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# THE EXISTENCE OF SOLUTIONS OF ONE NONLOCAL IN TIME PROBLEM FOR MULTIDIMENSIONAL WAVE EQUATIONS WITH POWER NONLINEARITY

**Abstract.** For multidimensional wave equations with power nonlinearity we investigate the question on the existence of solutions in a nonlocal in time problem whose particular case is a periodic case.

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## 1. Statement of the Problem

In the space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and $t$, in the cylindrical domain $D = \Omega \times (0, T)$, where $\Omega$ is some open Lipschitz domain in $\mathbb{R}^n$, we consider a nonlocal problem of finding a solution $u(x,t)$ of the equation

$$L_\lambda u := u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} + 2au_t + cu + \lambda |u|^\alpha u = F(x,t), \quad (x,t) \in D_T, \tag{1.1}$$

satisfying the homogeneous boundary condition

$$\Big( \frac{\partial u}{\partial \nu} + \sigma u \Big) \Big|_\Gamma = 0 \tag{1.2}$$

on the lateral boundary $\Gamma : \partial\Omega \times (0,T)$ of the cylinder $D_T$ and the homogeneous nonlocal conditions

$$\mathcal{K}_\mu u := u(x,0) - \mu u(x,T) = 0, \quad x \in \Omega, \tag{1.3}$$

$$\mathcal{K}_\mu u_t := u_t(x,0) - \mu u_t(x,T) = 0, \quad x \in \Omega, \tag{1.4}$$

where $F$ is the given function; $\alpha$, $\lambda$, $\mu$, $a$, $c$ and $\sigma$ are the given constants and $\alpha > 0$, $\lambda\mu \neq 0$; $\frac{\partial}{\partial \nu}$ is the derivative with respect to the outer normal to $\partial D_T$, $n \geq 2$.

*Remark* 1.1. A great number of works are devoted to the investigation of nonlocal problems. In the case of abstract evolution equations and partial differential equations of hyperbolic type, the nonlocal problems are studied in [1–13, 17, 21]. Note that for $|\mu| \neq 1$ it suffices to restrict ourselves to the case $|\mu| < 1$, since the case $|\mu| > 1$ reduces to the previous one if we pass from the variable $t$ to the variable $t' = T - t$. The case $|\mu| = 1$ will be treated in the final Section 4. In particular, the problem (1.1)–(1.4) for $\mu = 1$ can be treated as a periodic problem.

We introduce into consideration the following spaces of functions:

$$\overset{\circ}{C}{}^2_\mu(D_T) := \Big\{ v \in C^2(D_T) : \Big( \frac{\partial v}{\partial \nu} + \sigma v \Big) \Big|_\Gamma = 0, \ \mathcal{K}_\mu v = 0, \ \mathcal{K}_\mu v_t = 0 \Big\}, \tag{1.5}$$

$$\overset{\circ}{W}{}^1_{2,\mu}(D_T) := \Big\{ v \in W^1_2(D_T) : \ \mathcal{K}_\mu v = 0 \Big\}, \tag{1.6}$$

where $W^1_2(D_T)$ is the well-known Sobolev space consisting of functions of the class $L_2(D_T)$ whose all generalized first order derivatives belong likewise to $L_2(D_T)$, and the equality $\mathcal{K}_\mu v = 0$ is understood in a sense of the trace theory [16, p. 71].

**Definition 1.1.** Let $F \in L_2(\Omega_T)$. The function $u$ will be said to be a strong generalized solution of the problem (1.1)–(1.4) of the class $W^1_2$ in the domain $D_T$ if $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ and there exists a sequence of functions

$u_m \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$, and $L_\lambda u_m \to F$ in the space $L_2(D_T)$.

Note that the above definition of a solution of the problem (1.1)–(1.4) remains valid in a linear case, as well, that is for $\lambda = 0$.

*Remark* 1.2. Obviously, a classical solution of the problem (1.1)–(1.4) from the space $C^2(\overline{D}_T)$ is a strong generalized solution of that problem of the class $W_2^1$ in the domain $D_T$ in a sense of Definition 1.1.

*Remark* 1.3. It should be noted that even in a linear case, that is for $\lambda = 0$, the problem (1.1)–(1.4) is not always well-posed. For example, for $\lambda = a = c = 0$ and $|\mu| = 1$, the homogeneous problem corresponding to (1.1)–(1.4) may have infinite set of linearly independent solutions, whereas in order for the inhomogeneous problem to be solvable, it is necessary that a finite or an infinite set of conditions in the form of functional equalities imposed on the right-hand side $F$ of equation (1.1) be fulfilled (see Remark 4.1 below).

The present paper is organized as follows. In Section 2, for some conditions on the coefficients of the problem (1.1)–(1.4) an a priori estimate for a strong generalized solution of the class $W_2^1$ is proved. In Section 3, on the basis of the obtained a priori estimate it is proved that the problem (1.1)–(1.4) is solvable. In the last Section 4, as an application of the results obtained in the previous sections, we consider the case $|\mu| = 1$.

## 2. An a Priori Estimate of Solution of the Problem (1.1)–(1.4)

Consider the conditions

$$a \geq 0, \quad c \geq 0, \quad \sigma \geq 0. \tag{2.1}$$

**Lemma 2.1.** *Let $\lambda > 0$, $|\mu| < 1$, and let $F \in L_2(D_T)$ and conditions* (2.1) *be fulfilled. Then for a strong generalized solution $u$ of the problem* (1.1)–(1.4) *of the class $W_2^1$ in the domain $D_T$ in a sense of Definition* 1.1 *the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1 \|F\|_{L_2(D_T)} + c_2 \tag{2.2}$$

*with nonnegative constants $c_i = c_i(\lambda, \mu, \Omega, T)$, independent of $u$ and $F$, and $c_1 > 0$, is valid, whereas in a linear case, that is for $\lambda = 0$, if $\sigma > 0$, the constant $c_2 = 0$, and by virtue of* (2.2)*, a solution of the problem* (1.1)–(1.4) *is unique in a sense of Definition* 1.1.

*Proof.* Let $u$ be a strong generalized solution of the problem (1.1)–(1.4) of the class $W_2^1$ in the domain $D_T$. By Definition 1.1, there exists the sequence of functions $u_m \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ (see (1.5)) such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} = 0, \quad \lim_{m\to\infty} \|L_\lambda u_m - F\|_{L_2(D_T)} = 0. \tag{2.3}$$

Let us consider the function $u_m \in \overset{\circ}{C}{}^2_\mu(\overline{D}_T)$ as a solution of the problem

$$L_\lambda u_m = F_m, \quad (x,t) \in D_T, \tag{2.4}$$

$$\Big(\frac{\partial u_m}{\partial \nu} + \sigma u_m\Big)\Big|_{\Gamma} = 0, \quad \mathcal{K}_\mu u_m = 0, \quad \mathcal{K}_\mu u_{mt} = 0. \tag{2.5}$$

Here

$$F_m := L_\lambda u_m. \tag{2.6}$$

Multiplying both parts of equality (2.4) by $2u_{mt}$ and integrating with respect to the domain $D_\tau := D_T \cap \{t < \tau\}$, $0 < \tau \le T$, we obtain

$$\begin{aligned} &\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt - 2\int\limits_{D_\tau}\sum_{i=1}^n \frac{\partial^2 u_m}{\partial x_i^2}\,\frac{\partial u_m}{\partial t}\,dx\,dt \\ &+4a\int\limits_{D_\tau} u_{mt}^2\,dx\,dt + c\int\limits_{D_\tau}(u_m^2)_t\,dx\,dt + \frac{2\lambda}{\alpha+2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\,|u_m|^{\alpha+2}\,dx\,dt \\ &= 2\int\limits_{D_\tau} F_m u_{mt}\,dx\,dt. \end{aligned} \tag{2.7}$$

Assume $\omega_\tau := \{(x,t) \in \overline{D}_T:\ x \in \Omega,\ t = \tau\}$, $0 \le \tau \le T$, where $\omega_0$ and $\Omega_T$ are, respectively, the lower and upper bases of the cylindrical domain $D_T$. Let $\nu := (\nu_{x_1}, \nu_{x_2}, \dots, \nu_{x_n}, \nu_t)$ be the unit vector of the outer normal to $\partial D_\tau$. Since

$$\begin{gathered} \nu_{x_i}\big|_{\omega_\tau\cup\omega_0} = 0, \quad i = 1,\dots,n, \\ \nu_t\big|_{\Gamma_\tau := \Gamma\cap\{t\le\tau\}} = 0, \quad \nu_t\big|_{\omega_\tau} = 1, \quad \nu_t\big|_{\omega_0} = -1, \end{gathered}$$

therefore, taking into account (2.5) and integrating by parts, we have

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt = \int\limits_{D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\nu_t\,ds = \int\limits_{\omega_\tau} u_{mt}^2\,dx - \int\limits_{\omega_0} u_{mt}^2\,dx, \tag{2.8}$$

$$\begin{aligned} -2\int\limits_{D_\tau}\sum_{i=1}^n\frac{\partial^2 u_m}{\partial x_i^2}\,\frac{\partial u_m}{\partial t}\,dx\,dt &= \int\limits_{D_\tau}\sum_{i=1}^n\Big[(u_{mx_i}^2)_t - 2(u_{mx_i}u_{mt})_{x_i}\Big]\,dx\,dt \\ &= \int\limits_{\omega_\tau}\sum_{i=1}^n u_{mx_i}^2\,dx - \int\limits_{\omega_0}\sum_{i=1}^n u_{mx_i}^2\,dx - 2\int\limits_{\Gamma_\tau}\Big[\sum_{i=1}^n u_{mx_i}\nu_i\Big]u_{mt}\,ds \\ &= \int\limits_{\omega_\tau}\sum_{i=1}^n u_{mx_i}^2\,dx - \int\limits_{\omega_0}\sum_{i=1}^n u_{mx_i}^2\,dx + 2\int\limits_{\Gamma_\tau}\sigma u_m u_{mt}\,ds \\ &= \int\limits_{\omega_\tau}\sum_{i=1}^n u_{mx_i}^2\,dx - \int\limits_{\omega_0}\sum_{i=1}^n u_{mx_i}^2\,dx + \sigma\int\limits_{\Gamma_\tau}(u_m^2)_t\,ds \end{aligned}$$

$$= \int\limits_{\omega_\tau} \sum_{i=1}^{n} u_{mx_i}^2\, dx - \int\limits_{\omega_0} \sum_{i=1}^{n} u_{mx_i}^2\, dx + \sigma \int\limits_{\partial\omega_\tau} u_m^2\, ds - \sigma \int\limits_{\partial\omega_0} u_m^2\, ds, \tag{2.9}$$

$$\frac{2\lambda}{\alpha+2} \int\limits_{D_\tau} \frac{\partial}{\partial t} |u_m|^{\alpha+2}\, dx\, dt$$
$$= \frac{2\lambda}{\alpha+2} \int\limits_{\omega_\tau} |u_m|^{\alpha+2}\, dx - \frac{2\lambda}{\alpha+2} \int\limits_{\omega_0} |u_m|^{\alpha+2}\, dx, \tag{2.10}$$

$$\int\limits_{D_\tau} (u_m^2)_t\, dx\, dt = \int\limits_{\omega_\tau} u_m^2\, dx - \int\limits_{\omega_0} u_m^2\, dx.$$

Assuming

$$w_m(\tau) = \int\limits_{\omega_\tau} \Big[ c u_m^2 + u_{mt}^2 + \sum_{i=1}^{n} u_{mx_i}^2 + \frac{2\lambda}{\alpha+2} |u_m|^{\alpha+2} \Big]\, dx$$
$$+ \sigma \int\limits_{\partial\omega_\tau} u_m^2\, ds \tag{2.11}$$

by virtue of (2.8), (2.9), (2.10) and (2.7), we obtain

$$w_m(\tau) + 4a \int\limits_{D_\tau} u_{mt}^2\, dx\, dt = w_m(0) + 2 \int\limits_{D_\tau} F_m \frac{\partial u_m}{\partial t}\, dx\, dt. \tag{2.12}$$

Since $2F_m u_{mt} \le \varepsilon^{-1} F_m^2 + \varepsilon u_{mt}^2$ for every $\varepsilon = const > 0$, it follows from (2.12), owing to $a \ge 0$, that

$$w_m(\tau) \le w_m(0) + \varepsilon \int\limits_{D_\tau} u_{mt}^2\, dx\, dt + \varepsilon^{-1} \int\limits_{D_\tau} F_m^2\, dx\, dt. \tag{2.13}$$

Next, by virtue of (2.11), $\lambda > 0$ and $\sigma \ge 0$, we have

$$\int\limits_{D_\tau} u_{mt}^2\, dx\, dt = \int\limits_0^\tau \Big[ \int\limits_{\omega_s} u_{mt}^2\, dx \Big]\, ds \le \int\limits_0^\tau w_m(s)\, ds,$$

whence, with regard for (2.13), we obtain

$$w_m(\tau) \le \varepsilon \int\limits_0^\tau w_m(\xi)\, d\xi + w_m(0) + \varepsilon^{-1} \int\limits_{D_\tau} F_m^2\, dx\, dt, \quad 0 < \tau \le T. \tag{2.14}$$

Since $D_\tau \subset D_T$, $0 < \tau \le T$, the right-hand side of inequality (2.14) is a nondecreasing function of the variable $\tau$, and by Gronwall's lemma, it follows from (2.14) that

$$w_m(\tau) \le \Big[ w_m(0) + \varepsilon^{-1} \int\limits_{D_\tau} F_m^2\, dx\, dt \Big] e^{\varepsilon\tau}, \quad 0 < \tau \le T. \tag{2.15}$$

By virtue of (2.5), $\lambda > 0$, $\sigma \geq 0$, $|\mu| < 1$, $\alpha > 0$, from (2.12) we get

$$\begin{aligned}
w_m(0) = \int\limits_{\Omega} \Big[ c u_m^2(x,0) + u_{mt}^2(x,0) + \sum_{i=1}^{n} u_{mx_i}^2(x,0) \\
+ \frac{2\lambda}{\alpha+2} \, |u_m^2(x,0)|^{\alpha+2} \Big] \, dx + \sigma \int\limits_{\partial\Omega} u_m^2(x,0) \, ds \\
= \int\limits_{\Omega} \Big[ \mu^2 c u_m^2(x,T) + \mu^2 u_{mt}^2(x,T) + \mu^2 \sum_{i=1}^{n} u_{mx_i}^2(x,T) \\
+ \frac{2\lambda|\mu|^{\alpha+2}}{\alpha+2} \, |u_m(x,T)|^{\alpha+2} \Big] \, dx + \sigma \int\limits_{\partial\Omega} \mu^2 u_m^2(x,T) \, ds \leq \mu^2 w_m(T). \quad (2.16)
\end{aligned}$$

Using inequality (2.15) for $\tau = T$, by virtue of (2.16), we find that

$$w_m(0) \leq \mu^2 w_m(T) \leq \mu^2 \Big[ w_m(0) + \varepsilon^{-1} \int\limits_{D_\tau} F_m^2 \, dx \, dt \Big] e^{\varepsilon T}. \quad (2.17)$$

Since $|\mu| < 1$, we can choose a positive constant $\varepsilon = \varepsilon(\mu, T)$ so small that

$$\mu_1 = \mu^2 e^{\varepsilon T} < 1. \quad (2.18)$$

For example, in the capacity of $\varepsilon$ from (2.18) we can take the number $\varepsilon = \frac{1}{T} \ln(\frac{1}{|\mu|})$.

Owing to (2.18), from (2.17) we obtain

$$w(0) \leq (1-\mu_1)^{-1} \mu^2 \varepsilon^{-1} e^{\varepsilon T} \|F_m\|^2_{L_2(D_T)}. \quad (2.19)$$

Taking into account (2.19), from (2.15) we find that

$$w_m(\tau) \leq \gamma \|F_m\|^2_{L_2(D_T)}, \quad 0 < \tau \leq T. \quad (2.20)$$

Here

$$\gamma = \Big[ (1-\mu_1)^{-1} \mu^2 \varepsilon^{-1} e^{\varepsilon T} + \varepsilon^{-1} \Big] e^{\varepsilon T}, \quad \varepsilon = \frac{1}{T} \ln\Big(\frac{1}{|\mu|}\Big). \quad (2.21)$$

By virtue of $\lambda > 0$, $\alpha > 0$, $c \geq 0$, $\sigma \geq 0$ and (2.11), we have

$$\begin{aligned}
\int\limits_{\omega_\tau} u_m^2 \, dx = \int\limits_{\omega_\tau, \, |u_m| \leq 1} u_m^2 \, dx + \int\limits_{\omega_\tau, \, |u_m| > 1} u_m^2 \, dx \\
\leq \operatorname{mes} \Omega + \int\limits_{\omega_\tau, \, |u_m| > 1} |u_m|^{\alpha+2} \, dx \\
\leq \operatorname{mes} \Omega + \frac{\alpha+2}{2\lambda} \, w_m(\tau). \quad (2.22)
\end{aligned}$$

It follows from (2.11), (2.20) and (2.22) that

$$\int\limits_{\omega_\tau}\Big[u_m^2+u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big]\,dx \le \operatorname{mes}\Omega+\frac{\alpha+2}{2\lambda}\,w_m(\tau)+w_m(\tau)$$

$$=\operatorname{mes}\Omega+\Big(1+\frac{\alpha+2}{2\lambda}\Big)\gamma\|F_m\|^2_{L_2(D_T)},\quad 0<\tau\le T. \qquad (2.23)$$

By (2.23), we obtain

$$\|u_m\|^2_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)}=\int\limits_0^T\Big[\int\limits_{\omega_\tau}\Big(u_m^2+u_{mt}^2+\sum_{i=1}^n u_{mx_i}^2\Big)\,dx\Big]\,d\tau$$

$$\le T\operatorname{mes}\Omega+T\Big(1+\frac{\alpha+2}{2\lambda}\Big)\gamma\|F_m\|^2_{L_2(D_T)},\quad 0<\tau\le T. \qquad (2.24)$$

Taking from both parts of inequality (2.24) the square root and using the obvious inequality $(a^2+b^2)^{1/2}\le|a|+|b|$, we have

$$\|u_m\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)}\le c_1\|F_m\|_{L_2(D_T)}+c_2. \qquad (2.25)$$

Here, due to (2.21), for $\lambda>0$, we get

$$\begin{cases} c_1=\left(T\Big(1+\dfrac{\alpha+2}{2\lambda}\Big)\Big[(1-\mu_1)^{-1}\mu^2\varepsilon^{-1}e^{\varepsilon T}+\varepsilon^{-1}\Big]e^{\varepsilon T}\right)^{\frac12}, \\ \qquad \varepsilon=\dfrac{1}{T}\,\ln\Big(\dfrac{1}{|\mu|}\Big), \\ c_2=(T\operatorname{mes}\Omega)^{\frac12}. \end{cases} \qquad (2.26)$$

Bearing in mind limiting equalities (2.3) and passing in inequality (2.25) to the limit, as $m\to\infty$, we obtain (2.2). Thus Lemma 2.1 is proved for $\lambda>0$.

In a linear case, that is for $\lambda=0$, but $\sigma>0$, the proof of a priori estimate (2.2) with $c_2=0$ follows from the following reasoning. As is known, the norm of the space $W_2^1(\Omega)$ for $\sigma>0$ is equivalent to the norm

$$\|v\|^2=\int\limits_{\Omega}\sum_{i=1}^n v_{x_i}^2\,dx+\sigma\int\limits_{\partial\Omega}v^2\,ds$$

[18, p. 147] that is, in particular, there exists the positive constant $c_0=c_0(\Omega,\sigma)$ such that

$$\|v\|^2_{W_2^1(\Omega)}=\int\limits_{\Omega}\Big[v^2+\sum_{i=1}^n v_{x_i}^2\Big]\,dx$$

$$\le c_0\Big[\int\limits_{\Omega}\sum_{i=1}^n v_{x_i}^2\,dx+\sigma\int\limits_{\partial\Omega}v^2\,ds\Big]\quad\forall\,v\in W_2^1(\Omega). \qquad (2.27)$$

By (2.1),(2.27), instead of (2.22) and (2.23), with regard for (2.11), we will have

$$\int\limits_{\omega_\tau} \Big[u_m^2 + u_{mt}^2 + \sum_{i=1}^n u_{mx_i}^2\Big]\,dx \le \int\limits_{\omega_\tau} u_m^2\,dx + w_m(\tau) \le (c_0+1)w_m(\tau). \quad (2.28)$$

From (2.20) and (2.28), analogously to how we have obtained (2.24), it follows that

$$\|u_m\|^2_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \le \int\limits_0^T (c_0+1)w_m(\tau)\,d\tau \le T(c_0+1)\gamma\|F_m\|^2_{L_2(D_T)}. \quad (2.29)$$

Passing in inequality (2.29) to the limit, as $m \to \infty$, and taking into account (2.3), we obtain estimate (2.2) in which

$$\begin{cases} c_1 = \Big(T(c_0+1)\Big[(1-\mu_1)^{-1}\mu^2\varepsilon^{-1}e^{\varepsilon T} + \varepsilon^{-1}\Big]e^{\varepsilon T}\Big)^{\frac12}, \\ c_2 = 0, \end{cases} \quad (2.30)$$

what proves Lemma 2.1 in case $\lambda = 0$ and $\sigma > 0$. □

*Remark* 2.1. In Section 3, the question on the solvability of the problem (1.1)–(1.4) is reduced to that of finding a uniform with respect to the parameter $s \in [0,1]$ a priori estimate for a strong generalized solution of the equation

$$v_{tt} - \sum_{i=1}^n v_{x_i x_i} + s(c-a^2)v + s\lambda\exp(-\alpha a t)|v|^\alpha v = s\exp(at)F(x,t), \quad (x,t) \in D_T, \quad (2.31)$$

satisfying both the boundary condition

$$\Big(\frac{\partial v}{\partial \nu} + \sigma v\Big)\Big|_\Gamma = 0 \quad (2.32)$$

and the nonlocal conditions

$$(\mathcal{K}_{\mu_0} v)(x) = 0, \quad (\mathcal{K}_{\mu_0} v_t)(x) = 0, \;\; x \in \Omega, \quad (2.33)$$

where $\mu_0 = \mu\exp(-aT)$, $|\mu| < 1$, and the operator $\mathcal{K}_{\mu_0}$ for $\mu = \mu_0$ is defined in (1.3). To obtain a uniform with respect to $\tau$ a priori estimate for the solution of the problem (2.31)–(2.33) it is sufficient that instead of (2.1) be fulfilled the more bounded conditions

$$a \ge 0, \;\; c \ge a^2, \;\; \sigma > 0. \quad (2.34)$$

For this case, we present in the proof of Lemma 2.1 certain changes. Assuming

$$\widetilde{w}_m(\tau)=\int\limits_{\omega_\tau}\Big[s(c-a^2)v_m^2+v_{mt}^2+\sum_{i=1}^n v_{mx_i}^2+\frac{2s\lambda}{\alpha+2}\,\exp(-\alpha a\tau)|v_m|^{\alpha+2}\Big]\,dx \\ +\sigma\int\limits_{\partial\omega_\tau} v_m^2\,ds,$$

instead of equality (2.12) for $u_m$, in regard to the function $v_m$ we get

$$\widetilde{w}_m(\tau)+\frac{2s\lambda a}{\alpha+2}\int\limits_{D_\tau}\exp(-\alpha at)|v_m|^{\alpha+2}\,dx\,dt \\ =\widetilde{w}_m(0)+2s\int\limits_{D_\tau}\exp(at)F_m v_{mt}\,dx\,dt,$$

whence by virtue of $s\lambda a\geq 0$, $s\in[0,1]$, analogously to (2.13)–(2.15), we, respectively, obtain

$$\widetilde{w}_m(\tau)\leq\widetilde{w}_m(0)+\varepsilon\int\limits_{D_T}v_{mt}^2\,dx\,dt+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt,$$

$$\widetilde{w}_m(\tau)\leq\varepsilon\int\limits_0^T w_m(\xi)\,d\xi+\widetilde{w}_m(0)+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt,$$

$$\widetilde{w}_m(\tau)\leq\Big[\widetilde{w}_m(0)+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt\Big]e^{\varepsilon\tau},\quad 0<\tau\leq T.$$

Further, by (2.33), (2.34) and $\mu_0=\mu\exp(-at)$, $|\mu|<1$, taking into account the fact that

$$|\mu_0|^{\alpha+2}=|\mu_0|^2\exp(-\alpha aT)|\mu_0|^\alpha\exp(\alpha aT) \\ =|\mu_0|^2\exp(-\alpha aT)|\mu|^\alpha\leq|\mu_0|^2\exp(-\alpha aT),$$

we instead of (2.16) have

$$\widetilde{w}_m(0)=\int\limits_\Omega\Big[s(c-a^2)v_m^2(x,0)+v_{mt}^2(x,0)+\sum_{i=1}^n v_{mx_i}^2(x,0) \\ +\frac{2s\lambda}{\alpha+2}\,|v_m(x,0)|^{\alpha+2}\Big]\,dx+\sigma\int\limits_{\partial\Omega}v_m^2(x,0)\,ds \\ =\int\limits_\Omega\Big[\mu_0^2 s(c-a^2)v_m^2(x,T)+\mu_0^2 v_{mt}^2(x,T)$$

$$+\mu_0^2\sum_{i=1}^n v_{mx_i}^2(x,T)+\frac{2s\lambda|\mu_0|^{\alpha+2}}{\alpha+2}\,|v_m(x,T)|^{\alpha+2}\Big]\,dx$$

$$+\,\sigma\int\limits_{\partial\Omega}\mu_0^2 v_m^2(x,T)\,ds\le\mu_0^2\widetilde{w}_m(T).$$

Analogously, instead of (2.17)–(2.21) we, respectively, obtain

$$\widetilde{w}_m(0)\le\mu_0^2\widetilde{w}_m(T)\le\mu_0^2\bigg[\widetilde{w}_m(0)+\varepsilon^{-1}\exp(2aT)\int\limits_{D_T}F_m^2\,dx\,dt\bigg]e^{\varepsilon T},$$

$$\mu_2=\mu_0^2e^{\varepsilon T}<1,$$

$$\widetilde{w}_m(0)\le(1-\mu_2)^{-1}\mu_0^2\varepsilon^{-1}e^{\varepsilon T}\exp(2aT)\|F_m\|^2_{L_2(D_T)},$$

$$\widetilde{w}_m(\tau)\le\widetilde{\gamma}\|F_m\|^2_{L_2(D_T}, \ \ 0<\tau\le T,$$

$$\widetilde{\gamma}=\Big[(1-\mu_2)^{-1}\mu_0^2\varepsilon^{-1}e^{\varepsilon T}+\varepsilon^{-1}\Big]\exp(2a+\varepsilon)T,$$

where by virtue of $|\mu_0|\le|\mu|$, we can take in the capacity of $\varepsilon$ the same number $\varepsilon=\frac{1}{T}\ln(\frac{1}{|\mu|})$ as in (2.21). Next, analogously to how from (2.20) and (2.28) we have got a priori estimate (2.2) with the constants $c_1$ and $c_2$, from (2.30) we will have

$$\|v\|_{\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)}\le c_3\|F\|_{L_2(D_T)}, \tag{2.35}$$

where the positive constant

$$c_3=\Big\{T(c_0+1)\Big[(1-\mu_2)^{-1}\mu_0^2\varepsilon^{-1}e^{\varepsilon T}+\varepsilon^{-1}\Big]\exp(2a+\varepsilon)T\Big\}^{\frac{1}{2}} \tag{2.36}$$

does not depend on $v$, $F$ and on the parameter $s\in[0,1]$.

## 3. The Existence of a Solution of the Problem (1.1)–(1.4)

To prove that the problem (1.1)–(1.4) has a solution in case $|\mu|<1$, we will use the well-known facts dealing with the solvability of the following mixed problem

$$u_{tt}-\sum_{i=1}^n u_{x_ix_i}=F(x,t), \ \ (x,t)\in D_T, \tag{3.1}$$

$$\Big(\frac{\partial u}{\partial\nu}+\sigma u\Big)\Big|_\Gamma=0, \quad u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \ \ x\in\Omega, \tag{3.2}$$

where $F$, $\varphi$ and $\psi$ are the given functions, $\sigma=const>0$.

For $F\in L_2(D_T)$, $\varphi\in W_2^1(\Omega)$, $\psi\in L_2(\Omega)$ a unique generalized solution $u$ of the problem (3.1), (3.2) from the space $E_{2,1}(D_T)$ with the norm

$$\|v\|^2_{E_{2,1}(D_T)}=\sup_{0\le t\le T}\int\limits_\omega\Big[v^2+v_t^2+\sum_{i=1}^n v_{x_i}^2\Big]\,dx$$

is given by the formula [16, pp. 214, 226], [19, pp. 292, 294]

$$u = \sum_{k=1}^{\infty} \biggl( a_k \cos \mu_k t + b_k \sin \mu_k t \\ + \frac{1}{\mu_k} \int_0^t F_k(\tau) \sin \mu_k (t-\tau)\, d\tau \biggr) \varphi_k(x), \quad (3.3)$$

where $\lambda_k = -\mu_k^2$, $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots$ are eigen-functions and $\lim\limits_{k\to\infty} \mu_k = 0$, while $\varphi_k \in W_2^1(\Omega)$ are the corresponding eigen-functions of the spectral problem $\Delta w = \lambda w$, $(\frac{\partial w}{\partial \nu} + \sigma w)\big|_{\partial\Omega} = 0$ in the domain $\Omega$ ($\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$) which form simultaneously an orthonormalized basis in $L_2(\Omega)$ and orthogonal basis in $W_2^1(\Omega)$ in a sense of the scalar product

$$(v, w)_{W_2^1(\Omega)} = \int_\Omega \sum_{i=1}^{n} v_{x_i} w_{x_i}\, dx + \int_{\partial\Omega} \sigma v w\, ds$$

[16, p. 237], that is,

$$(\varphi_k, \varphi_l)_{L_2(\Omega)} = \delta_k^l, \quad (\varphi_k, \varphi_l)_{W_2^1(\Omega)} = -\lambda_k \delta_k^l, \ \ \delta_k^l = \begin{cases} 1, & l = k, \\ 0, & l \ne k. \end{cases} \quad (3.4)$$

Here

$$a_k = (\varphi, \varphi_k)_{L_2(\Omega)}, \quad b_k = \mu_k^{-1} (\psi, \varphi_k), \ \ k = 1, 2, \dots, \quad (3.5)$$

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t) \varphi_k(x), \\ F_k(t) = (F, \varphi_k)_{L_2(\omega_t)}, \ \ \omega_\tau := D_T \cap \{t = \tau\}, \quad (3.6)$$

and for the solution $u$ from (3.3) the estimate

$$\|u\|_{E_{2,1}(D_T)} \le \gamma \Bigl( \|F\|_{L_2(D_T)} + \|\varphi\|_{W_2^1(\Omega)} + \|\psi\|_{L_2(\Omega)} \Bigr) \quad (3.7)$$

with the positive constant $\gamma$, independent of $F$, $\varphi$ and $\psi$, is valid [16, pp. 214, 226].

Let us consider now the linear problem

$$L_0 u := \frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = F(x,t), \ \ (x,t) \in D_T, \quad (3.8)$$

$$\Bigl( \frac{\partial u}{\partial \nu} + \sigma u \Bigr)\Big|_\Gamma = 0, \quad (3.9)$$

$$u(x, 0) - \mu u(x, T) = 0, \quad u_t(x, 0) - \mu u_t(x, T) = 0, \ \ x \in \Omega, \quad (3.10)$$

corresponding to (1.1)–(1.4) in case $a = c = \lambda = 0$.

Show that for $|\mu| < 1$, for any $F \in L_2(D_T)$, there exists a unique strong generalized solution of the problem (3.8)–(3.10). Indeed, since the space of finite infinitely differentiable functions $C_0^\infty(D_T)$ is dense in $L_2(D_T)$, therefore for $F \in L_2(D_T)$ and for any natural number $m$ there exists the function $F_m \in C_0^\infty(D_T)$ such that

$$\|F_m - F\|_{L_2(D_T)} < \frac{1}{m}\,. \tag{3.11}$$

On the other hand, for the function $F_m$ in the space $L_2(D_T)$ the decomposition [16]

$$F_m(x,t) = \sum_{k=1}^{\infty} F_{m,k}(t)\varphi_k(x), \quad F_{m,k}(t) = (F_m, \varphi_k)_{L_2(\Omega)} \tag{3.12}$$

is valid.

Therefore there exists the natural number $\ell_m$, $\lim\limits_{m\to\infty} \ell_m = \infty$, such that for

$$\widetilde{F}_m(x,t) = \sum_{k=1}^{\ell_m} F_{m,k}(t)\varphi_k(x) \tag{3.13}$$

the inequality

$$\|\widetilde{F}_m - F_m\|_{L_2(D_T)} < \frac{1}{m} \tag{3.14}$$

holds.

It follows from (3.11) and (3.14) that

$$\lim_{m\to\infty} \|\widetilde{F}_m - F\|_{L_2(D_T)} = 0. \tag{3.15}$$

The solution $u = u_m$ of the problem (3.1), (3.2) for

$$\varphi = \sum_{k=1}^{\ell_m} \widetilde{a}_k \varphi_k, \quad \psi = \sum_{k=1}^{\ell_m} \mu_k \widetilde{b}_k \varphi_k, \quad F = \widetilde{F}_m$$

is given by formula (3.3) which with regard for (3.4)–(3.6) and (3.13) takes the form

$$u_m = \sum_{k=1}^{\ell_m} \Bigg( \widetilde{a}_k \cos \mu_k t + \widetilde{b}_k \sin \mu_k t + \frac{1}{\mu_k} \int_0^t F_{m,k}(\tau) \sin \mu_k (t-\tau)\, d\tau \Bigg) \varphi_k(x). \tag{3.16}$$

By the construction, the function $u_m$ from (3.16) satisfies equation (3.8) and the boundary condition (3.9) for $F = \widetilde{F}_m$ from (3.13).

Define now unknown coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ in such a way that the function $u_m$ from (3.16) likewise satisfy the nonlocal conditions (3.10). Towards this end, we substitute the right-hand side of (3.16) into equalities (3.10). As a result, taking into account that the system of functions $\{\varphi_k(x)\}$ forms the basis in $L_2(\Omega)$, to find coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$, we obtain the following system of linear algebraic equations

$$\begin{cases} (1-\mu\cos\mu_k T)\widetilde{a}_k - (\mu\sin\mu_k T)\widetilde{b}_k \\ \qquad = \dfrac{\mu}{\mu_k}\displaystyle\int_0^T F_{m,k}(\tau)\sin\mu_k(T-\tau)\,d\tau, \\ (\mu\mu_k\sin\mu_k T)\widetilde{a}_k + \mu_k(1-\mu\cos\mu_k T)\widetilde{b}_k \\ \qquad = \mu\displaystyle\int_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,d\tau, \end{cases} \tag{3.17}$$

$k = 1, 2, \dots, \ell_m$, whose solution is

$$\widetilde{a}_k = \Big[d_{1k}\mu\mu_k\sin\mu_k T - d_{2k}(1-\mu\cos\mu_k T)\Big]\Delta_k^{-1}, \quad k = 1, 2, \dots, \ell_m, \tag{3.18}$$

$$\widetilde{b}_k = \Big[d_{2k}(1-\mu\cos\mu_k T) - d_{1k}\mu\mu_k\sin\mu_k T\Big]\Delta_k^{-1}, \quad k = 1, 2, \dots, \ell_m. \tag{3.19}$$

Here

$$d_{1k} = \frac{\mu}{\mu_k}\int_0^T F_{m,k}(\tau)\sin\mu_k(T-\tau)\,d\tau,$$

$$d_{2k} = \mu\int_0^T F_{m,k}(\tau)\cos\mu_k(T-\tau)\,d\tau$$

and since $|\mu| < 1$, for the determinant $\Delta_k$ of system (3.17), we have

$$\Delta_k = \mu_k\Big[(1-\mu\cos\mu_k T)^2 + \mu^2\sin^2\mu_k T\Big] \geq \mu_k\big(1-|\mu|\big)^2 > 0. \tag{3.20}$$

Below, the Lipschitz domain $\Omega$ will be assumed to be such that the eigen-functions $\varphi_k \in C^2(\overline{\Omega})$, $k \geq 1$. For example, this fact will hold if $\partial\Omega \in C^{[\frac{n}{2}]+3}$ [18, p. 227]. This may take place also in the case of piecewise smooth Lipschitz domain, for example, for the parallelepiped $\Omega = \{x \in \mathbb{R}^n : |x_i| < a_i, \ i = 1, \dots, n\}$, the corresponding eigen-functions $\varphi_k \in C^\infty(\overline{\Omega})$ [19] (see also Remark 4.1). Thus, since $F_m \in C_0^\infty(D_T)$, by virtue of (3.12), the function $F_{m,k} \in C^2([0,T])$, and hence the function $u_m$ from (3.16) belongs to the space $C^2(\overline{D}_T)$. Next, by the construction, the function $u_m$ from (3.16) will belong to the space $\overset{\circ}{C}{}^2_\mu(D_T)$ which has been defined in (1.5), and

$$L_0 u_m = \widetilde{F}_m, \quad L_0(u_m - u_k) = \widetilde{F}_m - \widetilde{F}_k. \tag{3.21}$$

From (3.21) and a priori estimate (2.2) for $a = c = \lambda = 0$ in which by Lemma 2.1 the constant $c_2 = 0$, we have

$$\|u_m - u_k\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \leq c_1\|\widetilde{F}_m - \widetilde{F}_k\|_{L_2(D_T)}. \tag{3.22}$$

By virtue of (3.15), it follows from (3.22) that the sequence $u_m \in \overset{\circ}{C}{}^2_\mu(D_T)$ is fundamental in the whole space $\overset{\circ}{W}{}^1_{2,\mu}(D_T)$. Therefore there exists the function $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ such that by (3.15) and (3.21) the limiting equalities (2.3) are valid for $\lambda = 0$. The latter means that the function $u$ is a strong generalized solution of the problem (3.8)–(3.10). The uniqueness of that solution follows from a priori estimate (2.2) in which $\lambda = 0$ and the constant $c_2 = 0$, that is,

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\mu}(D_T)} \le c_1 \|f\|_{L_2(D_T)}. \tag{3.23}$$

*Remark* 3.1. Thus the linear problem (3.8)–(3.10) has a unique strong generalized solution $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ for which we can write $u = \Box_\mu^{-1}(F)$, where $\Box_\mu^{-1} : L_2(D_T) \to \overset{\circ}{W}{}^1_{2,\mu}(D_T)$ is the linear continuous operator whose norm by virtue of (3.23) admits the estimate

$$\|\Box_\mu^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}^1_{2,\mu}(D_T)} \le c_1. \tag{3.24}$$

*Remark* 3.2. Regarding a new unknown function $v := u\exp(at)$, the problem (1.1)–(1.4) can be written in the form

$$\begin{aligned} \widetilde{L}_\lambda v := v_{tt} - \sum_{i=1}^n v_{x_i x_i} + (c-a^2)v + \lambda \exp(-\alpha a t)|v|^\alpha v \\ = \exp(at)F(x,t), \quad (x,t) \in D_T, \end{aligned} \tag{3.25}$$

$$\Big(\frac{\partial v}{\partial \nu} + \sigma v\Big)\Big|_\Gamma = 0, \tag{3.26}$$

$$(\mathcal{K}_{\mu_0} v)(x) = 0, \quad (\mathcal{K}_{\mu_0} v_t)(x) = 0, \ \ x \in \Omega, \tag{3.27}$$

where $\mu_0 = \mu \exp(-aT)$. Note that the problems (1.1)–(1.4) and (3.25)–(3.27) are equivalent in a sense that $u$ is a strong generalized solution of the problem (1.1)–(1.4), if and only if $v$ is a strong generalized solution of the problem (3.25)–(3.27), that is $v \in \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$, and there exists the sequence of functions $v_m \in \overset{\circ}{C}{}^2_{\mu_0}(D_T)$ such that $v_m \to v$ in the space $\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$, and $\widetilde{L}_\lambda v_m \to \exp(at)F(x,t)$ in the space $L_2(D_T)$.

*Remark* 3.3. The embedding operator $I : W_2^1(D_T) \to L_q(D_T)$ is the linear, continuous, compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [16, p. 81]. At the same time, the Nemytski's operator $\mathcal{N} : L_q(D_T) \to L_2(D_T)$ acting by the formula $\mathcal{N}v = (c-a^2)v + \lambda\exp(-\alpha a t)|v|^\alpha v$ is continuous and bounded if $q \ge 2(\alpha+1)$ [14, p. 349], [15, pp. 66, 67]. Thus, if $\alpha < \frac{2}{n-1}$, that is $2(\alpha+1) < \frac{2(n+1)}{n-1}$, then there exists the number $q$ such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2(\alpha+1)$. Therefore, in this case the operator

$$\mathcal{N}_0 = \mathcal{N} I : \overset{\circ}{W}{}^1_{2,\mu_0}(D_T) \to L_2(D_T) \tag{3.28}$$

will be continuous and compact. Moreover, from $w \in \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$ it all the more follows that $\exp(-\alpha at)|v|^\alpha v \in L_2(D_T)$, and if $v_m \to v$ in the space $\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$, then $\exp(-\alpha at)|v_m|^\alpha v_m \to \exp(-\alpha at)|v|^\alpha v$ in the space $L_2(D_T)$.

*Remark* 3.4. Under the assumption that $a \geq 0$ and $|\mu| < 1$, we have $|\mu_0| < 1$, and taking into account Remarks 3.1 and 3.2, the function $v \in \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$ is a strong generalized solution of the problem (3.25)–(3.27), if and only if $v$ is a solution of the following functional equation

$$v = \square^{-1}_{\mu_0}\Big((a^2 - c)v - \lambda \exp(-\alpha at)|v|^\alpha v\Big) + \square^{-1}_{\mu_0}\big(\exp(at)F\big) \tag{3.29}$$

in the space $\overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$.

We rewrite equation (3.29) in the form

$$v = A_0 v := -\square^{-1}_{\mu_0}(\mathcal{N}_0 v) + \square^{-1}_{\mu_0}\big(\exp(at)F\big), \tag{3.30}$$

where the operator $\mathcal{N}_0 : \overset{\circ}{W}{}^1_{2,\mu_0}(D_T) \to L_2(D_T)$ from (3.28) is, by Remark 3.3, continuous and compact one. Consequently, owing to (3.24), the operator $A_0 : \overset{\circ}{W}{}^1_{2,\mu_0}(D_T) \to \overset{\circ}{W}{}^1_{2,\mu_0}(D_T)$ from (3.30) is likewise continuous and compact for $0 < \alpha < \frac{2}{n-1}$. At the same time, by Remarks 2.1, 3.2 and 3.4, if conditions (2.34) are fulfilled for every value of parameter $s \in [0,1]$ and for every solution $v$ of equation $v = sA_0 v$ with the parameter $s \in [01,]$, then a priori estimate (2.35) with nonnegative constant $c_3$ from (2.36), independent of $v$, $F$ and $s$, is valid. Therefore, by the Lerè–Schauder theorem [20, p. 375], equation (3.30), and hence by Remarks 3.2 and 3.4, the problem (1.1)–(1.4) has at least one solution $u \in \overset{\circ}{W}{}^1_{2,\mu}(D_T)$. Thus we have proved the following

**Theorem 3.1.** *Let $0 < \alpha < \frac{2}{n-1}$, $\lambda > 0$, $|\mu| < 1$ and conditions* (2.34) *be fulfilled. Then for any $F \in L_2(D_T)$ the problem* (1.1)*–*(1.4) *has at least one strong generalized solution of the class $W^1_2$ in the domain $D_T$ in a sense of Definition* 1.1*.*

## 4. The Case $|\mu| = 1$

Instead of conditions (2.1) we consider now the conditions

$$a > 0, \quad c \geq a^2, \quad \sigma > 0. \tag{4.1}$$

**Theorem 4.1.** *Let $0 < \alpha < \frac{2}{n-1}$, $\lambda > 0$, $|\mu| = 1$ and conditions* (4.1) *be fulfilled. Then for any $F \in L_2(D_T)$ the problem* (1.1)*–*(1.4) *has at least one strong generalized solution of the class $W^1_2$ in the domain $D_T$ in a sense of Definition* 1.1*.*

*Proof.* Regarding a new unknown function $v := u\exp(at)$, the problem (1.1)–(1.4) by Remark 3.2 reduces equivalently to the nonlocal problem (3.25)–(3.27), where by virtue of $a > 0$, for the number $\mu_0 = \mu\exp(-aT)$ we have $|\mu_0| < 1$. Therefore if the conditions of Theorem 4.1 are fulfilled, then repeating reasoning mentioned in proving Theorem 3.1 we can conclude that the problem (3.25)–(3.27) and hence the problem (1.1)–(1.4) has at least one strong generalized solution of the class $W_2^1$ in the domain $D_T$. □

*Remark* 4.1. It should be noted that for $|\mu| = 1$ the homogeneous problem corresponding to (1.1)–(1.4) may have even in a linear case, i.e., for $\lambda = 0$, a finite or even an infinite set of linearly independent solutions, if conditions (4.1) are violated, whereas for the solvability of that problem the function $F \in L_2(D_T)$ must satisfy, respectively, a finite or an ininite number of conditions of solvability of type $\ell(F) = 0$, where $\ell$ is the linear continuous functional in $L_2(D_T)$. Indeed, let us consider the case $\lambda = a = c = 0$, $\sigma = 1$. When $\mu = 1$, we denote by $\Lambda(1)$ a set of those $\mu_k$ from (3.3) for which the ratio $\frac{\mu_k T}{2\pi}$ is a natural number, i.e., $\Lambda(1) = \{\mu_k : \frac{\mu_k T}{2\pi} \in \mathbb{N}\}$. Formulas (3.18) and (3.19) for finding unknown coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ in the representation (3.16) have been obtained from the system of linear algebraic equations (3.17). In case $\lambda(1) \neq \varnothing$ and $\mu_k \in \Lambda(1)$, $\mu = 1$, the determinant of system (3.17) given by formula (3.20) is equal to zero. Moreover, in this case all coefficients $\widetilde{a}_k$ and $\widetilde{b}_k$ in the left-hand side of system (3.17) are equal to zero. Therefore, in accordance with (3.3), the homogeneous problem corresponding to (3.8), (3.9) and (3.10) is satisfied with the function

$$u_k(x,t) = (C_1\cos\mu_k t + C_2\sin\mu_k t)\varphi_k(x), \tag{4.2}$$

where $C_1$ and $C_2$ are arbitrary constant numbers, and in this case the necessary conditions for the solvability of the inhomogeneous problem (3.8)–(3.10) corresponding to $\mu_k \in \Lambda(1)$ are

$$\begin{aligned} \ell_{k,1}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x)\sin\mu_k(T-t)\,dx\,dt = 0,\\ \ell_{k,2}(F) &= \int\limits_{D_T} F(x,t)\varphi_k(x)\cos\mu_k(T-t)\,dx\,dt = 0. \end{aligned} \tag{4.3}$$

Analogously, in case $\mu = -1$, we denote by $\Lambda(-1)$ a set of those $\mu_k$ from (3.3) for which the ratio $\frac{\mu_k T}{\pi}$ is an odd natural number. For $\mu_k \in \Lambda(-1)$, $\mu = -1$, the function $u_k$ from (4.2) is, likewise, a solution of the homogeneous problem corresponding to (3.8)–(3.10), and conditions (4.3) are the necessary ones for solvability of that problem. For example, for $n = 2$, the eigen-numbers and eigen-functions of the spectral problem $\Delta w = \lambda w$, $(\frac{\partial w}{\partial\nu} + w)\big|_{\partial\Omega} = 0$ are

$$\lambda_k = -\frac{1}{4}\left[(2k_1-1)^2 + (2k_2-1)^2\right], \quad k = (k_1,k_2) \in \mathbb{N}^2,$$

$$\varphi_k(x_1,x_2)=d_k\big(\sin\widetilde{\mu}_{k_1}x_1+\widetilde{\mu}_{k_1}\cos\widetilde{\mu}_{k_1}x_1\big)\big(\sin\widetilde{\mu}_{k_2}x_2+\widetilde{\mu}_{k_2}\cos\widetilde{\mu}_{k_2}x_2\big),$$

where $\widetilde{\mu}_{k_i}=\frac{1}{2}(2k_i-1)$, $\mu_k=\frac{1}{2}\sqrt{(2k_1-1)^2+(2k_2-1)^2}$, and $d_k$ is the normalizing factor defined from the condition $\|\varphi_k\|_{L_2(\Omega)}=1$. It can be easily seen that if the number $T$ is such that $\frac{T}{2\sqrt{2}\,\pi}\in\mathbb{N}$, then for any $k=(k_1,k_2)$ such that $k_1=k_2$ we have $\mu_k\in\Lambda(1)$. In this case, i.e., for $\mu=1$ and $\frac{T}{2\sqrt{2}\,\pi}\in\mathbb{N}$, the homogeneous problem corresponding to (3.8)–(3.10) will have an infinite set of linearly independent solutions of type (4.2), and for the solvability of that problem it is necessary that an infinite number of conditions of type (4.3) for $k=(k_1,k_2)$ such that $k_1=k_2\in\mathbb{N}$ are fulfilled. The case $\mu=-1$ is considered analogously.

## References

1. S. Aizicovici and M. McKibben, Existence results for a class of abstract nonlocal Cauchy problems. *Nonlinear Anal.* **39** (2000), No. 5, Ser. A: Theory Methods, 649–668.
2. G. A. Avalishvili, Nonlocal in time problems for evolution equations of second order. *J. Appl. Anal.* **8** (2002), No. 2, 245–259.
3. G. Bogveradze and S. Kharibegashvili, On some nonlocal problems for a hyperbolic equation of second order on a plane. *Proc. A. Razmadze Math. Inst.* **136** (2004), 1–36.
4. A. Bouziani, On a class of nonclassical hyperbolic equations with nonlocal conditions. *J. Appl. Math. Stochastic Anal.* **15** (2002), No. 2, 135–153.
5. L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Appl. Anal.* **40** (1991), No. 1, 11–19.
6. D. Gordeziani and G. Avalishvili, Investigation of the nonlocal initial boundary value problems for some hyperbolic equations. *Hiroshima Math. J.* **31** (2001), No. 3, 345–366.
7. E. Hernández, Existence of solutions for an abstract second-order differential equation with nonlocal conditions. *Electron. J. Differential Equations* **2009**, No. 96, 10 pp.
8. S. S. Kharibegashvili, On the well-posedness of some nonlocal problems for the wave equation. (Russian) *Differ. Uravn.* **39** (2003), No. 4, 539–553, 575; translation in *Differ. Equ.* **39** (2003), No. 4, 577–592.
9. S. Kharibegashvili and B. Midodashvili, Some nonlocal problems for second order strictly hyperbolic systems on the plane. *Georgian Math. J.* **17** (2010), No. 2, 287–303.
10. S. Kharibegashvili and B. Midodashvili, Solvability of nonlocal problems for semilinear one-dimensional wave equations. *Electron. J. Differential Equations* **2012**, No. 28, 16 pp.
11. T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Phys.* **1** (1994), 1–144.
12. T. Kiguradze, On periodic in the plane solutions of nonlinear hyperbolic equations. *Nonlinear Anal.* **39** (2000), No. 2, Ser. A: Theory Methods, 173–185.
13. T. Kiguradze, On bounded and time-periodic solutions of nonlinear wave equations. *J. Math. Anal. Appl.* **259** (2001), No. 1, 253–276.
14. M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik, and P. E. Sobolevskiĭ, Integral operators in spaces of summable functions. (Russian) *Izdat. "Nauka", Moscow*, 1966.

15. A. Kufner and S. Fučík , Nonlinear differential equations. (Translated into Russian) *Izdat "Nauka", Moscow*, 1998; English original: Studies in Applied Mechanics, 2. *Elsevier Scientific Publishing Co., Amsterdam–New York*, 1980.
16. O. A. Ladyzhenskaya, Boundary value problems of mathematical physics. (Russian) *Izdat. "Nauka", Moscow*, 1973.
17. B. Midodashvili, A nonlocal problem for fourth order hyperbolic equations with multiple characteristics. *Electron. J. Differential Equations* **2002**, No. 85, 7 pp. (electronic).
18. V. P. Mikhaĭlov, Partial differential equations. (Russian) *Izdat. "Nauka", Moscow*, 1976.
19. S. G. Mikhlin, A course in mathematical physics. (Russian) *Izdat. "Nauka", Moscow*, 1968.
20. V. A. Trenogin, Functional analysis. (Russian) Second edition. *"Nauka", Moscow*, 1993.
21. X. Xue, Existence of solutions for semilinear nonlocal Cauchy problems in Banach spaces. *Electron. J. Differential Equations* **2005**, No. 64, 7 pp. (electronic).

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