# On the solvability of one boundary value problem for a class of higher-order nonlinear partial differential equations

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**Abstract.** The boundary value problem for a class of higher-order nonlinear partial differential equations is considered. The theorems on existence, uniqueness and nonexistence of solutions of this problem are proved.

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## 1. Statement of the problem

In the Euclidean space $\mathbb{R}^{n+1}$ of the variables $x = (x_1, x_2, ..., x_n)$ and $t$ we consider the nonlinear equation of the type

$$L_f u := \frac{\partial^{2(2k+1)} u}{\partial t^{2(2k+1)}} - \Delta^2 u + f(u, \nabla u) = F(x,t), \tag{1.1}$$

where $f$ and $F$ are given, and $u$ is an unknown real functions,
$\nabla := (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t})$, $\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$, and $k \geq 0$ is an integer number.

For the equation (1.1) we consider the boundary value problem: find in the cylindrical domain $D_T := \Omega \times (0,T)$, where $\Omega$ is a Lipschitz domain in $\mathbb{R}^n$, a solution $u(x,t)$ of that equation according to the boundary conditions

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{\Omega_0 \cup \Omega_T} = 0, \ i = 0, ..., 2k, \tag{1.2}$$

$$u|_{\Gamma_T} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_T} = 0, \tag{1.3}$$

where $\Gamma_T : \partial\Omega \times (0,T)$ is the lateral face of the cylinder $D_T$, $\Omega_0 : x \in \Omega$, $t = 0$ and $\Omega_T : x \in \Omega$, $t = T$ are the lower and upper bases of this cylinder, respectively, while $\frac{\partial}{\partial \nu}$ is a derivative with respect to the outer normal to the boundary $\partial D_T$ of the domain $D_T$. For $T=\infty$ we have $D_\infty=\Omega\times(0,\infty),\Gamma_\infty=\partial\Omega\times(0,\infty)$.

Below, for function $f=f(s_0,s_1,...,s_{n+1}),(s_0,s_1,...,s_{n+1})\in\mathbb{R}^{n+2}$ we assume that

$$f\in C(\mathbb{R}^{n+2}) \tag{1.4}$$

and

$$|f(s_0,s_1,...,s_{n+1})|\le M+\sum_{i=0}^{n+1}M_i|s_i|^{\alpha_i}\ \ \forall s=(s_0,s_1,...,s_{n+1})\in\mathbb{R}^{n+2}, \tag{1.5}$$

where $M,M_i,\alpha_i=const>0,\ i=0,1,...,n+1$.

To research of initial and mixed problems for the high order semilinear hyperbolic equations having a structure different from (1.1) is dedicated numerous literature (for example, see works [1], [3], [5], [8-10], [13-14], [18-19] and literature cited there). Some of the results in this direction have been discussed in workshop materials [6, 7].

Note that the left side part of the operator $L_f$ from (1.1), i.e. $L_0$ is a hyppoelliptic operator [4].

Denote by $C^{4,4k+2}(\overline{D}_T)$ the space of functions $u$ continuous in $\overline{D}_T$ and having there continuous partial derivatives $\partial_x^\beta u$, $\frac{\partial^l u}{\partial t^l}$, where $\partial_x^\beta=\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}...\partial x_n^{\beta_n}}$, $\beta=(\beta_1,...,\beta_n)$, $|\beta|=\sum_{i=1}^n\beta_i\le 4$; $l=1,...,4k+2$.

Let

$$C_0^{4,4k+2}(\bar{D}_T,\partial D_T):=\Big\{u\in C^{4,4k+2}(\overline{D}_T):\ u|_{\Gamma_T}=\frac{\partial u}{\partial\nu}\Big|_{\Gamma_T}=0,$$
$$\frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T}=0,\ i=0,...,2k\Big\}.$$

Let $u\in C_0^{4,4k+2}(\overline{D}_T,\partial D_T)$ be a classical solution of the problem (1.1), (1.2), (1.3). Multiplying both parts of the equation (1.1) by an arbitrary function $\varphi\in C_0^{4,4k+2}(\bar{D}_T,\partial D_T)$ and integrating the obtained equation by parts over the domain $D_T$, we obtain

$$-\int\limits_{D_T}\left[\frac{\partial^{2k+1}u}{\partial t^{2k+1}}\cdot\frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}}+\Delta u\cdot\Delta\varphi\right]dxdt+\int\limits_{D_T}f(u,\nabla u)\varphi dxdt=$$
$$=\int\limits_{D_T}F\varphi dxdt\ \ \forall\varphi\in C_0^{4,4k+2}(\bar{D}_T,\partial D_T). \tag{1.6}$$

We take the equality (1.6) as a basis for our definition of the weak generalized solution $u$ of the problem (1.1), (1.2), (1.3).

Introduce the Hilbert space $W_0^{2,2k+1}(D_T)$ as a completion of the classical space $C_0^{4,4k+2}(\overline{D}_T,\partial D_T)$ with respect to the norm

$$||u||^2_{W_0^{2,2k+1}(D_T)}=$$

$$= \int_{D_T} \left[ u^2 + \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 + \sum_{i,j=1}^{n} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 + \sum_{i=1}^{2k+1} \left( \frac{\partial^i u}{\partial t^i} \right)^2 \right] dxdt (1.7)$$

of the classical space $C_0^{4,4k+2}(\bar{D}_T, \partial D_T)$.

**Remark 1.1.** It follows from (1.7) that if $u \in W_0^{2,2k+1}(D_T)$, then $u \in \overset{o}{W}{}_2^1(D_T)$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^l u}{\partial t^l} \in L_2(D_T)$; $i,j = 1,...,n$; $l = 1,...,2k+1$. Here $W_2^m(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having up to the $m$-th order generalized derivatives from $L_2(D_T)$, and
$\overset{o}{W}{}_2^1(D_T) = \{u \in W_2^1(D_T) : \ u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory [12]. Moreover, in the case when the domain $\Omega$ is convex, implying that $D_T$ is also convex, since the following estimate is valid [12]

$$\int_{D_T} \left[ \sum_{i,j=1}^{n} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 + \sum_{i=1}^{n} \left( \frac{\partial^2 u}{\partial x_i \partial t} \right)^2 + \left( \frac{\partial^2 u}{\partial t^2} \right)^2 \right] dxdt \le$$

$$c \int_{D_T} \left[ \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial t^2} \right]^2 dxdt$$

$$\forall u \in \overset{o}{C}{}^2(\overline{D}_T, \partial D_T) := \{u \in C^2(\overline{D}_T) : u|_{\partial D_T} = 0\} \qquad (1.8)$$

with positive constant $c$, not dependent on $u$ and domain $D_T$, then from (1.7) we obtain continuous embedment of the spaces

$$W_0^{2,2k+1}(D_T) \subset W_2^2(D_T). \qquad (1.9)$$

Below, we assume that domain $\Omega$ is convex.

**Remark 1.2.** As it is known the space $W_2^2(D_T)$ is continuously and compactly embedded into $L_p(D_T)$ for $p < \frac{2(n+1)}{n-3}$ when $n > 3$ and for any $p \ge 1$ when $n = 2,3$; analogously, the space $W_2^1(D_T)$ is continuously and compactly embedded into $L_q(D_T)$ for $q < \frac{2(n+1)}{n-1}$ [14]. Therefore, taking into account continuous embeddment of spaces (1.9), the inequality (1.5) and due to the properties of the Nemitskii operators $N_i$, $i = 0, 1, ..., n+1$, acting by formulas $N_i v = |v|^{\alpha_i}$ [11], we obtain that the nonlinear operator

$$N : W_0^{2,2k+1}(D_T) \to L_2(D_T) \qquad (1.10)$$

acting by formula

$$Nu = f(u, \nabla u) \qquad (1.11)$$

is continuous and compact if the powers of nonlinearity $\alpha_i$ in the right side part of inequality (1.5) satisfy the following inequalities:

$$1 < \alpha_0 < \frac{n+1}{n-3} \text{ for } n > 3;\ \alpha_0 > 1 \text{ for } n = 2,3; \qquad (1.12)$$

$$1 < \alpha_i < \frac{n+1}{n-1},\ i = 1, ..., n+1,\ n \ge 2. \qquad (1.13)$$

Besides, from the remarks made above it follows that if $u \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$ then $f(u, \nabla u) \in L_2(D_T)$ and for $u_m \to u$ in the space $\overset{\circ}{W}{}^{2,2k+1}(D_T)$ we have $f(u_m, \nabla u_m) \to f(u, \nabla u)$ in the space $L_2(D_T)$.

**Definition 1.1**. Let function $f$ satisfy the conditions (1.4), (1.5), (1.12) and (1.13); $F \in L_2(D_T)$. The function $u \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$ is said to be a weak generalized solution of the problem (1.1), (1.2), (1.3), if for any $\varphi \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$ the integral equality (1.6) is valid.

## 2. Reduction of the problem (1.1), (1.2), (1.3) to the nonlinear functional equation

In the space $\overset{\circ}{C}{}^{4,4k+2}(\overline{D}_T, \partial D_T)$, together with scalar product

$$(u,v)_0 = \int_{D_T} \Bigg[ u \cdot v + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} + \\ + \sum_{i=1}^{2k+1} \frac{\partial^i u}{\partial t^i}\frac{\partial^i v}{\partial t^i} \Bigg] dxdt \tag{2.1}$$

with the norm $||u||_0 = ||u||_{\overset{\circ}{W}{}^{2,2k+1}(D_T)}$, defined by the right side part of the equality (1.7), let us introduce the following scalar product

$$(u,v)_1 = \int_{D_T} \left[ \frac{\partial^{2k+1} u}{\partial t^{2k+1}}\frac{\partial^{2k+1} v}{\partial t^{2k+1}} + \Delta u \cdot \Delta v \right] dxdt \tag{2.2}$$

with the norm

$$||u||_1^2 = \int_{D_T} \left[ \left( \frac{\partial^{2k+1} u}{\partial t^{2k+1}} \right)^2 + (\Delta u)^2 \right] dxdt, \tag{2.3}$$

where $u, v \in \overset{\circ}{C}{}^{4,4k+2}(\overline{D}_T, \partial D_T)$.

**Lemma 2.1**. The inequalities

$$c_1 ||u||_0 \le ||u||_1 \le c_2 ||u||_0 \quad \forall u \in \overset{\circ}{C}{}^{4,4k+2}(\overline{D}_T, \partial D_T) \tag{2.4}$$

hold, where the positive constants $c_1$ and $c_2$ do not depend on $u$.

**Proof**. If $u \in \overset{\circ}{C}{}^{4,4k+2}(\overline{D}_T, \partial D_T)$ then all the more the function $u(\cdot, t) \in C^2(\overline{\Omega})$, $u(\cdot, t)|_{\partial\Omega} = 0$ for fixed $t \in [0, T]$ and according to the known inequality [12]

$$\int_{\Omega} \left[ u^2(\cdot, t) + \sum_{i=1}^{n} \left( \frac{\partial u(\cdot, t)}{\partial x_i} \right)^2 \right] dx \le c_0 \int_{\Omega} \big( \Delta u(\cdot, t) \big)^2 dx, \tag{2.5}$$

where the positive constant $c_0 = c_0(\Omega)$ does not depend on $t \in [0, T]$. Integrating the inequality (2.5) with respect to $t \in [0, T]$ we obtain

$$\int_{D_T} \left[ u^2 + \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 \right] dxdt \le$$

$$c_0 \int_{D_T} (\Delta u)^2 dxdt \ \ \forall u \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T). \tag{2.6}$$

Now let us estimate the values

$$\int_{D_T} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2 dxdt, \ \ i = 1, ..., 2k,$$

by the value

$$\int_{D_T} \Big(\frac{\partial^{2k+1} u}{\partial t^{2k+1}}\Big)^2 dxdt.$$

Since $u \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$ satisfies the equalities (1.2), then

$$\frac{\partial^i u(\cdot,t)}{\partial t^i} = \frac{1}{(2k-i)!} \int_0^t (t-\tau)^{2k-i} \frac{\partial^{2k+1} u(\cdot,\tau)}{\partial t^{2k+1}} d\tau, \ \ i = 1, ..., 2k. \tag{2.7}$$

Using the Cauchy's inequality from (2.7) we have

$$\Big(\frac{\partial^i u(\cdot,t)}{\partial t^i}\Big)^2 \leq \frac{1}{((2k-i)!)^2} \int_0^t (t-\tau)^{2(2k-i)} d\tau \int_0^t \Big(\frac{\partial^{2k+1} u(\cdot,\tau)}{\partial t^{2k+1}}\Big)^2 d\tau =$$

$$= \frac{t^{4k-2i+1}}{((2k-i)!)^2 (4k-2i+1)} \int_0^t \Big(\frac{\partial^{2k+1} u(\cdot,\tau)}{\partial t^{2k+1}}\Big)^2 d\tau \leq$$

$$\leq T^{4k-2i+1} \int_0^T \Big(\frac{\partial^{2k+1} u(\cdot,\tau)}{\partial t^{2k+1}}\Big)^2 d\tau,$$

whence we obtain

$$\int_0^T \Big(\frac{\partial^i u(\cdot,t)}{\partial t^i}\Big)^2 dt \leq T^{4k-2i+2} \int_0^T \Big(\frac{\partial^{2k+1} u(\cdot,\tau)}{\partial t^{2k+1}}\Big)^2 d\tau, \ \ i = 1, ..., 2k. \tag{2.8}$$

Integrating both parts of the inequality (2.8) over domain $\Omega$ we get

$$\int_{D_T} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2 dxdt \leq T^{4k-2i+2} \int_{D_T} \Big(\frac{\partial^{2k+1} u}{\partial t^{2k+1}}\Big)^2 dxdt, \ \ i = 1, ..., 2k. \tag{2.9}$$

Finally, since the domain $\Omega$ is convex and thereafter the inequality (1.8) is valid, then from (1.7), (2.3), (2.6) and (2.9) it is clear that (2.4) is valid. The Lemma 2.1 is proved.

**Remark 2.1**. In view of the Lemma 2.1 by completion of the space $C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$ by the norm (2.3) we obtain the same Hilbert space $W_0^{2,2k+1}(D_T)$ with equivalent scalar products.

**Remark 2.2**. First let us consider the linear problem corresponded to (1.1), (1.2), (1.3), i.e. when $f = 0$. In this case for $F \in L_2(D_T)$ we introduce analogously a notion of a weak generalized solution $u \in W_0^{2,2k+1}(D_T)$ of this problem for which due to (1.6), (2.2) it is valid the following integral equality

$$(u,\varphi)_1 = \int_{D_T} \left[\frac{\partial^{2k+1} u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1} \varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta \varphi\right] dxdt = -\int_{D_T} F\varphi dxdt \tag{2.10}$$

$$\forall \varphi \in W_0^{2,2k+1}(D_T).$$

Taking into account (2.6) we have

$$\left|\int_{D_T} F\varphi dxdt\right| \leq ||F||_{L_2(D_T)}||\varphi||_{L_2(D_T)} \leq$$

$$\leq ||F||_{L_2(D_T)}||\varphi||_0 \leq c^{\frac{1}{2}}||F||_{L_2(D_T)}||\varphi||_1. \tag{2.11}$$

According to the Remark 2.1, (2.10) and (2.11) from the Riesz theorem it follows that there exists a unique function $u \in W_0^{2,2k+1}(D_T)$ which satisfies the equality (2.10) for any $\varphi \in W_0^{2,2k+1}(D_T)$ and for its norm the following estimate

$$||u||_1 \leq c_0^{\frac{1}{2}}||F||_{L_2(D_T)} \tag{2.12}$$

is valid. Thus, introducing the notation $u = L_0^{-1}F$, we find that to the linear problem corresponding to (1.1), (1.2), (1.3), i.e. for $f = 0$, there corresponds the linear bounded operator

$$L_0^{-1} : L_2\left(D_T\right) \to W_0^{2,2k+1}(D_T)$$

and for its norm the estimate

$$\left\|L_0^{-1}\right\|_{L_2(D_T)\to W_0^{2,2k+1}(D_T)} \leq c_0^{\frac{1}{2}} \tag{2.13}$$

holds by virtue of (2.12).

**Remark 2.3**. Taking into account Definition 1.1 and Remark 2.2, we can rewrite the equality (1.6), equivalent to the problem (1.1), (1.2), (1.3) in the form

$$u = L_0^{-1}\left[-f(u, \nabla u) + F\right] \tag{2.14}$$

in the Hilbert space $W_0^{2,2k+1}(D_T)$. Due to (1.11) equation (2.14) can be rewritten in the form

$$u = Au := L_0^{-1}(-Nu + F), \tag{2.15}$$

where in view of (2.13) and the Remark 1.2, if nonlinear function $f = f(u, \nabla u)$ satisfies conditions (1.5), (1.12) and (1.13), then the operator $A : W_0^{2,2k+1}(D_T) \to W_0^{2,2k+1}(D_T)$ from (2.15) will be continuous and compact [11].

Below we use well-known multiplicative inequality [12]

$$||v||_{p,G} \leq \beta||v_x||_{m,G}^{\tilde{\alpha}}||v||_{r,G}^{1-\tilde{\alpha}} \ \ \forall v \in \overset{0}{W}{}_2^1(G),\, G \subset \mathbb{R}^{n+1}, \tag{2.16}$$

$$\tilde{\alpha} = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} - \frac{1}{\tilde{m}}\right)^{-1},\ \tilde{m} = \frac{(n+1)m}{n+1-m},$$

whence taking into account [17]

$$\int_G |v|dG \leq (\operatorname{mes} G)^{1-\frac{1}{p}}||v||_{p,G}\,,\ p \geq 1,$$

for $G = D_T \subset \mathbb{R}^{n+1},\ r = 1, m = 2$ and $1 < p \leq \frac{2(n+1)}{n-1}$, where the constant $\beta = const > 0$ does not depend on domain $G$ and $v$, we have [6]

$$||v||_{L_p(D_T)} \leq \beta_0(\operatorname{mes} D_T)^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}}||v||_{\overset{0}{W}{}_2^1(D_T)} \ \ \forall v \in \overset{0}{W}{}_2^1(D_T), \tag{2.17}$$

where $\beta_0 = const > 0$ does not depend on $v$ and $T$.

Since $\mathrm{mes} D_T = T \mathrm{mes}\Omega$ then from (2.17) we have

$$||v||_{L_p(D_T)} \leq \beta_1 T^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}} ||v||_{\overset{0}{W}{}_2^1(D_T)} \quad \forall v \in \overset{0}{W}{}_2^1(D_T), \tag{2.18}$$

where $\beta_1 = \beta_0(\mathrm{mes}\Omega)^{\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2}}$, besides, it is easy to see that the condition $\frac{1}{p}+\frac{1}{n+1}-\frac{1}{2} > 0$ is equivalent to the condition $p < \frac{2(n+1)}{n-1}$.

Further, due to (1.7), (1.9), (2.4), if $u \in W_0^{2,2k+1}(D_T)$, then according to the well-known results on existence of traces on the domain boundary for the functions from the Sobolev spaces $W_2^k(D_T)$, $k \geq 1$, we have [12]

$$\frac{\partial u}{\partial t}, \ \frac{\partial u}{\partial x_i} \in \overset{0}{W}{}_2^1(D_T), \ i = 1, ..., n, \ \forall u \in W_0^{2,2k+1}(D_T).$$

Thus, if the inequalities (1.13) are fulfilled, in view of (1.7) and (2.18) we have

$$\left[\int_{D_T} |u_{x_i}|^{2\alpha_i} dxdt\right]^{\frac{1}{2}} = ||u_{x_i}||^{\alpha_i}_{L_{2\alpha_i}(D_T)} \leq \beta_1^{\alpha_i} T^{\alpha_i(\frac{1}{2\alpha_i}+\frac{1}{n+1}-\frac{1}{2})} ||u_{x_i}||^{\alpha_i}_{\overset{0}{W}{}_2^1(D_T)} \leq$$

$$\leq \beta_1^{\alpha_i} T^{\alpha_i(\frac{1}{2\alpha_i}+\frac{1}{n+1}-\frac{1}{2})} ||u||^{\alpha_i}_{W_0^{2,2k+1}(D_T)} = \beta_1^{\alpha_i} T^{\alpha_i(\frac{1}{2\alpha_i}+\frac{1}{n+1}-\frac{1}{2})} ||u||_0^{\alpha_i}, \tag{2.19}$$

$$i = 1, ..., n, \ \ \forall u \in W_0^{2,2k+1}(D_T),$$

analogously,

$$\left[\int_{D_T} |u_t|^{2\alpha_{n+1}} dxdt\right]^{\frac{1}{2}} \leq \beta_1^{\alpha_{n+1}} T^{\alpha_{n+1}(\frac{1}{2\alpha_{n+1}}+\frac{1}{n+1}-\frac{1}{2})} ||u||_0^{\alpha_{n+1}} \tag{2.20}$$

$$\forall u \in W_0^{2,2k+1}(D_T).$$

Note that due to (1.13) in the inequalities (2.19) and (2.20) we have

$$\gamma_i = \alpha_i\Big(\frac{1}{2\alpha_i} + \frac{1}{n+1} - \frac{1}{2}\Big) > 0, \ i = 1, ..., n+1. \tag{2.21}$$

At fulfillment of the condition (1.12) due to (2.18) and the well-known inequality [17]

$$||v||_{L_\alpha(G)} \leq (\mathrm{mes} G)^{1-\frac{\alpha}{p}} ||v||_{L_p(G)}, \ 0 \leq \alpha \leq p,$$

we have

$$\left[\int_{D_T} |u|^{2\alpha_0} dxdt\right]^{\frac{1}{2}} \leq \beta_0^{\alpha_0} T^{\gamma_0} ||u||_0^{\alpha_0} \ \forall u \in W_0^{2,2k+1}(D_T), \tag{2.22}$$

with positive constants $\beta_0$ and $\gamma_0$ which do not depend on $u$ and $T$.

Below we need the following refinement of the first inequality from (2.4). Due to (1.7), (1.8), (2.3), (2.6) and (2.9) we have

$$||u||_0^2 \leq \int_{D_T} \left[c_0(\Delta u)^2 + 2c\Big((\Delta u)^2 + \Big(\frac{\partial^2 u}{\partial t^2}\Big)^2\Big) + \sum_{i=1}^{2k+1} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2\right] dxdt \leq$$

$$\leq \int_{D_T}\left[(c_0+2c)(\Delta u)^2+2cT^{4k-2}\Big(\frac{\partial^{2k+1}u}{\partial t^{2k+1}}\Big)^2+\Big\{\sum_{i=1}^{2k+1}T^{4k-2i+2}\Big\}\Big(\frac{\partial^{2k+1}u}{\partial t^{2k+1}}\Big)^2\right]dxdt \leq$$

$$\leq \lambda^2(T)\int_{D_T}\Big[\Big(\frac{\partial^{2k+1}u}{\partial t^{2k+1}}\Big)^2+(\Delta u)^2\Big]dxdt =$$

$$= \lambda^2(T)||u||_1^2 \ \forall u \in W_0^{2,2k+1}(D_T), \tag{2.23}$$

where

$$\lambda(T)=\begin{cases}(c_0+4c+2k+1)^{\frac{1}{2}}, & T\leq 1,\\ (c_0+4c+2k+1)^{\frac{1}{2}}T^{2k}, & T>1.\end{cases} \tag{2.24}$$

Now, taking into account (1.5), (2.13), (2.19) - (2.24) we estimate $||Au||_{W_0^{2,2k+1}(D_T)}=||Au||_1$ from (2.15)

$$||Au||_{W_0^{2,2k+1}(D_T)} \leq ||L_0^{-1}||_{L_2(D_T)\to W_0^{2,2k+1}(D_T)}||-Nu+F||_{L_2(D_T)} \leq$$

$$\leq c_0^{\frac{1}{2}}||Nu||_{L_2(D_T)}+c_0^{\frac{1}{2}}||F||_{L_2(D_T)} \leq c_0^{\frac{1}{2}}\Big[\int_{D_T}\Big(M+M_0|u|^{\alpha_0}+\sum_{i=1}^{n}M_i|u_{x_i}|^{\alpha_i}+$$

$$+M_{n+1}|u_t|^{\alpha_{n+1}}\Big)^2dxdt\Big]^{\frac{1}{2}}+c_0^{\frac{1}{2}}||F||_{L_2(D_T)} \leq c_0^{\frac{1}{2}}\Big[\int_{D_T}(n+3)\big(M^2+M_0^2|u|^{2\alpha_0}+$$

$$+\sum_{i=1}^{n}M_i^2|u_{x_i}|^{2\alpha_i}+M_{n+1}^2|u_t|^{2\alpha_{n+1}}\Big)dxdt\Big]^{\frac{1}{2}}+c_0^{\frac{1}{2}}||F||_{L_2(D_T)} \leq$$

$$\leq (c_0(n+3))^{\frac{1}{2}}\Big[\Big(\int_{D_T}M^2dxdt\Big)^{\frac{1}{2}}+\Big(\int_{D_T}M_0^2|u|^{2\alpha_0}dxdt\Big)^{\frac{1}{2}}+$$

$$+\sum_{i=1}^{n}\Big(\int_{D_T}M_i^2|u_{x_i}|^{2\alpha_i}dxdt\Big)^{\frac{1}{2}}+\Big(\int_{D_T}M_{n+1}^2|u_t|^{2\alpha_{n+1}}dxdt\Big)^{\frac{1}{2}}\Big]+c_0^{\frac{1}{2}}||F||_{L_2(D_T)} \leq$$

$$\leq (c_0(n+3))^{\frac{1}{2}}\Big[\Big(M^2\mathrm{mes}D_T\Big)^{\frac{1}{2}}+M_0\beta_0^{\alpha_0}T^{\gamma_0}||u||_0^{\alpha_0}+\sum_{i=1}^{n+1}M_i\beta_1^{\alpha_i}T^{\gamma_i}||u||_0^{\alpha_i}\Big]+$$

$$+c_0^{\frac{1}{2}}||F||_{L_2(D_T)} \leq (c_0(n+3))^{\frac{1}{2}}\sum_{i=0}^{n+1}M_i\beta_i^{\alpha_i}T^{\gamma_i}\lambda^{\alpha_i}(T)||u||_1^{\alpha_i}+$$

$$+(c_0(n+3))^{\frac{1}{2}}(M^2\mathrm{mes}\Omega)^{\frac{1}{2}}+c_0^{\frac{1}{2}}||F||_{L_2(D_T)} =$$

$$=\sum_{i=0}^{n+1}a_i(T)||u||_1^{\alpha_i}+b(T) \ \ \forall u \in W_0^{2,2k+1}(D_T). \tag{2.25}$$

Here

$$a_i(T)=(c_0(n+3))^{\frac{1}{2}}M_i\beta_i^{\alpha_i}T^{\gamma_i}\lambda^{\alpha_i}(T), \ i=0,...,n+1, \tag{2.26}$$

$$b(T)=(c_0(n+3))^{\frac{1}{2}}(M^2\mathrm{mes}\Omega)^{\frac{1}{2}}T^{\frac{1}{2}}+c_0^{\frac{1}{2}}||F||_{L_2(D_T)}, \tag{2.27}$$

besides, in derivation of the estimate (2.25) we used the following obvious inequalities

$$\Big(\sum_{i=1}^{m} k_i\Big)^2 \le m\sum_{i=1}^{m} k_i^2, \quad \Big(\sum_{i=1}^{m} k_i^2\Big)^{\frac{1}{2}} \le \sum_{i=1}^{m} |k_i|.$$

Let us simplify the right hand part of the estimate (2.25). Since $\alpha_i > 1$, $i = 0, ..., n+1$, then for $||u_1|| \le 1$ we have $||u||_1^{\alpha_i} \le 1$, and for $||u||_1 > 1$ we have $||u||_1^{\alpha_i} \le ||u||_1^{\alpha}$, where

$$\alpha = \max_{0\le i\le n+1} \alpha_i > 1. \tag{2.28}$$

Therefore, from (2.25) we obtain

$$||Au||_{W_0^{2,2k+1}(D_T)} \le a(T)||u||_1^{\alpha} + b_1(T) \ \ \forall u \in W_0^{2,2k+1}(D_T), \tag{2.29}$$

where

$$a(T) = \sum_{i=0}^{n+1} a_i(T), \ \ b_1(T) = \sum_{i=0}^{n+1} a_i(T) + b(T), \tag{2.30}$$

where $a_i(T)$, $i = 0, ..., n+1$, and $b(T)$ are defined by the equalities (2.26) and (2.27).

## 3. The cases of existence and absence of solutions of the problem (1.1), (1.2), (1.3)

In this section, assuming that the function $F$ is defined in the domain $D_\infty$ and

$$F|_{D_T} \in L_2(D_T) \ \forall T > 0,$$

under the assumptions regarding the nonlinear function $f$ we prove the existence of positive number $T_0$ such that for $0 < T < T_0$ the problem (1.1), (1.2), (1.3) has at least one generalized solution $u \in W_0^{2,2k+1}(D_T)$ in the sense of Definition 1.1, while for sufficiently large $T$ this problem may not have a solution in the domain $D_T$. Generally speaking, the number $T_0$ depends on $F$.

According to the Remark 2.3 the solvability of the problem (1.1), (1.2), (1.3) is equivalent to the solvability of functional equation (2.15) in the Hilbert space $W_0^{2,2k+1}(D_T)$, in which operator $A$, acting by formula (2.15), is continuous and compact. For clarification of the question of solvability of functional equation (2.15) consider the following algebraic equation

$$az^{\alpha} + b_1 = z \tag{3.1}$$

with respect to unknown $z > 0$, where $a = a(T)$ and $b_1 = b_1(T)$, defined by equality (2.30) take part in the estimate (2.29) for the value $||Au||_{W_0^{2,2k+1}(D_T)}$.

For $T > 0$ due to (2.26), (2.27) and (2.30) it is clear that $a > 0$ and $b_1 \ge 0$. Simple analysis, analogous to that given in [17], for $\alpha = 3$ shows that:

1) for $b_1 = 0$ equation (3.1) together with root $z_1 = 0$ has only one positive root $z_2 = a^{-\frac{1}{\alpha-1}}$;

2) if $b_1 > 0$, then for $0 < b_1 < b_0$, where

$$b_0 = [\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}]a^{-\frac{1}{\alpha-1}}, \tag{3.2}$$

equation (3.1) has two positive roots $z_1$ and $z_2$, $0 < z_1 < z_2$. For $b = b_0$ these roots coincide and we have only one positive root

$$z_1 = z_2 = z_0 = (\alpha a)^{-\frac{1}{\alpha-1}};$$

3) for $b_1 > b_0$ equation (3.1) does not have nonnegative roots.

Note that in the case $0 < b_1 < b_0$ there hold the following inequalities

$$z_1 < z_0 = (\alpha a)^{-\frac{1}{\alpha-1}} < z_2.$$

In view of (2.24), (2.26), (2.27), (2.30) and (3.2) the condition $b_1 < b_0$ is equivalent to the condition

$$g(T) := a^{\frac{\alpha}{\alpha-1}}(T) + a^{\frac{1}{\alpha-1}}(T)\big[(c_0(n+3)M^2T\text{mes}\Omega)^{\frac{1}{2}} + c_0^{\frac{1}{2}}||F||_{L_2(D_T)}\big] <$$
$$< \alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}. \tag{3.3}$$

Since the Lebesque measure is absolutely continuous and $F \in L_{2,loc}(D_\infty)$, $F|_{D_T} \in L_2(D_T)$ $\forall T > 0$, and $\lim_{T\to 0} \text{mes} D_T = 0$, then

$$\lim_{T\to 0} ||F||_{L_2(D_T)} = 0. \tag{3.4}$$

Further, due to (2.21) and since $\gamma_0 > 0$ in (2.2), and also (2.26), (2.28), (2.30) and (3.3), (3.4) we have

$$\lim_{T\to 0} g(T) = 0. \tag{3.5}$$

At the same time due to (2.28) the right hand part of inequality (3.3) is positive. Therefore, there exists a positive number $T_0 = T_0(F)$ such that $b_1 < b_0$ when the condition

$$0 < T < T_0(F) \tag{3.6}$$

is fulfilled. Thus, if $T$ satisfies inequality (3.6), then operator

$$A : W_0^{2,2k+1}(D_T) \to W_0^{2,2k+1}(D_T),$$

acting by formula (2.15), maps the ball $B(0, z_2) := \{u \in W_0^{2,2k+1}(D_T) : ||u||_{W_0^{2,2k+1}(D_T)} \le z_2\}$ into itself, where $z_2 = z_2(T)$ is a maximal positive root of the equation (3.1). Indeed, if $u \in B(0, z_2)$, then due to (2.29) and (3.1) we have

$$||Au||_{W_0^{2,2k+1}(D_T)} \le a||u||_1^\alpha + b_1 \le a z_2^\alpha + b_1 = z_2.$$

Therefore, taking into account that operator $A$ is continuous and compact and maps closed convex ball $B(0, z_2) \subset W_0^{2,2k+1}(D_T)$ into itself, then according to the Schauder's theorem [2] the equation (2.15) has at least one solution $u$ from the space $W_0^{2,2k+1}(D_T)$.

Thus, the following theorem is valid.

**Theorem 3.1.** Let Lipschitz domain $\Omega$ be convex; function $f$ satisfy conditions (1.4), (1.5), (1.12), (1.13); function $F$ be defined in $D_\infty$ and $F|_{D_T} \in L_2(D_T)$ $\forall T > 0$. Then there exists a number $T_0 = T_0(F) > 0$ such that for any positive number $T < T_0$ the problem (1.1), (1.2), (1.3) has at least one weak generalized solution $u \in \overset{\circ}{W}{}^{2,2k+1}_0(D_T)$ in the sense of the Definition 1.1.

Now let us consider the cases of absence of a solution of the problem (1.1), (1.2), (1.3).

**Theorem 3.2**. Let $\Omega : |x| < 1$, function $F^0$ be defined in $D_\infty$ and for fixed $T > 0$, $F^0|_{D_T} \geq 0$, $||F^0|_{D_T}||_{L_2(D_T)} \neq 0$ and $F = \mu F^0$, $\mu = const > 0$. Then, if function $f = f(s_0, s_1, ..., s_{n+1})$ satisfies conditions (1.4), (1.5), (1.12), (1.13) and

$$f(s_0, s_1, ..., s_{n+1}) \leq -|s_0|^\alpha \ \ \forall (s_0, s_1, ..., s_{n+1}) \in \mathbb{R}^{n+2}, \tag{3.7}$$

where $\alpha = const > 1$, there exists a number $\mu_0 = \mu_0 \left(F^0, \alpha\right) > 0$ such that for $\mu > \mu_0$ the problem (1.1), (1.2), (1.3) cannot have a weak generalized solution in the space $W^{2,2k+1}_0(D_T)$.

**Proof**. Assume that the conditions of the theorem are fulfilled and the solution $u \in W^{2,2k+1}_0(D_T)$ of the problem (1.1), (1.2), (1.3) does exist for any fixed $\mu > 0$. Then the equality (1.6) takes the form

$$\int\limits_{D_T} f(u, \nabla u)\varphi dxdt = \int\limits_{D_T} \Big[\frac{\partial^{2k+1}u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta\varphi\Big] dxdt +$$

$$+\mu \int\limits_{D_T} F^0 \varphi dxdt \ \ \forall \varphi \in W^{2,2k+1}_0(D_T). \tag{3.8}$$

By integration by parts it can be easily verified that

$$\int\limits_{D_T} \Big[\frac{\partial^{2k+1}u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta\varphi\Big] dxdt = \int\limits_{D_T} u\Big[-\frac{\partial^{2(2k+1)}\varphi}{\partial t^{2(2k+1)}} + \Delta^2\varphi\Big] dxdt =$$

$$= -\int\limits_{D_T} uL_0\varphi dxdt \ \ \forall \varphi \in C^{4,4k+2}_0(\overline{D}_T, \partial D_T), \tag{3.9}$$

where the space $C^{4,4k+2}_0(\overline{D}_T, \partial D_T)$ is defined in the first section, besides, $C^{4,4k+2}_0(\overline{D}_T, \partial D_T) \subset W^{2,2k+1}_0(D_T)$.

In view of (3.9) we rewrite the equality (3.8) as follows

$$-\int\limits_{D_T} f(u, \nabla u)\varphi dxdt = \int\limits_{D_T} uL_0\varphi dxdt -$$

$$-\mu \int\limits_{D_T} F^0 \varphi dxdt \ \ \forall \varphi \in C^{4,4k+2}_0(\overline{D}_T, \partial D_T). \tag{3.10}$$

Below we will use the method of test functions [15]. As a test function we take $\varphi \in C^{4,4k+2}_0(\overline{D}_T, \partial D_T)$ such that $\varphi|_{D_T} > 0$. Due to (3.7) from (3.10) we have

$$\int_{D_T} |u|^\alpha \varphi dxdt \le \int_{D_T} uL_0\varphi dxdt - \mu \int_{D_T} F^0\varphi dxdt \ \ \forall \varphi \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T). \tag{3.11}$$

If in Young's inequality with parameter $\varepsilon > 0$

$$ab \le \frac{\varepsilon}{\alpha} a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} b^{\alpha'}; \ a, b \ge 0, \ \alpha' = \frac{\alpha}{\alpha - 1}$$

we take $a = |u|\varphi^{1/\alpha}$, $b = |L_0\varphi|/\varphi^{1/\alpha}$, then taking into account that $\alpha'/\alpha = \alpha' - 1$ we will have

$$|uL_0\varphi| = |u|\varphi^{1/\alpha} \frac{|L_0\varphi|}{\varphi^{1/\alpha}} \le \frac{\varepsilon}{\alpha}|u|^\alpha \varphi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}. \tag{3.12}$$

From (3.11), (3.12) we have the inequality

$$\Big(1 - \frac{\varepsilon}{\alpha}\Big) \int_{D_T} |u|^\alpha \varphi dxdt \le \frac{1}{\alpha'\varepsilon^{\alpha'-1}} \int_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dxdt - \mu \int_{D_T} F^0 \varphi dxdt,$$

whence for $\varepsilon < \alpha$ we get

$$\int_{D_T} |u|^\alpha \varphi dxdt \le \frac{\alpha}{(\alpha - \varepsilon)\, \alpha'\varepsilon^{\alpha'-1}} \int_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dxdt - \frac{\alpha\mu}{\alpha - \varepsilon} \int_{D_T} F^0\varphi dxdt. \tag{3.13}$$

Taking into account the equalities $\alpha' = \frac{\alpha}{\alpha-1}$, $\alpha = \frac{\alpha'}{\alpha'-1}$ and $\min\limits_{0<\varepsilon<\alpha} \frac{\alpha}{(\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = 1$ which is achieved at $\varepsilon = 1$, from (3.13) we find that

$$\int_{D_T} |u|^\alpha \varphi dxdt \le \int_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dxdt - \alpha'\mu \int_{D_T} F^0\varphi dxdt. \tag{3.14}$$

Note that it is not difficult to show the existence of a test function $\varphi$ such that

$$\varphi \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T), \ \ \varphi|_{D_T} > 0, \ \ \kappa_0 = \int_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} dxdt < +\infty. \tag{3.15}$$

Indeed, as it can be easily verified, the function

$$\varphi(x, t) = [(1 - |x|^2)t(T - t)]^m$$

for a sufficiently large positive $m$ satisfies conditions (3.15).

Since by the condition of the theorem $F^0 \in L_2(D_T)$, $\|F^0\|_{L_2(D_T)} \ne 0$, $F^0 \ge 0$, and $\mathrm{mes} D_T < +\infty$, due to the fact that $\varphi|_{D_T} > 0$ we will have

$$0 < \kappa_1 = \int_{D_T} F^0 \varphi dxdt < +\infty. \tag{3.16}$$

Denote by $g(\mu)$ the right-hand side of the inequality (3.14) which is a linear function with respect to $\mu$, and by (3.15), (3.16) we will have

$$g(\mu) < 0 \text{ for } \mu > \mu_0 \text{ and } g(\mu) > 0 \text{ for } \mu < \mu_0, \tag{3.17}$$

where

$$g(\mu) = \kappa_0 - \alpha' \mu \kappa_1, \ \ \mu_0 = \frac{\kappa_0}{\alpha' \kappa_1} > 0.$$

Owing to (3.17) for $\mu > \mu_0$, the right-hand side of the inequality (3.14) is negative, whereas the left-hand side of that inequality is nonnegative. The obtained contradiction proves the theorem.

**Remark 3.1.** Note that in the Theorem 3.2 for simplicity we assume $\Omega : |x| < 1$. However, this theorem is valid in more general case, when $\Omega$ represents a convex, sufficiently smooth domain. Our assumption was caused by the construction of a test function $\varphi$ satisfying conditions (3.15) according to the formula

$$\varphi(x,t) = [(1-|x|^2)t(T-t)]^m \tag{3.18}$$

for a sufficiently large positive $m$. If the boundary of the convex domain $\Omega$ is given by the equation $\partial\Omega : \omega(x) = 0$, where $\nabla_x \omega|_{\partial\Omega} \neq 0$, $\omega|_{\Omega} > 0$ and $\omega \in C^4(R^n)$, then, instead of the test function defined by formula (3.18), we should take

$$\varphi(x,t) = [\omega(x)t(T-t)]^m,$$

where $m$ is a sufficiently large positive number, and in this case the Theorem 3.2 remains valid.

## 4. One case of solvability of the problem (1.1), (1.2), (1.3)

Let function $f$ in the equation (1.1) depend only on variable $u$ and satisfy conditions

$$f \in C(\mathbb{R}), \ |f(u)| \leq M + M_0 |u|^{\alpha_0} \ \ \forall u \in \mathbb{R}, \tag{4.1}$$

where $\alpha_0 = const$ satisfies conditions (1.12). Consider the following condition

$$\lim_{|u|\to\infty} \sup \frac{f(u)}{u} \leq 0. \tag{4.2}$$

**Lemma 4.1**. Let Lipschitz domain $\Omega$ be convex,$F \in L_2(D_T)$ and the conditions (4.1), (4.2), (1.12) be fulfilled. Then for a weak generalized solution $u \in \overset{\circ}{W}{}^{2,2k+1}_0(D_T)$ of the problem (1.1), (1.2), (1.3) the a priori estimate

$$||u||_0 = ||u||_{W_0^{2,2k+1}(D_T)} \leq c_3 ||F||_{L_2(D_T)} + c_4 \tag{4.3}$$

is valid with constants $c_3 > 0$ and $c_4 \geq 0$, independent of $u$ and $F$.

**Proof**. Since $f \in C(\mathbb{R})$, then from (4.2) it follows that for each $\varepsilon > 0$ there exists a number $M_\varepsilon \geq 0$ such that

$$uf(u) \leq M_\varepsilon + \varepsilon u^2 \ \ \forall u \in \mathbb{R}. \tag{4.4}$$

Putting $\varphi = u \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$ in the equality (1.6) and taking into account (4.4) and (2.3), for any $\varepsilon > 0$ we obtain

$$||u||_1^2 = \int\limits_{D_T} uf(u)dxdt - \int\limits_{D_T} Fudxdt \le$$

$$\le M_\varepsilon \text{mes} D_T + \varepsilon \int_{D_T} u^2 dxdt + \int_{D_T} \Big(\frac{1}{4\varepsilon}F^2 + \varepsilon u^2\Big)dxdt =$$

$$= \frac{1}{4\varepsilon}||F||^2_{L_2(D_T)} + M_\varepsilon \text{mes} D_T + 2\varepsilon ||u||^2_{L_2(D_T)} \le$$

$$\le \frac{1}{4\varepsilon}||F||^2_{L_2(D_T)} + M_\varepsilon \text{mes} D_T + 2\varepsilon ||u||_0^2. \tag{4.5}$$

From (4.5) by virtue of (2.4) we have

$$c_1^2 ||u||_0^2 \le ||u||_1^2 \le \frac{1}{4\varepsilon}||F||^2_{L_2(D_T)} + M_\varepsilon \text{mes} D_T + 2\varepsilon ||u||_0^2,$$

whence for $\varepsilon = \frac{1}{4}c_1^2$ we obtain

$$||u||_0^2 \le 2c_1^{-4}||F||^2_{L_2(D_T)} + 2c_1^{-2} M_\varepsilon \text{mes} D_T.$$

From the last inequality follows (4.3) for $c_3 = 2c_1^{-4}$ and $c_4 = 2c_1^{-2} M_\varepsilon \text{mes} D_T$, where $\varepsilon = \frac{1}{4}c_1^2$. Lemma 4.1 is proved.

According to the Remark 2.3 the problem (1.1), (1.2), (1.3) is equivalent to the functional equation (2.15), where the operator $A$, acting in the Hilbert space $\overset{\circ}{W}{}^{2,2k+1}(D_T)$, is continuous and compact. At the same time according to the a priori estimate (4.3) of the Lemma 4.1 in which the constants $c_3 = 2c_1^{-4}$ and $c_4 = 2c_1^{-2} M_\varepsilon \text{mes} D_T, \varepsilon = \frac{1}{4}c_1^2$, for any parameter $\tau \in [0,1]$ and for every solution $u \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$ of the equation $u = \tau A u$ with the above-mentioned parameter the a priori estimate (4.3) is valid with the same constants $c_3 > 0$ and $c_4 \ge 0$, independent of $u$, $F$ and $\tau$. Therefore, by the Schaefer's fixed point theorem [19] the equation (2.15) and hence the problem (1.1), (1.2), (1.3) has at least one weak generalized solution $u$ from the space $\overset{\circ}{W}{}^{2,2k+1}(D_T)$. Thus the following theorem is valid.

**Theorem 4.1**. Let Lipschitz domain $\Omega$ be convex, the conditions (4.1), (4.2) and (1.12) be fulfilled. Then for any $F \in L_2(D_T)$ the problem (1.1), (1.2), (1.3) has at least one weak generalized solution $u \in \overset{\circ}{W}{}^{2,2k+1}(D_T)$.

## 5. The uniqueness of a solution of the problem (1.1), (1.2), (1.3)

**Theorem 5.1**. Let Lipschitz domain $\Omega$ be convex, $f = f(u)$ be a monotone function and satisfy the conditions (4.1), (1.12). Then for any $F \in L_2(D_T)$ the problem (1.1), (1.2), (1.3) cannot have more than one weak generalized solution in the space $\overset{\circ}{W}{}^{2,2k+1}(D_T)$.

**Proof**. Let $F \in L_2(D_T)$, and moreover, let $u_1$ and $u_2$ be two weak generalized solutions of the problem (1.1), (1.2), (1.3) from the space $\overset{\circ}{W}{}^{1,2k}_0(D_T)$, i.e., according to (1.6) the equalities

$$\int\limits_{D_T} \left[\frac{\partial^{2k+1} u_i}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta u_i \cdot \Delta\varphi\right] dxdt =$$

$$= \int\limits_{D_T} f(u_i)\varphi dxdt - \int_{D_T} F\varphi dxdt \;\; \forall\varphi \in W_0^{2,2k+1}(D_T), \tag{5.1}$$

are valid, $i = 1, 2$.

From (5.1), for the difference $v = u_2 - u_1$ we have

$$\int\limits_{D_T} \left[\frac{\partial^{2k+1} v}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta v \cdot \Delta\varphi\right] dxdt =$$

$$\int\limits_{D_T} \big(f(u_2) - f(u_1)\big)\varphi dxdt \;\; \forall\varphi \in W_0^{2,2k+1}(D_T). \tag{5.2}$$

Putting $\varphi = v \in W_0^{2,2k+1}(D_T)$ in the equality (5.2), due to (2.3) we obtain

$$||v||_1 = \int\limits_{D_T} \big(f(u_2) - f(u_1)\big)(u_2 - u_1) dxdt. \tag{5.3}$$

Since $f$ is the monotone function, we have

$$\big(f(s_2) - f(s_1)\big)(s_2 - s_1) \geq 0 \quad \forall s_1, s_2 \in \mathbb{R}. \tag{5.4}$$

From (2.4), (5.3) and (5.4) it follows that

$$c_1||v||_0 \leq ||v||_1 \leq 0,$$

whence we find that $v = 0$, i.e., $u_2 = u_1$, and hence the proof of the Theorem 5.1 is complete.

From Theorems 4.1 and 5.1, in its turn, it follows

**Theorem 5.2**. Let Lipschitz domain $\Omega$ be convex, $f$ be a monotone function and satisfy the conditions (4.1), (4.2) and (1.12). Then for any $F \in L_2(D_T)$ the problem (1.1), (1.2), (1.3) has a unique weak generalized solution in the space $W_0^{2,2k+1}(D_T)$.

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