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SOME REMARKS CONCERNING MONOTONE AND CONTINUOUS RESTRICTIONS OF REAL-VALUED FUNCTIONS

A. KHARAZISHVILI

Abstract. We consider those restrictions of real-valued functions, which have certain nice properties, e.g., continuity or monotonicity. We prove the non-existence of restrictions of such a kind in concrete situations and show close connections of this topic with some classical examples of sets and functions in real analysis (Luzin sets, Sierpiński sets, continuous nowhere approximately differentiable functions, Sierpiński-Zygmund functions, etc.).

რეზიუმე. სტატიაში განხილულია ნამდვილმნიშვნელობიანი ფუნქციების ისეთი შევიწროებები, რომლებსაც აქვთ კარგი ანალიზური თვისებები. მაგ., მონოტონურობა ან უწყვეტობა. ამავე დროს, ზოგიერთ კონკრეტულ შემთხვევაში დამკიცებულია ამ ტიპის შევიწროებების არარსებობა. ეს საკითხები უკავშირდება ნამდვილი ანალიზის ისეთ კლასიკურ სიმრავლეებსა და ფუნქციებს, როგორიცაა ლუზინის სიმრავლე, სერპინსკის სიმრავლე, უწყვეტი არსად აპროქსიმატულად დიფერენცირებადი ფუნქცია, სერპინსკი-ზიგმუნდის ფუნქცია და სხვა.

In various works devoted to behavior of real-valued functions, the restrictions to certain subsets of their domains are often considered, taking into account the circumstance that such restrictions may have much better descriptive properties than those of original functions (see, for instance, [2], [3], [8], [14]). Here we are going to present several results connected with restrictions of real-valued functions, which possess rather good structure from the view-point of real analysis. Among these properties the continuity or monotonicity of an appropriate restriction will be of primer interest for us. On the other hand, some exotic (pathological) functions will also be pointed out, for which no continuous or monotone restriction to a non-small subset can exist.

Our notation is fairly standard but, for the reader's convenience, we recall the meaning of certain abbreviations which will be used in the sequel:

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$\mathcal{P}(X)$ = the power set of a set X;
card(X) = the cardinality of a set X;
ω = the first infinite ordinal (cardinal);
$\mathbf{R}$ = the real line;
$\mathbf{c}$ = the cardinality of the continuum;
λ = the classical Lebesgue measure on $\mathbf{R}$;
λ^* = the outer measure associated with λ;
λ_* = the inner measure associated with λ;
dom(f) = the domain of a function f;
$f|X$ = the restriction of a function f to a set X;
cl(X) = the closure of a subset X of a topological space;
$C(f)$ = the set of all continuity points of a function f defined on a topological space;
$D(f)$ = the set of all discontinuity points of a function f defined on a topological space.

Let X and Y be any two sets. We say that f is a partial function acting from X into Y if f is a function whose graph is contained in $X \times Y$. In this case, the ordinary notation $f : X \to Y$ is used.

It is well known that the set $D(f)$ of all discontinuity points of a monotone function $f : \mathbf{R} \to \mathbf{R}$ is always (at most) countable and, conversely, if X is an arbitrary countable subset of $\mathbf{R}$, then there exists a monotone function $g : \mathbf{R} \to \mathbf{R}$ such that $D(g) = X$. The latter fact enables to construct many examples of monotone functions $g : \mathbf{R} \to \mathbf{R}$ with everywhere dense set $D(g)$ (cf. [4], [12]).

Also, it directly follows from the said above that any monotone function $f : \mathbf{R} \to \mathbf{R}$ is continuous at all points of a co-countable subset of $\mathbf{R}$, so f turns out to be continuous on a large subset of $\mathbf{R}$. An analogous fact holds true for any partial monotone function $f : \mathbf{R} \to \mathbf{R}$. In this more general case, the set $D(f)$ is again at most countable, so f is continuous at all points of a co-countable subset of dom(f).

In view of the above observation, it makes sense to consider a dual problem. Namely, having a continuous function $f : \mathbf{R} \to \mathbf{R}$, one can ask whether f is monotone on a certain large subset of $\mathbf{R}$. In general, the answer to this question is trivially negative. Indeed, if $f : \mathbf{R} \to \mathbf{R}$ is continuous but nowhere differentiable, then f cannot be monotone on a subset of $\mathbf{R}$ which is dense in some non-degenerate subinterval of $\mathbf{R}$ (indeed, it suffices to apply the classical Lebesgue theorem on the differentiability almost everywhere of any monotone function). So the posed question should be replaced by substantially weakened one, and we may try to study the following variant of this question: does there exist a non-small subset of $\mathbf{R}$ on which f is monotone?

It is well known that, having functions with nice descriptive structure, one can always obtain their monotone restrictions to certain non-empty perfect subsets of **R**. For the sake of completeness, we give a proof of this result.

Theorem 1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a Lebesgue measurable function* (*or let $f : \mathbf{R} \to \mathbf{R}$ have the Baire property*). *Then there exists a nonempty perfect set $P \subset \mathbf{R}$ such that the restriction $f|P$ is monotone on P.*

Proof. The argument presented below is fairly standard and is usually applied in many similar situations (cf., for instance, [1]). First of all, there exists a nonempty perfect subset T of **R** such that the restriction $f|T$ is continuous (this is true for all Lebesgue measurable functions f and for all those functions f, which possess the Baire property). Of course, we may assume that $\mathrm{diam}(T) < 1$, where the symbol $\mathrm{diam}(T)$ stands for the diameter of T. Let us denote $g = f|T$ and suppose that, for every nonempty perfect set $Q \subset T$, the restriction $g|Q$ is not decreasing. Then, by using ordinary induction, we can construct a dyadic system

$$(T_{i_1 i_2 \dots i_k})_{i_1 \in \{0,1\}, i_2 \in \{0,1\}, \dots, i_k \in \{0,1\}} \qquad (k < \omega)$$

of nonempty perfect subsets of T satisfying the following conditions:

(a) $T_\emptyset = T$;

(b) $T_{i_1 i_2 \dots i_k i_{k+1}} \subset T_{i_1 i_2 \dots i_k}$;

(c) $T_{i_1 i_2 \dots i_k 0} \cap T_{i_1 i_2 \dots i_k 1} = \emptyset$;

(d) $\mathrm{diam}(T_{i_1 i_2 \dots i_k}) < 1/2^k$;

(e) if $k < \omega$ and $(i_1, i_2, \dots, i_k) \prec (j_1, j_2, \dots, j_k)$, then $x < y$ and $g(x) < g(y)$ for all points $x \in T_{i_1 i_2 \dots i_k}$ and $y \in T_{j_1 j_2 \dots j_k}$, where $\preceq$ denotes the standard lexicographical ordering in the set of all k-sequences whose terms belong to $\{0, 1\}$.

As soon as the above dyadic system of sets is determined, we may put

$$P = \bigcap_{k<\omega} (\cup \{T_{i_1 i_2 \dots i_k} : (i_1, i_2, \dots i_k) \in \{0,1\}^k\}).$$

It follows from the construction of P that $g|P = f|P$ is an increasing function on P. This finishes the proof of Theorem 1. □

The natural question arises whether the preceding theorem admits further generalizations. In particular, one can ask whether it is possible to choose the above-mentioned nonempty perfect set P so that $\lambda(P) > 0$ would be fulfilled.

We will show that the answer to the posed question is negative. For our purpose, we need the existence of a continuous function $f : \mathbf{R} \to \mathbf{R}$, which is nowhere approximately differentiable. This fact was first established by Jarnik (see [6]). It is much deeper than the existence of a continuous nowhere differentiable function.

Lemma 1. *There exist continuous bounded functions acting from **R** into **R**, which are nowhere approximately differentiable.*

Remark 1. Actually, Jarnik proved that almost all (in the sense of the Baire category) functions from the Banach space $C[0,1]$ are nowhere approximately differentiable. Clearly, this result significantly strengthens the corresponding result of Banach and Mazurkiewicz for the usual differentiability (see, e.g., [10], [12]). Further investigations showed that analogous statements hold true for many kinds of so-called generalized derivatives.

Lemma 2. *There exists a continuous function* $f : \mathbf{R} \to \mathbf{R}$ *such that, for every closed set* $P \subset \mathbf{R}$ *with* $\lambda(P) > 0$*, the restriction* $f|P$ *is not monotone on* P*.*

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous, bounded and nowhere approximately differentiable function (see Lemma 1 above). We are going to show that f is the required one. Take any closed set $P \subset \mathbf{R}$ whose Lebesgue measure is strictly positive. We assert that $f|P$ cannot be monotone. Suppose otherwise, i.e., $f|P$ is either increasing or decreasing. Without loss of generality, we may assume that $f|P$ is increasing. Denote by

$$f^* : \mathbf{R} \to \mathbf{R}$$

some increasing function extending $f|P$ (the existence of f^* is obvious). According to the classical theorem of Lebesgue, f^* is differentiable almost everywhere. Consequently, there exists $x \in P$ such that x is a density point of P and f^* is differentiable at x. By virtue of the relation $f^*|P = f|P$, this circumstance immediately implies the fact that the original function f is approximately differentiable at x. But the latter contradicts the definition of f. The obtained contradiction finishes the proof. □

Now, we are ready to establish the following statement which essentially strengthens Lemma 2.

Theorem 2. *There exists a continuous function* $f : \mathbf{R} \to \mathbf{R}$ *such that, for every set* $X \subset \mathbf{R}$ *with* $\lambda^*(X) > 0$*, the restriction* $f|X$ *is not monotone on* X*.*

Proof. Take the same continuous, bounded and nowhere approximately differentiable function $f : \mathbf{R} \to \mathbf{R}$. Let X be any subset of $\mathbf{R}$ with $\lambda^*(X) > 0$. We must show that the restriction $f|X$ cannot be monotone. Suppose otherwise, i.e., $f|X$ is monotone. Denote by Y the closure of X. The set Y is Lebesgue measurable and, by virtue of the assumption $\lambda^*(X) > 0$, we immediately get $\lambda(Y) > 0$. Since f is continuous, we conclude that $f|Y$ is monotone, too, but this contradicts Lemma 2. Theorem 2 has thus been proved.

In view of Theorems 1 and 2, it makes sense to introduce the following definitions.

Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function and let $\mathcal{L}$ be a family of subsets of $\mathbf{R}$.

We say that f is relatively monotone (relatively continuous) with respect to $\mathcal{L}$ if there exists at least one set $X \in \mathcal{L} \cap \mathcal{P}(\mathrm{dom}(f))$, for which the restriction $f|X$ is monotone (continuous).

We say that f is absolutely non-monotone (absolutely discontinuous) with respect to $\mathcal{L}$ if f is not relatively monotone (relatively continuous) with respect to $\mathcal{L}$, i.e., there exists no set $X \in \mathcal{L} \cap \mathcal{P}(\mathrm{dom}(f))$, for which $f|X$ is monotone (continuous).

In this terminology, Theorem 1 says that every Lebesgue measurable (having the Baire property) function $f : \mathbf{R} \to \mathbf{R}$ is relatively monotone with respect to the family of all nonempty perfect subsets of $\mathbf{R}$.

On the other hand, Theorem 2 states that there exists a continuous function $f : \mathbf{R} \to \mathbf{R}$ which is absolutely non-monotone with respect to the family of all subsets of $\mathbf{R}$ having strictly positive outer Lebesgue measure.

□

Let us give several other examples illustrating the introduced notions.

Example 1. Any function $g : \mathbf{R} \to \mathbf{R}$ turns out to be relatively monotone with respect to the class of all countably infinite subsets of $\mathbf{R}$. This fact can easily be deduced from an infinite version of the well-known combinatorial theorem due to Ramsey and also admits a direct simple proof. Actually, if $(E, \preceq)$ and $(F, \preceq)$ are any two linearly ordered sets, E is infinite and $\phi : E \to F$ is a mapping, then there exists an infinite set $X \subset E$ for which the restriction $\phi|X$ is monotone. However, even in the case $E = F = \mathbf{R}$, one cannot assert that among such sets X there is an everywhere dense subset of $\mathbf{R}$.

Example 2. In view of the Lebesgue theorem on differentiability almost everywhere of monotone functions acting from $\mathbf{R}$ into itself, any continuous nowhere differentiable function is absolutely non-monotone with respect to the family of all nonempty open intervals in $\mathbf{R}$. Moreover, it is well known that there exist everywhere differentiable functions $g : \mathbf{R} \to \mathbf{R}$ which are absolutely non-monotone with respect to the same family (see, e.g., [5], [7], [16]).

Example 3. Any function $f : \mathbf{R} \to \mathbf{R}$ turns out to be relatively continuous with respect to the family of all countable everywhere dense subsets of $\mathbf{R}$. This classical result is due to Blumberg. It inspired many other restriction theorems in real analysis (cf. [2], [3], [8]). In some sense, the above-mentioned result cannot be strengthened because, under the Continuum Hypothesis, any Sierpiński-Zygmund function (see [10], [15]) is absolutely discontinuous (hence absolutely non-monotone) with respect to the class of all uncountable subsets of $\mathbf{R}$.

In our further considerations we will be dealing with Sierpiński-Zygmund functions, so it is reasonable to recall their definition here.

A function $f : \mathbf{R} \to \mathbf{R}$ is a Sierpiński-Zygmund function if, for every set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restriction $f|X$ is not continuous, i.e., the relation $D(f|X) \neq \emptyset$ holds true.

It immediately follows from this definition that any Sierpiński-Zygmund function is absolutely discontinuous (hence absolutely non-monotone) with respect to the family of all those subsets of $\mathbf{R}$ whose cardinalities are equal to $\mathbf{c}$.

In the classical construction of Sierpiński and Zygmund [15] their function f is defined by the method of transfinite recursion in such a manner that the graph of f almost avoids the graphs of all real-valued continuous functions whose domains are uncountable G_δ-subsets of $\mathbf{R}$. Actually, Sierpiński-Zygmund's construction admits a significant generalization to the case when a certain family of topologies on $\mathbf{R}$ is given instead of the standard Euclidean topology of $\mathbf{R}$. More precisely, we have the following statement.

Theorem 3. *Let E be a set of cardinality $\mathbf{c}$ and let $\{\mathcal{T}_i : i \in I\}$ be a family of topologies on E such that:*

(1) $\mathrm{card}(I) \leq \mathbf{c}$*;*

(2) *for each $i \in I$, the cardinality of the Borel σ-algebra $\mathcal{B}(E, \mathcal{T}_i)$ does not exceed $\mathbf{c}$.*

Then there exists a function $f : E \to \mathbf{R}$ such that, for any topology $\mathcal{T}_i$ $(i \in I)$, the corresponding mapping

$$f : (E, \mathcal{T}_i) \to \mathbf{R}$$

is a Sierpiński-Zygmund type function for the topological space $(E, \mathcal{T}_i)$.

Proof. The argument is similar to that of Sierpiński and Zygmund (cf. [10], [15]). First of all, notice that:

(*) for each $i \in I$ and for any partial continuous function $g : (E, \mathcal{T}_i) \to \mathbf{R}$, there exists a partial continuous function $g^* : (E, \mathcal{T}_i) \to \mathbf{R}$ extending g and defined on a Borel subset of $(E, \mathcal{T}_i)$.

Let α denote the least ordinal number of cardinality $\mathbf{c}$. Using conditions (1) and (2), we may define an α-sequence $\{g_\xi : \xi < \alpha\}$ of partial functions satisfying the following relation:

for any $i \in I$ and for any partial continuous mapping $g : (E, \mathcal{T}_i) \to \mathbf{R}$ whose domain is of cardinality $\mathbf{c}$ and belongs to $\mathcal{B}(E, \mathcal{T}_i)$, there exists an ordinal $\xi < \alpha$ such that $g = g_\xi$.

Now, let $\{x_\xi : \xi < \alpha\}$ be an injective enumeration of all points of E. In order to define the required f, we proceed by the method of transfinite recursion. Suppose that, for $\xi < \alpha$, the values $\{f(x_\zeta) : \zeta < \xi\}$ have already been determined. Consider the point x_ξ, choose a point

$$y \in \mathbf{R} \setminus \{g_\eta(x_\xi) : \eta < \xi, x_\xi \in \mathrm{dom}(g_\eta)\}$$

and put $f(x_\xi) = y$. Proceeding in this manner, we are able to define the function $f : E \to \mathbf{R}$. By using (*), it is not difficult to check that this f

is a Sierpiński-Zygmund function with respect to every topology $\mathcal{T}_i$, where $i \in I$. □

Remark 2. As shown by Roslanowski and Shelah [13], there are models of set theory in which the following phenomenon occurs: for any function f acting from $\mathbf{R}$ into $\mathbf{R}$, there exists a set $X \subset \mathbf{R}$ such that $\lambda^*(X) > 0$ and the restriction $f|X$ is continuous. A similar result was proved by Shelah [14] in terms of Baire category (instead of Lebesgue measure).

In the sequel, we will need the notion of a Luzin subset of $\mathbf{R}$ and the notion of a Sierpiński subset of $\mathbf{R}$.

Recall that $L \subset \mathbf{R}$ is a Luzin set if L is uncountable and, for every first category set $X \subset \mathbf{R}$, the relation $\mathrm{card}(X \cap L) \leq \omega$ holds true.

Recall also that $S \subset \mathbf{R}$ is a Sierpiński set if S is uncountable and, for every λ-measure zero set $X \subset \mathbf{R}$, the relation $\mathrm{card}(X \cap S) \leq \omega$ holds true.

It is well known that the Continuum Hypothesis implies the existence of Luzin and Sierpiński subsets of $\mathbf{R}$ (see, for instance, [10], [11], [12]).

In connection with Example 3, the following statement seems to be of interest.

Theorem 4. *Assuming the Continuum Hypothesis, there exists a function $g : \mathbf{R} \to \mathbf{R}$ which is not a Sierpiński-Zygmund function, but is absolutely non-monotone with respect to the family of all uncountable subsets of* $\mathbf{R}$.

Proof. Take an arbitrary continuous nowhere differentiable function $g_1 : \mathbf{R} \to \mathbf{R}$ and an arbitrary Sierpiński-Zygmund function $g_2 : \mathbf{R} \to \mathbf{R}$. Let L be a Luzin subset of $\mathbf{R}$. Define the function $g : \mathbf{R} \to \mathbf{R}$ as follows: $g(x) = g_1(x)$ if $x \in L$ and $g(x) = g_2(x)$ if $x \in \mathbf{R} \setminus L$.

Let us check that g is the required one. First of all, g is not a Sierpiński-Zygmund function, because $g|L = g_1|L$ is continuous. Let now X be an uncountable subset of $\mathbf{R}$. Only two cases are possible.

1. $\mathrm{card}(X \cap L) \leq \omega$. In this case, we have $\mathrm{card}(X \cap (\mathbf{R} \setminus L)) > \omega$. Since g_2 is a Sierpiński-Zygmund function, the restriction

$$g|(X \cap (\mathbf{R} \setminus L)) = g_2|(X \cap (\mathbf{R} \setminus L))$$

is not monotone, whence it follows that $g|X$ is not monotone.

2. $\mathrm{card}(X \cap L) > \omega$. Suppose for a while that $g|(X \cap L) = g_1|(X \cap L)$ is monotone. Then the continuous function g_1 should be monotone on the closure of $X \cap L$. Further, since g_1 is nowhere differentiable, the set $\mathrm{cl}(X \cap L)$ is necessarily nowhere dense in $\mathbf{R}$. Taking into account the fact that L is a Luzin set, this yields

$$\mathrm{card}(L \cap \mathrm{cl}(X \cap L)) \leq \omega,$$

which contradicts the inclusion $X \cap L \subset L \cap \mathrm{cl}(X \cap L)$ and the uncountability of $X \cap L$. The obtained contradiction completes the proof. □

Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function.

We shall say that f is countably continuous if $\mathrm{dom}(f)$ can be represented in the form $\mathrm{dom}(f) = \cup\{X_n : n < \omega\}$, where all restrictions $f|X_n$ $(n < \omega)$ are continuous.

We shall say that f is countably monotone if $\mathrm{dom}(f)$ can be represented in the form $\mathrm{dom}(f) = \cup\{X_n : n < \omega\}$, where all restrictions $f|X_n$ $(n < \omega)$ are monotone.

Example 4. It is easy to check that every countably monotone partial function $f : \mathbf{R} \to \mathbf{R}$ is also countably continuous. On the other hand, let $g : \mathbf{R} \to \mathbf{R}$ be any continuous nowhere differentiable function. Then it is not difficult to verify that g is not countably monotone.

The characteristic function of any subset of $\mathbf{R}$ is countably monotone (hence countably continuous). This circumstance indicates that there exist many countably monotone functions which are not Lebesgue measurable and do not possess the Baire property.

Any Sierpiński-Zygmud function is not countably continuous (hence is not countably monotone).

In order to present two further results concerning absolutely non-monotone functions, let us recall two important notions.

We say that a set $X \subset \mathbf{R}$ is categorically thick in $\mathbf{R}$ if, for every second category set $Y \subset \mathbf{R}$ possessing the Baire property, we have $X \cap Y \neq \emptyset$.

We say that a set $X \subset \mathbf{R}$ is thick in the sense of Lebesgue measure if, for every set $Y \in \mathrm{dom}(\lambda)$ with $\lambda(Y) > 0$, we have $X \cap Y \neq \emptyset$ (this is equivalent to the equality $\lambda_*(\mathbf{R} \setminus X) = 0$).

Theorem 1 implies, in particular, that every continuous function $f : \mathbf{R} \to \mathbf{R}$ is monotone on some set of cardinality continuum. At the same time, the following statement is valid.

Theorem 5. *Under the Continuum Hypothesis, there exists a partial function* $f : \mathbf{R} \to \mathbf{R}$ *such that:*

(1) $\mathrm{dom}(f)$ *is thick in* $\mathbf{R}$ *in the sense of Baire category;*

(2) f *is a restriction of some continuous function acting from* $\mathbf{R}$ *into itself;*

(3) f *is absolutely non-monotone with respect to the family of all uncountable subsets of* $\mathbf{R}$ (*consequently,* f *is not countably monotone*).

Proof. Let L be a Luzin subset of $\mathbf{R}$ thick in the sense of Baire category. The existence of such a subset can be obtained by a slight modification of the standard Luzin construction (cf. [10], [11], [12]). Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous nowhere differentiable function. Denote $f = g|L$. We assert that f is the required function. Indeed, take any uncountable set $X \subset \mathrm{dom}(f) = L$ and suppose that $f|X$ is monotone. Only two cases are possible.

(a) The set X is of first category in $\mathbf{R}$. Then, by virtue of the definition of L, we must have

$$\omega \geq \mathrm{card}(X \cap L) = \mathrm{card}(X) > \omega,$$

which yields a contradiction. We thus conclude that this case is impossible.

(b) The set X is of second category in $\mathbf{R}$. Let Y denote the closure of X. Since $g|X = f|X$ is monotone, and g is continuous, we get that $g|Y$ is also monotone. Further, Y is a second category closed set in $\mathbf{R}$, whence it follows that Y contains a nonempty open subinterval of $\mathbf{R}$ on which the function g must be monotone, too. But this contradicts the definition of g.

Theorem 5 has thus been proved. □

Remark 3. Relation (2) in the formulation of Theorem 5 can be strengthened by the following relation:

(2') f is a restriction of some everywhere differentiable function acting from $\mathbf{R}$ into itself.

Indeed, in the proof of Theorem 5 we might start with an everywhere differentiable function $g : \mathbf{R} \to \mathbf{R}$ which is absolutely non-monotone with respect to the family of all nonempty open intervals in $\mathbf{R}$ (cf. Example 2).

By utilizing a similar argument, one can obtain the following dual statement.

Theorem 6. *Under the Continuum Hypothesis, there exists a partial function* $f : \mathbf{R} \to \mathbf{R}$ *such that:*

(1) $\mathrm{dom}(f)$ *is thick in* $\mathbf{R}$ *in the sense of Lebesgue measure;*

(2) f *is a restriction of some continuous function acting from* $\mathbf{R}$ *into itself;*

(3) f *is absolutely non-monotone with respect to the family of all uncountable subsets of* $\mathbf{R}$ (*consequently,* f *is not countably monotone*).

Proof. The argument is analogous to the proof of Theorem 5. Let S be a Sierpiński subset of $\mathbf{R}$ thick in the sense of Lebesgue measure. The existence of such a subset can be obtained by a slight modification of the standard Sierpiński construction (cf. again [10], [11], [12]). Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous nowhere approximately differentiable function. Let us put $f = g|S$. Then f trivially satisfies relations (1), (2) and, similarly to the proof of Theorem 5, we obtain that f satisfies relation (3) as well.

Finally, let us return to the sets $C(f)$ and $D(f)$ of a partial function $f : \mathbf{R} \to \mathbf{R}$. The first of them is always of type G_δ in $\mathrm{dom}(f)$ and the second one is always of type F_σ in $\mathrm{dom}(f)$.

Conversely, if a set $Z \subset \mathbf{R}$ is of type F_σ in $\mathbf{R}$, then $Z = D(f)$ for some function $f : \mathbf{R} \to \mathbf{R}$ (see [10], [12]).

We have already mentioned that there are extremely discontinuous functions acting from $\mathbf{R}$ into $\mathbf{R}$. Among them, Sierpiński-Zygmund functions are of special interest and find nontrivial applications for constructing various counterexamples (cf. Theorem 4). However, having any partial function $f : \mathbf{R} \to \mathbf{R}$ with $\mathrm{card}(\mathrm{dom}(f)) > \omega$, one may assert that there is a restriction of f such that the set of all continuity points of this restriction is infinite. To formulate a more precise result in this direction, we need the following notion.

Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function. We shall say that $x \in \mathrm{dom}(f)$ is a strong continuity point for f if x is a condensation point of $\mathrm{dom}(f)$ and f is continuous at x. The set of all strong continuity points for f will be denoted by $C_0(f)$. □

Theorem 7. *Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function whose domain is uncountable. Then there exists an uncountable set $X \subset \mathrm{dom}(f)$ such that $\mathrm{card}(C_0(f|X)) \geq \omega$.*

The proof of this statement is quite easy and is omitted here (one should take into account the fact that the graph of f is an uncountable subset of the plane $\mathbf{R}^2$, hence this graph contains uncountably many condensation points of itself).

Remark 4. It is not difficult to see that the result analogous to Theorem 7 remains valid in more general situations, e.g., in the case of a partial function $g : E \to F$, where E and F are two separable metric spaces and $\mathrm{dom}(g)$ is uncountable.

The existence of Sierpiński-Zygmund functions shows that, under the Continuum Hypothesis, the inequality in Theorem 7 cannot be replaced by the strict inequality. In this context, the next simple example is also relevant.

Example 5. Let $\{q_n : n < \omega\}$ be an injective enumeration of all rational numbers. Define a function $f : \mathbf{Q} \to \mathbf{R}$ by putting

$$f(q_n) = n \qquad (n < \omega).$$

Let X be an infinite subset of $\mathbf{Q}$. It is easy to verify that if $x \in C(f|X)$, then x is an isolated point of X. In other words, no restriction of f to an infinite subset X of $\mathrm{dom}(f)$ can be continuous at an accumulation point of X.

Many results concerning restrictions of various functions to non-small subsets of the real line can be found in [2], [3], [4], [8], [10], [12].

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Author's address:

A. Razmadze Mathemetical Institute
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186
Georgia