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PIECEWISE AFFINE APPROXIMATIONS OF CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES AND GALE POLYHEDRA

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Let P be a polyhedron in the Euclidean space $\mathbf{R}^n$ and let $\phi : P \to [0,1]$ be a continuous function. Any triangulation of P allows us to construct a certain piecewise affine continuous approximation $\psi : P \to [0,1]$ of ϕ. Indeed, consider a triangulation $\{T_i : i \in I\}$ of P into n-dimensional simplices. If x is an arbitrary point of P, then there exists a simplex T_i such that $x \in T_i$. Let $x_{i,0}$, $x_{i,1}$,..., $x_{i,n}$ denote the vertices of T_i. Clearly, x admits a unique representation in the form

$$x = \alpha_0 x_{i,0} + \alpha_1 x_{i,1} + \cdots + \alpha_n x_{i,n},$$

where all α_i $(i = 0, 1, \ldots, n)$ are nonnegative real numbers whose sum is equal to 1. Putting

$$\psi(x) = \alpha_0 \phi(x_{i,0}) + \alpha_1 \phi(x_{i,1}) + \cdots + \alpha_n \phi(x_{i,n}),$$

we get the desired piecewise affine continuous approximation ψ of ϕ (the correctness of this definition is guaranteed, because $\{T_i : i \in I\}$ is a triangulation of P).

We thus see that the obtained approximating function ψ can be represented, e.g., in the form

$$\psi = \max\{\psi_i : i \in I\},$$

where each function $\psi_i : P \to [0,1]$ is affine on T_i and is identically equal to zero on $P \setminus T_i$. In such a case, all functions ψ_i $(i \in I)$ may be treated as "affine pieces" of ψ and it is natural to ask about the minimal possible value of the number of these pieces (or, equivalently, about the minimum of card(I)).

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Let us first consider the situation when a convex polygon P with $v(P) = v \geq 3$ vertices is given in the plane $\mathbf{R}^2$. Suppose that this polygon is arbitrarily dissected (decomposed) into finitely many triangles, i.e., the relation

$$P = \cup\{T_i : 1 \leq i \leq m\}$$

holds true, where all triangles T_i have pairwise disjoint interiors. Then we can assert that $m \geq v - 2$. Indeed, the sum of all interior angles of these triangles is equal to $\pi \cdot m$. As well known, the sum of all interior angles of P is equal to $\pi(v-2)$. But the latter sum is contained in the first one and, consequently, does not exceed it. So we may write

$$\pi(v-2) \leq \pi m, \quad v - 2 \leq m.$$

Moreover, by using induction on v, it is easy to show that there exists a triangulation of P consisting of exactly $v - 2$ triangles whose vertices are contained in the set of vertices of P.

The above-mentioned facts directly lead to the following conclusion.

For any convex polygon $P \subset \mathbf{R}^2$, denote by $s(P)$ the minimal cardinality of a dissection of P into triangles. Then $s(P) = v - 2$, where $v = v(P)$ is the number of vertices (equivalently, sides) of P. In particular, the value of $s(P)$ is completely determined by a canonical parameter associated with P. The role of such a parameter is played by the number of vertices (sides) of P.

Remark 1. For non-convex polygons, this conclusion fails to be true. For instance, it is not difficult to give an example of a simple polygon $Q \subset \mathbf{R}^2$ with 6 vertices, such that $s(Q) = 2$. On the other hand, let $P \subset \mathbf{R}^2$ be a simple polygon with $v = v(P)$ vertices and with some points x_1, x_2, $\ldots$, x_w lying in the interior of P. Consider an arbitrary triangulation $\{T_i : 1 \leq i \leq m\}$ of P such that the set of all vertices of this triangulation contains all vertices of P and all the points x_1, x_2, $\ldots$, x_w. Then the relation $v + 2w - 2 \leq m$ holds true. Moreover, the relation $v + 2w - 2 = m$ is valid if and only if the set of all vertices of $\{T_i : 1 \leq i \leq m\}$ is equal to the union of $\{x_1, x_2, \ldots, x_w\}$ with the set of all vertices of P.

If we deal with the three-dimensional Euclidean space $\mathbf{R}^3$, then a rather surprising circumstance occurs, namely, the number $v = v(P)$ of vertices of a convex polyhedron $P \subset \mathbf{R}^3$ is not sufficient to determine uniquely the analogous value:

$s(P) =$ the minimal cardinality of a dissection of P into three-dimensional simplices (i.e., tetrahedra).

This circumstance can readily be derived from the following example.

Example 1. Let P_1 be a prism whose base is a triangle and let P_2 be an octahedron. Obviously, the number of vertices of P_1 coincides with the number of vertices of P_2 and both of them are equal to 6. An easy argument shows that $s(P_1) = 3$ and $s(P_2) = 4$, so we infer that $s(P_1) \neq s(P_2)$. More generally, let P be a convex bi-pyramid in $\mathbf{R}^3$ with $4m$ facets (hence the number of vertices of P is equal to $2m + 2$). It is not hard to prove that $s(P) = 2m$. For this purpose, it suffices to observe that there is a family of $2m$ facets of P possessing the following property: any two facets of this family either have no common points or have only one common vertex. From the above-mentioned fact one immediately obtains that if Q is an octahedron, then $s(Q) = 4$.

Example 1 shows that if the number $v = v(P)$ of vertices of a convex polyhedron $P \subset \mathbf{R}^3$ is given, then we only can speak of some estimates for $s(P)$ described in terms of v. Taking into account the arbitrariness of P, it is natural to try to establish the validity of two inequalities of the type

$$g_1(v) \leq s(P) \leq g_2(v),$$

where the functions g_1 and g_2 have the same order of growth when v tends to infinity. In other words, we would like to have the inequalities

$$0 < liminf_{v \to +\infty}(g_1(v)/g_2(v)) \leq limsup_{v \to +\infty}(g_1(v)/g_2(v)) < +\infty.$$

In particular, if both functions g_1 and g_2 are polynomials (of a variable v) whose degrees coincide, then the situation may be regarded as sufficiently nice for our purpose.

It turns out that, in the case of $\mathbf{R}^3$, estimating functions g_1 and g_2 do exist and can be chosen to be polynomials of degree 1, i.e., g_1 and g_2 are affine functions of v. In order to demonstrate this circumstance, we need the classical (and widely known) Euler formula for an arbitrary convex polyhedron $P \subset \mathbf{R}^3$. Namely, recall that $v - e + f = 2$, where the symbol $v = v(P)$ denotes again the number of all vertices of P, the symbol $e = e(P)$ denotes the number of all edges of P, and $f = f(P)$ stands for the number of all facets of P (see, e.g., [2], [4], [5]).

Theorem 1. *For any convex polyhedron $P \subset \mathbf{R}^3$ with $v(P)$ vertices, the inequality $v(P) - 3 \leq s(P)$ holds true.*

This theorem shows that the function $g_1(v) = v - 3$ is a lower estimate for $s(P)$, where P has exactly $v = v(P)$ vertices. In fact, this is a precise lower estimate. To explain the situation in more details, let us introduce one definition (see [6]).

Let $P \subset \mathbf{R}^3$ be a convex polyhedron and let $z \in \mathbf{R}^3$ be a point not belonging to P. We shall say that the polyhedron $Q = conv(P \cup \{z\})$ is a primitive extension of P if there exists a facet D of P such that D is a triangle and $Q = P \cup conv(D \cup \{z\})$. In other words, Q is obtained by adding to P some tetrahedron whose base is one of the facets of P.

Let $\{P_0, P_1, \ldots, P_k\}$ be a finite sequence of polyhedra in $\mathbf{R}^3$. We shall say that this sequence is primitive if P_0 is a tetrahedron and, for each $i \in [0, k-1]$, the polyhedron P_{i+1} is a primitive extension of P_i.

A polyhedron P is called primitive if $P = P_k$ for some primitive sequence $\{P_0, P_1, \ldots, P_k\}$.

Actually, the proof of Theorem 1 yields that, for a convex polyhedron $P \subset \mathbf{R}^3$, the following two assertions are equivalent:

(a) $s(P) = v(P) - 3$;

(b) P is a primitive polyhedron.

Remark 2. It can easily be checked that the unit cube in the space $\mathbf{R}^3$ is a primitive polyhedron. Similarly to the said above, the notion of a primitive polyhedron can also be introduced for the space $\mathbf{R}^n$ where $n \geq 4$. It turns out that, for $n \geq 4$, the unit cube in $\mathbf{R}^n$ is not a primitive polyhedron (see again [6]).

Let us return to the case of $\mathbf{R}^3$ and let us try to find an appropriate affine function g_2 which will play the role of an upper estimate for $s(P)$. Indeed, we have

Theorem 2. *For any convex polyhedron $P \subset \mathbf{R}^3$ with $v(P)$ vertices, the inequality $s(P) \leq 2(v(P) - 2)$ holds true, so we may take $g_2(v) = 2(v - 2)$.*

The proof of this statement is also based on the Euler formula.

We thus conclude that, in the case of the three-dimensional Euclidean space $\mathbf{R}^3$, both estimating functions g_1 and g_2 can be chosen to be affine (i.e., linear). Clearly, for $\mathbf{R}^2$ we have a much simpler situation, namely, $g_1(v) = g_2(v) = v - 2$. Briefly speaking, if $n \leq 3$, then both functions g_1 and g_2 exist and are affine.

Dealing with the four-dimensional Euclidean space $\mathbf{R}^4$, we encounter the next surprise concerning natural analogues of the above-mentioned functions. To explain this extraordinary situation, we need the notion of Gale polyhedra (see [3]). This type of polyhedra was first indicated by C. Carathéodory in 1907 but his result was not widely recognized. In 1956, D. Gale rediscovered these polyhedra and gave a number of their applications.

In $\mathbf{R}^3$ every convex polyhedron with at least five vertices necessarily has two vertices such that the line segment determined by them is not an edge of the polyhedron. It was demonstrated by Carathéodory and Gale that in the space $\mathbf{R}^4$ there exists a convex polyhedron G which has arbitrarily many vertices and possesses the property that any two distinct vertices of G turn out to be the endpoints of some of its edge. The construction of such a polyhedron G is very clever and intriguing. We would like to recall it here.

Example 2. In $\mathbf{R}^4$ take a finite sequence of points

$$(t_1, t_1^2, t_1^3, t_1^4), \ (t_2, t_2^2, t_2^3, t_2^4), \ \ldots, \ (t_v, t_v^2, t_v^3, t_v^4),$$

where $v \geq 5$ is a natural number and $0 < t_1 < t_2 < \cdots < t_v$. It is easy to check that these points are in general position, i.e., no five of them belong to an affine hyperplane of $\mathbf{R}^4$. Denote by G the convex hull of these points and let us verify that G has the above-mentioned property. For this purpose, fix two distinct t_i and t_j and consider the polynomial

$$(t - t_i)^2(t - t_j)^2 = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + t^4.$$

Evidently, we may associate to this polynomial the affine hyperplane Γ in the space $\mathbf{R}^4$, defined as follows:

$$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + x_4 = 0.$$

The definition of this hyperplane directly implies that:

(1) both points $(t_i, t_i^2, t_i^3, t_i^4)$ and $(t_j, t_j^2, t_j^3, t_j^4)$ lie in Γ;

(2) all other points $(t_k, t_k^2, t_k^3, t_k^4)$ $(k \neq i,\ k \neq j)$ lie in one open half-space determined by Γ.

The relations (1) and (2) immediately give us that all taken v points are convexly independent (so they are vertices of G) and any two of them determine an edge of G.

By using an easy induction on $n \geq 4$, it can be shown that, for every natural number $v \geq n+1$, there exists a convex polyhedron $G \subset \mathbf{R}^n$ with v vertices such that any two distinct vertices of G are the end-points of some edge of G. In this manner, we get many convex polyhedra of Gale type in $\mathbf{R}^n$, where $n \geq 4$.

Let P be a convex polyhedron in $\mathbf{R}^4$. Similarly to the 2-dimensional and 3-dimensional cases, we may introduce the following number:

$s(P) =$ the minimal cardinality of a dissection of P into four-dimensional simplices.

Taking into account the above-mentioned property of Gale polyhedra, we deduce the next statement.

Theorem 3. *If G is an arbitrary Gale polyhedron in $\mathbf{R}^4$ with $v(G)$ vertices, then the inequality*

$$v(G)(v(G) - 1)/20 \leq s(G).$$

holds true.

Theorem 3 shows that the situations in $\mathbf{R}^3$ and $\mathbf{R}^4$ essentially differ from each other. Namely, it is not difficult to indicate a class $\mathcal{P}_n$ of convex polyhedra Q in the space $\mathbf{R}^n$ $(n \geq 2)$ with arbitrarily many vertices and such that $s(Q) \leq v(Q) - n$ for each $Q \in \mathcal{P}_n$. For instance, all primitive polyhedra are in $\mathcal{P}_n$. For $n = 2$, the class $\mathcal{P}_n$ coincides with the family of all convex polyhedra in $\mathbf{R}^n$. For $n \geq 3$, the class $\mathcal{P}_n$ is a proper (and rather poor) subfamily of the family of all convex polyhedra in $\mathbf{R}^n$. For $n \geq 4$, the existence of polyhedra from $\mathcal{P}_n$ and the existence of Gale polyhedra in $\mathbf{R}^n$

imply that there are no functions g_1 and g_2 of one variable, satisfying the inequalities

$$g_1(v(P)) \leq s(P) \leq g_2(v(P))$$

and such that

$$0 < \mathrm{liminf}_{v \to +\infty}(g_1(v)/g_2(v)) \leq \mathrm{limsup}_{v \to +\infty}(g_1(v)/g_2(v)) < +\infty.$$

A more thorough consideration leads to a somewhat deeper result. Let $n > 2$ be an even natural number, i.e., $n = 2m$, where $m \in \{2, 3, \ldots\}$. For any natural number $v \geq n + 1$, we may take a sequence of points in the space $\mathbf{R}^n$:

$$(t_1, t_1^2, \ldots, t_1^n), \ (t_2, t_2^2, \ldots, t_2^n), \ \ldots, \ (t_v, t_v^2, \ldots, t_v^n),$$

where $0 < t_1 < t_2 < \cdots < t_v$. Considering, as in Example 2, the polynomials

$$(t - t_{i_1})^2 (t - t_{i_2})^2 \ldots (t - t_{i_m})^2 = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n,$$

where $i_1, i_2, \ldots, i_m$ are pairwise distinct indices from the set $\{1, 2, \ldots, v\}$, we conclude that all these points are in general and convex position. The convex hull G of them is the polyhedron possessing the following property:

Every m-element subset of the set of all vertices of G is the set of all vertices of an $(m-1)$-dimensional simplex which is a face of G.

The existence of such a G yields a nontrivial consequence. Namely, by using an argument similar to the proof of Theorem 3, one can deduce the following fact.

Theorem 4. *There exists no upper estimate for $s(P)$ having the polynomial form with respect to n and $v = v(P)$, where P ranges over the class of all convex polyhedra in $\mathbf{R}^n$. More precisely, there exists no polynomial $h(n, v)$ of two variables n and v, such that $s(P) \leq h(n, v(P))$ for every convex polyhedron $P \subset \mathbf{R}^n$ with $v(P)$ vertices.*

Let us return to a continuous function $\phi : P \to [0, 1]$ defined on a polyhedron $P \subset \mathbf{R}^n$ and to its piecewise affine continuous approximation $\psi : P \to [0, 1]$ described at the beginning of the report. Recall that this ψ was expressed in the form

$$\psi = \max\{\psi_i : i \in I\}.$$

By virtue of Theorem 4, we may conclude that, in general, the number $\mathrm{card}(I)$ of "affine pieces" of such an approximation ψ is very large in comparison with the number n of variables of ϕ and the number $v(P)$ of vertices of P. Moreover, let $x_1, x_2, \ldots, x_w$ be some points lying in the interior of P and let $\{T_i : i \in I\}$ be a triangulation of P into simplices, corresponding to $\{\psi_i : i \in I\}$ and having the property that all points $x_1, x_2, \ldots, x_w$ are vertices of this triangulation. Supposing that $w = w(n, v)$ is a function of n and v, we again can assert that, in general, the growth of $\mathrm{card}(I)$ is non-polynomial. Indeed, only two cases are possible.

1. The growth of $w = w(n, v)$ is non-polynomial. In this case, we use the simple inequality $(w(n, v) + v)/(n + 1) \leq \mathrm{card}(I)$ and readily deduce that the growth of $\mathrm{card}(I)$ must be non-polynomial, too.

2. The growth of $w = w(n, v)$ is of polynomial character, i.e., $w(n, v) \leq p(n, v)$ for some fixed polynomial $p = p(n, v)$ with strictly positive coefficients. Now, assume that there exists a polynomial $h_0 = h_0(n, v + w)$ such that $\mathrm{card}(I) \leq h_0(n, v + w)$. Supposing, without loss of generality, that all coefficients of h_0 are strictly positive, we readily get

$$\mathrm{card}(I) \leq h_0(n, v + w) \leq h_0(n, v + p(n, v)) = h(n, v),$$

which contradicts Theorem 4.

Some other interesting consequences of the existence of Gale polyhedra in the Euclidean space $\mathbf{R}^n$, where $n \geq 4$, should also be mentioned. We present them in the next two examples.

Example 3. By applying the Poincaré duality to Gale polyhedra in $\mathbf{R}^4$, one can easily show that there exists a convex polyhedron P in the space $\mathbf{R}^4$ with arbitrarily many facets, which possesses the following property: any two facets of P have a common two-dimensional face of P. It follows from this result that, for every natural number k, there exists a family $\{P_j : j \in \{1, 2, \ldots, k\}\}$ of convex polyhedra in the space $\mathbf{R}^3$ such that the intersection of any two distinct members of this family is their common facet (hence is two-dimensional). It is useful to compare this fact with the circumstance that in $\mathbf{R}^2$ there are no five simple polygons any two of which have one-dimensional intersection (the latter follows directly from the non-planarity of Kuratowski's graph K_5).

In connection with the above example, see also [1], [7]; some other results in this direction are presented in [8].

Example 4. Let (V, E) be a graph (the symbol V denotes the set of all vertices and the symbol E stands for the set of all edges of this graph). A family $\{X_v : v \in V\}$ of sets is called a set-theoretic realization of (V, E) if the relation $X_v \cap X_u \neq \emptyset$ is satisfied if and only if $\{v, u\} \in E$. It can be shown that any graph admits a set-theoretic realization (E. Marczewski's theorem). By using the result of Example 3, one readily derives that, for any finite graph, there exists its set-theoretic realization consisting of convex polyhedra in $\mathbf{R}^3$. Notice, by the way, that convex polygons in the plane are not sufficient for set-theoretic realizations of finite graphs.

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