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ERGODIC COMPONENTS AND NONSEPARABLE EXTENSIONS OF INVARIANT MEASURES

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Let E be an uncountable set, G be a group of transformations of E and let μ be a nonzero σ-finite G-invariant (or, more generally, G-quasiinvariant) measure on E. As usual, we denote by $\mathrm{dom}(\mu)$ the σ-algebra of all μ-measurable subsets of E.

One of the main questions in the theory of invariant (quasiinvariant) measures can be formulated in the following general manner: how rich is the σ-algebra $\mathrm{dom}(\mu)$? In other words, we are interested in the question: how many subsets of E can belong to $\mathrm{dom}(\mu)$?

There are several results, in connection with this general problem, for those cases where the pair (E, G) satisfies some natural additional conditions. For instance, supposing that a transformation group G is uncountable and acts freely in E (or, more generally, acts almost freely in E with respect to μ), it can be shown that $\mathrm{dom}(\mu)$ differs from the power-set $P(E)$ of E (cf. [1], [4]). Note that a very particular case of this result gives a solution to one problem posed by J. C. Oxtoby (see [1]).

We thus claim that, under rather natural assumptions, the relation $\mathrm{dom}(\mu) \neq P(E)$ holds true. At the same time, there are examples of nonzero σ-finite G-invariant measures μ for which $\mathrm{card}(\mathrm{dom}(\mu)) = \mathrm{card}(P(E))$. Therefore, it makes sense to reformulate the main question in other terms, e.g. in terms of the space $L_2(\mu)$ canonically associated with μ.

Recall that $L_2(\mu)$ denotes the Hilbert space of all square integrable (with respect to μ) real-valued functions on E. The Hilbert dimension of $L_2(\mu)$ may be regarded as a certain characteristic of μ. If $L_2(\mu)$ is separable, then the original measure μ is called separable; otherwise, μ is called nonseparable. It is not difficult to verify that, in general, the Hilbert dimension of $L_2(\mu)$ does not exceed $\mathrm{card}(P(E))$.

The following more concrete problem seems to be important in the theory of invariant measures (dynamical systems).

Problem 1. Let E be an uncountable set, G be a group of transformations of E and let μ be a nonzero σ-finite G-invariant (or, more generally, G-quasiinvariant) measure on E. Does there exist a G-invariant (respectively, G-quasiinvariant) extension μ' of μ such that the Hilbert dimension of $L_2(\mu')$ takes the maximum value?

Also, one can formulate the following weaker version of Problem 1.

Problem 2. Let E be an uncountable set, G be a group of transformations of E and let μ be a nonzero σ-finite G-invariant (G-quasiinvariant) measure on E. Does there exist a nonseparable G-invariant (G-quasiinvariant) extension of μ?

It is natural to suppose in the formulation of both these problems that the pair (E, G) is a homogeneous space, i.e. the group G acts transitively in E.

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Here we are going to discuss some aspects of the above-mentioned problems and to present the corresponding result. First, let us make several historical remarks concerning nonseparable extensions of invariant measures.

Many years ago, E. Marczewski formulated the question whether there exists a nonseparable translation-invariant extension of the classical Lebesgue measure λ on the real line $\mathbf{R}$ (see, e.g., [8] and comments therein). Some time later the papers [3] and [7] were published in which this question was solved positively, i.e. it was shown in [3] and [7] that λ admits various nonseparable translation-invariant extensions. However, the methods applied in [3] and [7] essentially differ from each other. To explain it in more details, denote by the symbol $\mathbf{c}$ the cardinality of the continuum. The extension λ' of λ constructed in [3] is such that the Hilbert dimension of $L_2(\lambda')$ is equal to $2^{\mathbf{c}}$ (i.e. the Hilbert dimension of $L_2(\lambda')$ is maximal) and the extension λ'' of λ constructed in [7] is such that the Hilbert dimension of $L_2(\lambda'')$ is equal to $\mathbf{c}$, hence is not maximal. The method of [3] uses the techniques of σ-independent almost invariant thick sets. The method of [7] is based on the existence of an everywhere discontinuous group homomorphism acting from $\mathbf{R}$ into the infinite-dimensional torus $\mathbf{T}^{\mathbf{c}}$ and having thick graph in the product space $\mathbf{R}\times\mathbf{T}^{\mathbf{c}}$. Notice also that the construction presented in [3] admits a straightforward generalization to the case of an uncountable commutative compact metrizable group equipped with its Haar measure. This generalization is thoroughly considered in the well-known monograph by E. Hewitt and K. Ross [2].

One may suppose that special topological properties of the Haar (respectively, Lebesgue) measure play an important role in the above-mentioned constructions. However, we will show in the sequel that the situation is absolutely different. Actually, it turns out that topological concepts are inessential for constructing nonseparable invariant extensions of an invariant measure, and the key role is played by so-called ergodic components of this measure (see below the precise definition of ergodic components).

Let E be a set, G be a group of transformations of E and let μ be a nonzero σ-finite G-invariant (more generally, G-quasiinvariant) measure on E. We may assume, without loss of generality, that μ is also complete.

Recall that μ is metrically transitive (or ergodic) if, for every μ-measurable set $X \subset E$ with $\mu(X) > 0$, there exists a countable family $\{g_k : k \in K\} \subset G$ such that

$$\mu(E \setminus \cup\{g_k(X) : k \in K\}) = 0.$$

It is well known that the metrical transitivity (or ergodicity) of an invariant measure μ is closely connected with its uniqueness property. Indeed, if μ has the uniqueness property, then μ is necessarily metrically transitive. Conversely, if μ is complete, G is uncountable and acts almost freely in E with respect to μ, then the metrical transitivity of μ implies the uniqueness property of μ (cf. [4], [5]).

Example 1. The left Haar measure μ on a σ-compact locally compact group G is metrically transitive (ergodic). More precisely, let H be an everywhere dense subgroup of G. Then the same μ considered as an invariant measure with respect to H (which acts on G from the left) is also metrically transitive (see, for instance, [5]).

Example 2. It was shown in [5] that the nonseparable translation-invariant extension λ'' of λ, constructed in [7], is metrically transitive (ergodic).

A set $Y \subset E$ is called to be almost G-invariant with respect to μ if the equality $\mu(g(Y)\triangle Y) = 0$ holds true for all transformations $g \in G$.

Clearly, $Y \subset E$ is almost G-invariant with respect to μ if and only if $E \setminus Y$ is almost G-invariant with respect to μ. Moreover, the union of a countable family of almost G-invariant sets with respect to μ is almost G-invariant with respect to μ. Therefore, the class of all almost G-invariant sets with respect to μ forms a G-invariant σ-algebra of subsets of E.

Let Y be a μ-measurable almost G-invariant subset of E with $\mu(Y) > 0$. Then Y determines a nonzero σ-finite G-invariant (G-quasiinvariant) measure μ_Y by the formula:

$$\mu_Y(X) = \mu(X \cap Y) \quad (X \in \mathrm{dom}(\mu)).$$

If μ_Y is metrically transitive (ergodic), then we say that μ_Y is an ergodic component of μ.

Obviously, μ is metrically transitive if and only if μ_E is an ergodic component of μ.

Example 3. It can be shown that the nonseparable translation-invariant measure λ' on $\mathbf{R}$ does not have any ergodic component.

Below, the symbols μ^* and μ_* denote, as usual, the outer measure and the inner measure associated with μ.

A set $Z \subset E$ is called to be thick with respect to μ if $\mu_*(E \setminus Z) = 0$.

We say that a family $\{X_i : i \in I\}$ of subsets of E is admissible for μ if the following relations are satisfied:

(1) $\{X_i : i \in I\}$ is disjoint;

(2) $\mu^*(\cup\{X_i : i \in I\}) > 0$;

(3) for each index $i \in I$, we have $\mu(X_i) = 0$;

(4) for any subset J of I, the set $\cup\{X_i : i \in J\}$ is almost G-invariant with respect to μ.

It is convenient to introduce the notation:

$$a(\mu) = \inf\{\mathrm{card}(I) \ : \ \textit{there exists an admissible family } \{X_i \ : \ i \in I\} \textit{ for } \mu\}.$$

As usual, for any cardinal number a, we denote by $cf(a)$ the cofinality of a, i.e. the least cardinal b such that a admits a representation $a = \sum\{a_t : t \in b\}$, where all cardinals a_t are strictly less than a. Recall that $cf(a)$ is always a regular cardinal number.

It directly follows from the definition of $a(\mu)$ that $a(\mu)$ is an uncountable regular cardinal.

Example 4. It can be proved that there exists a translation-invariant extension ν of the Lebesgue measure λ such that $a(\nu) = \omega_1$. Moreover, if E is a vector space over the field $\mathbf{Q}$ of all rational numbers and $\mathrm{card}(E) \geq \mathbf{c}$, then there exists a σ-finite translation-invariant measure μ on E such that $a(\mu) = \omega_1$.

We need two auxiliary propositions.

Lemma 1. *Let a σ-finite G-quasiinvariant measure μ be complete and metrically transitive (ergodic) and let a set $Y \subset E$ be almost G-invariant with respect to μ. Then at least one of the following three relations is valid:*

1) $\mu(Y) = 0$;

2) $\mu(E \setminus Y) = 0$;

3) *both sets Y and $E \setminus Y$ are thick with respect to μ (consequently, both of them are nonmeasurable with respect to μ).*

Lemma 2. *Suppose that* $\mathrm{card}(E) = \mathrm{card}(G) = a$ *and G acts transitively in E. Let μ be a nonzero σ-finite G-invariant (G-quasiinvariant) measure on E such that $\mu_*(Z) = 0$ for all sets $Z \subset E$ with* $\mathrm{card}(Z) < a$.

If $cf(a) > \omega$, then μ can be extended to a G-invariant (G-quasiinvariant) measure on E for which there exists at least one admissible family of subsets of E.

Using the lemmas formulated above, we can prove the following statement.

Theorem 1. *Assume that the Generalized Continuum Hypothesis holds. Let E be a set, G be a group of transformations of E and let μ be a nonzero σ-finite G-invariant measure on E. Suppose that these five conditions are satisfied:*

1) $cf(\mathrm{card}(E)) > \omega$;
2) $\mathrm{card}(G) = \mathrm{card}(E)$ *and G acts transitively in E;*
3) *μ has at least one ergodic component;*
4) *$a(\mu)$ is less than the first strongly inaccessible cardinal;*
5) *$\mu_*(Z) = 0$ for all sets $Z \subset E$ with* $\mathrm{card}(Z) < \mathrm{card}(E)$.

Then there exists a G-invariant extension μ' of μ such that the Hilbert dimension of the space $L_2(\mu')$ is equal to $2^{a(\mu)}$.

The proof of Theorem 1 uses the technique of S. Ulam's transfinite matrices (see [9]) and uncountable σ-independent families of almost invariant thick sets (cf.[2], [3]).

As a direct consequence of Theorem 1, we have the next statement.

Theorem 2. *Assume the Generalized Continuum Hypothesis. Let E be a set, G be a group of transformations of E and let μ be a nonzero σ-finite G-invariant measure on E. Suppose also that:*

(1) $\mathrm{card}(E)$ *is less than the first strongly inaccessible cardinal and* $cf(\mathrm{card}(E)) > \omega$;
(2) $\mathrm{card}(G) = \mathrm{card}(E)$ *and G acts transitively in E;*
(3) *μ has at least one ergodic component;*
(4) *$\mu_*(Z) = 0$ for all sets $Z \subset E$ with* $\mathrm{card}(Z) < \mathrm{card}(E)$.

Then there exists a G-invariant extension μ' of μ such that the Hilbert dimension of $L_2(\mu')$ is equal to $2^{\mathbf{c}}$(in particular, μ' is a nonseparable extension of μ).

Remark 1. Unfortunately, the G-invariant extensions of μ whose existence is stated by Theorem 1 and Theorem 2 have no ergodic components, because the construction of those extensions is essentially based on the existence of uncountable σ-independent almost G-invariant thick subsets of E. In this connection, it would be interesting to find a general method of extending measures by means of which a given nonzero σ-finite G-invariant (respectively, G-quasiinvariant) measure μ with ergodic components can be extended to a nonseparable G-invariant (respectively, G-quasiinvariant) measure μ' with ergodic components.

Remark 2. As mentioned in Example 2, the measure λ'' constructed in [7] is metrically transitive. It was proved that λ'' can be extended to a translation-invariant measure λ''' on $\mathbf{R}$ such that the Hilbert dimension of $L_2(\lambda''')$ is equal to $2^{\mathbf{c}}$ (for more details, see [6]).

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