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# On extensions of partial functions

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## Abstract

The problem of extending partial functions is considered from the general viewpoint. Some aspects of this problem are illustrated by examples, which are concerned with typical real-valued partial functions (e.g. semicontinuous, monotone, additive, measurable, possessing the Baire property).


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In this note we would like to consider several facts from mathematical analysis concerning extensions of real-valued partial functions. Some of those facts are rather easy and are accessible to average-level students. But some of them are much deeper, important and have applications in various branches of mathematics. The best known example of this type is the famous Tietze–Urysohn theorem, which states that every real-valued continuous function defined on a closed subset of a normal topological space can be extended to a real-valued continuous function defined on the whole space (see, for instance, [8], Chapter 1, Section 14). Another example of this kind is the fundamental Hahn–Banach theorem on extending a continuous linear functional defined on a vector subspace of a given normed

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vector space (see any text-book of functional analysis). Obviously, many other interesting and important examples can be pointed out in this context.

Let $\mathbf{R}$ denote the real line and let $X$ be an arbitrary subset of $\mathbf{R}$. A function $f : X \to \mathbf{R}$ is called a partial function acting from $\mathbf{R}$ into $\mathbf{R}$. We may write $f : \mathbf{R} \to \mathbf{R}$ simply saying that $f$ is a partial function whose domain is contained in $\mathbf{R}$. As usual, we denote $\mathrm{dom}(f) = X$. If $Y$ is any subset of $\mathbf{R}$, then the symbol $f|Y$ stands for the restriction of $f$ to $Y$. The symbol $cl(Y)$ denotes the closure of $Y$.

Here we are interested in the following general question: does there exist an extension $f^* : \mathbf{R} \to \mathbf{R}$ of $f$ defined on the whole $\mathbf{R}$ and having some "nice" properties. In particular, we may require that $f^*$ would be continuous, semicontinuous, monotone, Borel measurable, Lebesgue measurable, or would have the Baire property. An analogous question can be posed for partial functions acting from a set $E$ into $\mathbf{R}$, where $E$ is assumed to be endowed with some additional structure. In such a case an extension $f^* : E \to \mathbf{R}$ must preserve (in an appropriate sense) that structure. It is clear that questions of this type frequently arise in mathematical analysis, general topology and abstract algebra. Therefore, this topic is of interest for large groups of mathematicians.

Below, we make a small list of results in this direction, comment on each of them or give a necessary explanation, and refer the reader to other related works, in which extensions of partial functions are considered more thoroughly (see, e.g., [8]). For our convenience, we present the material in the form of examples of statements on extensions of real-valued partial functions. We think that, in various lecture courses, it is useful to provide students with additional information concerning extensions of partial functions. Such an approach essentially helps them to see more vividly deep connections between different fields of mathematics. Besides, the students should know that the general problem of extending partial functions is important in all mathematics - it naturally appears in many branches of this discipline and finds numerous applications.

We start with a very simple result that can be included in any lecture course of real analysis, oriented to beginners.

**Example 1.** A partial function $f : \mathbf{R} \to \mathbf{R}$ admits a continuous extension $f^* : \mathbf{R} \to \mathbf{R}$ defined on $\mathbf{R}$ if and only if, for each open interval $]a, b[ \subset \mathbf{R}$, the restriction of $f$ to the set $\mathrm{dom}(f) \cap ]a, b[$ is uniformly continuous. More generally, let $\{]a_i, b_i[ : i \in I\}$ be a family of open intervals in $\mathbf{R}$ such that $\mathbf{R} = \bigcup\{]a_i, b_i[ : i \in I\}$. One can assert that a partial function $f : \mathbf{R} \to \mathbf{R}$ admits a continuous extension $f^*$ with $\mathrm{dom}(f^*) = \mathbf{R}$ if and only if the restrictions of $f$ to all sets $\mathrm{dom}(f) \cap ]a_i, b_i[$ are uniformly continuous.

The proof of this fact is very easy: it suffices to take into account that any continuous real-valued function defined on a closed bounded subinterval of $\mathbf{R}$ is uniformly continuous. Also, it is not difficult to give an example of a partial function $f : \mathbf{R} \to \mathbf{R}$ for which there exists a countable family $\{[a_i, b_i] : i \in I\}$ of segments such that $\bigcup\{[a_i, b_i] : i \in I\} = \mathbf{R}$ and all restrictions $f|(\mathrm{dom}(f) \cap [a_i, b_i])$ are uniformly continuous, but $f$ does not admit a continuous extension $f^*$ with $\mathrm{dom}(f^*) = \mathbf{R}$.

The second example is less trivial.

**Example 2.** Recall that a partial function $f : \mathbf{R} \to \mathbf{R}$ is upper (respectively, lower) semicontinuous if, for each $t \in \mathbf{R}$, the set $\{x \in \mathrm{dom}(f) : f(x) < t\}$ (respectively, the set

$\{x \in \mathrm{dom}(f) : f(x) > t\}$) is open in $\mathrm{dom}(f)$. For $f$ with $\mathrm{dom}(f) = \mathbf{R}$, this definition is equivalent to the following: $f$ is upper (respectively, lower) semicontinuous if and only if $\limsup_{y \to x} f(y) = f(x)$ (respectively, $\liminf_{y \to x} f(y) = f(x)$) for all $x \in \mathbf{R}$ (see [8], Chapter 1, Section 18). It is interesting to note that every bounded upper (lower) semicontinuous partial function admits an upper (lower) semicontinuous extension defined on $\mathbf{R}$. Let us formulate a more precise result in this direction. First, recall that a partial function $g : \mathbf{R} \to \mathbf{R}$ is locally bounded from above (from below) if, for each point $x \in \mathbf{R}$, there exists a neighborhood $U(x)$ such that $g|U(x)$ is bounded from above (from below). A partial function $g : \mathbf{R} \to \mathbf{R}$ is locally bounded if it is locally bounded from above and from below simultaneously.

We also recall that an upper semicontinuous function can take its values from the set $\mathbf{R} \cup \{-\infty\}$ and a lower semicontinuous function can take its values from the set $\mathbf{R} \cup \{+\infty\}$.

Now, let $f : \mathbf{R} \to \mathbf{R} \cup \{-\infty, +\infty\}$ be any partial function. The following two assertions are equivalent:

(a) $f$ admits an upper (lower) semicontinuous extension $f^*$ with $\mathrm{dom}(f^*) = cl(\mathrm{dom}(f))$;
(b) $f$ is upper (lower) semicontinuous and locally bounded from above (from below).

The equivalence of (a) and (b) implies the validity of the next two statements.

(i) Let $f : \mathbf{R} \to [a, b]$ be a partial upper semicontinuous function. Then there exists an upper semicontinuous function $f^* : \mathbf{R} \to [a, b]$ extending $f$.
(ii) Let $f : \mathbf{R} \to [a, b]$ be a partial lower semicontinuous function. Then there exists a lower semicontinuous function $f^* : \mathbf{R} \to [a, b]$ extending $f$.

It should be mentioned that the same results hold true in a more general situation, e.g., for partial semicontinuous bounded functions acting from a normal topological space $E$ into the real line $\mathbf{R}$ (of course, in this generalized case, the Tietze–Urysohn theorem has to be applied to $E$).

The next example deals with monotone extensions of partial functions acting from $\mathbf{R}$ into $\mathbf{R}$.

**Example 3.** Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function increasing on its domain. It is easy to show that $f$ can always be extended to an increasing function $f^*$ defined on some maximal (with respect to inclusion) subinterval of $\mathbf{R}$. Let us denote the above-mentioned maximal subinterval by $T$ and let $a = \inf T$, $b = \sup T$. If $a = -\infty$ and $b = +\infty$, then $f^*$ is the required increasing extension of $f$ defined on the whole $\mathbf{R}$.

If, for example, $a \neq -\infty$, then in view of the maximality of $T$, we must have $\inf_{t \in T} f(t) = -\infty$ and, therefore, $f$ cannot be extended to an increasing function acting from $\mathbf{R}$ into $\mathbf{R}$. Analogously, if $b \neq +\infty$, then in view of the maximality of $T$, we must have $\sup_{t \in T} f(t) = +\infty$ and, therefore, $f$ cannot be extended to an increasing function acting from $\mathbf{R}$ into $\mathbf{R}$. We see that in both these cases our partial function $f$ is not locally bounded.

A similar result holds true for a decreasing partial function $f : \mathbf{R} \to \mathbf{R}$. We thus obtain a necessary and sufficient condition for extending a given partial function to a monotone function acting from **R** into **R**. Namely, the following two assertions are equivalent:

(a) $f$ is extendable to a monotone function $f^*$ with $\mathrm{dom}(f^*) = \mathbf{R}$;
(b) $f$ is monotone and locally bounded.

Of course, there is no problem connected with extending monotone partial functions if we admit infinite values of functions under consideration (cf. Example 2). Indeed, in this case any monotone partial function

$$f : \mathbf{R} \to \mathbf{R} \cup \{-\infty, +\infty\}$$

can be extended to a monotone function $f^*$ with $\mathrm{dom}(f^*) = \mathbf{R}$.

**Example 4.** Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function. Suppose that $f$ is Borel on its domain, i.e., for every Borel set $B \subset \mathbf{R}$, the preimage $f^{-1}(B)$ is a Borel subset of $\mathrm{dom}(f)$. It can be proved that $f$ always admits a Borel extension $f^* : \mathbf{R} \to \mathbf{R}$ with $\mathrm{dom}(f^*) = \mathbf{R}$. However, the proof of this fact is not easy. It needs a certain classification of all Borel partial functions acting from **R** into **R**. This classification is due to Baire and, according to it, each Borel partial function $f$ has its own Baire order $\alpha = \alpha(f)$, where $\alpha$ is some countable ordinal number. For instance, the equality $\alpha(f) = 0$ means that $f$ is continuous on its domain. Taking into account the above-mentioned classification, the existence of $f^*$ can be established by using the method of transfinite induction on $\alpha$ (for more details, see, e.g., [8], Chapter 3, Section 35).

**Example 5.** Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function. The following two assertions are equivalent:

(a) $f$ admits an extension $f^*$ defined on **R** and measurable in the Lebesgue sense;
(b) there exists a Lebesgue measure zero set $A \subset \mathbf{R}$ such that the restriction of $f$ to $\mathrm{dom}(f)\backslash A$ is a Borel function on its domain.

This fact can be proved by using the well-known Luzin criterion for the Lebesgue measurability of real-valued functions (see, e.g., [13]).

A similar fact holds true for partial functions having the Baire property. Recall that the Baire property may be regarded as a certain topological analog of measurability (see [8,10,13]). Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function. The following two assertions are equivalent:

(i) $f$ admits an extension $f^*$ defined on **R** and having the Baire property;
(ii) there exists a first category set $B \subset \mathbf{R}$ such that the restriction of $f$ to $\mathrm{dom}(f)\backslash B$ is a Borel function on its domain.

In connection with the latter fact, let us remark that if $X$ is a second category subset of **R**, then there exists a function $f : X \to \mathbf{R}$, which cannot be extended to a function

$f^* : \mathbf{R} \to \mathbf{R}$ having the Baire property. This deep result is due to Novikov (see [12]). It is essentially based on the Axiom of Choice and some special facts from descriptive set theory (e.g., the separation principle for analytic sets). A more detailed discussion of this result is also given in Chapter 14 of [5].

**Example 6.** If $g : \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function (respectively, function having the Baire property), then there exists a nonempty perfect set $C \subset \mathbf{R}$ such that $g|C$ is monotone on its domain (see, e.g., [5], Chapter 4, Exercise 11).

Sierpiński and Zygmund proved in [15] (see also [8], Chapter 3, Section 35) that there exists a real-valued function $h$ defined on $\mathbf{R}$ and satisfying the following condition: for any set $X \subset \mathbf{R}$ of cardinality continuum, the restriction $h|X$ is not continuous. In particular, this condition readily implies that, for any set $X \subset \mathbf{R}$ of cardinality continuum, the restriction $h|X$ is not monotone on $X$ (indeed, it suffices to apply the fact that the set of all discontinuity points of any monotone partial function is at most countable).

Assuming the Continuum Hypothesis (or, more generally, Martin's Axiom), we have the following two statements.

(a) If $X \subset \mathbf{R}$ is of second category, then $h|X$ cannot be extended to a function on $\mathbf{R}$ having the Baire property (cf. Novikov's result mentioned above).
(b) If $X \subset \mathbf{R}$ is of strictly positive outer Lebesgue measure, then $h|X$ cannot be extended to a function on $\mathbf{R}$ measurable in the Lebesgue sense.

Note that the validity of (a) and (b) does not need the full power of Martin's Axiom. Actually, it suffices to assume that any subset of $\mathbf{R}$ of cardinality strictly less than the cardinality continuum is of first category (respectively, of Lebesgue measure zero).

Statement (a) directly implies that no Sierpiński–Zygmund function has the Baire property.

Statement (b) implies that no Sierpiński–Zygmund function is Lebesgue measurable.

Moreover, one can assert that a Sierpiński–Zygmund function is nonmeasurable with respect to the completion of any nonzero $\sigma$-finite diffused (i.e., vanishing at all singletons) Borel measure on $\mathbf{R}$. Sierpiński–Zygmund functions have other interesting and important properties. Many works are devoted to these functions (see, e.g., [2,5,8]).

**Example 7.** Consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers. Let $f : \mathbf{Q} \to \mathbf{Q}$ denote the identical mapping. Since $\mathbf{Q}$ is a vector subspace of $\mathbf{R}$ and $f$ is a partial linear functional, it admits a linear extension $f^* : \mathbf{R} \to \mathbf{Q}$ with $\mathrm{dom}(f^*) = \mathbf{R}$. Of course, the construction of $f^*$ is not effective: it is based on the Axiom of Choice or the Zorn Lemma. Thus, the obtained extension $f^*$ is a solution of the Cauchy functional equation, i.e., we have

$$f^*(x + y) = f^*(x) + f^*(y)$$

for all $x \in \mathbf{R}$ and $y \in \mathbf{R}$. At the same time, taking into account the relation $\mathrm{ran}(f^*) = \mathbf{Q}$, we see that $f^*$ is discontinuous at all points of $\mathbf{R}$, i.e., $f^*$ is a nontrivial solution of the Cauchy functional equation (cf. [5,7,10]). It is well known that all nontrivial solutions of the Cauchy functional equation are nonmeasurable with respect to the Lebesgue measure and do not possess the Baire property (see, for instance, [7] or [10]). In our case, it is easy to see that the set $\{x \in \mathbf{R} : f^*(x) = 0\}$ is not Lebesgue measurable and does not have the Baire property. Note that, among nontrivial solutions of the Cauchy functional equation, one can meet some Sierpiński–Zygmund functions (see Chapter 11 of [5]; cf. also [11]). In addition, it should be pointed out that nontrivial solutions of the Cauchy functional equation are successfully applied in some deep questions concerning equidecomposability of polyhedra lying in a finite-dimensional Euclidean space (see, e.g., [3]).

**Example 8.** One can expect that if a partial function $f : \mathbf{R} \to \mathbf{R}$ is defined on a small (in an appropriate sense) subset of $\mathbf{R}$, then $f$ admits extensions with "nice" properties as well as extensions with "bad" properties. Indeed, if $\mathrm{dom}(f)$ is of Lebesgue measure zero (respectively, is of first category), then $f$ trivially can be extended to a Lebesgue measurable function (respectively, to a function possessing the Baire property). Thus, in both these cases, we come to extensions of functions with "nice" properties.

We are going to present an example of an extension of a partial function with an extremely "bad" property from the measure-theoretic viewpoint. First, let us introduce two auxiliary notions.

We recall that a set $X \subset \mathbf{R}$ is universal measure zero if there exists no nonzero $\sigma$-finite diffused Borel measure on $X$. For example, every Luzin subset of $\mathbf{R}$ is universal measure zero (this property and various other properties of Luzin sets are considered in [5,7,8,10,13]).

We shall say that a function $g : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable if there exists no nonzero $\sigma$-finite diffused measure $\mu$ on $\mathbf{R}$ such that $g$ is measurable with respect to $\mu$. In this definition, the domain of $\mu$ may be an arbitrary $\sigma$-algebra of subsets of $\mathbf{R}$ containing all singletons. We thus see that absolutely nonmeasurable functions (if they exist) are of more pathological nature than well-known examples of Lebesgue nonmeasurable real-valued functions.

It is proved in [6] that the following two assertions are equivalent:

(a) a function $g$ is absolutely nonmeasurable;
(b) the set $\mathrm{ran}(g)$ is universal measure zero and, for each $t \in \mathbf{R}$, the set $g^{-1}(t)$ is at most countable.

Starting with this result, it is not hard to show that an absolutely nonmeasurable function acting from $\mathbf{R}$ into $\mathbf{R}$ exists if and only if there exists a universal measure zero subset of $\mathbf{R}$ of cardinality continuum. Also, we infer the validity of the next statement.

Assume the Continuum Hypothesis (or, more generally, Martin's Axiom). Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function. The following two assertions are equivalent:

(i) $f$ admits an absolutely nonmeasurable extension $f^*$ with $\mathrm{dom}(f^*) = \mathbf{R}$;
(ii) the set $\mathrm{ran}(f)$ is universal measure zero and the sets $f^{-1}(t)$ are countable for all $t \in \mathbf{R}$.

From the equivalence of (i) and (ii) we obtain that any partial function $f : \mathbf{R} \to \mathbf{R}$ defined on a countable subset of $\mathbf{R}$ admits an extension to an absolutely nonmeasurable function defined on $\mathbf{R}$.

Let us remark that, according to the well-known Blumberg theorem, any function acting from **R** into **R** can be regarded as an extension of some continuous partial function whose domain is a countable everywhere dense subset of **R** (for the proof, see, e.g., [5], Chapter 7). The existence of a Sierpiński–Zygmund function $h : \mathbf{R} \to \mathbf{R}$ shows that, under the Continuum Hypothesis, there is no uncountable set $X \subset \mathbf{R}$ such that $h|X$ is continuous. Consequently, $h$ cannot be considered as an extension of a continuous partial function defined on an uncountable set of points.

Let us also note that (under Martin's Axiom) there exist additive absolutely nonmeasurable functions, which simultaneously are Sierpiński–Zygmund functions. The construction of such functions is based on the fact that there exists a generalized Luzin subset of **R** being a vector space over **Q** (recall once more that extensive information about Luzin subsets of **R** is contained in [5,7,8,10,13]).

The next example is concerned with extensions of real-valued partial functions of two variables.

**Example 9.** Let $\lambda$ denote the Lebesgue measure on the real line **R**. Consider a function of two real variables $G : \mathbf{R} \times [0, 1] \to \mathbf{R}$. Recall that $G$ satisfies the Carathéodory conditions if the following two relations hold:

(i) for each $x \in \mathbf{R}$, the function $G(x, \cdot) : [0, 1] \to \mathbf{R}$ is continuous;
(ii) for each $y \in [0, 1]$, the function $G(\cdot, y) : \mathbf{R} \to \mathbf{R}$ is $\lambda$-measurable.

Functions of this type play an important role in the theory of differential equations, optimization, probability, etc.

It is well known that if $G$ satisfies the Carathéodory conditions, then $G$ is measurable with respect to the product $\sigma$-algebra $\mathrm{dom}(\lambda) \otimes \mathscr{B}([0, 1])$, where $\mathscr{B}([0, 1])$ denotes the $\sigma$-algebra of all Borel subsets of [0, 1].

Now, take a partial function $F : \mathbf{R} \times [0, 1] \to \mathbf{R}$, i.e., suppose that $\mathrm{dom}(F) \subset \mathbf{R} \times [0, 1]$. Suppose, in addition, that $F$ is measurable with respect to $\mathrm{dom}(\lambda) \otimes \mathscr{B}([0, 1])$. Then the following two assertions are equivalent:

(a) $F$ admits an extension $F^* : \mathbf{R} \times [0, 1] \to \mathbf{R}$ with $\mathrm{dom}(F^*) = \mathbf{R} \times [0, 1]$, satisfying the Carathéodory conditions;
(b) for each $x \in \mathbf{R}$, the partial function $F(x, \cdot)$ is uniformly continuous on its domain.

This result is rather deep (compare with the simplest Example 1). Indeed, to establish the equivalence of (a) and (b), one has to apply the Choquet theorem on capacities (see, e.g., [4]) and the theorem on measurable selectors due to Kuratowski and Ryll-Nardzewski [9]. It should be mentioned that the same result remains true if we replace the Lebesgue measure space $(\mathbf{R}, \mathrm{dom}(\lambda), \lambda)$ by an arbitrary $\sigma$-finite complete measure space $(\Omega, \mathrm{dom}(\mu), \mu)$ and consider partial functions of the form $F : \Omega \times [0, 1] \to \mathbf{R}$ (cf. [5], Chapter 15).

Note that the standard Wiener process $W : \mathbf{R}^{[0,1]} \times [0, 1] \to \mathbf{R}$, which is regarded as a mathematical model of the Brownian motion, yields a good example of a function of two variables, satisfying the Carathéodory conditions. Here $\mathbf{R}^{[0,1]}$ stands, as usual, for the space of all real-valued functions on the segment [0, 1] and this space is assumed to be equipped with the completion of the Wiener probability measure $\mu_w$. Actually, in this example we are dealing with a certain set $\Omega \subset \mathbf{R}^{[0,1]}$ of full $\mu_w$-measure such that, for any $t \in [0, 1]$, the partial function $W(\cdot, t)$ is measurable with respect to $\mu_w$ and, for any $\omega \in \Omega$, the trajectory $W(\omega, \cdot)$ is continuous on [0, 1].

Our last example is concerned with extensions of real-valued partial set-functions.

We recall that a set-function is any function whose domain is some family of sets. Equivalently, we may say that a set-function $f$ is a function whose domain is a subset of the power-set $\mathscr{P}(E)$ of some set $E$. Thus, $f$ can be treated as a partial function acting from $\mathscr{P}(E)$.

**Example 10.** The Lebesgue measure $\lambda$ is a real-valued function defined on some class of subsets of $\mathbf{R}$. It is well known that this class is proper, i.e., there are $\lambda$-nonmeasurable sets in $\mathbf{R}$. Therefore, $\lambda$ may be regarded as a partial function acting from the power-set of $\mathbf{R}$ into $\mathbf{R}$. Several constructions of $\lambda$-nonmeasurable subsets of $\mathbf{R}$ are widely presented in the literature (Vitali sets, Bernstein sets, nontrivial ultrafilters in the set of all natural numbers, etc.). Compare also Example 7 where a nonmeasurable set associated with a nontrivial solution of the Cauchy functional equation is indicated. It is natural to ask whether there exists a measure $\mu$ on $\mathbf{R}$ extending $\lambda$ and defined on the family of all subsets of $\mathbf{R}$. This problem was originally posed by Banach. Under some additional set-theoretical assumptions (e.g., the Continuum Hypothesis or Martin's Axiom), the answer is negative (see, for instance, [5,8,13]). However, it seems that this question is undecidable within the standard system of axioms of Zermelo–Fraenkel set theory.

It is not difficult to show that, for any countable disjoint family $\{X_i : i \in I\}$ of subsets of $\mathbf{R}$, there exists a measure $\lambda'$ on $\mathbf{R}$ extending $\lambda$ and such that $\{X_i : i \in I\} \subset \mathrm{dom}(\lambda')$ (see, e.g., [1]). We thus see that the countable additivity of $\lambda$ can be preserved under the assumption that the "points" $X_i$ $(i \in I)$, on which we extend $\lambda$, are pairwise disjoint. Actually, it is shown in [1] that the same result remains valid for any $\sigma$-finite measure $\mu$ given on an abstract set $E$ and for any disjoint family $\{Y_j : j \in J\}$ of subsets of $E$. In particular, we easily obtain from this result that, for every finite family $\{Z_1, Z_2, \ldots, Z_n\}$ of subsets of $E$, there exists a measure $\mu'$ extending $\mu$ and satisfying the relation $\{Z_1, Z_2, \ldots, Z_n\} \subset \mathrm{dom}(\mu')$. However, if we have an infinite sequence $\{Z_1, Z_2, \ldots, Z_n, \ldots\}$ of subsets of $E$, then we cannot assert, in general, that $\mu$ is extendable to a measure $\mu'$ for which all these subsets become $\mu'$-measurable. In other words, sometimes there are countably many "points" in the power-set of $E$, which do not admit further extension of a given nonzero $\sigma$-finite measure $\mu$. For instance, we always have such "bad" points $Z_1, Z_2, \ldots, Z_n, \ldots$ in the power-set of $E$, where $E$ is an arbitrary uncountable universal measure zero subset of $\mathbf{R}$. Note that the existence of an uncountable universal measure zero set $E \subset \mathbf{R}$ is well known (see, e.g., [8], Chapter 2, Section 24, where the classical Luzin-Sierpiński construction of such an $E$ is given, starting with a decomposition of an analytic non-Borel set into uncountably many Borel components). In this context, the works [14,16] should also be mentioned, in which a much stronger result is presented stating the existence of uncountable universally small subsets of the real line.

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