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ON THE SOLVABILITY OF A PROBLEM NONLOCAL IN TIME FOR A SEMILINEAR MULTIDIMENSIONAL WAVE EQUATION

ПРО РОЗВ'ЯЗНІСТЬ НЕЛОКАЛЬНОЇ ЗА ЧАСОМ ЗАДАЧІ ДЛЯ НАПІВЛІНІЙНОГО БАГАТОВИМІРНОГО ХВИЛЬОВОГО РІВНЯННЯ

We study a nonlocal (in time) problem for semilinear multidimensional wave equations. The theorems on existence and uniqueness of solutions of this problem are proved.

Вивчається нелокальна за часом задача для напівлінійних багатовимірних хвильових рівнянь. Доведено теореми про існування та єдиність росв'язків цієї задачі.

1. Introduction. In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \ldots, x_n)$ and t, in the cylindrical domain $D_T = \Omega \times (0, T)$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, consider a nonlocal problem of finding a solution $u(x, t)$ of the following equation:

$$L_\lambda u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \lambda f(x, t, u) = F(x, t), \quad (x, t) \in D_T, \tag{1.1}$$

satisfying the homogeneous boundary condition on the part of the boundary $\Gamma := \partial\Omega \times (0, T)$ of the cylinder D_T

$$u \big|_\Gamma = 0, \tag{1.2}$$

the initial condition

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \tag{1.3}$$

and the nonlocal condition

$$K_\mu u_t := u_t(x, 0) - \mu u_t(x, T) = \psi(x), \quad x \in \Omega, \tag{1.4}$$

where f, F, φ and ψ are given functions; λ and μ are given nonzero constants and $n \geq 2$.

To the study of nonlocal problems for partial differential equations there are devoted many papers. When a nonlocal problem is posed for abstract evolution equations and hyperbolic partial differential equations we would suggest the reader refer to works [1 – 15] and the references therein.

Note that the problem (1.1) – (1.4) in the work [15] is studied in the class of continuous functions for the case of one spatial variable, i.e., for $n = 1$. The method of investigation given in the work [15], based on the integral representation of the solution of corresponding linear problem, is useless for multidimensional case, i.e., for $n > 1$. In this work the problem (1.1) – (1.4) in the multidimensional case is studied in the Sobolev space $W_2^1(D_T)$, basing on expansions of the functions from the space $\overset{0}{W}{}_2^1(\Omega)$ in the basis, consisting of eigenfunctions of spectral problem $\Delta w = \tilde{\lambda} w$, $w \mid_{\partial\Omega} = 0$ and using embedding theorems in the Sobolev spaces. It must be noted also that if for $n = 1$ there is no need of any restriction on the behavior of function $f(x, t, u)$ with respect to variable u when $u \to \infty$, while in the case $n > 1$, we require of function $f(x,t,u)$ that for $u \to \infty$ it must have a growth not exceeding polynomial. Moreover, for using the embedding theorems in the Sobolev spaces we additionally require that the order of polynomial growth must be less than a certain value, which depends of the dimension of the space.

Below, on the function $f = f(x,t,u)$ we impose the following requirements:

$$f \in C(\overline{D}_T \times \mathbb{R}), \qquad |f(x,t,u)| \le M_1 + M_2|u|^\alpha, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{1.5}$$

where

$$0 \le \alpha = \mathrm{const} < \frac{n+1}{n-1}. \tag{1.6}$$

***Remark* 1.1.** The embedding operator $I\colon W_2^1(D_T) \to L_q(D_T)$ represents a linear continuous compact operator for $1 < q < \dfrac{2(n+1)}{n-1}$, when $n > 1$ [16]. At the same time the Nemitski operator $N:$ $L_q(D_T) \to L_2(D_T)$, acting by the formula $Nu = f(x,t,u)$ due to (1.5) is continuous and bounded if $q \ge 2\alpha$ [17]. Thus, since due to (1.6) we have $2\alpha < \dfrac{2(n+1)}{n-1}$, then there exists the number q such that $1 < q < \dfrac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Therefore, in this case the operator

$$N_0 = NI\colon \overset{0}{W}{}_2^1(D_T,\Gamma) \to L_2(D_T), \tag{1.7}$$

where $\overset{0}{W}{}_2^1(D_T,\Gamma) := \{w \in W_2^1(D_T)\colon w\,|_\Gamma = 0\}$, will be continuous and compact. Besides, from $u \in \overset{0}{W}{}_2^1(D_T,\Gamma)$ it follows that $f(x,t,u) \in L_2(D_T)$ and, if $u_m \to u$ in the space $\overset{0}{W}{}_2^1(D_T,\Gamma)$, then $f(x,t,u_m) \to f(x,t,u)$ in the space $L_2(D_T)$.

Definition 1.1. *Let function f satisfy the conditions* (1.5) *and* (1.6), $F \in L_2(D_T)$, $\varphi \in$ $\in \overset{0}{W}{}_2^1(\Omega) := \{v \in W_2^1(\Omega)\colon v\,|_{\partial\Omega} = 0\}$, $\psi \in L_2(\Omega)$. *We call a function u a generalized solution of the problem* (1.1)–(1.4), *if $u \in \overset{0}{W}{}_2^1(D_T,\Gamma)$ and there exists a sequence of functions* $u_m \in \overset{0}{C}{}^2(\overline{D}_T,\Gamma) := \big\{w \in C^2(\overline{D}_T)\colon w\,|_\Gamma = 0\big\}$ *such that*

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} = 0, \qquad \lim_{m\to\infty} \|L_\lambda u_m - F\|_{L_2(D_T)} = 0, \tag{1.8}$$

$$\lim_{m\to\infty} \|u_m\,|_{t=0} - \varphi\|_{\overset{0}{W}{}_2^1(\Omega)} = 0, \qquad \lim_{m\to\infty} \|K_\mu u_{mt} - \psi\|_{L_2(\Omega)} = 0. \tag{1.9}$$

It is obvious that a classical solution $u \in C^2(\overline{D}_T)$ of the problem (1.1)–(1.4) represents a generalized solution of this problem. It is easy to verify that a generalized solution of the problem (1.1)–(1.4) is a solution of the problem (1.1) in the sense of the theory of distributions. Indeed, let $F_m := L_\lambda u_m$, $\varphi_m := u_m\,|_{t=0}$, $\psi_m := K_\mu u_{mt}$. Multiplying the both sides of the equality $L_\lambda u_m = F_m$ by test function $w \in V := \big\{v \in \overset{0}{W}{}_2^1(D_T,\Gamma)\colon v(x,T) - \mu v(x,0) = 0,\ x \in \Omega\big\}$ and integrating in the domain D_T, after simple transformations, connected with integration by parts and the equality $w\,|_\Gamma = 0$, we get

$$\int_\Omega [u_{mt}(x,T)w(x,T) - u_{mt}(x,0)w(x,0)]dx +$$

$$+\int\limits_{D_T}\left[-u_{mt}w_t+\sum_{i=1}^{n}u_{mx_i}w_{x_i}+\lambda f(x,t,u_m)w\right]dx\,dt=\int\limits_{D_T}F_m w\,dx\,dt\quad\forall w\in V. \tag{1.10}$$

Due to $K_\mu u_{mt}=\psi_m(x)$ and $w(x,T)-\mu w(x,0)=0$, $x\in\Omega$, it is easy to see that $u_{mt}(x,T)w(x,T)-$ $-\,u_{mt}(x,0)w(x,0)=u_{mt}(x,T)(w(x,T)-\mu w(x,0))-\psi_m(x)w(x,0)=-\psi_m(x)w(x,0)$, $x\in\Omega$. Therefore, the equality (1.10) takes the form

$$-\int\limits_{\Omega}\psi_m(x)w(x,0)dx+$$

$$+\int\limits_{D_T}\left[-u_{mt}w_t+\sum_{i=1}^{n}u_{mx_i}w_{x_i}+\lambda f(x,t,u_m)w\right]dx\,dt=\int\limits_{D_T}F_m w\,dx\,dt\quad\forall w\in V. \tag{1.11}$$

In view of (1.5), (1.6) according to the Remark 1.1 we have $f(x,t,u_m)\to f(x,t,u)$ in the space $L_2(D_T)$, when $u_m\to u$ in the space $\overset{0}{W}{}_2^1(D_T,\Gamma)$. Therefore, due to (1.8) and (1.9), passing to the limit in the equality (1.11) for $m\to\infty$, we get

$$-\int\limits_{\Omega}\psi w(x,0)dx+\int\limits_{D_T}\left[-u_tw_t+\sum_{i=1}^{n}u_{mx_i}w_{x_i}+\lambda f(x,t,u)w\right]dx\,dt=\int\limits_{D_T}Fwdx\,dt\quad\forall w\in V. \tag{1.12}$$

Since $C_0^\infty(D_T)\subset V$, then from (1.12), integrating by parts, we have

$$\int\limits_{D_T}\big[u\Box w+\lambda f(x,t,u)w\big]dx\,dt=\int\limits_{D_T}Fw\,dx\,dt\quad\forall w\in C_0^\infty(D_T), \tag{1.13}$$

where $\Box:=\partial^2/\partial t^2-\sum_{i=1}^{n}\partial^2/\partial x_i^2$, and $C_0^\infty(D_T)$ is a space of finite infinitely differentiable functions in D_T. The equality (1.13), which is valid for any $w\in C_0^\infty(D_T)$, means that a generalized solution u of the problem (1.1)–(1.4) is a solution of the equation (1.1) in the sense of the theory of distributions, besides, since the trace operator $u\to u\,|_{t=0}$ is well defined in the space $\overset{0}{W}{}_2^1(D_T,\Gamma)$, and, particularly, is a continuous operator from the space $\overset{0}{W}{}_2^1(D_T,\Gamma)$ into the space $L_2(\Omega\times\{t=0\})$, then due to (1.8) and (1.9) we receive that the initial condition (1.3) is fulfilled in the sense of the trace theory, while the nonlocal condition (1.4) in the integral sense is taken into account in the equality (1.12), which is valid for all $w\in V$. Note also that if a generalized solution u belongs to the class $C^2(\overline{D}_T)$, then due to the standard reasoning, connected with the integral equality (1.12), which is valid for any $w\in V$ [16], we have that u is a classical solution of the problem (1.1)–(1.4), satisfying the equation (1.1), the boundary condition (1.2), the initial condition (1.3) and the nonlocal condition (1.4) pointwisely.

Note that even in the linear case, i.e., for $\lambda=0$, the problem (1.1)–(1.4) is not always well-posed. For example, when $\lambda=0$ and $|\mu|=1$, the corresponding to (1.1)–(1.4) homogeneous problem may have infinite number of linearly independent solutions (see the Remark 3.2).

The work is organized in the following way. In Section 2 we single out the class of semilinear equations (1.1), when for $|\mu|<1$ a priori estimate is valid for the generalized solution of the problem (1.1) – (1.4). In Section 3 on the basis of a priori estimate, received in the previous section, the solvability of the problem (1.1) – (1.4) is proved. Finally, in Section 4 we give the conditions imposed on the data of the problem, which provide the uniqueness of the solution of this problem.

2. A priori estimate of the solution of the problem (1.1) – (1.4). Let

$$g(x,t,u) = \int\limits_0^u f(x,t,s)ds, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}. \tag{2.1}$$

Consider the following conditions imposed on function $g = g(x,t,u)$:

$$g(x,t,u) \geq -M_3, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{2.2}$$

$$g_t \in C(\overline{D}_T \times \mathbb{R}), \qquad g_t(x,t,u) \leq M_4, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{2.3}$$

where $M_i = \text{const} \geq 0$, $i = 3, 4$.

Let us consider some classes of functions $f = f(x,t,u)$ frequently encountered in applications and which satisfy the conditions (1.5), (2.2) and (2.3):

1. $f(x,t,u) = f_0(x,t)\beta(u)$, where f_0, $\dfrac{\partial}{\partial t}f_0 \in C(\overline{D}_T)$ and $\beta \in C(\mathbb{R})$, $|\beta(u)| \leq \tilde{M}_1 + \tilde{M}_2|u|^\alpha$, $\tilde{M}_i = \text{const} \geq 0$, $\alpha = \text{const} \geq 0$. In this case $g(x,t,u) = f_0(x,t)\displaystyle\int_0^u \beta(s)ds$ and when $f_0 \geq 0$, $\dfrac{\partial}{\partial t}f_0 \leq 0$, $\displaystyle\int_0^u \beta(s)ds \geq -M$, $M = \text{const} \geq 0$, the conditions (1.5), (2.2) and (2.3) will be fulfilled.

2. $f(x,t,u) = f_0(x,t)|u|^\alpha \operatorname{sign} u$, where f_0, $\dfrac{\partial}{\partial t}f_0 \in C(\overline{D}_T)$ and $\alpha > 1$. In this case $g(x,t,u) = f_0(x,t)\dfrac{|u|^{\alpha+1}}{\alpha+1}$ and when $f_0 \geq 0$, $\dfrac{\partial}{\partial t}f_0 \leq 0$, the conditions (1.5), (2.2) and (2.3) will be also fulfilled.

Lemma 2.1. *Let* $\lambda > 0$, $|\mu| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{0}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$ *and the conditions* (1.5), (2.2), (2.3) *be fulfilled. Then for a generalized solution* u *of the problem* (1.1) – (1.4) *the following a priori estimate*

$$\|u\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} \leq c_1\|F\|_{L_2(D_T)} + c_2\|\varphi\|_{\overset{0}{W}{}_2^1(\Omega)} + c_3\|\psi\|_{L_2(\Omega)} + c_4\|\varphi\|^{\frac{\alpha+1}{2}}_{\overset{0}{W}{}_2^1(\Omega)} + c_5 \tag{2.4}$$

is valid with nonnegative constants $c_i = c_i(\lambda,\mu,\Omega,T,M_1,M_2,M_3,M_4)$ *not depending on* u, F, φ, ψ, *and* $c_i > 0$ *for* $i < 4$, *whereas in the linear case, i.e., when* $\lambda = 0$ *the constants* $c_4 = c_5 = 0$ *and due to* (2.4) *in this case we have the uniqueness of the solution of the problem* (1.1) – (1.4).

Proof. Let u be a generalized solution of the problem (1.1) – (1.4). In view of the Definition 1.1 there exists a sequence of the functions $u_m \in \overset{0}{C}{}^2(\overline{D}_T,\Gamma)$ such that the limit equalities (1.8), (1.9) are fulfilled.

Set

$$L_\lambda u_m = F_m, \quad (x,t) \in D_T, \tag{2.5}$$

$$u_m|_\Gamma = 0, \tag{2.6}$$

$$u_m(x,0)=\varphi_m(x), \quad x\in\Omega, \tag{2.7}$$

$$K_\mu u_{mt}=\psi_m(x), \quad x\in\Omega. \tag{2.8}$$

Multiplying both sides of the equation (2.5) by $2u_{mt}$ and integrating in the domain $D_\tau :=$ $:= D_T \cap \{t<\tau\}$, $0<\tau\le T$, due to the (2.1), we obtain

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2 dx\,dt-2\int\limits_{D_\tau}\sum_{i=1}^n\frac{\partial^2 u_m}{\partial x_i^2}\frac{\partial u_m}{\partial t}dx\,dt+2\lambda\int\limits_{D_\tau}\frac{d}{dt}\big(g(x,t,u_m(x,t)\big)dx\,dt-$$

$$-2\lambda\int\limits_{D_\tau}g_t(x,t,u_m(x,t))dx\,dt=2\int\limits_{D_\tau}F_m\frac{\partial u_m}{\partial t}dx\,dt. \tag{2.9}$$

Let $\omega_\tau := \{(x,t)\in\overline{D}_T : x\in\Omega,\ t=\tau\}$, $0\le\tau\le T$. Denote by $\nu := (\nu_{x_1},\nu_{x_2},\ldots,\nu_{x_n},\nu_t)$ the unit vector of the outer normal to ∂D_τ. Since $\nu_{x_i}\big|_{\omega_\tau\cup\omega_0}=0$, $i=1,\ldots,n$, $\nu_t\big|_{\Gamma_\tau=\Gamma\cap\{t\le\tau\}}=0$, $\nu_t\big|_{\omega_\tau}=1$, $\nu_t\big|_{\omega_0}=-1$, then, taking into account the equalities (2.6) and integrating by parts, we have

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2 dx\,dt=\int\limits_{\partial D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\nu_t ds=\int\limits_{\omega_\tau}u_{mt}^2dx-\int\limits_{\omega_0}u_{mt}^2dx, \tag{2.10}$$

$$-2\int\limits_{D_\tau}\frac{\partial^2 u_m}{\partial x_i^2}\frac{\partial u_m}{\partial t}dx\,dt=\int\limits_{D_\tau}[(u_{mx_i}^2)_t-2(u_{mx_i}u_{mt})_{x_i}]dx\,dt=$$

$$=\int\limits_{\omega_\tau}u_{mx_i}^2dx-\int\limits_{\omega_0}u_{mx_i}^2dx, \quad i=1,\ldots,n, \tag{2.11}$$

$$2\lambda\int\limits_{D_\tau}\frac{d}{dt}\big(g(x,t,u_m(x,t)\big)dxdt=2\lambda\int\limits_{\partial D_\tau}g(x,t,u_m(x,t))\nu_t ds=$$

$$=2\lambda\int\limits_{\omega_\tau}g(x,t,u_m(x,t))dx-2\lambda\int\limits_{\omega_0}g(x,t,u_m(x,t))dx. \tag{2.12}$$

In view of (2.10), (2.11), (2.12) from (2.9) we get

$$\int\limits_{\omega_\tau}\left[u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\right]dx=\int\limits_{\omega_0}\left[u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\right]dx-2\lambda\int\limits_{\omega_\tau}g(x,t,u_m(x,t))dx+$$

$$+2\lambda\int\limits_{\omega_0}g(x,t,u_m(x,t))dx+2\lambda\int\limits_{D_\tau}g_t(x,t,u_m(x,t))dx\,dt+2\int\limits_{D_\tau}F_m u_{mt}dx\,dt. \tag{2.13}$$

Let

$$w_m(\tau) := \int\limits_{\omega_\tau} \left[u_{mt}^2 + \sum_{i=1}^{n} u_{mx_i}^2 \right] dx. \tag{2.14}$$

Since $2F_m u_{mt} \leq \epsilon^{-1} F_m^2 + \epsilon u_{mt}^2$ for any $\epsilon = \text{const} > 0$, then due to the (2.2), (2.3) and (2.14) from (2.13) it follows that

$$w_m(\tau) \leq w_m(0) + 2\lambda M_3 \operatorname{mes} \Omega + 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))| dx +$$

$$+ 2\lambda M_4 \tau \operatorname{mes} \Omega + \epsilon \int\limits_{D_T} u_{mt}^2 dx\, dt + \epsilon^{-1} \int\limits_{D_T} F_m^2 dx\, dt. \tag{2.15}$$

Taking into account that

$$\int\limits_{D_\tau} u_{mt}^2 dx\, dt = \int\limits_0^\tau \left[\int\limits_{\omega_s} u_{mt}^2 dx \right] ds \leq \int\limits_0^\tau \left[\int\limits_{\omega_s} \left[u_{mt}^2 + \sum_{i=1}^{n} u_{mx_i}^2 \right] dx \right] ds = \int\limits_0^\tau w_m(s) ds,$$

from (2.15) we obtain

$$w_m(\tau) \leq \epsilon \int\limits_0^\tau w_m(s) ds + w_m(0) + 2\lambda (M_3 + M_4 \tau) \operatorname{mes} \Omega +$$

$$+ 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))| dx + \epsilon^{-1} \int\limits_{D_\tau} F_m^2 dx\, dt, \quad 0 < \tau \leq T. \tag{2.16}$$

Because of $D_\tau \subset D_T$, $0 < \tau \leq T$, then according to the Gronwall's lemma [18] from (2.16) it follows that

$$w_m(\tau) \leq \Bigg[w_m(0) + 2\lambda (M_3 + M_4 T) \operatorname{mes} \Omega +$$

$$+ 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))| dx + \epsilon^{-1} \int\limits_{D_T} F_m^2 dx\, dt \Bigg] e^{\epsilon \tau}, \quad 0 < \tau \leq T. \tag{2.17}$$

Using obvious inequality

$$|a+b|^2 = a^2 + b^2 + 2ab \leq a^2 + b^2 + \epsilon_1 a^2 + \epsilon_1^{-1} b^2 = (1+\epsilon_1) a^2 + (1+\epsilon_1^{-1}) b^2,$$

which is valid for any $\epsilon_1 > 0$, from (2.8) we have

$$|u_{mt}(x,0)|^2 = |\mu u_{mt}(x,T) + \psi_m(x)|^2 \leq |\mu|^2 (1+\epsilon_1) u_{mt}^2(x,T) + (1+\epsilon_1^{-1}) \psi_m^2(x). \tag{2.18}$$

From (2.18) we obtain

$$\int\limits_{\omega_0} u_{mt}^2 dx = \int\limits_{\Omega} |u_{mt}(x,0)|^2\, dx \le$$

$$\le |\mu|^2(1+\epsilon_1)\int\limits_{\Omega} u_{mt}^2(x,T)dx + (1+\epsilon_1^{-1})\int\limits_{\Omega} \psi_m^2(x)\, dx =$$

$$= |\mu|^2(1+\epsilon_1)\int\limits_{\omega_T} u_{mt}^2 dx + (1+\epsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}. \tag{2.19}$$

In view of (2.7), (2.14) from (2.17) we get

$$\int\limits_{\omega_T} u_{mt}^2 dx \le w_m(T) \le \left[\int\limits_{\omega_0} \sum_{i=1}^n \varphi_{mx_i}^2 dx + \int\limits_{\omega_0} u_{mt}^2 dx + M_5\right] e^{\epsilon T}, \tag{2.20}$$

where

$$M_5 = 2\lambda(M_3 + M_4 T)\,\mathrm{mes}\,\Omega + 2\lambda\int\limits_{\omega_0} |g(x,t,u_m(x,t))| dx + \epsilon^{-1}\int\limits_{D_T} F_m^2\, dx\, dt. \tag{2.21}$$

From (2.19) and (2.20) it follows that

$$\int\limits_{\omega_0} u_{mt}^2 dx \le |\mu|^2(1+\epsilon_1)\left[\int\limits_{\omega_0} \sum_{i=1}^n \varphi_{mx_i}^2 dx + \int\limits_{\omega_0} u_{mt}^2 dx + M_5\right] e^{\epsilon T} + (1+\epsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}. \tag{2.22}$$

Because $|\mu| < 1$, then positive constants ϵ and ϵ_1 can be chosen so small that

$$\mu_1 = |\mu|^2(1+\epsilon_1)e^{\epsilon T} < 1. \tag{2.23}$$

Due to (2.23) from (2.22) we obtain

$$\int\limits_{\omega_0} u_{mt}^2 dx \le (1-\mu_1)^{-1}\left[|\mu|^2(1+\epsilon_1)\left(\int\limits_{\omega_0} \sum_{i=1}^n \varphi_{mx_i}^2 dx + M_5\right) e^{\epsilon T} + (1+\epsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}\right] \le$$

$$\le (1-\mu_1)^{-1}\left[|\mu|^2(1+\epsilon_1)\left(\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} + M_5\right) e^{\epsilon T} + (1+\epsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}\right]. \tag{2.24}$$

From (2.7), (2.14) and (2.24) it follows that

$$w_m(0) = \int\limits_{\omega_0} \left[u_{mt}^2 + \sum_{i=1}^n \varphi_{mx_i}^2\right] dx \le \|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} +$$

$$+(1-\mu_1)^{-1}\left[|\mu|^2(1+\epsilon_1)\left(\|\varphi_m\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} + M_5\right) e^{\epsilon T} + (1+\epsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega)}\right]. \tag{2.25}$$

In view of (2.21), (2.25) from (2.17) we get

$$w_m(\tau) \le \Bigg\{ \|\varphi_m\|^2_{\overset{0}{W}{}^1_2(\Omega)} + (1-\mu_1)^{-1} \times$$

$$\times \Bigg[|\mu|^2(1+\epsilon_1)\Bigg(\|\varphi_m\|^2_{\overset{0}{W}{}^1_2(\Omega)} + 2\lambda(M_3+M_4T)\operatorname{mes}\Omega \, +$$

$$+ \, 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))|dx + \epsilon^{-1}\int\limits_{D_T} F_m^2 dx\,dt \Bigg) e^{\epsilon T} + (1+\epsilon_1^{-1})\|\psi_m\|^2_{L_2(\Omega} \Bigg] +$$

$$+ \, 2\lambda(M_3+M_4T)\operatorname{mes}\Omega + 2\lambda \int\limits_{\omega_0} |g(x,t,u_m(x,t))|dx + \epsilon^{-1}\int\limits_{D_T} F_m^2 dx\,dt \Bigg\} e^{\epsilon T} =$$

$$= \tilde\gamma_1 \|F_m\|^2_{L_2(D_T)} + \tilde\gamma_2 \|\varphi_m\|^2_{\overset{0}{W}{}^1_2(\Omega)} + \tilde\gamma_3 \|\psi_m\|^2_{L_2(\Omega)} + \tilde\gamma_4 \int\limits_{\omega_0} |g(x,t,u_m(x,t))|dx + \tilde\gamma_5. \tag{2.26}$$

Here

$$\begin{aligned}
&\tilde\gamma_1 = \epsilon^{-1} e^{\epsilon T}\big[(1-\mu_1)^{-1}(1+\epsilon_1)e^{\epsilon T} + 1\big],\\
&\tilde\gamma_2 = e^{\epsilon T}\big[1 + (1-\mu_1)^{-1}|\mu|^2(1+\epsilon_1)\big],\\
&\tilde\gamma_3 = (1-\mu_1)^{-1}(1+\epsilon_1^{-1})e^{\epsilon T},\\
&\tilde\gamma_4 = 2\lambda[(1-\mu_1)^{-1}|\mu|^2(1+\epsilon_1)+1]e^{\epsilon T},\\
&\tilde\gamma_5 = 2\lambda(M_3+M_4T)\operatorname{mes}\Omega[(1-\mu_1)^{-1}|\mu|^2(1+\epsilon_1)e^{\epsilon T}+1]e^{\epsilon T}.
\end{aligned} \tag{2.27}$$

Since for fixed τ the function $u_m(x,\tau) \in \overset{0}{W}{}^1_2(\Omega)$, then due to the Friedrichs inequality [16] we have

$$\int\limits_{\omega_\tau} \left[u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \right] dx \le c_0 w_m(\tau) = c_0 \int\limits_{\omega_\tau} \left[u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \right] dx, \tag{2.28}$$

where positive constant $c_0 = c_0(\Omega)$ does not depend on u_m.

From (2.26) and (2.28) it follows

$$\|u_m\|^2_{\overset{0}{W}{}^1_2(D_T,\Gamma)} = \int\limits_0^T \left[\int\limits_{\omega_\tau} \left(u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2 \right) dx \right] d\tau \le$$

$$\leq \int\limits_0^T c_0 w_m(\tau)d\tau \leq c_0 T\tilde{\gamma}_1 \|F_m\|^2_{L_2(D_T)} + c_0 T\tilde{\gamma}_2 \|\varphi_m\|^2_{\overset{0}{W}{}^1_2(\Omega)} + c_0 T\tilde{\gamma}_3 \|\psi_m\|^2_{L_2(\Omega)} +$$

$$+ c_0 T\tilde{\gamma}_4 \int\limits_\Omega \big|g(x,0,u_m(x,0))\big| dx + c_0 T\tilde{\gamma}_5. \tag{2.29}$$

Due to (2.1), (1.5) we have

$$|g(x,0,s)| \leq M_6 + M_7 |s|^{\alpha+1}, \tag{2.30}$$

where M_6 and M_7 are some nonnegative constants. Taking into account (2.30) from (2.29) we get

$$\|u_m\|^2_{\overset{0}{W}{}^1_2(D_T,\Gamma)} \leq c_0 T\tilde{\gamma}_1 \|F_m\|^2_{L_2(D_T)} + c_0 T\tilde{\gamma}_2 \|\varphi_m\|^2_{\overset{0}{W}{}^1_2(\Omega)} + c_0 T\tilde{\gamma}_3 \|\psi_m\|^2_{L_2(\Omega)} +$$

$$+ c_0 T\tilde{\gamma}_4 M_6 \operatorname{mes}\Omega + c_0 T\tilde{\gamma}_4 M_7 \int\limits_\Omega |u_m(x,0)|^{\alpha+1} dx + c_0 T\tilde{\gamma}_5. \tag{2.31}$$

Reasoning from the Remark 1.1, concerning the space $W^1_2(\Omega)$, in view of the equality $\dim\Omega = \dim D_T - 1 = n$ show that the embedding operator $I\colon W^1_2(\Omega) \to L_q(\Omega)$ is a linear continuous compact operator for $1 < q < \dfrac{2n}{n-2}$, when $n > 2$ and for any $q > 1$ when $n = 2$ [16]. At the same time the Nemitski operator $N_1\colon L_q(\Omega) \to L_2(\Omega)$, acting by the formula $N_1 u = |u|^{\frac{\alpha+1}{2}}$ is continuous and bounded if $q \geq 2\dfrac{\alpha+1}{2} = \alpha + 1$ [17]. Thus, if $\alpha + 1 < \dfrac{2n}{n-2}$, i.e., $\alpha < \dfrac{n+2}{n-2}$, which is fulfilled due to (1.6) since $\dfrac{n+1}{n-1} < \dfrac{n+2}{n-2}$, then there exists number q such that $1 < q < \dfrac{2n}{n-2}$ and $q \geq \alpha + 1$. Therefore, in this case the operator

$$N_2 = N_1 I\colon W^1_2(\Omega) \to L_2(\Omega)$$

will be continuous and compact. Thus due to (1.9), (2.7) it follows that

$$\lim_{m\to\infty} \int\limits_\Omega |u_m(x,0)|^{\alpha+1} dx = \int\limits_\Omega |\varphi(x)|^{\alpha+1} dx, \tag{2.32}$$

and also [16]

$$\int\limits_\Omega |\varphi(x)|^{\alpha+1} dx \leq C_1 \|\varphi\|^{\alpha+1}_{\overset{0}{W}{}^1_2(\Omega)} \tag{2.33}$$

with positive constant C_1, not dependent on $\varphi \in \overset{0}{W}{}^1_2(\Omega)$.

In view of (1.8), (1.9), (2.5) – (2.8), (2.32) and (2.33), passing in (2.31) to the limit for $m \to \infty$ we obtain

$$\|u\|^2_{\overset{0}{W}{}^1_2(D_T,\Gamma)} \leq c_0 T\tilde{\gamma}_1 \|F\|^2_{L_2(D_T)} + c_0 T\tilde{\gamma}_2 \|\varphi\|^2_{\overset{0}{W}{}^1_2(\Omega)} + c_0 T\tilde{\gamma}_3 \|\psi\|^2_{L_2(\Omega)} +$$

$$+c_0T\tilde{\gamma_4}M_7C_1\|\varphi\|^{\alpha+1}_{\overset{0}{W}{}^1_2(\Omega)} + c_0T(\tilde{\gamma_5}+\tilde{\gamma_4}M_6\operatorname{mes}\Omega). \tag{2.34}$$

Taking the square root from the both sides of the inequality (2.34) and using the obvious inequality $\left(\sum_{i=1}^k a_i^2\right)^{1/2} \le \sum_{i=1}^k |a_i|$ we finally get

$$\|u\|_{\overset{0}{W}{}^1_2(D_T,\Gamma)} \le c_1\|F\|_{L_2(D_T)} + c_2\|\varphi\|_{\overset{0}{W}{}^1_2(\Omega)} + c_3\|\psi\|_{L_2(\Omega)} + c_4\|\varphi\|^{\frac{\alpha+1}{2}}_{\overset{0}{W}{}^1_2(\Omega)} + c_5. \tag{2.35}$$

Here

$$\begin{gathered} c_1 = (c_0T\tilde{\gamma_1})^{1/2}, \qquad c_2 = (c_0T\tilde{\gamma_2})^{1/2}, \qquad c_3 = (c_0T\tilde{\gamma_3})^{1/2}, \\ c_4 = (c_0T\tilde{\gamma_4}M_7C_1)^{1/2}, \qquad c_5 = [c_0T(\tilde{\gamma_5}+\tilde{\gamma_4}M_6\operatorname{mes}\Omega)]^{1/2}, \end{gathered} \tag{2.36}$$

where $\tilde{\gamma_i}$, $1 \le i \le 5$, are defined in (2.27). In the linear case, i.e., for $\lambda = 0$, due to (2.27) the constants $\tilde{\gamma_4} = \tilde{\gamma_5} = 0$ and from (2.36) it follows that in the estimate (2.4) the constants $c_4 = c_5 = 0$. Whence it follows the uniqueness of the solution of the problem (1.1) – (1.4) in the linear case.

Lemma 2.1 is proved.

3. The existence of the solution of the problem (1.1) – (1.4). For the existence of the solution of the problem (1.1) – (1.4) in the case $|\mu| < 1$ we will use well known facts about the solvability of the following linear mixed problem [16]:

$$L_0u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \quad (x,t) \in D_T, \tag{3.1}$$

$$u\big|_\Gamma = 0, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \tilde{\psi}(x), \quad x \in \Omega, \tag{3.2}$$

where F, φ and $\tilde{\psi}$ are given functions.

For $F \in L_2(D_T)$, $\varphi \in \overset{0}{W}{}^1_2(\Omega)$, $\tilde{\psi} \in L_2(\Omega)$ the unique generalized solution u of the problem (3.1), (3.2) (in the sense of the equality (1.12) where $f = 0$, and the number $\mu = 0$ in the definition of the space V) from the class $E_{2,1}(D_T)$ with the norm [16]

$$\|v\|^2_{E_{2,1}(D_T)} = \sup_{0 \le \tau \le T} \int\limits_{\omega_\tau} \left[u^2 + u_t^2 + \sum_{i=1}^n u_{x_i}^2\right] dx$$

is given by formula [16]

$$u = \sum_{k=1}^\infty \left(a_k\cos\mu_k t + b_k \sin\mu_k t + \frac{1}{\mu_k}\int\limits_0^t F_k(\tau)\sin\mu_k(t-\tau)d\tau\right)\varphi_k(x), \tag{3.3}$$

where $\tilde{\lambda}_k = -\mu_k^2$, $0 < \mu_1 \le \mu_2 \le \ldots$, $\lim_{k\to\infty}\mu_k = \infty$ are the eigenvalues, while $\varphi_k \in \overset{0}{W}{}^1_2(\Omega)$ are the corresponding eigenfunctions of the spectral problem $\Delta w = \tilde{\lambda} w$, $w\,|_{\partial\Omega} = 0$ in the domain Ω $\left(\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\right)$, simultaneously forming orthonormal basis in $L_2(\Omega)$ and orthogonal basis in

$\overset{0}{W}{}_2^1(\Omega)$ in the sense of scalar product $(v,w)_{\overset{0}{W}{}_2^1(\Omega)} = \int\limits_{\Omega} \sum_{i=1}^{n} v_{x_i} w_{x_i} dx$ [16], i.e.,

$$(\varphi_k, \varphi_l)_{L_2(\Omega)} = \delta_k^l, \qquad (\varphi_k, \varphi_l)_{\overset{0}{W}{}_2^1(\Omega)} = -\lambda_k \delta_k^l, \qquad \delta_k^l = \begin{cases} 1, & l = k, \\ 0, & l \neq k. \end{cases} \tag{3.4}$$

Here

$$a_k = (\varphi, \varphi_k)_{L_2(\Omega)}, \qquad b_k = \mu_k^{-1} (\tilde{\psi}, \varphi_k)_{L_2(\Omega)}, \quad k = 1, 2, \ldots, \tag{3.5}$$

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t) \varphi_k(x), \qquad F_k(t) = (F, \varphi_k)_{L_2(\omega_t)}, \qquad \omega_\tau := D_T \cap \{t = \tau\}, \tag{3.6}$$

besides, for the solution u from (3.3) it is valid the following estimate [16, 19]:

$$\|u\|_{E_{2,1}(D_T)} \leq \gamma (\|F\|_{L_2(D_T)} + \|\varphi\|_{\overset{0}{W}{}_2^1(\Omega)} + \|\tilde{\psi}\|_{L_2(\Omega)}) \tag{3.7}$$

with positive constant γ,not dependent on F, φ and $\tilde{\psi}$.

Let us consider the linear problem corresponding to (1.1)–(1.4), i.e., the case when $\lambda = 0$:

$$L_0 u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \quad (x,t) \in D_T, \tag{3.8}$$

$$u|_\Gamma = 0, \qquad u(x,0) = \varphi(x), \qquad K_\mu u_t = \psi(x), \quad x \in \Omega. \tag{3.9}$$

Let us show that when $|\mu| < 1$ for any $F \in L_2(D_T)$, $\varphi \in \overset{0}{W}{}_2^1(\Omega)$ and $\psi \in L_2(\Omega)$ there exists a unique generalized solution of the problem (3.8), (3.9) in the sense of the Definition 1.1 for $\lambda = 0$. Indeed, for $\varphi \in \overset{0}{W}{}_2^1(\Omega)$ and $\psi \in L_2(\Omega)$ there are valid the expansions $\varphi = \sum_{k=1}^{\infty} a_k \varphi_k$ and $\psi = \sum_{k=1}^{\infty} d_k \varphi_k$ in the spaces $\overset{0}{W}{}_2^1(\Omega)$ and $L_2(\Omega)$, respectively, where $a_k = (\varphi, \varphi_k)_{L_2(\Omega)}$ and $d_k = (\psi, \varphi_k)_{L_2(\Omega)}$ [16]. Therefore, setting

$$\varphi_m = \sum_{k=1}^{m} a_k \varphi_k, \qquad \psi_m = \sum_{k=1}^{m} d_k \varphi_k, \tag{3.10}$$

we have

$$\lim_{m \to \infty} \|\varphi_m - \varphi\|_{\overset{0}{W}{}_2^1(\Omega)} = 0, \qquad \lim_{m \to \infty} \|\psi_m - \psi\|_{L_2(\Omega)} = 0. \tag{3.11}$$

Since the space of finite infinitely differentiable functions $C_0^\infty(D_T)$ is dense in the space $L_2(D_T)$, then for $F \in L_2(D_T)$ and any natural number m there exists a function $F_m \in C_0^\infty(D_T)$ such that

$$\|F_m - F\|_{L_2(D_T)} < \frac{1}{m}. \tag{3.12}$$

On the other hand, for function F_m in the space $L_2(D_T)$ there is valid the following expansion [16]:

$$F_m(x,t) = \sum_{k=1}^{\infty} F_{m,k}(t)\varphi_k(x), \qquad F_{m,k}(t) = (F_m, \varphi_k)_{L_2(\Omega)}. \tag{3.13}$$

Therefore, there exists a natural number l_m such that $\lim_{m\to\infty} l_m = \infty$ and for

$$\tilde{F}_m(x,t) = \sum_{k=1}^{l_m} F_{m,k}(t)\varphi_k(x) \tag{3.14}$$

the inequality

$$\|\tilde{F}_m - F_m\|_{L_2(D_T)} < \frac{1}{m} \tag{3.15}$$

is valid. From (3.12) and (3.15) it follows

$$\lim_{m\to\infty} \|\tilde{F}_m - F\|_{L_2(D_T)} = 0. \tag{3.16}$$

The solution $u = u_m$ of the problem (3.1), (3.2) for $\varphi = \varphi_{l_m}$, $\tilde{\psi} = \sum_{k=1}^{l_m} \tilde{d}_k\varphi_k$ and $F = \tilde{F}_m$, where φ_{l_m} and $\tilde{F}_m$ are defined in (3.10) and (3.14), is given by formula (3.3) which, due to (3.4)–(3.6), takes the form

$$u_m = \sum_{k=1}^{l_m} \left(a_k \cos\mu_k t + \frac{\tilde{d}_k}{\mu_k}\sin\mu_k t + \frac{1}{\mu_k}\int_0^t F_{m,k}(\tau)\sin\mu_k(t-\tau)d\tau \right)\varphi_k(x). \tag{3.17}$$

For determination of the coefficients $\tilde{d}_k$ let us substitute the right-hand side of the expression (3.17) into the equality $K_\mu u_{mt} = \psi_{l_m}(x)$, where ψ_{l_m} is defined in (3.10). Consequently, taking into account that the system of functions $\{\varphi_k(x)\}$ represents a basis in $L_2(\Omega)$ and $1-\mu\cos\mu_k T \neq 0$ for $|\mu|<1$, we obtain the following formulas:

$$\tilde{d}_k = \frac{1}{1-\mu\cos\mu_k T}\left[(\varphi_{l_m}, \varphi_k)_{L_2(\Omega)} - a_k\mu\mu_k\sin\mu_k T + \right.$$

$$\left. + \mu\int_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)d\tau \right], \quad k = 1,\ldots,l_m. \tag{3.18}$$

Below we assume that the Lipschitz domain Ω is such that eigenfunctions $\varphi_k \in C^2(\overline{\Omega})$, $k \geq 1$. For example, this will take place if $\partial\Omega \in C^{[n/2]+3}$ [19]. This fact will also take place in the case of a piecewisely smooth Lipschitz domain, e.g., for the parallelepiped $\Omega = \{x \in \mathbb{R}^n \colon |x_i| < a_i,\ i = 1,\ldots,n\}$ the correspondent eigenfunctions $\varphi_k \in C^\infty(\overline{\Omega})$ [20]. Therefore, since $F_m \in C_0^\infty(D_T)$, then due to (3.13) the function $F_{m,k} \in C^2([0,T])$, and consequently the function u_m from (3.17) belongs to the space $C^2(\overline{D}_T)$. Further, since $\varphi_k|_{\partial\Omega} = 0$, then due to (3.17) we have $u_m|_\Gamma = 0$, and thereby $u_m \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma)$, $m = 1,2,\ldots$.

According to the construction the function u_m from (3.17) satisfies the following equalities:

$$u_m|_\Gamma = 0, \qquad L_0 u_m = \tilde{F}_m, \qquad u_m(x,0) = \varphi_{l_m}(x), \qquad K_\mu u_{mt} = \psi_{l_m}(x), \quad x \in \Omega, \quad (3.19)$$

and thereby

$$(u_m - u_k)|_\Gamma = 0, \qquad L_0(u_m - u_k) = \tilde{F}_m - \tilde{F}_k, \qquad (u_m - u_k)(x,0) = (\varphi_{l_m} - \varphi_{l_k})(x),$$

$$K_\mu(u_{mt} - u_{kt}) = (\psi_{l_m} - \psi_{l_k})(x), \quad x \in \Omega.$$

Therefore, from a priori estimate (2.4), where for $\lambda = 0$ the coefficients $c_4 = c_5 = 0$, we obtain

$$\|u_m - u_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} \le c_1\|\tilde{F}_m - \tilde{F}_k\|_{L_2(D_T)} + c_2\|\varphi_{l_m} - \varphi_{l_k}\|_{\overset{0}{W}{}_2^1(\Omega)} + c_3\|\psi_{l_m} - \psi_{l_k}\|_{L_2(\Omega)}. \quad (3.20)$$

In view of (3.11) and (3.16) from (3.20) it follows that the sequence $u_m \in \overset{0}{C}{}^2(\overline{D}_T,\Gamma)$ is fundamental in the complete space $\overset{0}{W}{}_2^1(D_T,\Gamma)$. Therefore, there exists a function $u \in \overset{0}{W}{}_2^1(D_T,\Gamma)$ such that due to (3.11), (3.16) and (3.19) there are valid the limit equalities (1.8), (1.9) for $\lambda = 0$. The last means that the function u is a generalized solution of the problem (3.8), (3.9). The uniqueness of this solution follows from a priori estimate (2.4) where the constants $c_4 = c_5 = 0$ for $\lambda = 0$. Therefore, for the solution u of the problem (3.8), (3.9) we have $u = L_0^{-1}(F,\varphi,\psi)$, where $L_0^{-1}\colon L_2(D_T) \times \overset{0}{W}{}_2^1(\Omega) \times L_2(\Omega) \to \overset{0}{W}{}_2^1(D_T,\Gamma)$, which norm due to (2.4) can be estimated as follows:

$$\|L_0^{-1}\|_{L_2(D_T)\times\overset{0}{W}{}_2^1(\Omega)\times L_2(\Omega)\to\overset{0}{W}{}_2^1(D_T,\Gamma)} \le \gamma_0 = \max(c_1,c_2,c_3). \quad (3.21)$$

Due to the linearity of the operator $L_0^{-1}\colon L_2(D_T) \times \overset{0}{W}{}_2^1(\Omega) \times L_2(\Omega) \to \overset{0}{W}{}_2^1(D_T,\Gamma)$ we have a representation

$$L_0^{-1}(F,\varphi,\psi) = L_0^{-1}(F,0,0) + L_0^{-1}(0,\varphi,0) + L_0^{-1}(0,0,\psi) = L_{01}^{-1}(F) + L_{02}^{-1}(\varphi) + L_{03}^{-1}(\psi), \quad (3.22)$$

where $L_{01}^{-1}\colon L_2(D_T) \to \overset{0}{W}{}_2^1(D_T,\Gamma)$, $L_{02}^{-1}\colon \overset{0}{W}{}_2^1(\Omega) \to \overset{0}{W}{}_2^1(D_T,\Gamma)$ and $L_{03}^{-1}\colon L_2(\Omega) \to \overset{0}{W}{}_2^1(D_T,\Gamma)$ are linear continuous operators, besides, according to (3.21)

$$\|L_{01}^{-1}\|_{L_2(D_T)\to\overset{0}{W}{}_2^1(D_T,\Gamma)} \le \gamma_0, \qquad \|L_{02}^{-1}\|_{\overset{0}{W}{}_2^1(\Omega)\to\overset{0}{W}{}_2^1(D_T,\Gamma)} \le \gamma_0,$$

$$\|L_{03}^{-1}\|_{L_2(\Omega)\to\overset{0}{W}{}_2^1(D_T,\Gamma)} \le \gamma_0. \quad (3.23)$$

***Remark* 3.1.** Note, that for $F \in L_2(D_T)$, $\varphi \in \overset{0}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$, due to (1.5), (1.6), (3.21)–(3.23) and the Remark 1.1 the function $u \in \overset{0}{W}{}_2^1(D_T,\Gamma)$ is a generalized solution of the problem (1.1)–(1.4) if and only if, when u is a solution of the following functional equation

$$u = L_{01}^{-1}(-\lambda f(x,t,u)) + L_{01}^{-1}(F) + L_{02}^{-1}(\varphi) + L_{03}^{-1}(\psi) \quad (3.24)$$

in the space $\overset{0}{W}{}_2^1(D_T,\Gamma)$.

Rewrite the equation (3.24) in the form

$$u = A_0 u := -\lambda L_{01}^{-1}(N_0 u) + L_{01}^{-1}(F) + L_{02}^{-1}(\varphi) + L_{03}^{-1}(\psi), \tag{3.25}$$

where the operator $N_0\colon \overset{0}{W}{}_2^1(D_T,\Gamma) \to L_2(D_T)$ from (1.7), according to the Remark 1.1 is continuous and compact operator. Therefore, due to (3.23) the operator $A_0\colon \overset{0}{W}{}_2^1(D_T,\Gamma) \to \overset{0}{W}{}_2^1(D_T,\Gamma)$ from (3.25) is also continuous and compact. At the same time, according to the Lemma 2.1 and (2.36) for any parameter $\tau \in [0,1]$ and for any solution u of the equation $u = \tau A_0 u$ with the parameter τ it is valid the same a priori estimate (2.4) with nonnegative constants c_i, not dependent on u, F, φ, ψ and τ. Therefore, due to the Schaefer's fixed point theorem [21], the equation (3.25), and therefore, due to the Remark 3.1 the problem (1.1)–(1.4) has at least one solution $u \in \overset{0}{W}{}_2^1(D_T,\Gamma)$. Thus, we have proved the following theorem.

Theorem 3.1. *Let* $\lambda > 0$, $|\mu| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{0}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$; *the conditions* (1.5), (1.6), (2.2), (2.3) *be fulfilled. Then the problem* (1.1)–(1.4) *has at least one generalized solution.*

Remark **3.2.** Note that for $|\mu| = 1$, even in the linear case, i.e., for $f = 0$, the homogeneous problem corresponding to (1.1)–(1.4) may have finite or even infinite number of linearly independent solutions. Indeed, in the case $\mu = 1$ denote by $\Lambda(1)$ the set of points μ_k from (3.3), for which the ratio $\dfrac{\mu_k T}{2\pi}$ is a natural number, i.e., $\Lambda(1) = \left\{\mu_k : \dfrac{\mu_k T}{2\pi} \in \mathbb{N}\right\}$. If we search for a solution of the problem (3.8), (3.9) in the form of representation (3.3), then for determination of unknown coefficients b_k,contained in it, let us substitute the right-hand side of this representation into the equality $K_\mu u_t = \psi(x)$. As a result we have

$$\mu_k(1 - \mu\cos\mu_k T) b_k = (\psi, \varphi_k)_{L_2(\Omega)} - a_k\mu_k \sin\mu_k T + \int\limits_0^T F_k(\tau)\cos\mu_k(T-\tau)\,d\tau. \tag{3.26}$$

It is obvious, that when $\Lambda(1) \neq \varnothing$ and $\mu_k \in \Lambda(1)$, $\mu = 1$ we have $1 - \cos\mu_k T = 0$ and for $F = 0$, $\varphi = \psi = 0$ and thereby for $a_k = 0$, $F_k(\tau) = 0$ the equality (3.26) will be satisfied by any number b_k. Therefore, in accordance with (3.3) the function $u_k(x,t) = C\sin\mu_k t\varphi_k(x)$, $C = \mathrm{const} \neq 0$, satisfies the homogeneous problem corresponding to (3.8), (3.9). Analogously, in the case $\mu = -1$ denote by $\Lambda(-1)$ the set of points μ_k from (3.3) for which the ratio $\dfrac{\mu_k T}{\pi}$ is odd integer number. In this case $1 - \mu\cos\mu_k T = 0$ for $\mu_k \in \Lambda(-1)$, $\mu = -1$ and the function $u_k(x,t) = C\sin\mu_k t\varphi_k(x)$, $C = \mathrm{const} \neq 0$, is a nontrivial solution of the homogeneous problem corresponding to (3.8), (3.9). For example, when $n = 2$, $\Omega = (0,1)\times(0,1)$ the eigenvalues and eigenfunctions of the Laplace operator Δ are [20]

$$\lambda_k = -\pi^2(k_1^2 + k_2^2), \qquad \varphi_k(x_1,x_2) = \sin k_1\pi_1 x_1 \sin k_2\pi x_2, \quad k = (k_1,k_2),$$

i.e., $\mu_k = \pi\sqrt{k_1^2 + k_2^2}$. For $k_1 = p^2 - q^2$, $k_2 = 2pq$, where p and q are any integer numbers we obtain $\mu_k = \pi(p^2+q^2)$ [22]. In this case for $\dfrac{T}{2} \in \mathbb{N}$ we have $\dfrac{\mu_k T}{2\pi} = (p^2+q^2)\dfrac{T}{2} \in \mathbb{N}$ and according to the said above, when $\mu = 1$ the homogeneous problem corresponding to (3.8), (3.9) has infinite number of linearly independent solutions

$$u_{p,q}(x,t) = \sin\pi(p^2+q^2)t\sin\pi(p^2-q^2)x_1\sin 2\pi pqx_2 \quad \forall p,q \in \mathbb{N}. \tag{3.27}$$

Analogously, when $\mu = -1$ the solutions of the homogeneous problem corresponding to (3.8), (3.9) in the case when p is an even number, while q and T odd numbers, are the functions from (3.27).

4. The uniqueness of the solution of the problem (1.1)–(1.4). On the function f in the equation (1.1) let us impose the following additional requirements:

$$f, f'_u \in C(\overline{D}_T \times \mathbb{R}), \qquad |f'_u(x,t,u)| \le a + b|u|^\gamma, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{4.1}$$

where $a, b, \gamma = \text{const} \ge 0$.

It is obvious that from (4.1) we have the condition (1.5) for $\alpha = \gamma + 1$ and when $\gamma < \dfrac{2}{n-1}$ we have $\alpha = \gamma + 1 < \dfrac{n+1}{n-1}$ and, therefore, the condition (1.6) is fulfilled.

Theorem 4.1. *Let* $|\mu| < 1$, $F \in L_2(D_T)$, $\varphi \in \overset{0}{W}{}_2^1(\Omega)$, $\psi \in L_2(\Omega)$ *and the condition* (4.1) *be fulfilled for* $\gamma < \dfrac{2}{n-1}$, *and also hold the conditions* (2.2), (2.3). *Then there exists a positive number* $\lambda_0 = \lambda_0(F, f, \varphi, \psi, \mu, D_T)$ *such that for* $0 < \lambda < \lambda_0$ *the problem* (1.1)–(1.4) *can not have more than one generalized solution.*

Proof. Indeed, suppose that the problem (1.1)–(1.4) has two different generalized solutions u_1 and u_2. According to Definition 1.1 there exist sequences of functions $u_{jk} \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma)$, $j = 1, 2$, such that

$$\lim_{k\to\infty} \|u_{jk} - u_j\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} = 0, \qquad \lim_{k\to\infty} \|L_\lambda u_{jk} - F\|_{L_2(D_T)} = 0, \tag{4.2}$$

$$\lim_{k\to\infty} \|u_{jk}|_{t=0} - \varphi\|_{\overset{0}{W}{}_2^1(\Omega)} = 0, \qquad \lim_{k\to\infty} \|K_\mu u_{jkt} - \psi\|_{L_2(\Omega)} = 0, \quad j = 1, 2. \tag{4.3}$$

Let

$$w := u_2 - u_1, \qquad w_k := u_{2k} - u_{1k}, \qquad F_k := L_\lambda u_{2k} - L_\lambda u_{1k}, \tag{4.4}$$

$$g_k := \lambda(f(x,t,u_{1k}) - f(x,t,u_{2k})). \tag{4.5}$$

In view of (4.2), (4.3) and (4.4) it is easy to see that

$$\lim_{k\to\infty} \|w_k - w\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} = 0, \qquad \lim_{k\to\infty} \|F_k\|_{L_2(D_T)} = 0, \tag{4.6}$$

$$\lim_{k\to\infty} \|w_k|_{t=0}\|_{\overset{0}{W}{}_2^1(\Omega)} = 0, \qquad \lim_{k\to\infty} \|K_\mu w_{kt}\|_{L_2(\Omega)} = 0. \tag{4.7}$$

In view of (4.4), (4.5) the function $w_k \in \overset{0}{C}{}^2(\overline{D}_T, \Gamma)$ satisfies the following equalities:

$$\frac{\partial^2 w_k}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2 w_k}{\partial x_i^2} = \big(F_k + g_k\big)(x,t), \quad (x,t) \in D_T, \tag{4.8}$$

$$w_k|_\Gamma = 0, \tag{4.9}$$

$$w_k(x,0) = \tilde\varphi_k(x), \quad x \in \Omega, \tag{4.10}$$

$$K_\mu w_{kt} := w_{kt}(x,0) - \mu w_{kt}(x,T) = \tilde{\psi}_k(x), \quad x \in \Omega, \tag{4.11}$$

where $\tilde{\varphi}_k(x) := u_{2k}(x,0) - u_{1k}(x,0)$, $\tilde{\psi}_k(x) := K_\mu u_{2kt} - K_\mu u_{1kt}$.

First let us estimate the function g_k from (4.5). Taking into account the obvious inequality $|d_1 + d_2|^\gamma \leq 2^\gamma \max(|d_1|^\gamma, |d_2|^\gamma) \leq 2^\gamma(|d_1|^\gamma + |d_2|^\gamma)$ for $\gamma \geq 0$, due to (4.1) we have

$$|f(x,t,u_{2k}) - f(x,t,u_{1k})| = \left|(u_{2k} - u_{1k}) \int\limits_0^1 f'_u(x,t,u_{1k} + \tau(u_{2k} - u_{1k}))d\tau\right| \leq$$

$$\leq |u_{2k} - u_{1k}| \int\limits_0^1 (a + b|(1-\tau)u_{1k} + \tau u_{2k}|^\gamma)d\tau \leq a|u_{2k} - u_{1k}| +$$

$$+ 2^\gamma b|u_{2k} - u_{1k}|(|u_{1k}|^\gamma + |u_{2k}|^\gamma) = a|w_k| + 2^\gamma b|w_k|(|u_{1k}|^\gamma + |u_{2k}|^\gamma). \tag{4.12}$$

In view of (4.5) from (4.12) we obtain

$$\|g_k\|_{L_2(D_T)} \leq \lambda a\|w_k\|_{L_2(D_T)} + \lambda 2^\gamma b\| \, |w_k|(|u_{1k}|^\gamma + |u_{2k}|^\gamma)\|_{L_2(D_T)} \leq$$

$$\leq \lambda a\|w_k\|_{L_2(D_T)} + \lambda 2^\gamma b\|w_k\|_{L_p(D_T)}\|(|u_{1k}|^\gamma + |u_{2k}|^\gamma)\|_{L_q(D_T)}. \tag{4.13}$$

Here we used the Hölder's inequality [23]

$$\|v_1 v_2\|_{L_r(D_T)} \leq \|v_1\|_{L_p(D_T)}\|v_2\|_{L_q(D_T)},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ and in the capacity of p, q and r we take

$$p = 2\frac{n+1}{n-1}, \qquad q = n+1, \qquad r = 2. \tag{4.14}$$

Since $\dim D_T = n+1$, then according to the Sobolev embedding theorem [17] for $1 \leq p \leq$ $\leq \dfrac{2(n+1)}{n-1}$ we get

$$\|v\|_{L_p(D_T)} \leq C_p\|v\|_{W_2^1(D_T)} \quad \forall v \in W_2^1(D_T) \tag{4.15}$$

with positive constant C_p, not dependent on $v \in W_2^1(D_T)$.

Due to the condition of the theorem $\gamma < \dfrac{2}{n-1}$ and, therefore, $\gamma(n+1) < \dfrac{2(n+1)}{n-1}$. Thus, due to (4.14) from (4.15) we have

$$\|w_k\|_{L_p(D_T)} \leq C_p\|w_k\|_{W_2^1(D_T)}, \qquad p = \frac{2(n+1)}{n-1}, \quad k \geq 1, \tag{4.16}$$

$$\|(|u_{1k}|^\gamma + |u_{2k}|^\gamma)\|_{L_q(D_T)} \leq \| \, |u_{1k}|^\gamma\|_{L_q(D_T)} + \| \, |u_{2k}|^\gamma\|_{L_q(D_T)} =$$

$$= \| \, u_{1k}\|^\gamma_{L_{\gamma(n+1)}(D_T)} + \| \, u_{2k}\|^\gamma_{L_{\gamma(n+1)}(D_T)} \leq C^\gamma_{\gamma(n+1)}\big(\| \, u_{1k}\|^\gamma_{W_2^1(D_T)} + \| \, u_{2k}\|^\gamma_{W_2^1(D_T)}\big). \tag{4.17}$$

In view of the first equality of (4.2) there exists a natural number k_0 such that for $k \geq k_0$ we obtain

$$\|u_{ik}\|^{\gamma}_{W_2^1(D_T)} \leq \|u_i\|^{\gamma}_{W_2^1(D_T)} + 1, \qquad i = 1, 2, \quad k \geq k_0. \tag{4.18}$$

Further, in view of (4.16), (4.17) and (4.18) from (4.13) we get

$$\|g_k\|_{L_2(D_T)} \leq \lambda a \|w_k\|_{L_2(D_T)} + \lambda 2^{\gamma} b C_p C^{\gamma}_{\gamma(n+1)} (\|u_1\|^{\gamma}_{W_2^1(D_T)} +$$

$$+ \|u_2\|^{\gamma}_{W_2^1(D_T)} + 2) \|w_k\|_{W_2^1(D_T)} \leq \lambda M_8 \|w_k\|_{W_2^1(D_T)}, \tag{4.19}$$

where we have used the inequality $\|w_k\|_{L_2(D_T)} \leq \|w_k\|_{W_2^1(D_T)}$,

$$M_8 = a + 2^{\gamma} b C_p C^{\gamma}_{\gamma(n+1)} \left(\|u_1\|^{\gamma}_{W_2^1(D_T)} + \|u_2\|^{\gamma}_{W_2^1(D_T)} + 2 \right), \qquad p = 2\frac{n+1}{n-1}. \tag{4.20}$$

Since a priori estimate (2.4) is valid for $\lambda = 0$, then due to (2.27) and (2.36) in this estimate $c_4 = c_5 = 0$, and thereby for the solution w_k of the problem (4.8) – (4.11) the following estimate:

$$\|w_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} \leq c_1^0 \|F_k + g_k\|_{L_2(D_T)} + c_2^0 \|\tilde{\varphi}_k\|_{\overset{0}{W}{}_2^1(\Omega)} + c_3^0 \|\tilde{\psi}_k\|_{L_2(\Omega)} \tag{4.21}$$

is valid, where the constants c_1^0, c_2^0, c_3^0 do not depend on λ.

Because of $\|w_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} = \|w_k\|_{W_2^1(D_T)}$ and due to (4.19) from (4.21) we have

$$\|w_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} \leq c_1^0 \|F_k\|_{L_2(D_T)} + \lambda c_1^0 M_8 \|w_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} + c_2^0 \|\tilde{\varphi}_k\|_{\overset{0}{W}{}_2^1(\Omega)} + c_3^0 \|\tilde{\psi}_k\|_{L_2(\Omega)}. \tag{4.22}$$

Note that since for u_1 and u_2 it is valid a priori estimate (2.4), then the constant M_8 from (4.20) will depend on λ, F, f, φ, ψ, D_T, besides, due to (2.27) and (2.36) the value of M_8 continuously depends on λ for $\lambda \geq 0$ and

$$0 \leq \lim_{\lambda \to 0+} M_8 = M_8^0 < +\infty. \tag{4.23}$$

Due to (4.23) there exists a positive number $\lambda_0 = \lambda_0(F, f, \varphi, \psi, \mu, D_T)$ such that for

$$0 < \lambda < \lambda_0 \tag{4.24}$$

we obtain $\lambda c_1^0 M_8 < 1$. Indeed, let us fix arbitrarily a positive number ε_1. Then, due to (4.23), there exists a positive number λ_1, such that $0 \leq M_8 < M_8^0 + \varepsilon_1$ for $0 \leq \lambda < \lambda_1$. It is obvious that for $\lambda_0 = \min\left(\lambda_1, (c_1^0(M_8^0 + \varepsilon_1))^{-1}\right)$ the condition $\lambda c_1^0 M_8 < 1$ will be fulfilled. Therefore, in the case (4.24) from (4.22) we get

$$\|w_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} \leq (1 - \lambda c_1^0 M_8)^{-1} \left[c_1^0 \|F_k\|_{L_2(D_T)} + c_2^0 \|\tilde{\varphi}_k\|_{\overset{0}{W}{}_2^1(\Omega)} + c_3^0 \|\tilde{\psi}_k\|_{L_2(\Omega)} \right], \quad k \geq k_0. \tag{4.25}$$

From (4.2) and (4.4) it follows that $\lim_{k\to\infty} \|w_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} = \|u_2 - u_1\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)}$. On the other hand due to (4.6), (4.7) and (4.10), (4.11) from (4.25) we have $\lim_{k\to\infty} \|w_k\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} = 0$. Thus $\|u_2 - u_1\|_{\overset{0}{W}{}_2^1(D_T,\Gamma)} = 0$, i.e., $u_2 = u_1$, which leads to contradiction.

Theorem 4.1 is proved.

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