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ON A ZAREMBA TYPE PROBLEM FOR NONLINEAR WAVE EQUATIONS IN THE ANGULAR DOMAINS

In a plane of independent variables x and t we consider a nonlinear wave equation

$$Lu := u_{tt} - u_{xx} + f(u) = F(x,t), \tag{1}$$

where $f = f(s)$ is a given real nonlinear with respect to the variable $s \in \mathbb{R}$ function, $F = F(x,t)$ is a given and $u = u(x,t)$ is an unknown real function, and it is assumed that f and F are continuous functions of their arguments.

By $D : \gamma_2(t) < x < \gamma_1(t),\ t > 0$ we denote the angular domain lying inside of the characteristic angle $\Lambda : t > |x|$ and bounded by noncharacteristic curves $\gamma_i : x = \gamma_i(t)$, $t \geq 0$, $i = 1, 2$, of the class C^2, coming out of the origin $O(0,0)$.

Assume $D_T := D \cap \{t < T\}$ and $\gamma_{i,T} := \gamma_i \cap \{t \leq T\}$, $T > 0$, $i = 1, 2$.

For equation (1) we consider the Darboux type problem when the oblique derivative of a solution is given on $\gamma_{1,T}$, while on $\gamma_{2,T}$ a solution itself of (1). The problem is formulated as follows: Find in the domain D_T a solution $u = u(x,t)$ of that equation under the boundary conditions

$$(l_1 u_x + l_2 u_t)|_{\gamma_{1,T}} = 0, \tag{2}$$

$$u|_{\gamma_{2,T}} = 0, \tag{3}$$

where l_1 and l_2 are the given continuous on γ_1 functions, and $(|l_1| + |l_2|)|_{\gamma_1} \neq 0$.

Note that in the linear case, that is, when the function f in equation (1) is linear with respect to s and instead of the boundary conditions (2), (3) there take place the conditions

$$(\alpha_i u_x + \beta_i u_t)|_{\gamma_{i,T}} = 0, \quad i = 1, 2; \quad u(0,0) = 0, \tag{4}$$

the problem (1), (4) in the domain D_T has been the subject of investigation in [1–6]. It should also be noted that the problem (1)–(3) is equivalent to the problem (1), (4) when direction (α_2, β_2) coincides with that of the tangent to the curve at every its point. In the case of equation (1) with power

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nonlinearity when homogeneous Dirichlet conditions are taken on γ_1 and γ_2 and one of these curves γ_1, or γ_2 is a characteristic, this problem has been investigated in [7–9], but in the case where both curves are noncharacteristic rays, the problem has been studied in [10]. The particular case of boundary conditions (2), (3) of the type $u_x|_{\gamma_{1,T}} = 0$, $u|_{\gamma_{2,T}} = 0$, where $\gamma_{1,T} : x = 0$, $0 \le t \le T$, and $\gamma_{2,T} : x = -t$, $0 \le t \le T$ are the characteristics of equation (1) with power nonlinearity, is investigated in [11, 12], but when $\gamma_{2,T} : x = -kt$, $0 \le t \le T$, where $0 < k = \text{const} < 1$, it is studied in [13]. In the case of equation (1) with power nonlinearity, when γ_1 and γ_2 are the noncharacteristic curves, the problem is considered in [14]. As is pointed out in [1, 6], analogous problems arise in mathematical modeling of small harmonic oscillations of a wedge in a supersonic flow, as well as of a string in a cylinder filled up with a viscous liquid.

In the present work we study a more general case of a nonlinear function f than the case of a power function $f = \lambda|s|^\alpha s$ considered in the above-mentioned works, when $\gamma_i : x = -k_i t$, $t \ge 0$ are noncharacteristic rays and $k_i = \text{const}$, $i = 1, 2$, $0 \le k_1 < k_2 < 1$.

Assume $\overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T) := \{ v \in C^2(\overline{D}_T) : (l_1 v_x + l_2 v_t)|_{\gamma_{1,T}} = 0,\ v|_{\gamma_{2,T}} = 0 \}$, $\gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Definition 1. Let $f \in C(\mathbb{R})$, $F \in C(\overline{D}_T)$; $l_1, l_2 \in C(\gamma_{1,T})$. The function u is said to be a strong generalized solution of the problem (1)–(3) of the class C in the domain D_T if $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ such that $u_n \to u$ and $Lu_n \to F$ in the space $C(\overline{D}_T)$, as $n \to \infty$.

Remark 1. Obviously, a classical solution of the problem (1)–(3) from the space $\overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ is a strong generalized solution of that problem in the sense of Definition 1.

Definition 2. Let $f \in C(\mathbb{R})$, $F \in C(\overline{D}_\infty)$; $l_1, l_2 \in C(\gamma_{1,\infty})$. We say that the problem (1)–(3) is globally solvable in the class C if for any finite $T > 0$ this problem has at least one strong generalized solution of the class C in the domain D_T.

Definition 3. Let $f \in C(\mathbb{R})$, $F \in C(\overline{D}_\infty)$; $l_1, l_2 \in C(\gamma_{1,\infty})$. The function $u \in C(\overline{D}_\infty)$ is said to be a global strong generalized solution of the problem (1)–(3) of the class C in the domain D_∞, if for any finite $T > 0$ the function $u|_{D_T}$ is a strong generalized solution of that problem of the class C in the domain D_T.

Definition 4. Let $f \in C(\mathbb{R})$, $F \in C(\overline{D}_\infty)$; $l_1, l_2 \in C(\gamma_{1,\infty})$. We say that the problem (1)–(3) is locally solvable in the class C if there exists a positive number $T_0 = T_0(F)$ such that for $T \le T_0$ this problem has at least one strong generalized solution of the class C in the domain D_T.

Remark 2. Below, it will be assumed that the direction (l_1, l_2) of the derivative appearing in the boundary condition (2) is not characteristic one, corresponding to a family of characteristics $x + t = \text{const}$ of equation (1), that is,

$$(l_1 + l_2)\big|_{\gamma_1} \neq 0.$$

By α we denote a nonobtuse angle which forms the ray γ_2 with the characteristic ray $x + t = 0$, $t \geq 0$, and by β we denote a nonobtuse angle lying between the directions of the vector $l = (l_1, l_2)$ and the characteristic ray $x - t = 0$, $t \geq 0$ at the point $O(0,0)$.

Theorem 1. *Let $f \in C(\mathbb{R})$; $l_1, l_2 \in C^1(\gamma_{1,\infty})$. Then for any $F \in C(\overline{D}_\infty)$, the problem* (1)–(3) *is locally solvable in the class C, i.e., there exists a positive number $T_0 = T_0(F)$ such that for $T \leq T_0$ this problem has at least one strong generalized solution of the class C in the domain D_T, and this problem in case $\alpha + \beta > \frac{\pi}{2}$ has an infinite set of linearly independent strong generalized solutions of the class C in the domain D_T for $T \leq T_0$.*

Suppose

$$g(s) := \int\limits_0^s f(s_1)ds_1, \quad s \in \mathbb{R}.$$

Consider the conditions

$$g(s) \geq -M_1 - M_2 s^2, \quad s \in \mathbb{R}, \quad M_i := \text{const} \geq 0, \quad i = 1, 2, \tag{5}$$

and

$$[(l_1^2 + l_2^2)\nu_t + 2l_1 l_2 \nu_x](P) \geq 0, \quad P \in \gamma_1, \tag{6}$$

imposed, respectively, on the nonlinear function f and on the geometric characteristics of the curve γ_1 and the vector l, where (ν_x, ν_t) is the unit vector of the outer normal to ∂D_T at the point P.

Remark 3. Here we present certain classes of functions f appearing frequently in applications and satisfying the condition (5):

1. $f \in C(\mathbb{R})$, $f(s) sign\ s \geq 0$, $s \in \mathbb{R}$. In particular, when $f(s) = |s|^\alpha sign\ s$, $s \in \mathbb{R}$, where $\alpha = \text{const} > 0$, $\alpha \neq 1$. In this case $g(s) = \frac{|s|^{\alpha+1}}{\alpha+1}$, $s \in \mathbb{R}$.
2. $f = ce^s$, $c = \text{const} > 0$. In this case $g(s) = c(e^s - 1)$, $s \in \mathbb{R}$.

Remark 4. It can be easily verified that the inequality $\alpha + \beta < \frac{\pi}{2}$ follows from the validity of the condition (6) at the point O. It should also be noted that when (2) is a Neumann homogeneous boundary condition, i.e., $l = (\nu_x, \nu_t)$, then the condition (6) will be fulfilled.

Theorem 2. *Let $f \in C(\mathbb{R})$; $l_1, l_2 \in C^1(\gamma_1)$ and the conditions* (5), (6) *be fulfilled. Then for any $F \in C(\overline{D}_T)$, the problem* (1)–(3) *has at least one strong generalized solution of the class C in the domain D_T.*

Corollary 1. *Let $f \in C(\mathbb{R})$; $l = (\nu_x, \nu_t)$ and the condition* (5) *be fulfilled. Then for any $F \in C(\overline{D}_T)$, the problem* (1)–(3) *has at least one strong generalized solution of the class C in the domain D_T.*

Corollary 2. *Let $f \in C(\mathbb{R})$; $l_1, l_2 \in C^1(\gamma_1)$ and the conditions* (5), (6) *be fulfilled. Then for any $F \in C(\overline{D}_\infty)$, the problem* (1)–(3) *is globally solvable in the class C.*

According to the definition, the function f satisfies the local Lipschitz condition on $\mathbb{R}$ if

$$|f(s_2) - f(s_1)| \leq m(r)|s_2 - s_1|, \quad |s_i| \leq r, \quad i = 1, 2, \tag{7}$$

where $m(r) := \mathrm{const} \geq 0$.

Theorem 3. *Let $f \in C(\mathbb{R})$ satisfy the condition* (7), *$F \in C(\overline{D}_T)$; $l_1, l_2 \in C(\gamma_1)$ and the condition* (6) *be fulfilled. Then the problem* (1)–(3) *may have no more than one strong generalized solution of the class C.*

Corollary 3. *If* (7) *holds and the conditions of Theorem* 2 *are fulfilled, then for any $F \in C(\overline{D}_T)$, the problem* (1)–(3) *has a unique strong generalized solution of the class C in the domain D_T, and in addition, this problem will have a unique global strong generalized solution of the class C in the domain D_T.*

Theorem 4. *If the conditions of Theorem* 2 *are fulfilled, then a strong generalized solution u of the problem* (1)–(3) *of the class C in the domain D_T belongs to the space $C^1(\overline{D}_T)$, and under additional requirement that $f \in C^1(\mathbb{R})$, $F \in C^1(\overline{D}_T)$ this solution belongs to the space $C^2(\overline{D}_T)$, i.e., it will be classical, and in both cases the conditions* (2) *and* (3) *are fulfilled pointwise.*

Corollary 4. *Let $f \in C^1(\mathbb{R})$; $l_1, l_2 \in C^1(\gamma_1)$ and the conditions* (5), (6) *be fulfilled. Then for any $F \in C^1(\overline{D}_T)$, the problem* (1)–(3) *in the domain D_T has a unique classical solution.*

Corollary 5. *Let $f \in C^1(\mathbb{R})$; $l_1, l_2 \in C^1(\gamma_1)$ and the conditions* (5), (6) *be fulfilled. Then for any $F \in C^1(\overline{D}_\infty)$, the problem* (1)–(3) *has a unique global classical solution $u \in C^2(\overline{D}_\infty)$.*

Note that if the condition (5) *is violated, the problem* (1)–(3) *may turn out to be globally unsolvable.*

Theorem 5. *Let $l_1, l_2 \in C(\gamma_1)$ and the function $f \in C(\mathbb{R})$ satisfy the condition*

$$f(s) \leq -\lambda |s|^{\alpha+1}, \quad s \in \mathbb{R}; \quad \lambda, \ \alpha := \mathrm{const} > 0.$$

Then if $F \in C(\overline{D}_\infty)$; $F \geq 0$ $F(x,t) \geq ct^{-m}$ and $t \geq 1$ for $t \geq 1$, where $c := \mathrm{const} > 0$ and $0 < m := \mathrm{const} \leq 2$,then there exists the positive number $T_0 = T_0(F)$ such that for $T > T_0$, the problem (1)–(3) *cannot have a strong generalized solution of the class C in the domain D_T.*

Remark 5. In the case if in equation (1) instead of a nonlinear term $f(u)$ there appears a nonlinear dissipative term of the type $g(u)u_t$, the scheme of investigation of the problem under consideration changes in principle. For example, it is not enough to have in the class C only one a priori estimate. On the basis of that eatimate, using the method of characteristics, we can get a priori estimate of solution now in the class C^1. Next, proceeding from that a priori estimate and from the local solvability (see, for e.g., [15]), we will be able to prove global solvability of this problem in the class C^1.

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