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ScienceDirect
Expo. Math. 27 (2009) 161 – 169
EXPOSITIONES MATHEMATICAE
www.elsevier.de/exmath

On sums of real-valued functions with extremely thick graphs

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Received 23 July 2008; received in revised form 8 September 2008

Abstract

We consider some properties of those functions acting from the real line **R** into itself, whose graphs are extremely thick subsets of the Euclidean plane $\mathbf{R}^2$. The structure of sums of such functions is studied and the obtained results are applied to certain measure extension problems.

MSC 2000: 28 A05; 28 D05

Keywords: Extremely thick set; Universally extendable σ-algebra; Universal measure zero set; Quasi-invariant measure

Let **R** (=$\mathbf{R}^1$) denote the real line and let λ (=λ_1) be the standard Lebesgue measure on **R** (we consider λ only on the Borel σ-algebra $\mathscr{B}(\mathbf{R})$ of **R**). It is well known that there exists a function $f : \mathbf{R} \to \mathbf{R}$ whose graph $Gr(f)$ is thick in the plane $\mathbf{R}^2$ with respect to the two-dimensional Lebesgue measure $\lambda_2 = \lambda \otimes \lambda$ on $\mathbf{R}^2$. In other words, for such an f, the equality $(\lambda_2)_*(\mathbf{R}^2 \backslash Gr(f)) = 0$ holds true, where $(\lambda_2)_*$ denotes, as usual, the inner measure associated with λ_2.

Recall that the first example of a function with λ_2-thick graph was constructed by Sierpiński with the aid of the method of transfinite recursion (see [1,10]). Functions of this type should be treated as pathological ones from the measure-theoretical and

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0723-0869/$ - see front matter
doi:10.1016/j.exmath.2008.10.003

topological viewpoint, because they are nonmeasurable with respect to the completion of λ and are discontinuous everywhere on $\mathbf{R}$. Besides, the set $Gr(f) \subset \mathbf{R}^2$ is nonmeasurable with respect to the completion of λ_2.

However, even for such pathological functions certain positive properties can be established. For instance, it turns out that any $f : \mathbf{R} \to \mathbf{R}$ with λ_2-thick graph becomes measurable with respect to an appropriate extension of λ. In this paper, we are going to present a nontrivial application of functions with thick graphs to some particular versions of the general measure extension problem.

The notation used in the paper is fairly standard.

If E is a set and $\mathscr{E}$ is a family of subsets of E, then the symbol $\sigma(\mathscr{E})$ denotes the σ-algebra in E generated by $\mathscr{E}$.

If μ is a measure on E, then $dom(\mu)$ stands for the σ-algebra of all μ-measurable sets.

μ^* denotes the outer measure associated with a given measure μ.

μ_* denotes the inner measure associated with a given measure μ.

ω stands for the first infinite cardinal number and $\mathbf{c}$ stands for the cardinality of the continuum.

Let E be a set and let $\mathscr{E}$ be a family of subsets of E. We shall say that a set $G \subset E$ is $\mathscr{E}$-thick in E if $G \cap Z \neq \emptyset$ for every $Z \in \mathscr{E}$.

Example 1. Let E be equipped with a σ-finite measure μ and let $\mathscr{E} = \{Z \in dom(\mu) : \mu(Z) > 0\}$. Then any $\mathscr{E}$-thick set X is usually called a μ-thick subset of E. Clearly, the μ-thickness of a set $X \subset E$ is equivalent to the equality $\mu_*(E \setminus X) = 0$.

If $E = \mathbf{R}$ and X is a Bernstein subset of E (see, e.g., [1]), then both sets X and $E \setminus X$ are μ-thick for any nonzero σ-finite continuous (i.e., vanishing at all singletons) Borel measure μ on E, whence it follows that both X and $E \setminus X$ are nonmeasurable with respect to the completion of μ.

Example 2. Let E_1 and E_2 be two separable metric spaces equipped with their Borel σ-algebras $\mathscr{B}(E_1)$ and $\mathscr{B}(E_2)$, respectively. Consider the product space $E = E_1 \times E_2$. Obviously, we have $\mathscr{B}(E) = \mathscr{B}(E_1) \otimes \mathscr{B}(E_2)$. Let us put

$$\mathscr{E} = \{Z \in \mathscr{B}(E) : card(pr_1(Z)) > \omega\}.$$

We shall say that a set $G \subset E$ is extremely thick if G is $\mathscr{E}$-thick in E.

Let μ be an arbitrary σ-finite Borel measure on E such that the produced marginal measure μ_1 defined by

$$\mu_1(X) = \mu(X \times E_2) \quad (X \in \mathscr{B}(E_1))$$

is continuous. In this case, one can easily see that any extremely thick subset G of E is μ-thick as well.

Below, one important method of extending measures will be essentially used (cf., for instance [7]).

Let $(E_1, \mathscr{A}_1)$ and $(E_2, \mathscr{A}_2)$ be two measurable spaces, μ be a σ-finite measure on the product σ-algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$ and let $g : E_1 \to E_2$ be a mapping whose graph $G = Gr(g)$

is μ-thick in the product space $E = E_1 \times E_2$. For each set $Z \in \mathscr{A}_1 \otimes \mathscr{A}_2$, we put

$$Z'_g = \{x \in E_1 : (x, g(x)) \in Z\}$$

and introduce the class of sets

$$\mathscr{A}'_1 = \{Z'_g : Z \in \mathscr{A}_1 \otimes \mathscr{A}_2\}.$$

Further, we define

$$\mu'_1(Z'_g) = \mu(Z) \quad (Z'_g \in \mathscr{A}'_1).$$

It is not difficult to verify the validity of the following relations:

(1) $\mathscr{A}'_1$ is a σ-algebra of subsets of E_1 containing $\mathscr{A}_1$;
(2) the functional μ'_1 is well defined on $\mathscr{A}'_1$ and is a measure extending the marginal measure μ_1 on $\mathscr{A}_1$ produced by μ; in particular, if $\mu = \nu_1 \otimes \nu_2$ and ν_2 is a probability measure on $\mathscr{A}_2$, then μ'_1 turns out to be an extension of ν_1;
(3) the mapping g is $(\mathscr{A}_2, \mathscr{A}'_1)$-measurable, i.e., for any $Y \in \mathscr{A}_2$, we have $g^{-1}(Y) \in \mathscr{A}'_1$.

Example 3. As in Example 2, let E_1 and E_2 be two separable metric spaces equipped with their Borel σ-algebras $\mathscr{A}_1 = \mathscr{B}(E_1)$ and $\mathscr{A}_2 = \mathscr{B}(E_2)$, respectively. Consider the product space $E = E_1 \times E_2$ equipped with its Borel σ-algebra $\mathscr{B}(E) = \mathscr{A}_1 \otimes \mathscr{A}_2$.

Let $\mathscr{A}$ be a σ-algebra of subsets of E_1 such that $\mathscr{B}(E_1) \subset \mathscr{A}$. We shall say that $\mathscr{A}$ is universally extendable if every σ-finite continuous Borel measure on E_1 can be extended to a measure defined on $\mathscr{A}$.

Let $g : E_1 \to E_2$ be a mapping. It directly follows from Example 2 and the above-mentioned relations (1)–(3) that if $G = Gr(g)$ is extremely thick in E, then the σ-algebra $\mathscr{A}'_1$ is universally extendable.

The introduced concepts of an extremely thick set and of a universally extendable σ-algebra will be applied in the concrete situation where $E_1 = E_2 = \mathbf{R}$ and $\mathbf{R}$ is equipped with its Borel σ-algebra $\mathscr{B}(\mathbf{R})$.

The following statement may be regarded as a far going generalization of Sierpiński's result which was pointed out in the beginning of the paper.

Theorem 1. *Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary function. There exist functions $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$ such that $f = f_1 + f_2$ and both sets $Gr(f_1)$ and $Gr(f_2)$ are extremely thick in the plane $\mathbf{R}^2$.*

Proof. We argue by the method of transfinite induction. Let α be the least ordinal number of cardinality $\mathbf{c}$. Let $\{\Xi_1, \Xi_2, \Xi_3\}$ be a partition of $[0, \alpha[$ such that

$$card(\Xi_1) = card(\Xi_2) = card(\Xi_3) = \mathbf{c}.$$

Further, let $\{B_\xi : \xi \in \Xi_1\}$, $\{B_\xi : \xi \in \Xi_2\}$ and $\{B_\xi : \xi \in \Xi_3\}$ be three families of Borel subsets of $\mathbf{R}^2$ satisfying the following conditions:

(a) $card(pr_1(B_\xi)) > \omega$ for all ordinals $\xi \in \Xi_1 \cup \Xi_2$;

(b) any Borel set $B \subset \mathbf{R}^2$ with $card(pr_1(B)) > \omega$ belongs to the intersection $\{B_\xi : \xi \in \Xi_1\} \cap \{B_\xi : \xi \in \Xi_2\}$;
(c) $B_\xi = \mathbf{R}^2$ for every ordinal $\xi \in \Xi_3$.

Let $\preccurlyeq$ be some well-ordering of $\mathbf{R}$ which is isomorphic to α.

We will construct by the method of transfinite recursion an α-sequence $\{x_\xi : \xi < \alpha\}$ of points of $\mathbf{R}$ and two corresponding α-sequences of values $\{f_1(x_\xi) : \xi < \alpha\}$ and $\{f_2(x_\xi) : \xi < \alpha\}$.

Suppose that, for an ordinal $\xi < \alpha$, the partial ξ-sequences

$$\{x_\zeta : \zeta < \xi\}, \quad \{f_1(x_\zeta) : \zeta < \xi\}, \quad \{f_2(x_\zeta) : \zeta < \xi\}$$

have already been defined. Consider three possible cases.

1. $\xi \in \Xi_1$. In this case, taking into account that $pr_1(B_\xi)$ is an uncountable analytic set in $\mathbf{R}$, we have $card(pr_1(B_\xi)) = \mathbf{c}$. So there exists a point $x \in pr_1(B_\xi) \setminus \{x_\zeta : \zeta < \xi\}$. Consequently, $(x, y) \in B_\xi$ for some point $y \in \mathbf{R}$. We put:

$$x_\xi = x, \quad f_1(x_\xi) = y, \quad f_2(x_\xi) = f(x_\xi) - f_1(x_\xi).$$

2. $\xi \in \Xi_2$. Similarly to the previous case, there exists a point $x \in pr_1(B_\xi) \setminus \{x_\zeta : \zeta < \xi\}$ and, consequently, $(x, y) \in B_\xi$ for some point $y \in \mathbf{R}$. We put

$$x_\xi = x, \quad f_2(x_\xi) = y, \quad f_1(x_\xi) = f(x_\xi) - f_2(x_\xi).$$

3. $\xi \in \Xi_3$. In this case, let x be the least element (with respect to the well-ordering $\preccurlyeq$) of the nonempty set $pr_1(B_\xi) \setminus \{x_\zeta : \zeta < \xi\}$. We put $x_\xi = x$ and define the values $f_1(x_\xi)$ and $f_2(x_\xi)$ so that the equality

$$f(x_\xi) = f_1(x_\xi) + f_2(x_\xi)$$

is satisfied (obviously, there are many possibilities to do this).

Proceeding in such a manner, we come to the three α-sequences $\{x_\xi : \xi < \alpha\}$, $\{f_1(x_\xi) : \xi < \alpha\}$ and $\{f_2(x_\xi) : \xi < \alpha\}$. In view of the described construction, the sets $Gr(f_1)$ and $Gr(f_2)$ are extremely thick in $\mathbf{R}^2$. It suffices only to show that $\mathbf{R} = \{x_\xi : \xi < \alpha\}$. But this equality holds true because of the relations

$$card(\Xi_3) = \mathbf{c}, (\forall x \in \mathbf{R})(card(\{y : y \preccurlyeq x\}) < \mathbf{c}).$$

Theorem 1has thus been proved. □

An analogous argument enables to establish the following statement.

Theorem 2. *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is an arbitrary additive function* (*i.e., f is a homomorphism of the additive group* $\mathbf{R}$ *into itself*). *Then there exist two additive functions $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$ such that $f = f_1 + f_2$ and both sets $Gr(f_1)$ and $Gr(f_2)$ are extremely thick in* $\mathbf{R}^2$.

The scheme of the proof remains essentially the same as before. Namely, we construct by the method of transfinite recursion a linearly independent (over the field **Q** of all rational numbers) α-sequence $\{x_\xi : \xi < \alpha\}$ of points of **R** with two corresponding α-sequences $\{f_1(x_\xi) : \xi < \alpha\}$ and $\{f_2(x_\xi) : \xi < \alpha\}$ such that

$$f(x_\xi) = f_1(x_\xi) + f_2(x_\xi) \quad (\xi < \alpha)$$

and then conclude that the vector space (over **Q**) generated by $\{x_\xi : \xi < \alpha\}$ coincides with the whole **R**, whence it follows that the equality $f(x) = f_1(x) + f_2(x)$ is valid for each point $x \in \mathbf{R}$.

As a straightforward consequence of Theorem 2, we obtain that any standard linear function $f : \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = ax \quad (x \in \mathbf{R}),$$

where $a \in \mathbf{R}$, can be represented as the sum of two additive functions whose graphs are extremely thick subsets of $\mathbf{R}^2$.

Theorems 1 and 2 are applicable to certain measure extension problems. The formulation of those problems goes back to Banach and Marczewski (see, e.g., [2–5,7,8,11,12]). Sometimes, it is more convenient to reformulate measure extension problems of this kind in terms of universally extendable σ-algebras (see Example 3). The next statement illustrates such an approach.

Theorem 3. *Let $\{\mathscr{L}_i : i \in I\}$ be a family of σ-algebras of subsets of* **R**. *Then the following three relations are equivalent*:

(1) *there exists a function $f : \mathbf{R} \to \mathbf{R}$ which is absolutely nonmeasurable with respect to $\{\mathscr{L}_i : i \in I\}$, i.e., for each $i \in I$, this f is not $\mathscr{L}_i$-measurable*;
(2) *there exist two countably generated and universally extendable σ-algebras $\mathscr{S}_1$ and $\mathscr{S}_2$ of subsets of* **R** *such that, for each $i \in I$, we have $\mathscr{S}_1 \cup \mathscr{S}_2 \not\subset \mathscr{L}_i$*;
(3) *there exists a countably generated σ-algebra $\mathscr{S}$ of subsets of* **R** *such that $\mathscr{B}(\mathbf{R}) \subset \mathscr{S}$ and, for any $i \in I$, we have $\mathscr{S} \not\subset \mathscr{L}_i$.*

Proof. (1) $\Rightarrow$ (2). Let $f : \mathbf{R} \to \mathbf{R}$ satisfy relation (1). By virtue of Theorem 1, this f can be represented in the form $f = f_1 + f_2$, where both functions $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$ have extremely thick graphs in $\mathbf{R}^2$. Let us put:

$$\mathscr{S}_1 = \{Z'_{f_1} : Z \in \mathscr{B}(\mathbf{R}^2)\}, \quad \mathscr{S}_2 = \{Z'_{f_2} : Z \in \mathscr{B}(\mathbf{R}^2)\}.$$

Both σ-algebras $\mathscr{S}_1$ and $\mathscr{S}_2$ are countably generated and universally extendable (cf. Example 3). Consider now the σ-algebra $\mathscr{S} = \sigma(\mathscr{S}_1 \cup \mathscr{S}_2)$ which also is countably generated and contains $\mathscr{B}(\mathbf{R})$. Clearly, the functions f_1 and f_2 are $\mathscr{S}$-measurable. Therefore, $f = f_1 + f_2$ is $\mathscr{S}$-measurable, too. Consequently, we may write $f^{-1}(\mathscr{B}(\mathbf{R})) \subset \mathscr{S}$. But this inclusion immediately implies that $\mathscr{S}_1 \cup \mathscr{S}_2 \not\subset \mathscr{L}_i$ for any $i \in I$, i.e., relation (2) holds.

(2) $\Rightarrow$ (3). This implication is trivial.

(3) $\Rightarrow$ (1). Suppose that $\mathscr{S}$ satisfies relation (3). It is easy to see that there exists a countable family $\{X_n : n < \omega\} \subset \mathscr{S}$ separating the points of **R** and generating $\mathscr{S}$. For

any $n < \omega$ and for each point $x \in \mathbf{R}$, let us define: $f_n(x) = 1$ if $x \in X_n$ and $f_n(x) = 0$ if $x \in \mathbf{R} \setminus X_n$. So we get the injective Marczewski function

$$f : \mathbf{R} \to \{0, 1\}^{\omega},$$

where $f = \{f_n : n < \omega\}$. Since the Cantor space $\{0, 1\}^{\omega}$ is topologically contained in $\mathbf{R}$, we may treat f as a function acting from $\mathbf{R}$ into $\mathbf{R}$. This function has the property that

$$f^{-1}(\{t \in \{0, 1\}^{\omega} : t_n = 1\}) = X_n \quad (n < \omega),$$

whence it follows that $f^{-1}(\mathscr{B}(\mathbf{R})) = \mathscr{S}$ and, consequently, for any $i \in I$, the function f cannot be $\mathscr{L}_i$-measurable. This shows that relation (1) holds for f.

The theorem has thus been proved. □

Remark 1. Let $\mathscr{L}$ be a σ-algebra of subsets of $\mathbf{R}$. We shall say that $\mathscr{L}$ is admissible if all singletons in $\mathbf{R}$ belong to $\mathscr{L}$ and there exists at least one nonzero σ-finite continuous measure μ with $dom(\mu) = \mathscr{L}$.

Let $\{\mathscr{L}_i : i \in I\}$ denote the family of all admissible σ-algebras of subsets of $\mathbf{R}$. Then assertion (1) of Theorem 3 is not deducible within **ZFC** theory. Indeed, if $\mathbf{c}$ is a real-valued measurable cardinal (i.e., is measurable in the Ulam sense [13]), then there exists a measure ν on $\mathbf{R}$ extending λ and defined on the power set $\mathscr{P}(\mathbf{R})$. In this case, we have $\mathscr{P}(\mathbf{R}) \in \{\mathscr{L}_i : i \in I\}$ and it is evident that there are no functions acting from $\mathbf{R}$ into $\mathbf{R}$ and absolutely nonmeasurable with respect to $\{\mathscr{L}_i : i \in I\}$. On the other hand, the existence of such functions is implied by certain additional set-theoretical assertions (e.g., the Continuum Hypothesis or Martin's Axiom).

By using Theorem 2, one can obtain (within **ZFC**) a result similar to Theorem 3, in terms of translation-invariant and translation-quasi-invariant extensions of λ.

Recall that a measure μ on $\mathbf{R}$ defined on some translation-invariant σ-algebra of subsets of $\mathbf{R}$ is translation-quasi-invariant if the family of all μ-measure zero sets is preserved under the action of the group of all translations of $\mathbf{R}$. Clearly, this property of μ is much weaker than the standard translation-invariance property.

For our further purposes, three auxiliary propositions are needed.

Lemma 1. *There exists an additive function $f : \mathbf{R} \to \mathbf{R}$ which is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation-quasi-invariant measures on* $\mathbf{R}$. *In other words, there is no nonzero σ-finite translation-quasi-invariant measure μ on* $\mathbf{R}$ *such that f is μ-measurable.*

Proof. We recall that a subset X of $\mathbf{R}$ is universal measure zero if, for any σ-finite continuous Borel measure μ on $\mathbf{R}$, the equality $\mu^*(X) = 0$ holds true. It is well known that there exist uncountable universal measure zero subsets of $\mathbf{R}$ (see, e.g., [3,9,11,12]). Moreover, there are uncountable universal measure zero vector subspaces (over $\mathbf{Q}$) of $\mathbf{R}$ (see, for instance [9] where a much stronger result is presented). Let V be an uncountable vector subspace (over $\mathbf{Q}$) of $\mathbf{R}$ which simultaneously is universal measure zero. Obviously, there exists a surjective homomorphism $f : \mathbf{R} \to V$, which can be regarded as an additive function acting from $\mathbf{R}$ into itself. We assert that f is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation-quasi-invariant measures on $\mathbf{R}$. To show this, suppose otherwise, i.e., suppose that f is μ-measurable for some nonzero σ-finite translation-quasi-invariant measure μ on $\mathbf{R}$. We may assume, without loss of generality, that μ is a probability measure. Let us put

$$\nu(Y) = \mu(f^{-1}(Y)) \quad (Y \in \mathscr{B}(\mathbf{R})).$$

Clearly, ν is a Borel probability measure on $\mathbf{R}$ such that $\nu^*(V) = 1$. The uncountability of V and the translation-quasi-invariance of μ imply $\mu(f^{-1}(y)) = 0$ for each $y \in \mathbf{R}$ and, consequently, ν is a continuous measure. But V is universal measure zero, so we must have $\nu^*(V) = 0$ contradicting the equality $\nu^*(V) = 1$. The obtained contradiction ends the proof. $\square$

Below, the symbol $\mathbf{T}$ ($=\mathbf{S}_1 \subset \mathbf{R}^2$) denotes the one-dimensional unit torus regarded as a compact commutative group equipped with its probability Haar measure.

Lemma 2. *Let $\phi : \mathbf{R} \to \mathbf{T}$ be the canonical surjective homomorphism defined by the formula*

$$\phi(x) = (\cos(x), \sin(x)) \quad (x \in \mathbf{R}).$$

The following two assertions are valid:

(1) *if the graph of a function $g : \mathbf{R} \to \mathbf{R}$ is extremely thick in $\mathbf{R}^2$, then the graph of $\phi \circ g$ is extremely thick in $\mathbf{R} \times \mathbf{T}$*;
(2) *if $f : \mathbf{R} \to \mathbf{R}$ is as in the proof of Lemma 1, then $\phi \circ f$ is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation-quasi-invariant measures on $\mathbf{R}$.*

Lemma 3. *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be any two translation-invariant σ-algebras of subsets of $\mathbf{R}$. Then the σ-algebra $\mathscr{S} = \sigma(\mathscr{S}_1 \cup \mathscr{S}_2)$ is also translation-invariant.*

We omit easy proofs of Lemmas 2 and 3.

Theorem 4. *There exist two countably generated, universally extendable, translation-invariant σ-algebras $\mathscr{S}_1$ and $\mathscr{S}_2$ of subsets of $\mathbf{R}$ satisfying the following relations*:

(1) *there is a translation-invariant measure μ_1 which extends λ and whose domain coincides with $\mathscr{S}_1$*;
(2) *there is a translation-invariant measure μ_2 which extends λ and whose domain coincides with $\mathscr{S}_2$*;
(3) *there exists no translation-quasi-invariant measure extending λ and defined on the σ-algebra $\mathscr{S} = \sigma(\mathscr{S}_1 \cup \mathscr{S}_2)$.*

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be as in the proof of Lemma 1. According to Theorem 2, f can be represented in the form $f = f_1 + f_2$, where $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$ are additive functions whose graphs are extremely thick in $\mathbf{R}^2$. Let $\phi : \mathbf{R} \to \mathbf{T}$ be as in Lemma 2. We define:

$$h = \phi \circ f, \quad h_1 = \phi \circ f_1, \quad h_2 = \phi \circ f_2.$$

Obviously, $h = h_1 + h_2$. In view of assertion (1) of Lemma 2, the homomorphisms h_1 and h_2 have extremely thick graphs in the product space $\mathbf{R} \times \mathbf{T}$. Denoting by $\mathscr{B}(\mathbf{R} \times \mathbf{T})$ the Borel σ-algebra of $\mathbf{R} \times \mathbf{T}$, let us put:

$$\mathscr{S}_1 = \{Z'_{h_1} : Z \in \mathscr{B}(\mathbf{R} \times \mathbf{T})\}, \quad \mathscr{S}_2 = \{Z'_{h_2} : Z \in \mathscr{B}(\mathbf{R} \times \mathbf{T})\}.$$

The σ-algebras $\mathscr{S}_1$ and $\mathscr{S}_2$ are the required ones. Indeed, applying the measure extension construction to $\lambda \otimes v$, where v is the Haar probability measure on $\mathbf{T}$, we infer that h_1 and h_2 become measurable with respect to appropriate translation-invariant extensions μ_1 and μ_2 of λ such that $dom(\mu_1) = \mathscr{S}_1$ and $dom(\mu_2) = \mathscr{S}_2$ (cf. [7]). It suffices only to verify that $\mathscr{S} = \sigma(\mathscr{S}_1 \cup \mathscr{S}_2)$ does not admit a translation-quasi-invariant extension of λ (note that, by virtue of Lemma 3, $\mathscr{S}$ is a translation-invariant σ-algebra of subsets of $\mathbf{R}$). Suppose otherwise, i.e., suppose that there exists a translation-quasi-invariant measure μ on $\mathbf{R}$ extending λ and satisfying the equality $dom(\mu) = \mathscr{S}$. Then the homomorphism $h = h_1 + h_2$ becomes measurable with respect to μ. But this yields a contradiction with assertion (2) of Lemma 2. The contradiction obtained completes the proof. □

Remark 2. Under some additional set-theoretical assumptions (e.g., assuming that there exists a generalized Luzin subset of $\mathbf{R}$), Theorem 4 can be essentially strengthened. Namely, the countably generated σ-algebras $\mathscr{S}_1$ and $\mathscr{S}_2$ can be chosen so that both of them admit translation-invariant extensions of λ but the σ-algebra $\mathscr{S} = \sigma(\mathscr{S}_1 \cup \mathscr{S}_2)$ does not admit a nonzero σ-finite continuous measure. For more details, see [6].

The main results of the first version of this paper were formulated and proved in terms of functions with λ_2-thick graphs and extensions of λ. The referee kindly informed the author that the corresponding proofs work in a more general situation, in terms of functions with extremely thick graphs and universally extendable σ-algebras. The author is very grateful to the referee for his valuable remarks and suggestions.

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