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ALMOST MEASURABLE REAL-VALUED FUNCTIONS AND EXTENSIONS OF THE LEBESGUE MEASURE

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We consider the concept of almost measurable real-valued functions, which is similar to the concept of almost continuous functions introduced by Stallings in [9] (see also [7]).

Let $\mathbf{R}$ denote the real line and let λ_n stand for the n-dimensional Lebesgue measure on the Euclidean space $\mathbf{R}^n$.

We say that a function $f : \mathbf{R} \to \mathbf{R}$ is almost measurable if, for every λ_2-measurable set $V \subset \mathbf{R}^2$ containing the graph of f, there exists a λ_1-measurable function $g : \mathbf{R} \to \mathbf{R}$ whose graph is also contained in V.

The following facts are easy consequences of this definition.

(a) Any Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}$ is almost measurable.

(b) If $f : \mathbf{R} \to \mathbf{R}$ is almost measurable and its graph is λ_2-measurable, then f is Lebesgue measurable.

In connection with (b), it should be mentioned that if the graph $Gr(f)$ of a function $f : \mathbf{R} \to \mathbf{R}$ is λ_2-measurable, then it is of λ_2-measure zero (because in $\mathbf{R}^2$ there are uncountably many pairwise disjoint translates of $Gr(f)$).

Using the classical Luzin-Jankov-von Neumann theorem on measurable selectors (see, e.g., [4]), we obtain the following characterization of almost measurable functions.

Theorem 1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function and let D denote some λ_2-measurable hull of the graph of f. The following two assertions are equivalent:*

(1) *f is almost measurable;*

(2) *there exists a disjoint covering $\{X_1, X_2\}$ of $\mathbf{R}$ by two λ_1-measurable sets such that the restriction $f|X_1$ is Lebesgue measurable and, for each point $x \in X_2$, the inequality $\lambda_1^*(\{y \in \mathbf{R} : (x, y) \in D\}) > 0$ holds true.*

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It is well known that there exists a function $f : \mathbf{R} \to \mathbf{R}$ whose graph is a thick subset of the plane $\mathbf{R}^2$, i.e. we have $(\lambda_2)_*(\mathbf{R}^2 \setminus Gr(f)) = 0$. Clearly, such an f is not Lebesgue measurable. One of the earliest examples of a function of this type was constructed by W. Sierpiński with the aid of the method of transfinite recursion (see, for instance, [1], [8]). All such (at first sight, pathological) functions turn out to be almost measurable in the sense of our definition. Indeed, Theorem 1 immediately implies

Theorem 2. *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function whose graph is thick with respect to the measure λ_2. Then f is almost measurable.*

Any almost measurable function becomes measurable with respect to an appropriate extension of the Lebesgue measure λ_1. To show this, we need a measure extension construction which can be successfully applied to a wide class of σ-finite measures (cf. [5], [2]).

Lemma. *Let X be a λ_1-measurable subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function and let D denote some λ_2-measurable hull of $Gr(f)$. Suppose that, for each point $x \in X$, the relation*

$$0 < \lambda_1^*(\{y \in \mathbf{R} : (x, y) \in D\}) < +\infty$$

holds true. Then there exists a measure λ_1' on X such that:

(1) *λ_1' extends the restriction of λ_1 to the σ-algebra of all Lebesgue measurable subsets of X;*

(2) *f is measurable with respect to λ_1'.*

Using this lemma and Theorem 1, we obtain the following statement.

Theorem 3. *Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary almost measurable function. Then there exists a measure λ_1' on $\mathbf{R}$ such that:*

(1) *λ_1' extends λ_1;*

(2) *f is measurable with respect to λ_1'.*

Remark 1. In connection with Theorem 3, it should be noticed that there exist many functions $g : \mathbf{R} \to \mathbf{R}$ which are measurable with respect to certain extensions of λ_1 but are not almost measurable.

Theorem 4. *Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be two functions. Suppose that f is almost measurable and g is Lebesgue measurable. Then their sum $f + g$ is almost measurable.*

Theorem 5. *There exist two functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ satisfying the following relations:*

(1) *f and g are additive;*

(2) *the graphs of f and g are thick with respect to λ_2;*

(3) *$ran(f + g)$ coincides with the set $\mathbf{Q}$ of all rational numbers.*

In particular, both f and g are almost measurable but their sum $f + g$ is not almost measurable.

Under additional set-theoretical assumptions, Theorem 5 can be essentially strengthened. Let $\mathbf{c}$ denote the cardinality of the continuum. Recall that $X \subset \mathbf{R}$ is a generalized Luzin set in $\mathbf{R}$ if $card(X) = \mathbf{c}$ and $card(X \cap Y) < \mathbf{c}$ for every first category set $Y \subset \mathbf{R}$. The existence of generalized Luzin sets easily follows from Martin's Axiom (usually denoted by **MA**). Under the same axiom, all generalized Luzin sets turn out to be universal measure zero (cf. [4], [6], [8]).

Assuming **MA**, there exist two functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that:

1) f and g are additive;
2) the sets $Gr(f)$ and $Gr(g)$ are thick with respect to λ_2;
3) $f + g$ is injective;
4) $ran(f + g)$ is a generalized Luzin set in $\mathbf{R}$.

The construction of f and g (by means of the method of transfinite recursion) is presented in [3]. Theorem 2 and relation 2) imply that both functions f and g are almost measurable. Relations 3) and 4) yield that the function $f+g$ is absolutely nonmeasurable, i.e. $f+g$ is nonmeasurable with respect to any nonzero σ-finite diffused measure on $\mathbf{R}$ (for more details, see [3]).

We thus conclude that, under **MA**, the sum of two almost measurable functions can be extremely bad from the measure-theoretical point of view.

Remark 2. The argument used in the proofs of the above-mentioned statements shows also that, for any function $h : \mathbf{R} \to \mathbf{R}$, there exist two functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that $h = f + g$ and both sets $Gr(f)$ and $Gr(g)$ are thick with respect to λ_2.

Analogously, for any additive function $h : \mathbf{R} \to \mathbf{R}$, there exist two additive functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that $h = f + g$ and both sets $Gr(f)$ and $Gr(g)$ are thick with respect to λ_2.

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