standard arguments, we obtain the inequality [25, p. 63]

$$\int_{D_\tau} \omega_{nm}^2\,dx\,dt \le \tau^2\int_{D_\tau}\biggl(\frac{\partial \omega_{nm}}{\partial t}\biggr)^2\,dx\,dt. \tag{5.11}$$

It follows from (5.9)–(5.11) that

$$w_{nm}(\tau) \le (1 + 4M^2(A) + 4\tau^2A^2c^2(A))\int_{D_\tau}\biggl(\frac{\partial \omega_{nm}}{\partial t}\biggr)^2\,dx\,dt + 2\int_{D_\tau} f_{nm}^2\,dx\,dt$$

$$\leq (1 + 4M^2(A) + 4T^2c^2(A)) \int_0^\tau w_{nm}(\sigma)\, d\sigma + 2\int_{D_T} f_{nm}^2\, dx\, dt.$$

Therefore, by Gronwall's lemma [24, p. 13 (Russian transl.)], we can write

$$w_{nm}(\tau) \leq c_2 \|f_{nm}\|^2_{L_2(D_T)}, \qquad 0 < \tau \leq T, \tag{5.12}$$

where

$$c_2 := 2\exp(1 + 4M^2(A) + 4T^2A^2c^2(A))T.$$

Arguing in the same way as in the derivation of estimate (2.13), taking into account the obvious inequality

$$\|f_{nm}\|^2_{L_2(D_T)} \leq \|f_{nm}\|^2_{C(\overline{D}_T)} \operatorname{mes} D_T,$$

and using (5.12), for $(x, t) \in \overline{D}_T$, we can write

$$|\omega_{nm}(x,t)|^2 \leq t w_{nm}(t) \leq Tc_2 \operatorname{mes} D_T \|f_{nm}\|^2_{C(\overline{D}_T)} = \frac{c_2T^3}{2}\|f_{nm}\|^2_{C(\overline{D}_T)}.$$

This yields

$$\|\omega_{nm}\|_{C(\overline{D}_T)} \leq T\sqrt{\frac{c_2T}{2}}\|f_{nm}\|_{C(\overline{D}_T)}. \tag{5.13}$$

By the definition of the function ω_{nm} and, in view of the first equality, it is easy to see that

$$\lim_{n,m\to\infty} \|\omega_{nm}\|_{C^1(\overline{D}_T)} = \|u^2 - u^1\|_{C^1(\overline{D}_T)}$$

and, particularly,

$$\lim_{n,m\to\infty} \|\omega_{nm}\|_{C(\overline{D}_T)} = \|u^2 - u^1\|_{C(\overline{D}_T)}.$$

Therefore, in inequality (5.13), passing to the limit as $n, m \to \infty$ and using (5.7), we obtain

$$\|u^2 - u^1\|_{C(\overline{D}_T)} = 0, \qquad \text{i.e.,}\quad u^1 = u^2.$$

Lemma 5.1 is proved. □

6. THE CASE OF THE BLOW-UP OF THE GLOBAL SOLUTION OF PROBLEM (1.1), (1.2)

In this section, we show that the violation of condition (2.1) can lead to the blow-up of the global solution of problem (1.1), (1.2) in the sense of Definition 1.2. Indeed, let $g(x, t, s) = -|s|^\alpha s$, $s \in \mathbb{R}$, and let $\alpha > -1$ (nonlinearity exponent).

Lemma 6.1. *Let u be a strong generalized solution of problem* (1.1), (1.2) *of class C^1 in the domain D_T in the sense of Definition* 1.1. *Then the following integral equality holds*:

$$\int_{D_T} u\Box\varphi\, dx\, dt = \int_{D_T} |u|^\alpha u u_t \varphi\, dx\, dt + \int_{D_T} f\varphi\, dx\, dt \tag{6.1}$$

for any function φ such that

$$\varphi \in C^2(\overline{D}_T), \qquad \varphi|_{t=T} = 0, \qquad \varphi_t|_{t=T} = 0, \qquad \varphi_x|_{\gamma_{2,T}} = 0. \tag{6.2}$$

Proof. By the definition of a strong generalized solution u of problem (1.1), (1.2) of class C^1 in the domain D_T, the function u belongs to $C^1(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$ such that relations (2.5) and (2.6) hold for $g = -|s|^\alpha s$. Set $f_n := Lu_n$. Let us multiply both sides of the equality $Lu_n = f_n$ by the function φ, and integrate the resulting equality over the domain D_T. Integrating by parts the left-hand side of this equality, using (6.2) and conditions (1.2), we see that

$$\int_{D_T} u_n \Box \varphi \, dx\, dt = \int_{D_T} |u_n|^\alpha u_n u_{nt} \varphi \, dx\, dt + \int_{D_T} f_n \varphi \, dx\, dt.$$

In this equality, passing to the limit as $n \to \infty$ and using (2.5) and (2.6), we obtain (6.1). This lemma 6.1 is proved. $\square$

Consider a function $\varphi^0 := \varphi^0(x,t)$ such that

$$\begin{gathered} \varphi^0 \in C^2(\overline{D}_\infty), \qquad \varphi^0 + \varphi^0_t \le 0, \qquad \varphi^0|_{D_{T=1}} > 0, \\ \varphi^0_x|_{\gamma_{2,\infty}} = 0, \qquad \varphi^0|_{t \ge 1} = 0 \end{gathered} \tag{6.3}$$

and also the number

$$\kappa_0 := \int_{D_{T=1}} \frac{|\Box \varphi^0|^{p'}}{|\varphi^0|^{p'-1}} \, dx\, dt < +\infty, \qquad p' = \frac{\alpha+2}{\alpha+1}. \tag{6.4}$$

It is easy to verify that, for sufficiently large positive constants n and m, the function φ^0 satisfying conditions (6.3) and (6.4) can be taken as the function

$$\varphi^0(x,t) = \begin{cases} x^n(1-t)^m, & (x,t) \in D_{T=1}, \\ 0, & t \ge 1. \end{cases}$$

Set $\varphi_T(x,t) := \varphi^0(x/T, t/T)$, $T > 0$. In view of (6.3), it is easy to see that

$$\begin{gathered} \varphi_T \in C^2(\overline{D}_T), \qquad \varphi_T + T\frac{\partial \varphi_T}{\partial t} \le 0, \qquad \varphi_T|_{D_T} > 0, \\ \frac{\partial \varphi_T}{\partial x}\bigg|_{\gamma_{2,T}} = 0, \qquad \varphi_T|_{t=T} = 0, \qquad \frac{\partial \varphi_T}{\partial t}\bigg|_{t=T} = 0. \end{gathered} \tag{6.5}$$

For a given f, consider the function

$$\zeta(T) := \int_{D_T} f \varphi_T \, dx\, dt, \qquad T > 0. \tag{6.6}$$

The following theorem on the blow-up of the global solution of problem (1.1), (1.2) is valid.

Theorem 6.1. *Let*

$$g(x,t,s) = -|s|^\alpha s, \qquad s \in \mathbb{R}, \quad \alpha > -1,$$

let $f \in C(\overline{D}_\infty)$, and suppose that $f \ge 0$ in the domain D_∞. Then if

$$\liminf_{T \to +\infty} \zeta(T) > 0, \tag{6.7}$$

then there exists a positive number $T^* := T^*(f)$ such that, for $T > T^*$, problem (1.1), (1.2) cannot have a strong generalized solution u of class C^1 in the domain D_T.

Proof. Suppose that, under the assumptions of this theorem, there exists a strong generalized solution u of problem (1.1), (1.2) of class C^1 in the domain D_T. Then, by Lemma 6.1, equality (6.1) holds; in view of (6.5), φ in this equality can be taken as the function $\varphi = \varphi_T$ i.e.,

$$\int_{D_T} u \Box \varphi_T \, dx\, dt = \int_{D_T} |u|^\alpha u u_t \varphi_T \, dx\, dt + \int_{D_T} f \varphi_T \, dx\, dt. \tag{6.8}$$

By (1.2) and (6.5), we have

$$\int_{D_T} |u|^\alpha u u_t \varphi_T \,dx\,dt = \frac{1}{\alpha+2}\int_{D_T} \varphi_T \frac{\partial}{\partial t}|u|^{\alpha+2}\,dx\,dt$$
$$= -\frac{1}{\alpha+2}\int_{D_T}|u|^{\alpha+2}\frac{\partial \varphi_T}{\partial t}\,dx\,dt \geq \frac{1}{(\alpha+2)T}\int_{D_T}|u|^{\alpha+2}\varphi_T\,dx\,dt.$$

Hence, using (6.6), from (6.8) we obtain

$$\frac{1}{pT}\int_{D_T}|u|^p\varphi_T\,dx\,dt \leq \int_{D_T} u\Box\varphi_T\,dx\,dt - \zeta(T), \qquad p := \alpha+2 > 1. \tag{6.9}$$

If, in Young's inequality with parameter $\varepsilon > 0$,

$$ab \leq \frac{\varepsilon}{p}a^p + \frac{1}{p'\varepsilon^{p'-1}}b^{p'}, \qquad a, b \geq 0, \quad \frac{1}{p}+\frac{1}{p'} = 1, \quad p > 1,$$

we put

$$a = |u|\varphi_T^{1/p}, \qquad b = \frac{|\Box\varphi_T|}{\varphi_T^{1/p}},$$

then, using the equality $p'/p = p'-1$, we obtain

$$|u\Box\varphi_T| = |u|\varphi_T^{1/p}\frac{|\Box\varphi_T|}{\varphi_T^{1/p}} \leq \frac{1}{pT}|u|^p\varphi_T + \frac{T^{p'-1}}{p'}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}.$$

In view of (6.9) and the last inequality, we can write

$$\frac{1-\varepsilon}{p}\int_{D_T}\varphi_T\,dx\,dt \leq \frac{1}{p'\varepsilon^{p'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \zeta(T),$$

whence we have

$$\int_{D_T}|u|^p\varphi_T\,dx\,dt \leq \frac{p}{(1-\varepsilon)p'\varepsilon^{p'-1}}\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p}{1-\varepsilon}\zeta(T).$$

Taking into account the equalities $p' = p/(p-1)$, $p = p'/(p-1)$, and

$$\min_{0<\varepsilon<1}\frac{p}{(1-\varepsilon)p'\varepsilon^{p'-1}} = p^{p'},$$

where the minimum is realized at $\varepsilon = 1/p$, from the last inequality we obtain

$$\int_{D_T}|u|^p\varphi_T\,dx\,dt \leq p^{p'}\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p}{1-\varepsilon}\zeta(T). \tag{6.10}$$

Since $\varphi_T(x,t) := \varphi^0(x/T, t/T)$, using (6.3), (6.4) and making the change of variables $x = Tx_1$, $t = Tt_1$, we can easily verify the relations

$$\int_{D_T}\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt = T^{-2(p'-1)}\int_{D_{T=1}}\frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx_1\,dt_1 = T^{-2(p'-1)}\kappa_0.$$

Hence, using (6.5), from (6.10) we obtain

$$0 \leq \frac{\kappa_0}{p'T^{p'-1}} - \zeta(T). \tag{6.11}$$

Since $p' = p/(p-1) > 1$, it follows that $-2(p'-1) < 0$ and, in view of (6.4), we will have

$$\lim_{T\to+\infty}\frac{\kappa_0}{p'T^{p'-1}} = 0.$$

Therefore, in view of (6.7), there exists a positive number $T^* := T^*(f)$ such that, for $T > T^*$, the right-hand side of inequality (6.11) is negative, while the left-hand side of this inequality is nonnegative. This implies that if u is a strong generalized solution of problem (1.1), (1.2) of class C^1 in the domain D_T, then necessarily $T \le T^*$, which proves Theorem 6.1. $\square$

Remark 6.1. It is easy to verify that if

$$f \in C(\overline{D}_\infty), \quad f \ge 0, \qquad f(x,t) \ge ct^{-m} \quad \text{for } t \ge 1,$$

where $c = \mathrm{const} > 0$ and $0 \le m = \mathrm{const} \le 2$, then condition (6.7) will hold, and thus, for $g = -|s|^\alpha s$, $s \in \mathbb{R}$, $\alpha > -1$, problem (1.1), (1.2) does not have a strong generalized solution u of class C^1 in the domain D_T for sufficiently large T. Indeed, in (6.6), introducing the transformation of the independent variables x and t by the formulas $x = Tx_1$, $t = Tt_1$ and assuming that $T > 1$, after a few manipulations, we obtain

$$\begin{aligned}\zeta(T) &= T^2 \int_{D_{T=1}} f(Tx_1, Tt_1)\varphi^0(x_1,t_1)\,dx_1\,dt_1 \\ &\ge cT^{2-m} \int_{D_{T=1}\cap\{t_1 \ge T^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1 \\ &\qquad + T^2 \int_{D_{T=1}\cap\{t_1 < T^{-1}\}} f(Tx_1, Tt_1)\varphi^0(x_1,t_1)\,dx_1\,dt_1.\end{aligned}$$

Further, let $T_1 > 1$ be an arbitrary fixed number. Then, using the last inequality for the function ζ, we obtain

$$\zeta(T) \ge cT^{2-m} \int_{D_{T=1}\cap\{t_1 \ge T^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1 \ge c \int_{D_{T=1}\cap\{t_1 \ge T_1^{-1}\}} t_1^{-m}\varphi^0(x_1,t_1)\,dx_1\,dt_1,$$

if $T \ge T_1 > 1$. The last inequality immediately yields condition (6.7).

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