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A NONSEPARABLE EXTENSION OF THE LEBESGUE MEASURE WITHOUT NEW NULLSETS

Abstract

Under the Continuum Hypothesis, it is shown that there exists a nonseparable extension of the Lebesgue measure on the real line whose nullsets coincide with the nullsets in the Lebesgue sense.

Let E be a set, $\mathcal{S}$ be a σ-algebra of subsets of E containing all one-element subsets (singletons) of E, and let μ be a nonzero σ-finite continuous (i.e., vanishing at all singletons) measure on $\mathcal{S}$. The general measure extension problem is to extend μ to a maximally large class of subsets of E. According to Ulam's theorem (see, for instance, [8] or [5], Chapter 5), it is consistent with the axioms of set theory that the domain of any extension μ' of μ cannot coincide with the power set of E (for instance, this is so if $card(E)$ is smaller than the first inaccessible cardinal number). Consequently, there always exists a set $X \subset E$ such that $X \notin dom(\mu')$. An easy argument shows that μ' can be extended to a measure μ'' so that X becomes μ''-measurable (see, e.g., Example 2 below). Thus, assuming that there are no large cardinals, one can infer that there are no maximal extensions of the original measure μ.

In this context, a similar result should be mentioned, which states that if $\{X_i : i \in I\}$ is an arbitrary partition of E, then there exists a measure ν on E extending μ and satisfying the relation

$$\{X_i : i \in I\} \subset dom(\nu).$$

For the proof of this result, see [2] or Cor. 2 in [1], p. 3. It should be noticed that measures in [1] and [2] are assumed to be probability ones, but the same argument works for arbitrary σ-finite measures. Actually, we do not need the

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above-mentioned result in our further consideration, but it also shows that, having any finite family of subsets of E, one can always extend μ to a measure which measures all these subsets (it suffices to consider the finite partition of E generated by the given family).

An analogous assertion fails to be true for arbitrary countable families of subsets of E. Moreover, the existence of a Luzin set L on the real line $\mathbb{R}$ having the cardinality of the continuum (see, e.g., [5], Chapter 20) implies that there is a countably generated σ-algebra of subsets of $\mathbb{R}$ containing all singletons and not admitting any nonzero σ-finite continuous measure. Indeed, it can easily be shown that every σ-finite continuous Borel measure given on a separable metric space is concentrated on some first category subset of that space. Since all first category subsets of L are at most countable, it follows that L does not admit a nonzero σ-finite continuous measure on the countably generated Borel σ-algebra of L. Now, using a one-to-one correspondence between L and $\mathbb{R}$, we obtain the required σ-algebra of subsets of $\mathbb{R}$. In a certain sense, a Luzin set L is a small subset of $\mathbb{R}$ because it has outer measure zero with respect to any Borel σ-finite continuous measure on $\mathbb{R}$. Recall that the existence of Luzin sets needs some additional set-theoretical axioms. For instance, the Continuum Hypothesis readily implies that there are Luzin subsets of $\mathbb{R}$ (see [5], Chapter 20). Similarly to this, Martin's Axiom implies that there are so-called generalized Luzin subsets of $\mathbb{R}$. In this connection, it should be underlined that the existence of analogous small subsets of $\mathbb{R}$ having cardinality ω_1 can be established within the theory **ZFC** (see, e.g., [6] or [9]).

All measures considered below are assumed to be nonzero, σ-finite, and continuous. If the need arises, we can additionally suppose, without loss of generality, that a measure under consideration is also complete (replacing it by its completion).

For any complete measure μ, we denote by $\mathcal{I}(\mu)$ the σ-ideal of all μ-measure zero sets (in short, μ-nullsets).

The symbol μ^* (respectively, μ_*) denotes the outer (respectively, inner) measure associated with μ.

The symbol λ stands for the Lebesgue measure on $\mathbb{R}$ (recall that λ is a complete measure).

The symbols ω and $\mathbf{c}$ denote, respectively, the least infinite cardinal and the cardinality of the continuum.

Example 1. A well-known method of extending a given complete measure μ is based on adding to $\mathcal{I}(\mu)$ some new sets, which are nonmeasurable with respect to μ and whose inner μ-measure is equal to zero (cf. [7]). Proceeding in this way, we come to a σ-ideal $\mathcal{I}'$ which properly contains $\mathcal{I}(\mu)$ and all of whose elements are of inner μ-measure zero. Then we consider the σ-algebra $\mathcal{S}'$ generated by $dom(\mu)$ and $\mathcal{I}'$. Any element U of this σ-algebra admits a representation $U = (X \cup Y) \setminus Z$ where $X \in dom(\mu)$, $Y \in \mathcal{I}'$ and $Z \in \mathcal{I}'$. We define a functional μ' on $\mathcal{S}'$ by the formula

$$\mu'(U) = \mu'((X \cup Y) \setminus Z) = \mu(X).$$

A straightforward verification shows that functional μ' is well defined and that μ' is also a measure on $\mathcal{S}'$ extending the initial measure μ.

Example 2. The method of extending measures described in Example 1 has a weak side. Indeed, from the viewpoint of the theory of Boolean algebras, the complete measure μ and its extension μ' are the same. Nevertheless, slightly changing the above method, we can achieve some difference between a measure and its extension if both of them are considered on the corresponding quotient Boolean algebras. For this purpose, let us take any set $T \subset E$ nonmeasurable with respect to a complete measure μ. Obviously, we must have $\mu_*(T) < \mu^*(T)$. If T_0 denotes a μ-measurable kernel of T and T_1 stands for a μ-measurable hull of T, then $\mu(T_1 \setminus T_0) > 0$, and the set $T \setminus T_0$ being a subset of $T_1 \setminus T_0$ satisfies the equalities

$$\mu_*(T \setminus T_0) = \mu_*((T_1 \setminus T_0) \setminus (T \setminus T_0)) = 0.$$

So we may assume (replacing, if necessary, T by $T \setminus T_0$ and E by $T_1 \setminus T_0$) that

$$\mu_*(T) = \mu_*(E \setminus T) = 0.$$

Let $\mathcal{S}'$ denote the σ-algebra of all those subsets U of E, which admit a representation

$$U = (X \cap T) \cup (Y \cap (E \setminus T)),$$

where $X \in dom(\mu)$ and $Y \in dom(\mu)$. Define a functional μ' on $\mathcal{S}'$ by the formula

$$\mu'(U) = (1/2)(\mu(X) + \mu(Y)).$$

As earlier, μ' is well defined and turns out to be a measure on $\mathcal{S}'$ extending μ. Also, since T is μ'-measurable, μ' strictly extends μ. Moreover, we see that the quotient Boolean algebra associated with μ is properly contained in the quotient Boolean algebra associated with μ'. At the same time, we have the equality $\mathcal{I}(\mu') = \mathcal{I}(\mu)$; i.e., μ' does not produce new nullsets.

Both of these constructions do not essentially change the metrical structure of μ. One can observe that the metric space associated with a measure μ and the metric space associated with its extension μ' obtained by using any of the two described constructions have the same topological weight. Therefore, if the original measure μ is separable (i.e., its metric space is separable), then the extended measure μ' is separable, too. Kakutani and Oxtoby [3] gave a construction of a nonseparable extension of the Lebesgue measure λ on the real line $\mathbb{R}$. Another construction of this kind was presented by Kodaira and Kakutani [4]. Both of those extensions of λ are invariant under the group of all isometries of $\mathbb{R}$.

It should be observed that the extension of λ obtained by Kakutani and Oxtoby has character $2^{\mathbf{c}}$ and necessarily yields new nullsets; i.e., there appear nullsets which are not of Lebesgue measure zero. This is so because the method of Kakutani and Oxtoby uses an uncountable independent family of subsets of $\mathbb{R}$ such that the intersection of any countable subfamily of this family is a Lebesgue nonmeasurable set which becomes a nullset with respect to the extended measure (for more details, see [3]).

The extension of λ obtained by Kodaira and Kakutani in [4] has character $\mathbf{c}$. By applying their method, one can also obtain new nullsets (see, for instance, Theorem 2 below). In this connection, the following question seems to be interesting: does there exist a nonseparable extension of λ whose nullsets are precisely the λ-nullsets? The question is of interest in view of the following circumstance: in many topics of real analysis, measure theory, and probability, the inner structure of a measure under consideration does not play any role, and only the induced concept "almost everywhere" is essential. The standard example of this type is a well-known theorem of Lebesgue stating that a bounded function $f : [a, b] \to \mathbb{R}$ is integrable in the Riemann sense if and only if f is continuous almost everywhere on $[a, b]$. The just mentioned Lebesgue theorem does not need the notion of the Lebesgue measure. For its proof, it completely suffices to apply the notion of a nullset in the Lebesgue sense. Numerous other examples of this kind can be pointed out.

Our goal is to demonstrate (under the Continuum Hypothesis) that there exists a nonseparable extension of λ which yields no new nullsets. It should be noticed that our argument may be regarded as a certain combination of the method of Kodaira and Kakutani [4] with the method of Luzin by means of which he proved the existence of his set on the real line (see, e.g., [5], Chapter 20).

We begin with some preliminary considerations.

Let $(E_1, \mathcal{S}_1, \mu_1)$ and $(E_2, \mathcal{S}_2, \mu_2)$ be two measure spaces, such that μ_1 is σ-finite and μ_2 is a probability measure. Let $f : E_1 \to E_2$ be a mapping, and let $G(f)$ denote the graph of f. Suppose that $G(f)$ is $(\mu_1 \times \mu_2)$-thick in the product set $E_1 \times E_2$; i.e., suppose that

$$(\mu_1 \times \mu_2)_*((E_1 \times E_2) \setminus G(f)) = 0.$$

For any set $Z \in dom(\mu_1 \times \mu_2)$, let us define

$$Z' = \{x \in E_1 : (x, f(x)) \in Z\}.$$

Further, introduce a class of sets

$$\mathcal{S}'_1 = \{Z' : Z \in dom(\mu_1 \times \mu_2)\},$$

and define a functional

$$\mu'_1(Z') = (\mu_1 \times \mu_2)(Z) \quad (Z \in dom(\mu_1 \times \mu_2)).$$

Then $\mathcal{S}'_1$ is a σ-algebra of subsets of E_1, and the functional μ'_1 is a measure on $\mathcal{S}'_1$ extending μ_1 (cf. [4], p. 576). In addition, the mapping f turns out to be measurable with respect to the σ-algebras $\mathcal{S}'_1$ and $\mathcal{S}_2$; i.e., for any set $Y \in \mathcal{S}_2$, we have $f^{-1}(Y) \in \mathcal{S}'_1$.

The remarks just made are rather simple, but they will be useful for our further constructions.

Lemma 1. *If a measure μ_1 is nonzero and a measure μ_2 is nonseparable, then the measure μ'_1 is nonseparable, too.*

Proof. Take any set $X \in dom(\mu_1)$ with $0 < \mu_1(X) < +\infty$. The nonseparability of μ_2 implies that, for some $\varepsilon > 0$, there exists an uncountable family $\{Y_i : i \in I\}$ of μ_2-measurable sets such that

$$\mu_2(Y_i \triangle Y_j) > \varepsilon \quad (i \in I,\ j \in I,\ i \neq j).$$

Obviously, we have

$$\mu'_1((X \times Y_i)' \triangle (X \times Y_j)') = (\mu_1 \times \mu_2)(X \times (Y_i \triangle Y_j)) =$$

$$\mu_1(X) \cdot \mu_2(Y_i \triangle Y_j) > \mu_1(X) \cdot \varepsilon \quad (i \in I,\ j \in I,\ i \neq j),$$

whence it follows that μ'_1 is nonseparable. □

Let J be a set with $card(J) = \mathbf{c}$. For any index $j \in J$, denote by ν_j the restriction of the Lebesgue measure λ to the Borel σ-algebra of $[0,1]$. Thus, ν_j is a Borel probability measure on $[0,1]$. Let ν stand for the product measure $\prod_{j \in J} \nu_j$.

The following two auxiliary propositions are easy and well known, but, for the sake of completeness, we give their short proofs here.

Lemma 2. *The cardinality of $dom(\nu)$ is equal to $\mathbf{c}$, and the topological weight of the metric space associated with ν is also equal to $\mathbf{c}$. In particular, ν is a nonseparable measure.*

Proof. Any set $Z \in dom(\nu)$ can be represented in the form $B \times [0,1]^{J \setminus J_0}$, where J_0 is a countable subset of J (certainly, depending on Z) and B is a Borel subset of $[0,1]^{J_0}$. Taking into account the equality $\mathbf{c}^\omega = \mathbf{c}$, this readily implies that $card(dom(\nu)) = \mathbf{c}$.

Further, for each $j \in J$, let us denote

$$Z_j = [0,1/2]_j \times [0,1]^{J \setminus \{j\}}.$$

Then we have

$$\nu(Z_j \triangle Z_k) = 1/2 \quad (j \in J,\ k \in J,\ j \neq k),$$

whence it follows that the topological weight of the metric space associated with ν is equal to $\mathbf{c}$ (in other words, ν has character $\mathbf{c}$). This ends the proof of Lemma 2. □

Lemma 3. *Let α be an infinite cardinal number satisfying the relation $\alpha^\omega = \alpha$, and let $(E_1, \mathcal{S}_1)$ and $(E_2, \mathcal{S}_2)$ be two measurable spaces such that $card(\mathcal{S}_1) \leq \alpha$ and $card(\mathcal{S}_2) \leq \alpha$. Then the cardinality of the product σ-algebra $\mathcal{S}_1 \otimes \mathcal{S}_2$ does not exceed α, either.*

Proof. The product σ-algebra $\mathcal{S}_1 \otimes \mathcal{S}_2$ is generated by the family of sets Z having the form $Z = X \times Y$, where $X \in \mathcal{S}_1$ and $Y \in \mathcal{S}_2$. The cardinality of this family does not exceed $\alpha \cdot \alpha = \alpha$. Taking into account the equality $\alpha^\omega = \alpha$, we come to the required result. □

The next lemma plays a key role in our further consideration.

Lemma 4. *Assume the Continuum Hypothesis. There exists a mapping*

$$f : \mathbb{R} \to [0,1]^J$$

satisfying the following relations:

(1) the graph of f is thick with respect to the product measure $\lambda \times \nu$;

(2) for any $(\lambda \times \nu)$-measure zero set Z, the set $Z' = \{x : (x, f(x)) \in Z\}$ is of λ-measure zero.

Proof. The required mapping f will be constructed by transfinite recursion.

In what follows, the symbol λ_0 stands for the restriction of the Lebesgue measure to the Borel σ-algebra of $\mathbb{R}$.

Let α denote the least ordinal number of cardinality $\mathbf{c}$. According to our assumption, $\alpha = \omega_1$; i.e., $card(\xi) \leq \omega$ for each ordinal $\xi < \alpha$.

Let $\preceq$ denote a well-ordering of $\mathbb{R}$ which is isomorphic to α.

Applying Lemmas 2 and 3, we deduce the equality

$$card(dom(\lambda_0 \times \nu)) = \mathbf{c}.$$

Let $\{Z_\xi : \xi < \alpha\}$ be the family of all those $(\lambda_0 \times \nu)$-measurable sets whose measure is strictly positive. We may suppose, without loss of generality, that the range of this family coincides with the range of $\{Z_\xi : \xi < \alpha,\ \xi \text{ is odd}\}$.

Let $\{T_\xi : \xi < \alpha\}$ be an enumeration of all those $(\lambda_0 \times \nu)$-measurable sets whose measure is equal to zero.

We are going to define an α-sequence $\{(x_\xi, y_\xi) : \xi < \alpha\}$ of points of the product space $\mathbb{R} \times [0,1]^J$. Suppose that, for an ordinal $\xi < \alpha$, the partial ξ-sequence $\{(x_\zeta, y_\zeta) : \zeta < \xi\}$ has already been constructed. Consider two cases.

(a). The ordinal ξ is even. In this case, denote by x the $\preceq$-least element of the set $\mathbb{R} \setminus \{x_\zeta : \zeta < \xi\}$ and, for each ordinal $\zeta < \xi$, define

$$T_\zeta(x) = \{y : (x, y) \in T_\zeta\}.$$

Also, define a set Ξ by the formula

$$\Xi = \{\zeta : \zeta < \xi \ \&\ \nu(T_\zeta(x)) = 0\}.$$

Obviously, we have $\nu(\cup\{T_\zeta(x) : \zeta \in \Xi\}) = 0$. Consequently,

$$[0,1]^J \setminus \cup\{T_\zeta(x) : \zeta \in \Xi\} \neq \emptyset.$$

Choose a point $y \in [0,1]^J \setminus \cup\{T_\zeta(x) : \zeta \in \Xi\}$, and put $(x_\xi, y_\xi) = (x, y)$.

(b). The ordinal ξ is odd. For each ordinal $\zeta < \xi$, define

$$T^0_\zeta = \{x \in \mathbb{R} : \nu(T_\zeta(x)) > 0\}.$$

Since $(\lambda_0 \times \nu)(T_\zeta) = 0$, we have $\lambda(T^0_\zeta) = 0$. Consider the set Z_ξ. Taking into account that $(\lambda_0 \times \nu)(Z_\xi) > 0$ and applying the Fubini theorem, we can find a point $x \in \mathbb{R} \setminus \cup\{T^0_\zeta : \zeta < \xi\}$ satisfying the relation $\nu(Z_\xi(x)) > 0$. Choose a point $y \in Z_\xi(x) \setminus \cup\{T_\zeta(x) : \zeta < \xi\}$, and put $(x, y) = (x_\xi, y_\xi)$.

Thus, in both cases (a) and (b), we have defined (x_ξ, y_ξ). Proceeding in this manner, we are able to construct the α-sequence $\{(x_\xi, y_\xi) : \xi < \alpha\}$ of points of $\mathbb{R} \times [0,1]^J$. Now, let us put

$$f(x_\xi) = y_\xi \qquad (\xi < \alpha).$$

We assert that the function f is the required one. Indeed, the equality

$$\mathbb{R} = \{x_\xi : \xi < \alpha\}$$

holds true (because the well-ordering $\preceq$ is isomorphic to α). Therefore, the domain of f coincides with $\mathbb{R}$. The thickness of $G(f)$ is a straightforward consequence of the relations

$$(x_\xi, y_\xi) \in Z_\xi \quad (\xi < \alpha,\ \xi \text{ is odd}).$$

Finally, let us show that the inclusion

$$\{x : (x, f(x)) \in T_\xi\} \subset T^0_\xi \cup \{x_\zeta : \zeta \leq \xi\}$$

is valid for each ordinal $\xi < \alpha$. Take any $(x, f(x)) \in T_\xi$. According to our construction, $(x, f(x)) = (x_\eta, y_\eta)$ for some $\eta < \alpha$. Consequently,

$$(x_\eta, y_\eta) \in T_\xi, \quad y_\eta \in T_\xi(x_\eta).$$

If $\eta \leq \xi$, then there is nothing to prove. Suppose now that $\xi < \eta$ and consider two cases.

(i). $\nu(T_\xi(x_\eta)) = 0$. If η is even, then our construction yields $y_\eta \notin T_\xi(x_\eta)$. If η is odd, then our construction also yields $y_\eta \notin T_\xi(x_\eta)$. Therefore, this case is impossible for $\xi < \eta$.

(ii). $\nu(T_\xi(x_\eta)) > 0$. This relation immediately implies $x_\eta \in T^0_\xi$, which yields the desired result.

The proof of Lemma 4 is thus completed. □

Theorem 1. *Under the Continuum Hypothesis, there exists a nonseparable measure λ' on the real line $\mathbb{R}$ extending λ such that $\mathcal{I}(\lambda') = \mathcal{I}(\lambda)$.*

Proof. In view of Lemma 4, there is a mapping

$$f : \mathbb{R} \to [0, 1]^J$$

whose graph is thick in the product space $\mathbb{R} \times [0, 1]^J$. For any $Z \in dom(\lambda_0 \times \nu)$, we put

$$Z' = \{x : (x, f(x)) \in Z\}, \quad \lambda'(Z') = (\lambda_0 \times \nu)(Z).$$

As said earlier, the functional λ' is well defined and is a measure extending λ_0. Obviously, the completion of λ' extends the Lebesgue measure on $\mathbb{R}$. We preserve the same notation for the completion of λ'. By virtue of Lemmas 1 and 2, λ' is a nonseparable measure. More precisely, we can assert that the character of λ' is equal to $\mathbf{c}$. This ends the proof of the theorem. □

Remark 1. Theorem 1 was established by assuming the Continuum Hypothesis. One of the referees kindly informed the author that the statement of this theorem remains valid under Martin's Axiom and even under a weaker assumption on cardinal invariants associated with $\mathcal{I}(\nu)$ and $\mathcal{I}(\lambda)$. The same referee also mentioned that the existence of a nonseparable extension of λ without new nullsets cannot be proved within **ZFC** theory. In this connection, it should be underlined that there is (within **ZFC**) a proper separable extension of λ whose nullsets are identical with the λ-nullsets (cf. Example 2).

Remark 2. As pointed out at the end of the proof of Theorem 1, the character of λ' is equal to $\mathbf{c}$. We do not know whether it is possible to show (at least, in some models of set theory) the existence of an extension of λ whose character is strictly greater than $\mathbf{c}$ and whose nullsets coincide with the nullsets in the Lebesgue sense.

Let $(E, \mathcal{S}, \mu)$ be a probability measure space. The argument presented before Lemma 1 shows that if a mapping $g : \mathbb{R} \to E$ is given whose graph is thick in $\mathbb{R} \times E$, then this mapping produces the measure λ_g which is an extension of the Lebesgue measure λ.

Theorem 2. *Suppose that $card(\mathcal{S}) \leq \mathbf{c}$. Then there exists a mapping $h : \mathbb{R} \to E$ satisfying the following conditions:*

(1) the graph of h is thick in the product space $\mathbb{R} \times E$;

(2) some set of λ_h-measure zero is thick in $\mathbb{R}$ with respect to λ; in particular, $\mathcal{I}(\lambda_h) \neq \mathcal{I}(\lambda)$.

Proof. Let B be a Bernstein subset of $\mathbb{R}$. We recall that, according to the definition of Bernstein sets, both of the sets B and $\mathbb{R} \setminus B$ are totally imperfect in $\mathbb{R}$, which readily implies the equalities

$$card(B) = card(\mathbb{R} \setminus B) = \mathbf{c},$$

$$\lambda_*(B) = \lambda_*(\mathbb{R} \setminus B) = 0.$$

In particular, B and $\mathbb{R} \setminus B$ are not measurable in the Lebesgue sense. Choose any point $y \in E$ and take the constant mapping $h_0 : B \to \{y\}$. By using the method of transfinite recursion, a mapping

$$h_1 : \mathbb{R} \setminus B \to E \setminus \{y\}$$

can easily be constructed such that the graph $G(h_1)$ is thick in the product space $\mathbb{R} \times E$ (cf. the proof of Theorem 1). Let h stand for the common extension of h_0 and h_1. Clearly, the graph of h is also thick in $\mathbb{R} \times E$. Consider the set $\mathbb{R} \times \{y\}$. Since the measure λ is σ-finite and $\mu(\{y\}) = 0$, we get

$$(\lambda \times \mu)(\mathbb{R} \times \{y\}) = 0.$$

Consequently, we must have

$$\lambda_h(\{x : (x, h(x)) \in \mathbb{R} \times \{y\}\}) = 0.$$

Therefore, $\{x : (x, h(x)) \in \mathbb{R} \times \{y\}\} \in \mathcal{I}(\lambda_h)$. But it is easy to see the validity of the relation

$$B = \{x : (x, h(x)) \in \mathbb{R} \times \{y\}\},$$

whence it follows that the set $\{x : (x, h(x)) \in \mathbb{R} \times \{y\}\}$ is thick with respect to λ. Moreover, the above-mentioned set being a Bernstein subset of $\mathbb{R}$ is thick with respect to any σ-finite continuous Borel measure on $\mathbb{R}$.

This completes the proof of Theorem 2.

□

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References

[1] A. Ascherl, J. Lehn, *Two principles for extending probability measures*, Manuscripta Math., **21** (1977), 43–50.

[2] D. Bierlein, *Über die Fortsetzung von Wahrscheinlichkeitsfeldern*, Z. Wahrsch. Verw. Gebiete, **1** (1962), 28–46.

[3] S. Kakutani, J. C. Oxtoby, *Construction of a nonseparable invariant extension of the Lebesgue measure space*, Ann. of Math., **52** (1950), 580–590.

[4] S. Kodaira, S. Kakutani, *A nonseparable translation-invariant extension of the Lebesgue measure space*, Ann. of Math., **52** (1950), 574–579.

[5] J. C. Oxtoby, *Measure and Category*, Springer-Verlag, Berlin, 1971.

[6] W. F. Pfeffer, K. Prikry, *Small spaces*, Proc. London Math. Soc., **58** (1989), 417–438.

[7] E. Szpilrajn (E. Marczewski), *Sur l'extension de la mesure lebesguienne*, Fund. Math., **25** (1935), 551–558.

[8] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math., **16** (1930), 140–150.

[9] P. Zakrzewski, *On a construction of universally small sets*, Real Anal. Exchange, **28(1)** (2002/03), 221–226.