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PARTIAL DIFFERENTIAL EQUATIONS

# Time-Periodic Problem for a Weakly Nonlinear Telegraph Equation with Directional Derivative in the Boundary Condition

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**Abstract**—We study a time-periodic problem for the wave equation with a power-law nonlinearity and with a directional derivative in the boundary condition. We study the existence, uniqueness, and absence of solutions of the problem.

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## 1. STATEMENT OF THE PROBLEM

In the strip

$$\Omega := \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, \ t \in \mathbb{R}\},$$

in the domain of the independent variables $x$ and $t$, consider the problem of finding a solution $U(x,t)$ of the telegraph equation with a power-law nonlinearity of the form

$$L_\lambda U := U_{tt} - U_{xx} + 2aU_t + cU + \lambda |U|^\alpha U = F(x,t), \qquad (x,t) \in \Omega, \tag{1.1}$$

with the homogeneous Poincaré boundary condition

$$\gamma_1 U_x(0,t) + \gamma_2 U_t(0,t) + \gamma_3 U(0,t) = 0, \qquad t \in \mathbb{R}, \tag{1.2}$$

for $x = 0$, the homogeneous Dirichlet boundary condition

$$U(l,t) = 0, \qquad t \in \mathbb{R}, \tag{1.3}$$

for $x = l$, and the periodicity condition with respect to the variable $t$,

$$U(x, t+T) = U(x,t), \qquad x \in [0,l], \qquad t \in \mathbb{R}, \tag{1.4}$$

with constant real coefficients $a$, $c$, and $\gamma_i$, $i = 1, 2, 3$, and a parameter $\lambda \neq 0$; moreover, $\gamma_1\gamma_2 \neq 0$. Here $T :=$ const $> 0$, $\alpha :=$ const $> 0$, and $F$ and $U$ are the given and unknown real functions $T$-periodic in time.

**Remark 1.1.** Since, by assumption, $\gamma_1\gamma_2 \neq 0$, it follows that the boundary condition (1.2) can be represented in the form

$$\gamma U_x(0,t) + U_t(0,t) + kU(0,t) = 0, \qquad t \in \mathbb{R},$$

where $\gamma := \gamma_1\gamma_2^{-1} \neq 0$ and $k := \gamma_3\gamma_2^{-1}$.

**Remark 1.2.** Set $\Omega_T := \Omega \cap \{0 < t < T\}$ and $f := F|_{\overline{\Omega}_T}$. One can readily see that if $U \in C^2(\overline{\Omega})$ is a classical solution of problem (1.1)–(1.4), then, by virtue of Remark 1.1, the function $u := U|_{\overline{\Omega}_T}$ is a classical solution of the nonlocal problem

$$L_\lambda u = f(x,t), \qquad (x,t) \in \Omega_T, \tag{1.5}$$

$$\gamma u_x(0,t) + u_t(0,t) + ku(0,t) = 0, \qquad 0 \le t \le T \qquad (\gamma \ne 0), \tag{1.6}$$

$$u(l,t) = 0, \qquad 0 \le t \le T, \tag{1.7}$$

$$(B_0 u)(x) = 0, \qquad (B_0 u_t)(x) = 0, \qquad x \in [0,l], \tag{1.8}$$

where $(B_0 w)(x) := w(x,0) - w(x,T)$ and $x \in [0,l]$, and conversely, if $f \in C(\overline{\Omega}_T)$ and $u \in C^2(\overline{\Omega}_T)$ is a classical solution of problem (1.5)–(1.8), then the function $U \in C^2(\overline{\Omega})$ that is the extension of the function $u$ $T$-periodic with respect to time from the domain $\Omega_T$ into the strip $\Omega$ is a classical solution of problem (1.1)–(1.4) provided that $f(x,0) = f(x,T)$, $x \in [0,l]$. Thus, instead of problem (1.1)–(1.4), we study problem (1.5)–(1.8).

**Definition 1.1.** Let $f \in C(\overline{\Omega}_T)$ be a given function. Set

$$\Gamma_1: \ x = 0, \qquad 0 \le t \le T, \qquad \Gamma_2: \ x = l, \qquad 0 \le t \le T.$$

A function $u$ is called a *strong generalized solution* of problem (1.5)–(1.8) of the class $C$ if it belongs to $C(\overline{\Omega}_T)$ and there exists a sequence of functions

$$u_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2; k) := \{w \in C^2(\overline{\Omega}_T): \ (\gamma w_x + w_t + kw)|_{\Gamma_1} = 0, \ w|_{\Gamma_2} = 0\},$$

such that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C(\overline{\Omega}_T)$ and $B_0 u_n \to 0$ and $B_0 u_{nt} \to 0$ in the spaces $C^1([0,l])$ and $C([0,l])$, respectively, as $n \to \infty$.

**Remark 1.3.** Obviously, a classical solution of problem (1.5)–(1.8) in the class $C^2(\overline{\Omega}_T)$ is a strong generalized solution of this problem in the class $C$.

Note that a wide set of publications (e.g., see [1–16] and the bibliography therein) deal with a periodic problem for nonlinear hyperbolic equations with boundary conditions of the Dirichlet or Robin type. In the present paper, we study the time-periodic problem (1.5)–(1.8) in which the direction of the derivative in the boundary condition does not coincide with the direction of the normal. Here the periodic problem is reduced to one time-nonlocal problem, and an a priori estimate (see Section 2) is proved for its solution. In the proof of the existence theorem, we use representations of solutions of the Cauchy, Goursat, and Darboux problems in various parts of the considered domain (see Section 3). In Section 4, we prove the uniqueness of the solution of problem (1.5)–(1.8), and in Section 5 we study the absence of a solution of that problem in the class of nonnegative functions.

## 2. A PRIORI ESTIMATE FOR A SOLUTION OF PROBLEM (1.5)–(1.8)

For the new unknown function

$$v := \varrho(\varepsilon,t)u, \qquad \varrho(\varepsilon,t) := \exp(\varepsilon t), \qquad (x,t) \in \Omega_T, \tag{2.1}$$

we rewrite problem (1.5)–(1.8) in the form

$$\begin{aligned}\Phi_\lambda(\varepsilon)v &:= v_{tt} - v_{xx} + 2(a-\varepsilon)v_t + (c + \varepsilon^2 - 2\varepsilon a)v + \lambda\varrho(-\alpha\varepsilon,t)|v|^\alpha v \\ &= \varrho(\varepsilon,t)f(x,t), \quad (x,t) \in \Omega_T,\end{aligned} \tag{2.2}$$

$$\gamma v_x(0,t) + v_t(0,t) + (k-\varepsilon)v(0,t) = 0, \qquad 0 \le t \le T \qquad (\gamma \ne 0), \tag{2.3}$$

$$v(l,t) = 0, \qquad 0 \le t \le T, \tag{2.4}$$

$$(B_\varepsilon v)(x) = 0, \qquad (B_\varepsilon v_t)(x) = 0, \qquad x \in [0,l], \tag{2.5}$$

$$(B_\varepsilon w)(x) := w(x,0) - \varrho(-\varepsilon,T)w(x,T), \qquad x \in [0,l], \tag{2.6}$$

where $\varepsilon = \varepsilon(a,c,k)$ is a sufficiently small positive number, which will be specified below.

Note that the function $u$ is a strong generalized solution of problem (1.5)–(1.8) in the class $C$ in the sense of Definition 1.1 if and only if the function $v$ is a strong generalized solution of problem (2.2)–(2.5) in the class $C$; i.e., the function $v$ belongs to $C(\overline{\Omega}_T)$, and there exists a sequence of functions $v_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2; k-\varepsilon)$ such that

$$\lim_{n\to\infty} \|v_n - v\|_{C(\overline{\Omega}_T)} = 0, \qquad \lim_{n\to\infty} \|\Phi_\lambda(\varepsilon)v_n - \varrho(\varepsilon,\cdot)f\|_{C(\overline{\Omega}_T)} = 0, \tag{2.7}$$

$$\lim_{n\to\infty} \|B_\varepsilon v_n\|_{C^1([0,l])} = 0, \qquad \lim_{n\to\infty} \|B_\varepsilon v_{nt}\|_{C([0,l])} = 0. \tag{2.8}$$

We introduce the condition

$$a > 0, \qquad c > 0; \qquad \gamma < 0, \qquad k > 0. \tag{2.9}$$

**Lemma 2.1.** *Let $\lambda > 0$, and let conditions (2.9) be satisfied. Then a strong generalized solution $u$ of problem (1.5)–(1.8) in the class $C$ satisfies the a priori estimate*

$$\|u\|_{C(\overline{\Omega}_T)} \le c_1 \|f\|_{C(\overline{\Omega}_T)} \tag{2.10}$$

*with a positive constant $c_1 = c_1(a, c, k, l, T)$ independent of the functions $u$ and $f$.*

**Proof.** Let $v$ be a strong generalized solution of problem (2.2)–(2.5) in the class $C$; i.e., $v$ belongs to $C(\overline{\Omega}_T)$, and there exists a sequence of functions

$$v_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2; k-\varepsilon)$$

such that the limit relations (2.7) and (2.8) hold.

Let us treat the function $v_n$ as a solution of the problem

$$\Phi_\lambda(\varepsilon)v_n = \varrho(\varepsilon,t)f_n(x,t), \qquad (x,t) \in \Omega_T, \tag{2.11}$$

$$\gamma v_{nx}(0,t) + v_{nt}(0,t) + (k-\varepsilon)v_n(0,t) = 0, \qquad 0 \le t \le T \qquad (\gamma \ne 0), \tag{2.12}$$

$$v_n(l,t) = 0, \qquad 0 \le t \le T, \tag{2.13}$$

$$(B_\varepsilon v_n)(x) = \varphi_n(x), \qquad (B_\varepsilon v_{nt})(x) = \psi_n(x), \qquad x \in [0,l], \tag{2.14}$$

where

$$f_n := \varrho(-\varepsilon,t)\Phi_\lambda(\varepsilon)v_n, \qquad \varphi_n := B_\varepsilon v_n, \qquad \psi_n := B_\varepsilon v_{nt}. \tag{2.15}$$

By multiplying both sides of relation (2.11) by $2v_{nt}$ and by integrating the resulting relation over the domain $\Omega_\tau := \{(x,t) \in \Omega_T : \ 0 < t < \tau\}$, $0 < \tau \le T$, we obtain

$$\int\limits_{\Omega_\tau} (v_{nt}^2)_t \,dx\,dt - 2\int\limits_{\Omega_\tau} v_{nxx}v_{nt}\,dx\,dt + 4(a-\varepsilon)\int\limits_{\Omega_\tau} v_{nt}^2\,dx\,dt + (c+\varepsilon^2-2\varepsilon a)\int\limits_{\Omega_\tau}(v_n^2)_t\,dx\,dt$$
$$+ \frac{2\lambda}{\alpha+2}\int\limits_{\Omega_\tau} \varrho(-\alpha\varepsilon,t)(|v_n|^{\alpha+2})_t\,dx\,dt = 2\int\limits_{\Omega_\tau} \varrho(\varepsilon,t)f_n v_{nt}\,dx\,dt. \tag{2.16}$$

Set $\omega_\tau$: $0 \le x \le l$, $t = \tau$; $\Gamma_{1,\tau}$: $x = 0$, $0 \le t \le \tau$, and $\Gamma_{2,\tau}$: $x = l$, $0 \le t \le \tau$; $0 \le \tau \le T$. Let $\nu := (\nu_x, \nu_t)$ be the unit outward normal to $\partial\Omega_\tau$. Since

$$\nu_x|_{\omega_0\cup\omega_\tau} = 0, \quad \nu_x|_{\Gamma_1} = -1, \quad \nu_x|_{\Gamma_2} = 1, \quad \nu_t|_{\Gamma_1\cup\Gamma_2} = 0, \quad \nu_t|_{\omega_0} = -1, \quad \nu_t|_{\omega_\tau} = 1,$$

it follows that, by using relations (2.12) and (2.13), by taking into account the relations $v_{nx}(0,t) = -\gamma^{-1}[v_{nt}(0,t) + (k-\varepsilon)v_n(0,t)]$, $0 \le t \le T$, and $v_{nt}(l,t) = 0$, $0 \le t \le T$, and by performing integration by parts on the left-hand side in relation (2.16), we obtain

$$\int_{\Omega_\tau}(v_{nt}^2)_t\,dx\,dt=\int_{\partial\Omega_\tau}v_{nt}^2\nu_t\,ds=\int_{\omega_\tau}v_{nt}^2\,dx-\int_{\omega_0}v_{nt}^2\,dx,$$

$$-2\int_{\Omega_\tau}v_{nxx}v_{nt}\,dx\,dt=\int_{\Omega_\tau}[(v_{nx}^2)_t-2(v_{nx}v_{nt})_x]\,dx\,dt$$

$$=\int_{\omega_\tau}v_{nx}^2\,dx-\int_{\omega_0}v_{nx}^2\,dx-2\int_{\Gamma_{1,\tau}\cup\Gamma_{2,\tau}}v_{nx}v_{nt}\nu_x\,dt$$

$$=\int_{\omega_\tau}v_{nx}^2\,dx-\int_{\omega_0}v_{nx}^2\,dx-2\gamma^{-1}\int_{\Gamma_{1,\tau}}[v_{nt}+(k-\varepsilon)v_n]v_{nt}\,dt$$

$$=\int_{\omega_\tau}v_{nx}^2\,dx-\int_{\omega_0}v_{nx}^2\,dx-2\gamma^{-1}\int_{\Gamma_{1,\tau}}v_{nt}^2\,dt-\gamma^{-1}(k-\varepsilon)\int_{\Gamma_{1,\tau}}(v_n^2)_t\,dt$$

$$=\int_{\omega_\tau}v_{nx}^2\,dx-\int_{\omega_0}v_{nx}^2\,dx-2\gamma^{-1}\int_{\Gamma_{1,\tau}}v_{nt}^2\,dt-\gamma^{-1}(k-\varepsilon)[v_n^2(0,\tau)-v_n^2(0,0)],$$

$$\int_{\Omega_\tau}(v_n^2)_t\,dx\,dt=\int_{\partial\Omega_\tau}v_n^2\nu_t\,ds=\int_{\omega_\tau}v_n^2\,dx-\int_{\omega_0}v_n^2\,dx,$$

$$\int_{\Omega_\tau}\varrho(-\alpha\varepsilon,t)(|v_n|^{\alpha+2})_t\,dx\,dt=\int_{\partial\Omega_\tau}\varrho(-\alpha\varepsilon,t)|v_n|^{\alpha+2}\nu_t\,ds+\alpha\varepsilon\int_{\Omega_\tau}\varrho(-\alpha\varepsilon,t)|v_n|^{\alpha+2}\,dx\,dt$$

$$=\varrho(-\alpha\varepsilon,\tau)\int_{\omega_\tau}|v_n|^{\alpha+2}\,dx-\int_{\omega_0}|v_n|^{\alpha+2}\,dx+\alpha\varepsilon\int_{\Omega_\tau}\varrho(-\alpha\varepsilon,t)|v_n|^{\alpha+2}\,dx\,dt.$$

Next, by virtue of these relations, from (2.16), we have

$$\int_{\omega_\tau}\left[(c+\varepsilon^2-2\varepsilon a)v_n^2+v_{nx}^2+v_{nt}^2+\frac{2\lambda}{\alpha+2}\varrho(-\alpha\varepsilon,\tau)|v_n|^{\alpha+2}\right]dx-\gamma^{-1}(k-\varepsilon)v_n^2(0,\tau)$$
$$+4(a-\varepsilon)\int_{\Omega_\tau}v_{nt}^2\,dx\,dt+\frac{2\lambda\alpha\varepsilon}{\alpha+2}\int_{\Omega_\tau}\varrho(-\alpha\varepsilon,t)|v_n|^{\alpha+2}\,dx\,dt-2\gamma^{-1}\int_{\Gamma_{1,\tau}}v_{nt}^2\,dt$$
$$=\int_{\omega_0}\left[(c+\varepsilon^2-2\varepsilon a)v_n^2+v_{nx}^2+v_{nt}^2+\frac{2\lambda}{\alpha+2}|v_n|^{\alpha+2}\right]dx-\gamma^{-1}(k-\varepsilon)v_n^2(0,0)$$
$$+2\int_{\Omega_\tau}\varrho(\varepsilon,t)f_nv_{nt}\,dx\,dt. \tag{2.17}$$

By virtue of conditions (2.9), we choose a number $\varepsilon=\varepsilon(a,c,k)>0$ small enough to ensure that

$$c+\varepsilon^2-2\varepsilon a\ge 0,\qquad k-\varepsilon\ge 0,\qquad a-\varepsilon\ge 0, \tag{2.18}$$

for example, $\varepsilon=\min(c/(2a),a/2,k/2)$.

By setting

$$w_{n,\lambda}(\tau):=\int_{\omega_\tau}\left[(c+\varepsilon^2-2\varepsilon a)v_n^2+v_{nx}^2+v_{nt}^2+\frac{2\lambda}{\alpha+2}\varrho(-\alpha\varepsilon,\tau)|v_n|^{\alpha+2}\right]dx$$
$$-\gamma^{-1}(k-\varepsilon)v_n^2(0,\tau),\qquad 0\le\tau\le T, \tag{2.19}$$

and by using inequalities (2.18) and $\lambda > 0$, from relation (2.17), we obtain

$$w_{n,\lambda}(\tau) \le w_{n,\lambda}(0) + 2\int\limits_{\Omega_\tau} \varrho(\varepsilon, t) f_n v_{nt}\, dx\, dt. \tag{2.20}$$

Since the inequality

$$2\varrho(\varepsilon, t) f_n v_{nt} \le \frac{\varrho(2\varepsilon, T)}{\varepsilon_1} f_n^2 + \varepsilon_1 v_{nt}^2 \tag{2.21}$$

holds for any $\varepsilon_1 = \text{const} > 0$, it follows from (2.20) that

$$w_{n,\lambda}(\tau) \le w_{n,\lambda}(0) + \frac{\varrho(2\varepsilon, T)}{\varepsilon_1} \int\limits_{\Omega_\tau} f_n^2\, dx\, dt + \varepsilon_1 \int\limits_{\Omega_\tau} v_{nt}^2\, dx\, dt. \tag{2.22}$$

By virtue of condition (2.18), notation (2.19), and the inequalities $\gamma < 0$ and $\lambda > 0$, we obtain the relation

$$\int\limits_{\Omega_\tau} v_{nt}^2\, dx\, dt = \int\limits_0^\tau \left[ \int\limits_{\omega_t} v_{nt}^2\, dx \right] dt \le \int\limits_0^\tau w_{n,\lambda}(t)\, dt,$$

which, together with inequality (2.22), implies that

$$w_{n,\lambda}(\tau) \le \varepsilon_1 \int\limits_0^\tau w_{n,\lambda}(t)\, dt + w_{n,\lambda}(0) + \frac{\varrho(2\varepsilon, T)}{\varepsilon_1} \int\limits_{\Omega_\tau} f_n^2\, dx\, dt, \qquad 0 \le \tau \le T.$$

The last inequality, together with the Gronwall lemma and the relation $\Omega_\tau \subset \Omega_T$, implies that

$$w_{n,\lambda}(\tau) \le [w_{n,\lambda}(0) + lT\varepsilon_1^{-1} \varrho(2\varepsilon, T) \|f_n\|^2_{C(\overline{\Omega}_T)}] \varrho(\varepsilon_1, \tau); \tag{2.23}$$

here we have used the obvious inequality

$$\|f_n\|^2_{L_2(\Omega_T)} \le lT \|f_n\|^2_{C(\overline{\Omega}_T)}.$$

Below we use the well-known inequalities (e.g., see [17, p. 67])

$$\begin{gathered} |a_1 + a_2|^2 = a_1^2 + a_2^2 + 2a_1 a_2 \le a_1^2 + a_2^2 + \varepsilon_2 a_1^2 + \varepsilon_2^{-1} a_2^2 = (1 + \varepsilon_2) a_1^2 + (1 + \varepsilon_2^{-1}) a_2^2, \\ |a_1 + a_2|^{\alpha+2} \le (1 + \varepsilon_2)|a_1|^{\alpha+2} + C(\alpha, \varepsilon_2)|a_2|^{\alpha+2}, \end{gathered} \tag{2.24}$$

which hold for any $\varepsilon_2 > 0$ and for arbitrary $a_1$ and $a_2$, where $C(\alpha, \varepsilon_2)$ is a positive constant.

By using notation (2.1), (2.6), and (2.19), condition (2.14), and inequalities (2.24), we obtain

$$\begin{aligned} w_{n,\lambda}(0) &= \int\limits_{\omega_0} \left[ (c + \varepsilon^2 - 2\varepsilon a) v_n^2 + v_{nx}^2 + v_{nt}^2 + \frac{2\lambda}{\alpha + 2} |v_n|^{\alpha+2} \right] dx - \gamma^{-1}(k - \varepsilon) v_n^2(0, 0) \\ &= \int\limits_{\omega_T} \Big[ (c + \varepsilon^2 - 2\varepsilon a)[\varrho(-\varepsilon, T) v_n + \varphi_n]^2 + [\varrho(-\varepsilon, T) v_{nx} + \varphi_{nx}]^2 + [\varrho(-\varepsilon, T) v_{nt} + \psi_n]^2 \\ &\quad + \frac{2\lambda}{\alpha + 2} |\varrho(-\varepsilon, T) v_n + \varphi_n|^{\alpha+2} \Big] dx - \gamma^{-1}(k - \varepsilon)[\varrho(-\varepsilon, T) v_n(0, T) + \varphi_n(0)]^2 \\ &\le (1 + \varepsilon_2) \varrho^2(-\varepsilon, T) w_{n,\lambda}(T) + \beta_{n,\lambda} = (1 + \varepsilon_2) \varrho(-2\varepsilon, T) w_{n,\lambda}(T) + \beta_{n,\lambda}, \end{aligned} \tag{2.25}$$

where

$$\beta_{n,\lambda} := C_1(\alpha,\varepsilon_2)\left\{\int\limits_{\omega_T}\left[(c+\varepsilon^2-2\varepsilon a)\varphi_n^2+\varphi_{nx}^2+\psi_n^2+\frac{2\lambda}{\alpha+2}|\varphi_n|^{\alpha+2}\right]dx-\gamma^{-1}(k-\varepsilon)\varphi_n^2(0)\right\},$$

$C_1(\alpha,\varepsilon_2) =$ const $> 0$.

One can readily see that

$$\lim_{n\to\infty}\beta_{n,\lambda}=0 \tag{2.26}$$

by virtue of relations (2.8) and (2.15). Then it follows from inequalities (2.23) and (2.25) that

$$w_{n,\lambda}(0)\le(1+\varepsilon_2)\varrho(-2\varepsilon,T)[w_{n,\lambda}(0)+lT\varepsilon_1^{-1}\varrho(2\varepsilon,T)\|f_n\|^2_{C(\overline{\Omega}_T)}]\varrho(\varepsilon_1,T)+\beta_{n,\lambda}. \tag{2.27}$$

By virtue of the equation $\varepsilon=\varepsilon(a,c,k)>0$ and notation (2.1), the positive numbers $\varepsilon_i = \varepsilon_i(\varepsilon,a,c,k,T)$, $i=1,2$, can be chosen small enough to ensure that

$$\mu := (1+\varepsilon_2)\varrho(-2\varepsilon,T)\varrho(\varepsilon_1,T)=(1+\varepsilon_2)\varrho(-2\varepsilon+\varepsilon_1,T)<1. \tag{2.28}$$

For example, since $e^x > 1+x$ for $x>0$, one can take $\varepsilon_1=\varepsilon$ and $\varepsilon_2=\varepsilon T$, where $\varepsilon = \min(c/(2a), a/2, k/2)$. Then, by using the estimate (2.28), from relation (2.27), we obtain the inequality

$$w_{n,\lambda}(0)\le\frac{lT\mu\varrho(2\varepsilon,T)}{\varepsilon_1(1-\mu)}\|f_n\|^2_{C(\overline{\Omega}_T)}+\frac{\beta_{n,\lambda}}{1-\mu}.$$

Relation (2.23), together with the last inequality, implies that

$$w_{n,\lambda}(\tau)\le\left[\frac{lT\varrho(2\varepsilon,T)}{\varepsilon_1}\|f_n\|^2_{C(\overline{\Omega}_T)}+\beta_{n,\lambda}\right]\frac{\varrho(\varepsilon_1,T)}{1-\mu}. \tag{2.29}$$

By taking into account conditions (2.13) and (2.18), notation (2.19), and the relations $\gamma<0$ and $\lambda>0$ and by using the Schwarz inequality, for any $(x,\tau)\in\Omega_T$, we obtain

$$|v_n(x,\tau)|^2=\left|\int\limits_x^l v_{nx}(\xi,\tau)\,d\xi\right|^2\le\int\limits_x^l 1^2\,d\xi\int\limits_x^l v_{nx}^2(\xi,\tau)\,d\xi\le l\int\limits_0^l v_{nx}^2(\xi,\tau)\,d\xi=l\int\limits_{\omega_\tau}v_{nx}^2\,dx\le lw_{n,\lambda}(\tau),$$

and hence

$$|v_n(x,\tau)|\le[lw_{n,\lambda}(\tau)]^{1/2},\qquad (x,\tau)\in\Omega_T. \tag{2.30}$$

Next, by virtue of the obvious inequality $\sqrt{a_1^2+a_2^2}\le\sqrt{|a_1|}+\sqrt{|a_2|}$, it follows from relations (2.29) and (2.30) that

$$|v_n(x,\tau)|\le c_1\|f_n\|_{C(\overline{\Omega}_T)}+c_2\sqrt{\beta_{n,\lambda}},\qquad (x,\tau)\in\Omega_T. \tag{2.31}$$

Here $c_2 := c_2(a,c,k,l,T) =$ const $>0$ and $c_1 := l\varrho(\varepsilon+2^{-1}\varepsilon_1,T)\sqrt{T/(\varepsilon_1(1-\mu))}$, where $\varepsilon_1=\varepsilon$; i.e.,

$$c_1 := l\varrho\left(\frac{3\varepsilon}{2},T\right)\left(\frac{T}{\varepsilon(1-\mu)}\right)^{1/2}; \tag{2.32}$$

moreover, if $a>0$, $c\ge a^2$, and $k\ge a$, then one can take the number $a$ for $\varepsilon$. It follows from inequality (2.31) that

$$\|v_n\|_{C(\overline{\Omega}_T)}\le c_1\|f_n\|_{C(\overline{\Omega}_T)}+c_2\sqrt{\beta_{n,\lambda}}.$$

Since the limit relation

$$\lim_{n\to\infty}\|f_n-f\|_{C(\overline{\Omega}_T)}=0$$

holds by virtue of relations (2.7) and (2.15), it follows that, by passing in the last inequality to the limit as $n \to \infty$ and by using (2.26), we obtain

$$\|v\|_{C(\overline{\Omega}_T)} \le c_1 \|f\|_{C(\overline{\Omega}_T)}. \tag{2.33}$$

This, together with notation (2.1) and the inequality $\varepsilon > 0$, implies the estimates

$$\|u\|_{C(\overline{\Omega}_T)} \le \|v\|_{C(\overline{\Omega}_T)} \le c_1 \|f\|_{C(\overline{\Omega}_T)}$$

with constant $c_1 := c_1(a, c, k, l, T) > 0$ defined in (2.32). The proof of Lemma 2.1 is complete.

**Remark 2.1.** In the linear case, i.e., in the case where the parameter $\lambda$ in Eq. (1.1) is zero, in a similar way, we introduce the notion of a strong generalized solution of problem (1.5)–(1/8) in the class $C$, which satisfies the same a priori estimate (2.10) under conditions (2.9) as shown by the argument presented in the proof of Lemma 2.1. Since problem (1.5)–(1.8) is linear, it follows that the strong generalized solution of this problem is unique in the class $C$.

**Remark 2.2.** It follows from the proof of Lemma 2.1 that if condition (2.9) is satisfied, then, for diminishing coefficients of Eq. (1.5), the constant $c_1$ occurring in the a priori estimates (2.10) and (2.33) can infinitely grow; for example, if $a \to 0+$, then $\varepsilon \to 0+$ because $0 < \varepsilon \le a$, and, by virtue of (2.32), we have $\lim_{\varepsilon\to 0+} c_1 = +\infty$. At the same time, the problem on the solvability of problem (1.5)–(1.8) in Section 3 is reduced to the problem of the derivation of a uniform, with respect to the parameter $\tau \in [0, 1]$, a priori estimate for a strong generalized solution of the equation

$$v_{tt} - v_{xx} + \tau(c - a^2)v + \tau\lambda\varrho(-\alpha a, t)|v|^\alpha v = \tau\varrho(a, t) f(x, t), \qquad (x, t) \in \overline{\Omega}_T, \tag{2.34}$$

satisfying the boundary and nonlocal conditions (2.3)–(2.5) for $\varepsilon = a$. To obtain a uniform, with respect to $\tau$, a priori estimate for the solution of problem (2.34), (2.3)–(2.5) for $\varepsilon = a$, it suffices to replace conditions (2.9) by the following more restrictive conditions:

$$a > 0, \qquad c \ge a^2, \qquad \gamma < 0, \qquad k \ge a. \tag{2.35}$$

Indeed, by considering the case of conditions (2.35) and by reproducing the argument used in the solution of problem (2.2)–(2.5) in Lemma 2.1 with $\varepsilon = a$ for the solution of problem (2.34), (2.3)–(2.5) with $\varepsilon = a$, we obtain the a priori estimate (2.33) with the constant

$$c_1 := l\varrho\left(\frac{3a}{2}, T\right)\left(\frac{T}{a(1-\mu)}\right)^{1/2}, \tag{2.36}$$

independent of $\tau \in [0, 1]$, where $\mu = (1 + aT)\varrho(-a, T) < 1$, because $\varrho(a, T) = e^{aT} > 1 + aT$.

## 3. REDUCTION OF PROBLEM (1.5)–(1.8) TO A NONLINEAR INTEGRAL EQUATION. SOLVABILITY OF PROBLEM (1.5)–(1.8)

In notation (2.1) with $\varepsilon = a$, i.e.,

$$v := \varrho(a, t)u, \qquad (x, t) \in \Omega_T, \tag{3.1}$$

we rewrite problem (1.5)–(1.8) in the form

$$\Phi_\lambda(a)v = v_{tt} - v_{xx} + (c - a^2)v + \lambda\varrho(-\alpha a, t)|v|^\alpha v = \varrho(a, t) f(x, t), \qquad (x, t) \in \Omega_T, \tag{3.2}$$

$$\gamma v_x(0, t) + v_t(0, t) + (k - a)v(0, t) = 0, \qquad 0 \le t \le T \qquad (\gamma \ne 0), \tag{3.3}$$

$$v(l, t) = 0, \qquad 0 \le t \le T, \tag{3.4}$$

$$(B_a v)(x) = 0, \qquad (B_a v_t)(x) = 0, \qquad x \in [0, l]. \tag{3.5}$$

Note that a function $u$ is a strong generalized solution of problem (1.5)–(1.8) in the class $C$ in the sense of Definition 1.1 if and only if the function $v$ is a strong generalized solution of problem (3.2)–(3.5) in the class $C$; i.e., $v$ belongs to $C(\overline{\Omega}_T)$, and there exists a function sequence $v_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2; k-a)$ such that

$$\lim_{n\to\infty} \|v_n - v\|_{C(\overline{\Omega}_T)} = 0, \qquad \lim_{n\to\infty} \|\Phi_\lambda(a)v_n - \varrho(a,\cdot)f\|_{C(\overline{\Omega}_T)} = 0,$$
$$\lim_{n\to\infty} \|B_a v_n\|_{C^1([0,l])} = 0, \qquad \lim_{n\to\infty} \|B_a v_{nt}\|_{C([0,l])} = 0.$$

In what follows, we study the solvability of problem (3.2)–(3.5) for the case in which $T = l$. To this end, we consider several auxiliary linear problems retaining the same notation $v$ and $f$ for the unknown function and the right-hand side of the equation.

**Problem I.** Find a function $v \in C^2(\overline{\Omega}_l)$ satisfying the equation

$$\Box v := v_{tt} - v_{xx} = f(x,t), \qquad (x,t) \in \Omega_l, \tag{3.6}$$

the boundary conditions

$$v(0,t) = \nu(t), \qquad v(l,t) = 0, \qquad 0 \le t \le l, \tag{3.7}$$

and the initial conditions

$$v(x,0) = \varphi(x), \qquad v_t(x,0) = \psi(x), \qquad x\imath n[0,l], \tag{3.8}$$

where $f \in C^1(\overline{\Omega}_l)$, $\nu \in C^2([0,l])$, $\varphi \in C^2([0,l])$, and $\psi \in C^1([0,l])$ are given functions satisfying the matching conditions

$$\nu(0) = \varphi(0), \quad \nu'(0) = \psi(0), \quad \nu''(0) - \varphi''(0) = f(0,0), \quad \varphi(l) = \psi(l) = 0, \quad \varphi''(l) = -f(l,0).$$

It is known that problem (3.8)–(3.10) is well posed; below we represent its solution $v \in C^2(\overline{\Omega}_l)$ in a convenient form. To this end, we split the domain $\Omega_l$, that is, the square with the vertices $O(0,0)$, $A_1(0,l)$, $A_2(l,l)$, and $A_3(l,0)$, into four right triangles $D_1 := \Delta OO_1A_3$, $D_2 := \Delta OO_1A_1$, $D_3 := \Delta A_3O_1A_2$, and $D_4 := \Delta O_1A_1A_2$, where the point $O_1(l/2, l/2)$ is the center of the square $\Omega_l$. It is well known that the solution of Problem I in the triangle $D_1$ is defined by the formula [18, pp. 158, 162]

$$v(x,t) = \frac{1}{2}[\varphi(x-t) + \varphi(x+t)] + \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega^1_{x,t}} f(\xi,\tau)\,d\xi\,d\tau, \qquad (x,t) \in D_1, \tag{3.9}$$

where $\Omega^1_{x,t}$ is the triangle with vertices $(x,t)$, $(x-t,0)$, and $(x+t,0)$.

To obtain a solution of Problem I in the triangles $D_2$, $D_3$, and $D_4$, we use the following relation (e.g., see [18, p. 173; 19; 20]) :

$$v(P) = v(P_1) + v(P_2) - v(P_3) + \frac{1}{2}\int\limits_{PP_1P_2P_3} f(\xi,\tau)\,d\xi\,d\tau, \tag{3.10}$$

which holds for any characteristic rectangle $PP_1P_2P_3 \subset \overline{\Omega}_l$ for Eq. (3.6), where $P$ and $P_3$, as well as $P_1$ and $P_2$, are opposite vertices of that rectangle; moreover, the ordinate of the point $P$ exceeds the ordinates of the remaining points.

Indeed, if the point $(x,t)$ belongs to $D_2$, then, by using relation (3.10) for the characteristic rectangle with vertices $P(x,t)$, $P_1(0,t-x)$, $P_2(t,x)$, and $P_3(t-x,0)$ and formula (3.9) for the point

$P_2(t,x)\in D_1$, we obtain

$$
\begin{aligned}
v(x,t) &= v(P_1)+v(P_2)-v(P_3)+\frac{1}{2}\int\limits_{PP_1P_2P_3} f(\xi,\tau)\,d\xi\,d\tau \\
&= \nu(t-x)+\frac{1}{2}[\varphi(t-x)+\varphi(t+x)]+\frac{1}{2}\int\limits_{t-x}^{t+x}\psi(\tau)\,d\tau \\
&\quad +\frac{1}{2}\int\limits_{\Omega^1_{t,x}} f(\xi,\tau)\,d\xi\,d\tau-\varphi(t-x)+\frac{1}{2}\int\limits_{PP_1P_2P_3} f(\xi,\tau)\,d\xi\,d\tau \\
&= \nu(t-x)+\frac{1}{2}[\varphi(t+x)-\varphi(t-x)]+\frac{1}{2}\int\limits_{t-x}^{t+x}\psi(\tau)\,d\tau \\
&\quad +\frac{1}{2}\int\limits_{\Omega^2_{x,t}} f(\xi,\tau)\,d\xi\,d\tau, \qquad (x,t)\in D_2.
\end{aligned}
\tag{3.11}
$$

Here $\Omega^2_{x,t}$ is the quadrangle $PP_1P_3\widetilde{P}_2$, and $\widetilde{P}_2=\widetilde{P}_2(x+t,0)$.

In a similar way, we obtain

$$
v(x,t)=\frac{1}{2}[\varphi(x-t)-\varphi(2l-x-t)]+\frac{1}{2}\int\limits_{x-t}^{2l-x-t}\psi(\tau)\,d\tau+\frac{1}{2}\int\limits_{\Omega^3_{x,t}} f(\xi,\tau)\,d\xi\,d\tau, \quad (x,t)\in D_3, \tag{3.12}
$$

and

$$
\begin{aligned}
v(x,t) &= \nu(t-x)-\frac{1}{2}[\varphi(t-x)+\varphi(2l-t-x)]+\frac{1}{2}\int\limits_{t-x}^{2l-t-x}\psi(\tau)\,d\tau \\
&\quad +\frac{1}{2}\int\limits_{\Omega^4_{x,t}} f(\xi,\tau)\,d\xi\,d\tau, \qquad (x,t)\in D_4,
\end{aligned}
\tag{3.13}
$$

where $\Omega^3_{x,t}$ is the quadrangle with the vertices $P^3(x,t)$, $P^3_1(l,x+t-l)$, $P^3_2(x-t,0)$, and $P^3_3(2l-x-t,0)$ and $\Omega^4_{x,t}$ is the pentagon with the vertices $P^4(x,t)$, $P^4_1(0,t-x)$, $P^4_2(t-x,0)$, $P^4_3(2l-x-t,0)$, and $P^4_4(l,x+t-l)$.

**Problem II.** Find a solution $v\in C^2(\overline{\Omega}_l)$ of Eq. (3.6) for $f\in C^1(\overline{\Omega}_l)$ satisfying the boundary and nonlocal conditions (3.3)–(3.5).

In what follows, we show that, under the conditions

$$
a>0, \qquad \gamma\neq\frac{1-\varrho(-2a,l)}{1+\varrho(-2a,l)}, \tag{3.14}
$$

Problem II has a unique classical solution $v\in C^2(\overline{\Omega}_l)$ for $f\in C^1(\overline{\Omega}_l)$ and a unique strong generalized solution (by analogy with Definition 1.1) $v$ in the class $C$ for $f\in C(\overline{\Omega}_l)$; moreover, in both cases, the solution can be represented by quadratures via the same formula.

**Remark 3.1.** One can readily see that condition (2.9) implies condition (3.14).

To construct a solution $v$ of Problem II by quadratures, we treat it as a solution of Problem I and assume for now that the functions

$$
\varphi(x):=v(x,0), \qquad \psi(x):=v_t(x,0), \qquad \nu(t):=v(0,t), \qquad 0\le x,t\le l, \tag{3.15}
$$

on the right-hand sides in conditions (3.7) and (3.8) are unknown.

**Remark 3.2.** By taking into account the structure of the domain $\Omega^2_{x,t}$, for $(x,t)\in D_2$, we obtain

$$\frac{1}{2}\int\limits_{\Omega^2_{x,t}} f(\xi,\tau)\,d\xi\,d\tau=\frac{1}{2}\int\limits_{0}^{t-x}d\tau\int\limits_{-x+t-\tau}^{x+t-\tau}f(\xi,\tau)\,d\xi+\frac{1}{2}\int\limits_{t-x}^{t}d\tau\int\limits_{x-t+\tau}^{x+t-\tau}f(\xi,\tau)\,d\xi.$$

Therefore, by virtue of formula (3.11), we have the relations

$$v_x(x,t)=-\nu'(t-x)+\frac{1}{2}[\varphi'(t+x)+\varphi'(t-x)+\psi(t+x)+\psi(t-x)]$$
$$+\frac{1}{2}\int\limits_0^t f(x+t-\tau,\tau)\,d\tau+\frac{1}{2}\int\limits_0^{t-x} f(t-x-\tau,\tau)\,d\tau-\frac{1}{2}\int\limits_{t-x}^{t} f(x-t+\tau,\tau)\,d\tau. \quad (3.16)$$

Likewise, we obtain

$$v_t(x,t)=\nu'(t-x)+\frac{1}{2}[\varphi'(t+x)-\varphi'(t-x)+\psi(t+x)-\psi(t-x)]$$
$$+\frac{1}{2}\int\limits_0^t f(x+t-\tau,\tau)\,d\tau-\frac{1}{2}\int\limits_0^{t-x} f(t-x-\tau,\tau)\,d\tau$$
$$+\frac{1}{2}\int\limits_{t-x}^{t} f(x-t+\tau,\tau)\,d\tau,\qquad (x,t)\in D_2. \quad (3.17)$$

**Remark 3.3.** Formulas for the derivatives $v_x$ and $v_t$ similar to (3.16) and (3.17) hold for the solution of Problem I, that is, problem (3.6)–(3.8) in the remaining domains $D_1$, $D_3$, and $D_4$.

By setting $x=0$ in formula (3.16) and by taking into account the boundary condition (3.3) for the unknown functions $\nu$, $\varphi$, and $\psi$, from (3.15), we obtain the equation

$$(\gamma-1)\nu'(t)+(a-k)\nu(t)=\gamma\left[\varphi'(t)+\psi(t)+\int\limits_0^t f(t-\tau,\tau)\,d\tau\right],\qquad 0\le t\le l. \quad (3.18)$$

In the domain $D_4$, the representation (3.13) for the function $v$ with $t=l$ has the form

$$v(x,l)=\nu(l-x)-\varphi(l-x)+f_1(x,l),\qquad 0\le x\le l, \quad (3.19)$$

where

$$f_1(x,t):=\frac{1}{2}\int\limits_{\Omega^4_{x,t}} f(\xi,\tau)\,d\xi\,d\tau. \quad (3.20)$$

**Remark 3.4.** By taking into account the structure of the domain $\Omega^4_{x,t}$, for $(x,t)\in D_4$, we obtain

$$2f_1(x,t)=\int\limits_0^{x+t-l}d\tau\int\limits_{t-x-\tau}^{\tau+2l-x-t}f(\xi,\tau)\,d\xi+\int\limits_{x+t-l}^{t-x}d\tau\int\limits_{t-x-\tau}^{x+t-\tau}f(\xi,\tau)\,d\xi+\int\limits_{t-x}^{t}d\tau\int\limits_{x-t+\tau}^{x+t-\tau}f(\xi,\tau)\,d\xi,$$

whence, by using the differentiation, we obtain

$$2f_{1x}(x,l)=-\int\limits_0^x f(l-x+\tau,\tau)\,d\tau+\int\limits_0^{l-x} f(l-x-\tau,\tau)\,d\tau$$
$$+\int\limits_x^l f(x+l-\tau,\tau)\,d\tau-\int\limits_{l-x}^{l} f(x-l+\tau,\tau)\,d\tau \quad (3.21)$$

and

$$2f_{1t}(x,l) = -\int\limits_0^x f(l-x+\tau,\tau)\,d\tau - \int\limits_0^{l-x} f(l-x-\tau,\tau)\,d\tau + \int\limits_x^l f(x+l-\tau,\tau)\,d\tau + \int\limits_{l-x}^l f(x-l+\tau,\tau)\,d\tau. \qquad (3.22)$$

By substituting the function (3.19) into the first condition in (3.5), for the unknown functions $\varphi$ and $\nu$ from (3.15), we obtain the equation

$$\varphi(x) + \varrho(-a,l)\varphi(l-x) - \varrho(-a,l)\nu(l-x) = \psi_0(x), \qquad 0 \le x \le l, \qquad (3.23)$$

where

$$\psi_0(x) := \varrho(-a,l)f_1(x,l), \qquad 0 \le x \le l. \qquad (3.24)$$

By replacing $x$ in Eq. (3.23) by $l-x$, we obtain the equation

$$\varrho(-a,l)\varphi(x) + \varphi(l-x) - \varrho(-a,l)\nu(x) = \psi_0(l-x), \qquad 0 \le x \le l. \qquad (3.25)$$

By virtue of the first condition in (3.14) and notation (2.1), we have $\varrho(-a,l) < 1$, which, after the elimination of $\varphi(l-x)$ from system (3.23), (3.25),

$$\varphi(x) = \frac{\psi_0(x) - \varrho(-a,l)\psi_0(l-x) + \varrho(-a,l)\nu(l-x) - \varrho(-2a,l)\nu(x)}{1-\varrho(-2a,l)}, \qquad 0 \le x \le l. \qquad (3.26)$$

Next, by differentiating relation (3.13) with respect to the variable $t$, we obtain the representation

$$v_t(x,t) = \nu'(t-x) - \frac{1}{2}[\varphi'(t-x) - \varphi'(2l-t-x)] - \frac{1}{2}[\psi(2l-t-x) + \psi(t-x)] + f_{1t}(x,t),$$

whence for $t = l$ we have

$$v_t(x,l) = \nu'(l-x) - \psi(l-x) + f_{1t}(x,l), \qquad 0 \le x \le l. \qquad (3.27)$$

By substituting the derivative (3.27) into the second condition in (3.5), for the unknown functions $\psi$ and $\nu$ in (3.15), we obtain the equation

$$\psi(x) + \varrho(-a,l)\psi(l-x) - \varrho(-a,l)\nu'(l-x) = \psi_1(x), \qquad 0 \le x \le l. \qquad (3.28)$$

Here

$$\psi_1(x) := \varrho(-a,l)f_{1t}(x,l), \qquad 0 \le x \le l. \qquad (3.29)$$

By replacing $x$ in Eq. (3.28) by $l-x$, we obtain

$$\varrho(-a,l)\psi(x) + \psi(l-x) - \varrho(-a,l)\nu'(x) = \psi_1(l-x), \qquad 0 \le x \le l. \qquad (3.30)$$

By following the derivation of the representation (3.26) and by eliminating $\psi(l-x)$ from system (3.28), (3.30), we obtain

$$\psi(x) = \frac{\psi_1(x) - \varrho(-a,l)\psi_1(l-x) + \varrho(-a,l)\nu'(l-x) - \varrho(-2a,l)\nu'(x)}{1-\varrho(-2a,l)}, \qquad 0 \le x \le l. \qquad (3.31)$$

By virtue of relations (3.26) and (3.31), we have

$$\varphi'(t) + \psi(t) = \frac{\psi_0'(t) + \psi_1(t) + \varrho(-a,l)\psi_0'(l-t) - \varrho(-a,l)\psi_1(l-t) - 2\varrho(-2a,l)\nu'(t)}{1-\varrho(-2a,l)}. \qquad (3.32)$$

By using relations (3.21), (3.22), (3.24), and (3.29), one can readily show that

$$\psi_0'(t)+\psi_1(t)=\varrho(-a,l)[f_{1x}(t,l)+f_{1t}(t,l)] = \varrho(-a,l)\left[-\int_0^t f(l-t+\tau,\tau)\,d\tau+\int_t^l f(t+l-\tau,\tau)\,d\tau\right]$$

and

$$\psi_0'(l-t)-\psi_1(l-t)=\varrho(-a,l)[f_{1x}(l-t,l)-f_{1t}(l-t,l)] = \varrho(-a,l)\left[\int_0^t f(t-\tau,\tau)\,d\tau-\int_t^l f(-t+\tau,\tau)\,d\tau\right].$$

By taking into account these relations, by substituting the expression (3.32) into relation (3.18), and by using (3.14), we obtain

$$\nu'(t)+b\nu(t)=h(t),\qquad 0\le t\le l. \tag{3.33}$$

Here

$$b:=\frac{(a-k)[1-\varrho(-2a,l)]}{\gamma-1+\varrho(-2a,l)+\gamma\varrho(-2a,l)}, \tag{3.34}$$

$$h(t):=\frac{\gamma}{\gamma-1+\varrho(-2a,l)+\gamma\varrho(-2a,l)}\left[-\varrho(-a,l)\int_0^t f(l-t+\tau,\tau)\,d\tau + \varrho(-a,l)\int_t^l f(t+l-\tau,\tau)\,d\tau-\varrho(-2a,l)\int_t^l f(-t+\tau,\tau)\,d\tau+\int_0^t f(t-\tau,\tau)\,d\tau\right]. \tag{3.35}$$

By virtue of condition (3.5) and the notation $\nu$ in (3.15), a solution of Eq. (3.33) should satisfy the nonlocal conditions

$$\nu(0)=\varrho(-a,l)\nu(l), \tag{3.36}$$

$$\nu'(0)=\varrho(-a,l)\nu'(l). \tag{3.37}$$

One can readily see that the unique solution of problem (3.33), (3.36) has the form

$$\nu(t)=\frac{\varrho(-b,t)}{\varrho(a+b,l)-1}\left[\varrho(a+b,l)\int_0^t \varrho(b,s)h(s)\,ds+\int_t^l \varrho(b,s)h(s)\,ds\right],\qquad 0\le t\le l, \tag{3.38}$$

where $b$ and $h$ are determined by relations (3.34) and (3.35).

**Remark 3.5.** One can readily see that the function $h$ occurring in (3.35) satisfies the nonlocal condition $h(0)=\varrho(-a,l)h(l)$ similar to (3.36), which, in turn, with regard to relations (3.33) and (3.36), implies the nonlocal condition (3.37).

**Remark 3.6.** One can also readily see that if $f\in C^1(\overline{\Omega}_l)$ [respectively, $f\in C(\overline{\Omega}_l)$], then, by virtue of relation (3.35), the function $\nu$ occurring in the representation (3.38) belongs to $C^2([0,l])$ [respectively, $\nu\in C^1([0,l])$]; therefore, by virtue of relations (3.20)–(3.22), (3.24), and (3.29), the functions $\varphi$ and $\psi$ defined by relations (3.26) and (3.31) also belong to the classes $C^2([0,l])$ [respectively, $C^1([0,l])$] and $C^1([0,l])$ [respectively, $C([0,l])$].

**Remark 3.7.** By virtue of the representations (3.9), (3.11), (3.12), and (3.13) in the domains $D_1$, $D_2$, $D_3$, and $D_4$, respectively, with the functions $\varphi$, $\psi$, and $\nu$ replaced by their expressions from formulas (3.26), (3.31), and (3.38), the solution $v$ of Problem II, that is, problem (3.6), (3.3)–(3.5), can be represented in the form

$$v = \Box^{-1} f, \tag{3.39}$$

where, by virtue of Remarks 3.2–3.6, the linear integral operator

$$\Box^{-1} : \ C^1(\overline{\Omega}_l) \to C^2(\overline{\Omega}_l) \tag{3.40}$$

is continuous. The same representations, together with the above remarks, imply that the linear integral operator

$$\Box^{-1} : \ C(\overline{\Omega}_l) \to C^1(\overline{\Omega}_l) \tag{3.41}$$

occurring in (3.39) is continuous as well. In addition, hence it follows that Problem II has a unique classical solution $v \in C^2(\overline{\Omega}_l)$ for $f \in C^1(\overline{\Omega}_l)$, and this solution admits the representation (3.39).

**Remark 3.8.** Note that since the space $C^1(\overline{\Omega}_l)$ is compactly embedded in $C(\overline{\Omega}_l)$ [21, p. 135 of the Russian translation], it follows from (3.41) that the linear operator $\Box^{-1} : \ C(\overline{\Omega}_l) \to C(\overline{\Omega}_l)$ is a compact operator.

**Remark 3.9.** If condition (3.14) is satisfied and $f \in C(\overline{\Omega}_l)$, then the linear problem II has a unique strong generalized solution $v$ in the class $C$. Indeed, let us show that the solution is given by the function (3.39). Since the space $C^1(\overline{\Omega}_l)$ is dense in the space $C(\overline{\Omega}_l)$ [22, p. 37 of the Russian translation], it follows that there exists a function sequence $f_n \in C^1(\overline{\Omega}_l)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{C(\overline{\Omega}_l)} = 0. \tag{3.42}$$

By virtue of the representations (3.39) and (3.40), the function $v_n = \Box^{-1} f_n \in C^2(\overline{\Omega}_l)$ is a classical solution of the linear problem II for $f = f_n$; moreover, $v_n \to v = \Box^{-1} f$ and $\Box v_n = f_n \to f$ as $n \to \infty$ in the space $C(\overline{\Omega}_l)$ by virtue of (3.41) and (3.42). Therefore, the function (3.39) is a strong generalized solution of problem II in the class $C$. Now let us show that this problem has no other generalized solutions. Indeed, if $\widetilde{v}$ is another strong generalized solution of Problem II in the class $C$, then, by definition, there exists a function sequence $\widetilde{v}_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2; k - a)$ such that

$$\lim_{n\to\infty} \|\widetilde{v}_n - \widetilde{v}\|_{C(\overline{\Omega}_l)} = 0, \qquad \lim_{n\to\infty} \|\Box\widetilde{v}_n - f\|_{C(\overline{\Omega}_l)} = 0. \tag{3.43}$$

By setting $f_n = \Box\widetilde{v}_n$ and by using (3.39), we obtain $\widetilde{v}_n = \Box^{-1} f_n$. By passing in the last relation to the limit as $n \to \infty$ and by taking into account relations (3.41) and (3.43), we obtain $\widetilde{v} = \Box^{-1} f$, i.e., $\widetilde{v} = v$. The latter contradicts the above-stipulated assumption. Therefore, if condition (3.14) is valid and $f \in C(\overline{\Omega}_l)$, then the linear problem II has a unique strong generalized solution $v$ in the class $C$, which admits the representation (3.39) and, by virtue of (3.41), belongs to the class $C^1(\overline{\Omega}_l)$. Under condition (3.14), the representations (3.39) and (3.41) imply that the strong generalized solution $v = \Box^{-1} f$ of Problem II in the class $C$ satisfies the estimate

$$\|\Box^{-1} f\|_{C(\overline{\Omega}_l)} \le c_0 \|f\|_{C(\overline{\Omega}_l)}, \tag{3.44}$$

where $c_0 = c_0(a, \gamma, k, l) > 0$.

**Remark 3.10.** As was mentioned above, by virtue of the transformation (3.1), problem (1.5)–(1.8) is equivalent to problem (3.2)–(3.5); moreover, by virtue of condition (3.14) and the above Remarks 3.6–3.9, for $f \in C(\overline{\Omega}_l)$ the function $v$ is a strong generalized solution of problem (3.2)–(3.5) in the class $C$ if and only if $v$ is a continuous solution of the nonlinear integral equation

$$v = Kv := \Box^{-1}\{(a^2 - c)v - \lambda\varrho(-\alpha a, t)|v|^\alpha v + \varrho(a, t) f(x, t)\}. \tag{3.45}$$

Note also that, by virtue of (3.41), a continuous solution $v$ of the nonlinear integral equation (3.45) actually belongs to the class $C^1(\overline{\Omega}_l)$; and if $f \in C^1(\overline{\Omega}_l)$, then, by virtue of (3.40), the function $v$ belongs to $C^2(\overline{\Omega}_l)$ and is a classical solution of problem (3.2)–(3.5).

**Remark 3.11** The operator $K: C(\overline{\Omega}_l) \to C(\overline{\Omega}_l)$ in (3.45) is continuous and compact, because the nonlinear operator $N: C(\overline{\Omega}_l) \to C(\overline{\Omega}_l)$ acting by the rule

$$Nv = (a^2 - c)v - \lambda\varrho(-\alpha a, t)|v|^\alpha v + \varrho(a, t)f(x, t)$$

is bounded and continuous and, by virtue of Remark 3.8, the linear operator $\Box^{-1}: C(\overline{\Omega}_l) \to C(\overline{\Omega}_l)$ is compact. By virtue of Remark 2.2, for $\tau \in [0, 1]$, each solution $v \in C(\overline{\Omega}_l)$ of the equation $v = \tau K v$ is a solution of problem (2.34), (2.3)–(2.5) for $\varepsilon = a$, which, under condition (2.35), satisfies the a priori estimate (2.33) with a positive constant $c_1$ defined by (2.36) for $T = l$ and independent of $\tau \in [0, 1]$. Therefore, by the Leray–Schauder theorem [23, p. 375], Eq. (3.45) has at least one solution $v \in C(\overline{\Omega}_l)$.

By virtue of the above remarks, we have thereby proved the following assertion.

**Theorem 3.1.** *Let* $\lambda > 0$, *let condition* (2.35) *be satisfied, and let* $f \in C(\overline{\Omega}_l)$. *Then problem* (1.5)–(1.8) *has at least one strong generalized solution* $u$ *in the class* $C$ *in the sense of Definition* 1.1, *which belongs to the space* $C^1(\overline{\Omega}_l)$; *moreover, if* $f \in C^1(\overline{\Omega}_l)$, *then this solution is classical.*

## 4. UNIQUENESS OF THE SOLUTION OF PROBLEM (1.5)–(1.8)

Let the following condition be satisfied:

$$|a^2 - c| < 1/c_0, \qquad 0 < \lambda < \lambda_0, \tag{4.1}$$

where $\lambda_0 := (1 - c_0|a^2 - c|)(c_0 M_0)^{-1}$, $M_0 := (1 + \alpha)(2c_1\|f\|_{C(\overline{\Omega}_l)})^\alpha$, and $c_0$ and $c_1$ are the constants occurring in (3.44) and (2.32), respectively.

**Theorem 4.1.** *Let* $T = l$, *let conditions* (2.35) *and* (4.1) *be satisfied, and let* $f \in C(\overline{\Omega}_l)$. *Then problem* (1.5)–(1.8) *has at most one strong generalized solution in the class* $C$.

**Proof.** Since problem (1.5)–(1.8) is equivalent to problem (3.2)–(3.5), it suffices to consider the problem on the uniqueness of the solution of the latter problem. Suppose that problem (3.2)–(3.5) has two possible distinct strong generalized solutions $v^1$ and $v^2$ in the class $C$. By Definition 1.1, there exists a sequence of functions $v_n^i \in \overset{\circ}{C}{}^2(\overline{\Omega}_l, \Gamma_1, \Gamma_2; k - a)$ such that, in particular, the following limit relations hold:

$$\lim_{n\to\infty} \|v_n^i - v^i\|_{C(\overline{\Omega}_l)} = 0, \qquad \lim_{n\to\infty} \|\Phi_\lambda(a)v_n^i - \varrho(a, \cdot)f\|_{C(\overline{\Omega}_l)} = 0. \tag{4.2}$$

Set $\omega_n := v_n^2 - v_n^1$. One can readily see that the function $\omega_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_l, \Gamma_1, \Gamma_2; k - a)$ is a classical solution of the problem

$$\Box\omega_n = (f_n + g_n)(x, t), \qquad (x, t) \in \Omega_l, \tag{4.3}$$

$$\gamma\omega_{nx}(0, t) + \omega_{nt}(0, t) + (k - a)\omega_n(0, t) = 0, \qquad 0 \le t \le l \qquad (\gamma \ne 0), \tag{4.4}$$

$$\omega_n(l, t) = 0, \qquad 0 \le t \le l, \tag{4.5}$$

$$(B_a\omega_n)(x) = 0, \qquad (B_a\omega_{nt})(x) = 0, \qquad x \in [0, l]. \tag{4.6}$$

Here

$$f_n := \Phi_\lambda(a)v_n^2 - \Phi_\lambda(a)v_n^1, \tag{4.7}$$

$$\begin{aligned} g_n &:= (a^2 - c)\omega_n - \lambda\varrho(-\alpha a, t)(|v_n^2|^\alpha v_n^2 - |v_n^1|^\alpha v_n^1) \\ &= (a^2 - c)\omega_n - \lambda(1 + \alpha)\varrho(-\alpha a, t)\left(\int_0^1 |v_n^1 + s(v_n^2 - v_n^1)|^\alpha\, ds\right)\omega_n, \end{aligned} \tag{4.8}$$

where we have used the obvious relation

$$\varphi(x_2) - \varphi(x_1) = (x_2 - x_1) \int\limits_0^1 \varphi'[x_1 + s(x_2 - x_1)]\, ds$$

for the function $\varphi(x) := |x|^\alpha x$, $x \in \mathbb{R}$, with $x_1 = v_n^1$ and $x_2 = v_n^2$.

By virtue of Remark 3.9 and the estimate (3.44), for the solution $\omega_n$ of problem (4.3)–(4.6), we have

$$\|\omega_n\|_{C(\overline{\Omega}_l)} \le c_0 \|f_n + g_n\|_{C(\overline{\Omega}_l)} \le c_0 \|f_n\|_{C(\overline{\Omega}_l)} + c_0 \|g_n\|_{C(\overline{\Omega}_l)}. \tag{4.9}$$

It follows from relations (4.2) and (4.7) that

$$\lim_{n\to\infty} \|f_n\|_{C(\overline{\Omega}_l)} = 0. \tag{4.10}$$

By virtue of the a priori estimate (2.33), the solutions $v^1$ and $v^2$ of problem (3.2)–(3.5) satisfy the inequalities

$$\|v^i\|_{C(\overline{\Omega}_l)} \le c_1 \|f\|_{C(\overline{\Omega}_l)}, \qquad i = 1, 2.$$

This, together with the inequalities $\alpha > 0$ and $a > 0$, implies that

$$(1+\alpha)\varrho(-\alpha a, t) \int\limits_0^1 |v_n^1 + s(v_n^2 - v_n^1)|^\alpha\, ds \le M_0, \tag{4.11}$$

where $M_0$ is the constant occurring in condition (4.1).

It follows from relations (4.8), (4.9), and (4.11) that

$$\|\omega_n\|_{C(\overline{\Omega}_l)} \le c_0 \|f_n\|_{C(\overline{\Omega}_l)} + c_0(|a^2 - c| + \lambda M_0) \|\omega_n\|_{C(\overline{\Omega}_l)},$$

which, together with condition (4.1), implies the estimate

$$\|\omega_n\|_{C(\overline{\Omega}_l)} \le \frac{1}{M_0(\lambda_0 - \lambda)} \|f_n\|_{C(\overline{\Omega}_l)}. \tag{4.12}$$

Since $\lim_{n\to\infty} \|\omega_n\|_{C(\overline{\Omega}_l)} = \|v^2 - v^1\|_{C(\overline{\Omega}_l)}$ by virtue of relations (4.2), it follows that, by taking into account relation (4.10) and by passing in the estimate (4.12) to the limit as $n \to \infty$, we obtain $\|v^2 - v^1\|_{C(\overline{\Omega}_l)} \le 0$, i.e., $v^1 = v^2$. The last assertion contradicts the above assumption. The proof of Theorem 4.1 is complete.

## 5. THE ABSENCE OF SOLUTION OF PROBLEM (1.5)–(1.8) IN THE CLASS OF NONNEGATIVE FUNCTIONS

Below, by using the method of test functions [13, pp. 10–12], we show that if the condition $\lambda > 0$ is violated, then problem (1.5)–(1.8) has at most one strong nonnegative generalized solution in the class $C$ in the sense of Definition 1.1.

**Lemma 5.1.** *Let $u \ge 0$ be a strong generalized solution of problem* (1.5)–(1.8) *in the class $C$ in the sense of Definition* 1.1. *Then one has the integral relation*

$$\int\limits_{\Omega_T} u\Box\varphi\, dx\, dt - 2a \int\limits_{\Omega_T} u\varphi_t\, dx\, dt + c \int\limits_{\Omega_T} u\varphi\, dx\, dt = -\lambda \int\limits_{\Omega_T} |u|^{\alpha+1}\varphi\, dx\, dt + \int\limits_{\Omega_T} f\varphi\, dx\, dt \tag{5.1}$$

*for any test function $\varphi$ such that*

$$\varphi \in C^2(\overline{\Omega}_T), \qquad \varphi|_{\partial\Omega_T} = 0, \qquad \nabla\varphi|_{\partial\Omega_T} = 0, \tag{5.2}$$

*where $\nabla := (\partial/\partial x, \partial/\partial t)$.*

**Proof.** By the definition of a strong generalized solution $u$ of problem (1.5)–(1.8) in the class $C$ in the sense of Definition 1.1, the function $u$ belongs to $C(\overline{\Omega}_T)$, and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2; k)$ such that, in particular, the following limit relations are true:

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{\Omega}_T)} = 0, \qquad \lim_{n\to\infty} \|L_\lambda u_n - f\|_{C(\overline{\Omega}_T)} = 0. \tag{5.3}$$

Set $f_n := L_\lambda u_n$. We multiply both the sides of the relation $L_\lambda u_n = f_n$ by the function $\varphi$ and integrate the resulting relation over the domain $\Omega_T$. After the integration by parts on the left-hand side in this relation with regard to the inclusion (5.2), we obtain

$$\int\limits_{\Omega_T} u_n \Box\varphi\,dx\,dt - 2a\int\limits_{\Omega_T} u_n\varphi_t\,dx\,dt + c\int\limits_{\Omega_T} u_n\varphi\,dx\,dt = -\lambda\int\limits_{\Omega_T} |u_n|^\alpha u_n\varphi\,dx\,dt + \int\limits_{\Omega_T} f_n\varphi\,dx\,dt.$$

By using relations (5.3), by passing in the last relation to the limit as $n \to \infty$, and by taking into account the inequality $u \ge 0$, we obtain the integral relation (5.1). The proof of the lemma is complete.

Let us introduce a function $\varphi_0 := \varphi_0(x, t)$ such that

$$\varphi_0 \in C^2(\overline{\Omega}_T), \qquad \varphi_0|_{\Omega_T} > 0, \qquad \varphi_0|_{\partial\Omega_T} = 0, \qquad \nabla\varphi_0|_{\partial\Omega_T} = 0 \tag{5.4}$$

and

$$\kappa_0 := \int\limits_{\Omega_T} \frac{|\Box\varphi_0 - 2a\varphi_{0t} + c\varphi_0|^{p'}}{\varphi_0^{p'-1}}\,dx\,dt < +\infty, \qquad p' = 1 + \frac{1}{\alpha}. \tag{5.5}$$

By a simple verification, one can show that, for a function $\varphi_0$ satisfying conditions (5.4) and (5.5), one can take the function

$$\varphi_0(x, t) = [xt(l - x)(l - t)]^m, \qquad (x, t) \in \Omega_T,$$

for sufficiently large $m > 0$.

If $\lambda < 0$, then, by Lemma 5.1 and relation (5.1), where the function $\varphi = \varphi_0$ can be taken for the function $\varphi$ with regard to conditions (5.4), we have

$$|\lambda|\int\limits_{\Omega_T} u^{\alpha+1}\varphi_0\,dx\,dt \le \int\limits_{\Omega_T} u|\Box\varphi_0 - 2a\varphi_{0t} + c\varphi_0|\,dx\,dt - \int\limits_{\Omega_T} f\varphi_0\,dx\,dt. \tag{5.6}$$

The absence of a solution of problem (1.5)–(1.8) is proved in the following assertion.

**Theorem 5.1.** *Let $\lambda < 0$, $\alpha > 0$, and $f = \beta f_0$, where $\beta =$ const $> 0$, $f_0 \in C(\overline{\Omega}_T)$, $f_0 \ge 0$, and $f_0 \not\equiv 0$. Then there exists a positive number $\beta_0 = \beta_0(\alpha, \lambda, f_0)$ such that for $\beta > \beta_0$ problem* (1.5)–(1.8) *has no strong generalized solution $u \ge 0$ in the class $C$ in the sense of Definition* 1.1.

**Proof.** If we set $a = u\varphi_0^{1/p}$ and $b = |\Box\varphi_0 - 2a\varphi_{0t} + c\varphi_0|\varphi_0^{-1/p}$ in the Young inequality

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{p'\varepsilon^{p'-1}}b^{p'}; \qquad a, b \ge 0, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \qquad p := \alpha + 1 > 1,$$

with the parameter $\varepsilon > 0$, then, by virtue of the relation $p'/p = p' - 1$, we obtain the inequality

$$u|\Box\varphi_0 - 2a\varphi_{0t} + c\varphi_0| = u\varphi_0^{1/p}|\Box\varphi_0 - 2a\varphi_{0t} + c\varphi_0|\varphi_0^{-1/p} \le \frac{\varepsilon}{p}u^p\varphi_0 + \frac{|\Box\varphi_0 - 2a\varphi_{0t} + c\varphi_0|^{p'}}{p'\varepsilon^{p'-1}\varphi_0^{p'-1}}. \tag{5.7}$$

By virtue of relations (5.5)–(5.7), we have the inequality

$$\left(|\lambda| - \frac{\varepsilon}{p}\right) \int\limits_{\Omega_T} u^p \varphi_0 \, dx \, dt \le \frac{\kappa_0}{p' \varepsilon^{p'-1}} - \beta \kappa_1, \tag{5.8}$$

where, by condition (5.4) and Theorem 5.1, the constant $\kappa_1$ is given by the relation

$$0 < \kappa_1 := \int\limits_{\Omega_T} f_0 \varphi_0 \, dx \, dt < +\infty.$$

Inequality (5.8) with $\varepsilon < |\lambda| p$ implies that

$$\int\limits_{\Omega_T} u^p \varphi_0 \, dx \, dt \le \frac{p \kappa_0}{(|\lambda| p - \varepsilon) p' \varepsilon^{p'-1}} - \frac{p \beta \kappa_1}{|\lambda| p - \varepsilon}. \tag{5.9}$$

By taking account of the relations $p' = p/(p-1)$, $p = p'/(p'-1)$, and

$$\min_{0 < \varepsilon < |\lambda| p} (p/((|\lambda| p - \varepsilon) p' \varepsilon^{p'-1})) = 1/|\lambda|^{p'}$$

(the equality is attained for $\varepsilon = |\lambda|$), from inequality (5.9), we obtain the estimate

$$\int\limits_{\Omega_T} u^p \varphi_0 \, dx \, dt \le \frac{\kappa_0}{|\lambda|^{p'}} - \frac{p' \beta \kappa_1}{|\lambda|}. \tag{5.10}$$

Obviously,

$$\chi(\beta) < 0 \quad \text{if} \quad \beta > \beta_0 \quad \text{and} \quad \chi(\beta) > 0 \quad \text{if} \quad \beta < \beta_0, \tag{5.11}$$

where

$$\chi(\beta) := \frac{\kappa_0}{|\lambda|^{p'}} - \frac{p' \beta \kappa_1}{|\lambda|}, \qquad \beta_0 := \frac{\kappa_0}{p' \kappa_1 |\lambda|^{p'-1}}.$$

It remains to note that the left-hand side of inequality (5.10) is nonnegative, while the right-hand side of the same inequality is negative for $\beta > \beta_0$ by virtue of condition (5.11). The latter implies that if $\beta > \beta_0$, then problem (1.5)–(1.8) has no strong generalized solutions in the class of nonnegative functions in the sense of Definition 1.1. The proof of the theorem is complete.

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