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FINITE FAMILIES OF NEGLIGIBLE SETS AND INVARIANT EXTENSIONS OF THE LEBESGUE MEASURE

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There are many interesting problems in the general theory of invariant measures and, in particular, in the theory of translation-invariant extensions of the classical Lebesgue measure given on a finite-dimensional Euclidean space (see, for instance, [1], [2], [3], [4], [5], and [10]). One of the problems of this type will be considered below. It has a certain combinatorial character.

We shall use the following fairly standard notation:

ω = the set of all natural numbers (and, simultaneously, the cardinality of this set);

ω_1 = the least uncountable cardinal number;

$\mathbf{R}$ = the real line;

$\mathbf{c}$ = the cardinality of the continuum;

$\mathbf{R}^n$ = the n-dimensional Euclidean space (so $\mathbf{R} = \mathbf{R}^1$);

$\mathrm{dom}(\mu)$ = the domain of a given σ-finite measure μ (i.e., the σ-algebra of all μ-measurable sets).

λ = the one-dimensional Lebesgue measure on $\mathbf{R}$;

λ_n = the n-dimensional Lebesgue measure on $\mathbf{R}^n$ (so $\lambda_1 = \lambda$).

Let E be a base set, μ be a σ-finite measure defined on some σ-algebra of subsets of E, and let $\{A_1, A_2, \dots, A_k\}$ be a finite family of subsets of E. It is well known that there always exists a measure μ' on E extending μ and such that all sets $A_1, A_2, \dots, A_k$ are μ'-measurable.

In contrast with this situation, if the original measure μ is invariant under a group G of transformations of E, then we cannot assert, in general, that there exists an extension μ' of μ which also is invariant under G and for which all given sets $A_1, A_2, \dots, A_k$ are μ'-measurable. Even for $k = 1$, it may happen that the single set A_1 turns out to be nonmeasurable with respect to any G-invariant extension of μ.

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For instance, if E coincides with the real line $\mathbf{R}$ and $\mu = \lambda$, then the classical construction of Vitali [9] yields a set $V \subset \mathbf{R}$ which is nonmeasurable with respect to every translation-invariant extension of λ (or, in other words, V turns out to be absolutely nonmeasurable with respect to the class of all translation-invariant extensions of λ).

Example 1. In [2] two sets $A_1 \subset \mathbf{R}$ and $A_2 \subset \mathbf{R}$ were constructed, which satisfy the following conditions:

(1) there exists a translation-invariant extension μ_1 of λ such that $\mu_1(A_1) = 0$;

(2) there exists a translation-invariant extension μ_2 of λ such that $\mu_2(A_2) = 0$;

(3) for every nonzero σ-finite translation-invariant measure μ on $\mathbf{R}$, the set $A_1 \cup A_2$ is nonmeasurable with respect to μ.

In particular, (3) implies that there exists no nonzero σ-finite translation-invariant measure ν on $\mathbf{R}$ such that both sets A_1 and A_2 are ν-measurable.

The natural question arises whether it is possible to generalize the above-mentioned Example 1 to the case of several subsets of the real line. The main goal of this report is to establish an analogous result for finitely many subsets $A_1, A_2, \dots, A_k$ of $\mathbf{R}$, where k is an arbitrary natural number greater than 2. Actually, it will be shown below that an old theorem of Sierpiński [6], concerning a certain logical equivalent of the Continuum Hypothesis, enables to give a positive answer to this question.

We need one notion from the theory of invariant measures.

Let E be a set and let G be a group of transformations of E. We recall (see, e.g., [3] and [4]) that a set $X \subset E$ is G-negligible in E if the following two conditions hold:

(a) there exists a nonzero σ-finite G-quasi-invariant measure μ_0 on E such that $X \in \mathrm{dom}(\mu_0)$;

(b) for any σ-finite G-quasi-invariant measure μ on E, we have the implication

$$X \in \mathrm{dom}(\mu) \Rightarrow \mu(X) = 0.$$

Some properties of G-negligible sets are discussed in [3] and [4]. In particular, the following auxiliary proposition is formulated therein.

Lemma 1. *Let Γ_1 and Γ_2 be two commutative groups and suppose that $\phi : \Gamma_1 \to \Gamma_2$ is a surjective homomorphism.*

If Y is a Γ_2-negligible subset of Γ_2, then $X = \phi^{-1}(Y)$ is a Γ_1-negligible subset of Γ_1.

We also need the next three auxiliary statements.

Lemma 2. *Let G and H be two commutative groups and let* $\mathrm{card}(H) > \omega$. *Consider the direct sum $G + H$. Let X be a subset of $G + H$ such that* $\mathrm{card}((g + H) \cap X) < \omega$ *for each element $g \in G$.*

Then X is a $(G + H)$-negligible subset of $G + H$.

Lemma 3. *Let $k \geq 2$ be a natural number and let G be a vector space over the field $\mathbf{Q}$ of rational numbers representable in the form of a direct sum*

$$G = G_1 + G_2 + \cdots + G_k,$$

where all G_i $(i = 1, 2, \dots, k)$ are vector subspaces of G of cardinality ω_1.

Then subsets $Y_1, Y_2, \dots, Y_k$ of G can be found such that:

(a) *for each index $i \in \{1, 2, \dots, k\}$, the union $Y_1 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cdots \cup Y_k$ is a G-negligible set in G;*

(b) *there exists a countable family $\{g_m : m < \omega\}$ of elements from G for which we have*

$$\cup\{g_m(Y_1 \cup Y_2 \cup \cdots \cup Y_k) : m < \omega\} = G.$$

In particular, there is no nonzero σ-finite G-invariant measure ν on G such that all sets $Y_1, Y_2, \dots, Y_k$ are ν-measurable.

The proof of Lemma 3 is based on some ideas of Sierpiński which he used in establishing the equivalence of the Continuum Hypothesis to the existence of certain decompositions of $\mathbf{R}^2$ and $\mathbf{R}^3$ (see [6], [7], and [8]).

Lemma 4. *For each natural number $n \geq 1$ and for each natural number $k \geq 2$, the Euclidean space $\mathbf{R}^n$ can be represented in the form of a direct sum*

$$G_1 + G_2 + \ldots + G_k + H,$$

where all G_i $(i = 1, 2, \dots, k)$ and H are vector spaces over the field $\mathbf{Q}$ of rational numbers and the following conditions are satisfied:

(a) $\mathrm{card}(G_1) = \mathrm{card}(G_2) = \cdots = \mathrm{card}(G_k) = \omega_1$*;*

(b) $\mathrm{card}(H) = \mathbf{c}$*;*

(c) *H is a λ_n-thick subset of $\mathbf{R}^n$, i.e., for any λ_n-measurable set $Z \subset \mathbf{R}^n$ with $\lambda_n(Z) > 0$, we have $Z \cap H \neq \emptyset$.*

With the aid of the above-mentioned lemmas we obtain the main statement of this report.

Theorem. *Let $n > 0$ and $k \geq 2$ be two natural numbers. Then subsets $A_1, A_2, \dots, A_k$ of the Euclidean space $\mathbf{R}^n$ can be found such that:*

(1) *for each index $i \in \{1, 2, \dots, k\}$, the set*

$$A_1 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup \cdots \cup A_k$$

is $\mathbf{R}^n$-negligible in $\mathbf{R}^n$;

(2) *for each index* $i \in \{1, 2, \dots, k\}$, *there is a complete translation-invariant extension* μ_i *of* λ_n *satisfying the equality*

$$\mu_i(A_1 \cup \dots \cup A_{i-1} \cup A_{i+1} \cup \dots \cup A_k) = 0$$

and, consequently, all sets $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_k$ *turn out to be measurable with respect to* μ_i*;*

(3) *there exists no nonzero* σ*-finite translation-invariant measure* μ *on* $\mathbf{R}^n$ *for which all sets* $A_1, A_2, \dots, A_k$ *are* μ*-measurable.*

Example 2. Let us consider the Euclidean plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ and let a set $X \subset \mathbf{R}^2$ be such that $\mathrm{card}(X \cap (\{t\} \times \mathbf{R})) < \omega$ for all $t \in \mathbf{R}$. Then, according to Lemma 2, X is $\mathbf{R}^2$-negligible in $\mathbf{R}^2$. At the same time, there exists a set $Z \subset \mathbf{R}^2$ which satisfies the relation $\mathrm{card}(Z \cap (\{t\} \times \mathbf{R})) \leq \omega$ for any $t \in \mathbf{R}$, but which is not $\mathbf{R}^2$-negligible in $\mathbf{R}^2$ (see, for instance, [2] or [4] where a much stronger result is presented).

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