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A combinatorial problem on translation-invariant extensions of the Lebesgue measure

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It is proved that, for every natural number $k \geq 2$, there exist k subsets of the real line such that any $k-1$ of them can be made measurable with respect to a translation-invariant extension of the Lebesgue measure, but there is no nonzero σ-finite translation-quasi-invariant measure for which all of these k subsets become measurable. In connection with this result, a related open problem is posed.

Let E be a base set and let μ be a nonzero σ-finite measure defined on some σ-algebra of subsets of E. The general measure extension problem is to extend the given μ to a maximally wide class of subsets of E. If μ is continuous (i.e., $\mu(\{x\}) = 0$ for all $x \in E$), then this problem turns out to be closely connected with the theory of large cardinals and additional set-theoretical axioms.

For example, suppose that E coincides with the real line $\mathbf{R}$ and μ coincides with the standard one-dimensional Lebesgue measure λ on this line. Then, as is well known (see, e.g., [8,14]), the following two assertions are equivalent:

(1) there exists an extension of λ defined on the family of all subsets of $\mathbf{R}$;

(2) there exists a nonzero σ-finite continuous measure defined on the family of all subsets of $\mathbf{R}$.

On the other hand, according to the classical result of Ulam [14], if the cardinality of the continuum $\mathbf{c} = \mathrm{card}(\mathbf{R})$ is strictly less than the first uncountable weakly inaccessible cardinal, then there exists no nonzero σ-finite continuous measure defined on the family of all subsets of $\mathbf{R}$. Moreover, under

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Martin's Axiom, there are countably many sets $\{Z_1, Z_2, \dots, Z_k, \dots\}$ contained in $\mathbf{R}$ and such that no nonzero σ-finite continuous measure ν on $\mathbf{R}$ can make all these sets to be measurable with respect to ν.

Let E be a base set, μ be a σ-finite measure defined on some σ-algebra of subsets of E, and let $\{X_1, X_2, \dots, X_k\}$ be a finite family of subsets of E. It is well known that there always exists a measure μ' on E extending μ and such that all sets $X_1, X_2, \dots, X_k$ are μ'-measurable. In contrast to this situation, if the original measure μ is invariant under a group G of transformations of E, then we cannot assert, in general, that there exists an extension μ' of μ which also is invariant under G and for which all given sets $X_1, X_2, \dots, X_k$ are μ'-measurable. Even for $k = 1$, it may happen that the single set X_1 turns out to be nonmeasurable with respect to every G-invariant extension of μ.

For instance, if E again coincides with the real line $\mathbf{R}$ and $\mu = \lambda$, then the classical construction of Vitali [15] yields a set $V \subset \mathbf{R}$ which is nonmeasurable with respect to every translation-invariant extension of λ (or, in other words, V turns out to be absolutely nonmeasurable with respect to the class of all translation-invariant extensions of λ).

At the same time, it was established by several authors that there are many subsets of $\mathbf{R}$ which become measurable with respect to certain translation-invariant extensions of λ (see, e.g., [1–7,9,13,16]). Moreover, it was proved that there exists even a nonseparable translation-invariant extension ν of λ (see [2,3,5,7]). Clearly, the domain of such a ν contains in itself a rich class of subsets of $\mathbf{R}$ which are not measurable with respect to λ.

Some delicate problems of the theory of translation-invariant extensions of λ were discussed in the literature (see, for instance, [1,5,6,9,13,16]). One of the problems of this type will be considered below. It is of a certain combinatorial character.

We shall use the following fairly standard notation:

$\omega =$ the set of all natural numbers (and, simultaneously, the cardinality of this set);

$\omega_1 =$ the least uncountable cardinal number;

$\mathbf{Q} =$ the field of all rational numbers;

$\mathbf{R} =$ the real line;

$\mathbf{c} =$ the cardinality of the continuum;

$\mathbf{R}^n =$ the n-dimensional Euclidean space (so $\mathbf{R} = \mathbf{R}^1$);

$\mathrm{dom}(\mu) =$ the domain of a given σ-finite measure μ (i.e., the σ-algebra of all μ-measurable sets);

$\lambda =$ the one-dimensional Lebesgue measure on $\mathbf{R}$;

$\lambda_n =$ the n-dimensional Lebesgue measure on $\mathbf{R}^n$ (so $\lambda_1 = \lambda$).

First of all, we would like to recall the following fact.

Example 1. In [4] two sets $A_1 \subset \mathbf{R}$ and $A_2 \subset \mathbf{R}$ were constructed, which satisfy the following conditions:

(1) there exists a translation-invariant extension μ_1 of λ such that $\mu_1(A_1) = 0$;

(2) there exists a translation-invariant extension μ_2 of λ such that $\mu_2(A_2) = 0$;

(3) there exists no nonzero σ-finite translation-invariant measure ν on $\mathbf{R}$ such that both sets A_1 and A_2 are ν-measurable.

Actually, it was demonstrated in [4] that, for any nonzero σ-finite translation-invariant measure μ on $\mathbf{R}$, the set $A_1 \cup A_2$ is nonmeasurable with respect to μ.

The natural question arises whether it is possible to generalize the above-mentioned Example 1 to the case of several subsets of the real line. The main goal of this paper is to establish an analogous result for finitely many subsets $A_1, A_2, \dots, A_k$ of $\mathbf{R}$, where k is an arbitrary natural number greater than 2. In fact, it will be shown below that by combining techniques of Hamel bases with the argument used in the proof of an old theorem of Sierpiński [10] concerning a certain logical equivalent of the Continuum Hypothesis, one can get a positive answer to this question.

Let E be a base set, G be a group of transformations of E, and let μ be a σ-finite measure defined on some G-invariant σ-algebra of subsets of E. We recall that μ is G-quasi-invariant if, for any $g \in G$ and $X \in \mathrm{dom}(\mu)$, the relation

$$\mu(X) = 0 \Leftrightarrow \mu(g(X)) = 0$$

holds true.

Obviously, the quasi-invariance of measures is a much weaker property than the ordinary invariance of measures.

In what follows, we need one notion from the theory of quasi-invariant measures.

Let E be again a base set and let G be a group of transformations of E. We recall (see, e.g., [5] or [6]) that a set $X \subset E$ is G-negligible in E if the following two conditions are satisfied:

(a) there exists a nonzero σ-finite G-quasi-invariant measure μ_0 on E such that $X \in \mathrm{dom}(\mu_0)$;

(b) for any σ-finite G-quasi-invariant measure μ on E, we have the implication

$$X \in \mathrm{dom}(\mu) \Rightarrow \mu(X) = 0.$$

Some properties of G-negligible sets are discussed in [4–6]. In particular, the following auxiliary proposition is formulated therein.

Lemma 1. *Let $(\Gamma_1, +)$ and $(\Gamma_2, +)$ be two commutative groups and suppose that $\phi : \Gamma_1 \to \Gamma_2$ is a surjective homomorphism. If Y is a Γ_2-negligible subset of Γ_2, then $X = \phi^{-1}(Y)$ is a Γ_1-negligible subset of Γ_1.*

The proof of Lemma 1 follows directly from the definition of negligible sets, so is omitted here (cf. [6]). Notice by the way that the commutativity of the groups Γ_1 and Γ_2 is not essential in the formulation of this lemma. However, below we will be dealing only with commutative groups (even with vector spaces over $\mathbf{Q}$), so the presented formulation is sufficient for our further purposes.

We also need some other auxiliary statements.

Lemma 2. *Let $(G, +)$ be a commutative group and let a nonempty set $X \subset G$ be such that $D + X \neq G$ for every countable set $D \subset G$. Denote by $\mathcal{S}$ the G-invariant σ-algebra of subsets of G, generated by X and the family of all countable subsets of G. Then there exists a continuous G-invariant probability measure μ on $\mathcal{S}$ satisfying the equality $\mu(X) = 0$.*

Further, let $\mathcal{T}$ be a σ-algebra of subsets of G and let ν be a σ-finite measure on $\mathcal{T}$ such that $\nu_*(D+X) = 0$ for any countable set $D \subset G$. Let $\mathcal{R}$ denote the σ-algebra of subsets of G, generated by $\mathcal{S} \cup \mathcal{T}$. Then there exists a unique σ-finite measure θ on $\mathcal{R}$ satisfying the relations:

(a) *$\theta(Y \cap Z) = \mu(Y)\nu(Z)$ for all $Y \in \mathcal{S}$ and $Z \in \mathcal{T}$;*

(b) *if $\mathcal{T}$ is G-invariant, then $\mathcal{R}$ is also G-invariant;*

(c) *if ν is G-invariant, then θ is also G-invariant;*

(d) *if ν is G-quasi-invariant, then θ is also G-quasi-invariant.*

Proof. The first part of this lemma is almost trivial. Indeed, it immediately follows from the assumption on X that the family of all those sets in G, which can be covered by countably many translates of X, forms a G-invariant σ-ideal $\mathcal{J}$. Each member of the σ-algebra $\mathcal{S}$ either belongs to $\mathcal{J}$ or is the complement of a set belonging to $\mathcal{J}$. Now, for any set $Y \in \mathcal{S}$, put $\mu(Y) = 0$ if $Y \in \mathcal{J}$, and $\mu(Y) = 1$ if $G \setminus Y \in \mathcal{J}$. It can easily be seen that the probability measure μ defined in this manner is continuous and G-invariant.

To establish the second part of the lemma, consider the product measure $\nu \otimes \mu$, the diagonal $\Delta = \{(g, g) : g \in G\} \subset G \times G$ and the canonical bijection $\phi : G \to \Delta$ given by

$$\phi(g) = (g, g) \quad (g \in G).$$

Obviously, we have $\mathcal{R} = \phi^{-1}(\mathcal{T} \otimes \mathcal{S})$. Since μ is a two-valued probability measure and $\nu_*(Z) = 0$ for each $Z \in \mathcal{S}$ with $\mu(Z) = 0$, we easily infer that Δ is $(\nu \otimes \mu)$-thick in $G \times G$. As is well known, in this case there exists a unique σ-finite measure θ on $\mathcal{R}$ satisfying the equality

$$(\nu \otimes \mu)(P) = \theta(\phi^{-1}(P)) \quad (P \in \mathcal{T} \otimes \mathcal{S}).$$

This equality directly implies relation (a). The relations (b), (c), and (d) are also readily verified. □

Lemma 3. *Let $(H, +)$ be a commutative group equipped with a σ-finite H-quasi-invariant measure μ and let X be a μ-measurable subset of H. Then there exists a countable subgroup H' of H such that the set*

$$X' = \cup\{h' + X : h' \in H'\}$$

is μ-almost H-invariant, i.e., for any $h \in H$, we have $\mu((h + X') \triangle X') = 0$.

Proof. If $\mu(X) = 0$, then there is nothing to prove. So let us consider the case when $\mu(X) > 0$ (only this case is of interest to us in what follows). Suppose to the contrary that there exists no countable subgroup H' of H with the required property and define by transfinite recursion an increasing ω_1-sequence of μ-measurable subsets of H. Namely, put $X_0 = X$. Assume that, for an ordinal $\xi < \omega_1$, the partial ξ-sequence $\{X_\zeta : \zeta < \xi\}$ has already been defined in such a way that every set X_ζ is of the form

$$X_\zeta = \cup\{h_{k,\zeta} + X : k < \omega\}$$

for some countable family $\{h_{k,\zeta} : k < \omega\} \subset H$. Consider the set

$$Y_\xi = \cup\{X_\zeta : \zeta < \xi\}.$$

Clearly, Y_ξ can be represented in a similar form (because ξ is a countable ordinal). According to our assumption, there exists an element $h \in H$ such that

$$\mu((h + Y_\xi)\triangle Y_\xi) > 0,$$

which implies that either $\mu((h + Y_\xi) \setminus Y_\xi) > 0$ or $\mu((-h + Y_\xi) \setminus Y_\xi) > 0$. Now, let us put

$$X_\xi = Y_\xi \cup (h + Y_\xi) \cup (-h + Y_\xi).$$

Proceeding in this manner, we get the ω_1-sequence $\{X_\xi : \xi < \omega_1\}$ of μ-measurable sets, which increases by inclusion and $\mu(X_{\xi+1} \setminus X_\xi) > 0$ for every ordinal $\xi < \omega_1$. But the latter contradicts the σ-finiteness of μ. The obtained contradiction ends the proof. □

Obviously, the assertion of Lemma 3 remains true in a more general situation where a σ-finite measure μ is given on a base set E and this μ is quasi-invariant under some group G of transformations of E (in short, G-quasi-invariant). Moreover, the argument presented above shows that the assertion of the lemma holds true for any G-quasi-invariant measure μ on E which satisfies the countable chain condition (in general, such a measure does not need to be σ-finite).

Below, having two commutative groups $(G, +)$ and $(H, +)$, we will consider their direct sum $G+H$ which, in fact, may be identified with the product group $G \times H$. Naturally, under such an identification $(G, +)$ is regarded as the subgroup $G \times \{0\}$ of $G \times H$ and $(H, +)$ is regarded as the subgroup $\{0\} \times H$ of $G \times H$. Analogously, having several commutative groups $(G_1, +), (G_2, +), \ldots, (G_k, +)$, we will identify their direct sum $G_1 + G_2 + \cdots + G_k$ with the product group $G_1 \times G_2 \times \cdots \times G_k$.

Lemma 4. *Let $(G, +)$ and $(H, +)$ be two commutative groups and let* $\mathrm{card}(H) > \omega$. *Consider the direct sum $G + H$. Let X be a subset of $G + H$ such that* $\mathrm{card}((g + H) \cap X) < \omega$ *for each element $g \in G$. Then X is a $(G + H)$-negligible subset of $G + H$.*

Proof. It readily follows from the described property of X that, for every countable family $\{g_i + h_i : i < \omega\} \subset G + H$, the inequality

$$\mathrm{card}((G + H) \setminus \cup\{g_i + h_i + X : i < \omega\}) > \omega$$

is valid. In view of Lemma 2, this implies that there exists a probability continuous $(G + H)$-invariant measure μ_0 on $G + H$ such that $X \in \mathrm{dom}(\mu_0)$ and $\mu_0(X) = 0$.

Now, let μ be any σ-finite $(G + H)$-quasi-invariant measure on $G + H$ such that $X \in \mathrm{dom}(\mu)$. We have to show that $\mu(X) = 0$. Suppose to the contrary that $\mu(X) > 0$. Since our μ is H-quasi-invariant, by virtue of Lemma 3 we can find a countable subgroup H' of H for which the set

$$X' = H' + X = \cup\{h' + X : h' \in H'\}$$

turns out to be almost H-invariant with respect to μ, i.e., the equality

$$\mu(X'\triangle(h + X')) = 0$$

holds true for every $h \in H$. Taking into account this equality, we infer that, for any countable family $\{h_j : j < \omega\} \subset H$, the relation

$$\mu(\cap\{h_j + X' : j < \omega\}) = \mu(X') > 0$$

is valid and, consequently,

$$\cap\{h_j + X' : j < \omega\} \neq \emptyset.$$

But, if a countable family $\{h_j : j < \omega\} \subset H$ is chosen satisfying the condition

$$h_j - h_r \notin H' \quad (j \in J,\ r \in J,\ j \neq r),$$

then, keeping in mind the definition of X, it is not difficult to verify that

$$\cap\{h_j + X' : j < \omega\} = \emptyset,$$

which yields a contradiction with the said above. The obtained contradiction finishes the proof. □

Lemma 5. *Let $(G, \|\cdot\|)$ be a normed vector space over $\mathbf{Q}$ with $\mathrm{card}(G) > 1$ and let $\{B_m : m < \omega\}$ be a countable family of balls in G. Then there exists a disjoint countable family $\{P_j : j \in J\}$ of subsets of G satisfying the following relations:*

(1) *each set P_j is a translate of some ball B_m;*

(2) *for any ball B_m, there are infinitely many indices $j \in J$ such that the set P_j is a translate of B_m.*

We omit a simple proof of the above assertion (the required disjoint family $\{P_j : j \in J\}$ can easily be constructed by ordinary recursion).

Lemma 6. *Let $G \neq \{0\}$ and H be two vector spaces over $\mathbf{Q}$ and let $G + H$ be their direct sum. Suppose that a set $X \subset G + H$ is given such that the inequality $\mathrm{card}(X \cap (g + H)) \leq \omega$ holds for every $g \in G$. Then there exists a set $Y \subset G + H$ satisfying the following conditions:*

(a) $\mathrm{card}(Y \cap (g + H)) \leq 1$ *for every $g \in G$;*

(b) $X \subset \cup\{g_i + Y : i \in I\}$ *for some countable family $\{g_i : i \in I\} \subset G$.*

Proof. We may treat G as a normed vector space over $\mathbf{Q}$. Indeed, denote by $\{z_\xi : \xi \in \Xi\}$ any Hamel basis of G. Each element $g \in G$ admits a unique representation in the form $g = \sum\{q_\xi z_\xi : \xi \in \Xi\}$, where all coefficients q_ξ belong to $\mathbf{Q}$ and only finitely many of them differ from zero. Putting

$$\|g\| = \sum\{|q_\xi| : \xi \in \Xi\},$$

we get a norm on G.

For every natural number $m > 0$, let $B_m = \{g \in G : \|g\| \leq m\}$ denote the ball in G with center at zero and with radius m. According to Lemma 5, there exists a disjoint countable family $\{P_j : j \in J\}$ of subsets of G satisfying the following relations:

(1) each set P_j is a G-translate of some ball B_m;

(2) for any ball B_m, there are infinitely many indices $j \in J$ such that the set P_j is a G-translate of B_m.

Let $J(m)$ denote the family of all those indices $j \in J$ for which P_j is a G-translate of B_m, i.e., $j \in J(m)$ if and only if there exists an element $g_{m,j} \in G$ such that $P_j = g_{m,j} + B_m$. Clearly, the sets $J(m)$ $(0 < m < \omega)$ are countably infinite, pairwise disjoint and their union coincides with J.

Consider the family of sets $\{X \cap (B_m + H) : 0 < m < \omega\}$. Obviously, we have

$$X = \cup\{X \cap (B_m + H) : 0 < m < \omega\}.$$

For any $g \in B_m$, the set $X \cap (g + H)$ is at most countable. This implies that $X \cap (B_m + H)$ can be represented in the form

$$X \cap (B_m + H) = \cup\{Y_{m,j} : j \in J(m)\},$$

where $\mathrm{card}(Y_{m,j} \cap (g + H)) \leq 1$ for each $j \in J(m)$ and for each $g \in B_m$. Now, we put

$$Y = \cup\{g_{m,j} + Y_{m,j} : j \in J(m), 0 < m < \omega\}.$$

It is not difficult to check that the set Y satisfies the conditions (a) and (b) of the lemma, which completes the proof. □

Lemma 7. *Let $k \geq 2$ be a natural number and let $(G, \|\cdot\|)$ be a vector space over* **Q** *representable in the form of a direct sum*

$$G = G_1 + G_2 + \cdots + G_k,$$

where all G_i $(i = 1, 2, \ldots, k)$ are vector subspaces of G of cardinality ω_1.

Then subsets $Y_1, Y_2, \ldots, Y_k$ of G can be found such that:

(1) *for each index $i \in \{1, 2, \ldots, k\}$, the union $Y_1 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cdots \cup Y_k$ is a G-negligible set in G;*

(2) *there exists a countable family $\{g_m : m < \omega\}$ of elements from G for which we have*

$$\cup\{g_m + (Y_1 \cup Y_2 \cup \cdots \cup Y_k) : m < \omega\} = G.$$

Consequently, there is no nonzero σ-finite G-quasi-invariant measure ν on G such that all sets $Y_1, Y_2, \ldots, Y_k$ are ν-measurable.

Proof. The argument is based on some ideas of Sierpiński which he used in establishing the equivalence of the Continuum Hypothesis to the existence of certain decompositions of $\mathbf{R}^2$ and $\mathbf{R}^3$ (see [10–12]).

Without loss of generality, we may suppose that the subspaces $G_1, G_2, \ldots, G_k$ are well-ordered by ordering relations which are isomorphic to ω_1. So let

$$\xi_i : G_i \to \omega_1 \quad (i = 1, 2, \ldots, k)$$

denote the corresponding isomorphisms. Consequently, if $x_i \in G_i$, where $i \in \{1, 2, \ldots, k\}$, then $\xi_i(x_i)$ denotes the countable ordinal corresponding to x_i with respect to the isomorphism ξ_i between G_i and ω_1.

In addition, we may assume that every G_i $(i = 1, 2, \ldots, k)$ is a normed vector space over **Q** (see the proof of Lemma 6).

Now, let us consider the sets X_i $(i = 1, 2, \ldots, k)$ defined as follows:

$$X_i = \{x_1 + x_2 + \cdots + x_k \in G : \xi_i(x_i) = \max(\xi_1(x_1), \xi_2(x_2), \ldots, \xi_k(x_k))\}.$$

Clearly, we have

$$G = X_1 \cup X_2 \cup \cdots \cup X_k.$$

Furthermore, each set X_i possesses the following property: for any $x_i \in G_i$, the set

$$X_i \cap (G_1 + \cdots + G_{i-1} + x_i + G_{i+1} + \cdots + G_k)$$

is at most countable. Applying Lemma 6, we come to a family $\{Y_1, Y_2, \ldots, Y_k\}$ of subsets of G such that:

(a) each set Y_i is uniform with respect to $G_1 + \cdots + G_{i-1} + G_{i+1} + \cdots + G_k$, i.e., for any $x_i \in G_i$, we have

$$\mathrm{card}(Y_i \cap (G_1 + \cdots + G_{i-1} + x_i + G_{i+1} + \cdots + G_k)) \leq 1;$$

(b) each set X_i can be covered by a countable family of translates of Y_i.

Notice now that relation (b) directly implies relation (2). It remains to show that relation (1) is also true. Observe that, for each integer $i \in [1, k]$ and for any element

$$x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k \in G_1 + \cdots + G_{i-1} + G_{i+1} + \cdots + G_k,$$

the set

$$(Y_1 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_k) \cap (x_1 + \cdots + x_{i-1} + G_i + x_{i+1} + \cdots + x_k)$$

consists of at most $k - 1$ elements. So, by virtue of Lemma 4, we conclude that the union $Y_1 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_k$ is G-negligible in G. Lemma 7 has thus been proved. □

Lemma 8. *For any natural numbers $n \geq 1$ and $k \geq 2$, the Euclidean space $\mathbf{R}^n$ can be represented in the form of a direct sum*

$$\mathbf{R}^n = G_1 + G_2 + \cdots + G_k + H,$$

where all G_i $(i = 1, 2, \ldots, k)$ and H are vector spaces over $\mathbf{Q}$ and the following conditions are fulfilled:

(1) $\mathrm{card}(G_1) = \mathrm{card}(G_2) = \cdots = \mathrm{card}(G_k) = \omega_1$;

(2) $\mathrm{card}(H) = \mathbf{c}$;

(3) *H is a λ_n-thick subset of $\mathbf{R}^n$.*

Proof. We use the technique of Hamel bases and the standard argument based on the method of transfinite induction. Namely, we identify $\mathbf{c}$ with the first ordinal number of cardinality continuum and denote by $\mathcal{B}$ the family of all Borel subsets of $\mathbf{R}^n$ having strictly positive λ_n-measure. Since $\mathrm{card}(\mathcal{B}) = \mathbf{c}$, we can represent $\mathcal{B}$ in the form $\{B_\xi : \xi < \mathbf{c}\}$ where $B_\xi = B_{\xi+1}$ for all ordinals $\xi < \mathbf{c}$. Further, for any set $T \subset \mathbf{R}^n$, denote by $\mathrm{span}_{\mathbf{Q}}(T)$ the linear span (over the rationals) of T. Obviously, the relation $\mathrm{card}(T) < \mathbf{c}$ implies the relation $\mathrm{card}(\mathrm{span}_{\mathbf{Q}}(T)) < \mathbf{c}$. Taking this circumstance into account and applying transfinite recursion, we are able to construct a family of points $\{x_\xi : \xi < \mathbf{c}\} \subset \mathbf{R}^n$ such that

$$x_\xi \in B_\xi \setminus \mathrm{span}_{\mathbf{Q}}(\{x_\zeta : \zeta < \xi\}) \quad (\xi < \mathbf{c}).$$

Let Ξ denote the set of all even ordinals strictly less than $\mathbf{c}$ and let Ξ' stand for the set of all odd ordinals strictly less than $\mathbf{c}$. Since $\mathrm{card}(\Xi) = \mathbf{c}$, there are pairwise disjoint subsets $\Xi_1, \Xi_2, \ldots, \Xi_k$ of Ξ such that

$$\mathrm{card}(\Xi_1) = \mathrm{card}(\Xi_2) = \cdots = \mathrm{card}(\Xi_k) = \omega_1.$$

Now, let us put $G' = \mathrm{span}_{\mathbf{Q}}(\{x_\xi : \xi \in \Xi'\})$ and

$$G_i = \mathrm{span}_{\mathbf{Q}}(\{x_\xi : \xi \in \Xi_i\}) \quad (i = 1, 2, \ldots, k).$$

Then G' and all G_i $(i = 1, \ldots, k)$ are vector spaces over $\mathbf{Q}$ and

$$\mathrm{card}(G_1) = \mathrm{card}(G_2) = \cdots = \mathrm{card}(G_k) = \omega_1.$$

Moreover, keeping in mind that the family of points $\{x_\xi : \xi < \mathbf{c}\}$ is linearly independent over $\mathbf{Q}$, we infer that the sum $G' + G_1 + \cdots + G_k$ is direct. Further, there exists a vector space $F \subset \mathbf{R}^n$ over $\mathbf{Q}$ satisfying the relations

$$F \cap (G' + G_1 + \cdots + G_k) = \{0\}, \quad F + (G' + G_1 + \cdots + G_k) = \mathbf{R}^n.$$

Denote $H = F + G'$. So we come to a representation of $\mathbf{R}^n$ in the form of a direct sum:

$$\mathbf{R}^n = G_1 + G_2 + \cdots + G_k + H.$$

Since $B_\xi = B_{\xi+1}$ for any ordinal $\xi < \mathbf{c}$, the family of points $\{x_\xi : \xi \in \Xi'\}$ is λ_n-thick in $\mathbf{R}^n$. Taking into account the relations

$$\{x_\xi : \xi \in \Xi'\} \subset G' \subset H,$$

we conclude that H is also λ_n-thick, which proves Lemma 8. □

With the aid of the above-mentioned lemmas, we are able to obtain the two main statements of this paper.

Theorem 1. *Let $n > 0$ and $k \geq 2$ be two natural numbers. Then subsets $A_1, A_2, \ldots, A_k$ of the Euclidean space $\mathbf{R}^n$ can be found such that:*

(1) *for each index $i \in \{1, 2, \ldots, k\}$, the set*

$$A_1 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup \cdots \cup A_k$$

is $\mathbf{R}^n$-negligible in $\mathbf{R}^n$;

(2) *for each index* $i \in \{1, 2, \dots, k\}$*, there is a complete translation-invariant extension* μ_i *of* λ_n *satisfying the equality*

$$\mu_i(A_1 \cup \dots \cup A_{i-1} \cup A_{i+1} \cup \dots \cup A_k) = 0$$

and, consequently, all sets $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_k$ *turn out to be measurable with respect to* μ_i*;*

(3) *there exists no nonzero* σ*-finite translation-quasi-invariant measure* μ *on* $\mathbf{R}^n$ *for which all sets* $A_1, A_2, \dots, A_k$ *are* μ*-measurable.*

Proof. Consider a representation of $\mathbf{R}^n$ in the form of a direct sum

$$\mathbf{R}^n = G_1 + G_2 + \dots + G_k + H,$$

where G_i $(i = 1, 2, \dots, k)$ and H are as in Lemma 8. Denote

$$G = G_1 + G_2 + \dots + G_k$$

and let $Y_1, Y_2, \dots, Y_k$ be subsets of G as in Lemma 7. Further, define the sets

$$A_i = Y_i + H \quad (i = 1, 2, \dots, k).$$

In view of Lemmas 1 and 7, all sets

$$B_i = A_1 \cup \dots \cup A_{i-1} \cup A_{i+1} \cup \dots \cup A_k \quad (i = 1, 2, \dots, k)$$

are $\mathbf{R}^n$-negligible. Moreover, keeping in mind the λ_n-thickness of H, we readily derive that, for any countable family $\{g_m : m < \omega\} \subset \mathbf{R}^n$, the set $\cup\{g_m + B_i : m < \omega\}$ has inner λ_n-measure zero. Then the standard construction of extending invariant measures (see, e.g., [1,2,5,9,13] or Lemma 2) enables us to conclude that there exists a translation-invariant complete extension μ_i of λ_n such that $\mu_i(B_i) = 0$, so all sets $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_k$ become measurable with respect to μ_i. On the other hand, since G can be covered by countably many G-translates of the set $Y_1 \cup Y_2 \cup \dots \cup Y_k$, the whole space $\mathbf{R}^n$ can also be covered by countably many G-translates of the set $A_1 \cup A_2 \cup \dots \cup A_k$. It easily follows from the above-mentioned circumstance that there exists no nonzero σ-finite translation-quasi-invariant measure μ on $\mathbf{R}^n$ such that

$$\{A_1, A_2, \dots, A_k\} \subset \mathrm{dom}(\mu).$$

This ends the proof of Theorem 1. □

Theorem 2. *Let* $n > 0$ *and* $k \geq 2$ *be two natural numbers. Then subsets* $C_1, C_2, \dots, C_k$ *of the Euclidean space* $\mathbf{R}^n$ *can be found such that:*

(1) *for each index* $i \in \{1, 2, \dots, k\}$*, the set* C_i *is* $\mathbf{R}^n$*-negligible in* $\mathbf{R}^n$*;*

(2) *for each index* $i \in \{1, 2, \dots, k\}$*, there is a complete translation-invariant extension* μ_i *of* λ_n *satisfying the relation* $C_i \in \mathrm{dom}(\mu_i)$ *and, consequently, the relation* $\mu_i(C_i) = 0$*;*

(3) *if* i *and* j *are any two distinct indices from* $\{1, 2, \dots, k\}$*, then there exists no nonzero* σ*-finite translation-quasi-invariant measure* μ *on* $\mathbf{R}^n$ *for which the sets* C_i *and* C_j *are* μ*-measurable.*

Proof. We preserve the notation used in the proof of Theorem 1. Let us put

$$C_i = B_i \quad (i \in \{1, 2, \dots, k\}).$$

We already know that each set C_i is $\mathbf{R}^n$-negligible in $\mathbf{R}^n$ and there exists a complete translation-invariant extension μ_i of λ_n such that $\mu_i(C_i) = 0$. Further, for any two distinct indices i and j from $\{1, 2, \dots, k\}$, we have

$$C_i \cup C_j = A_1 \cup A_2 \cup \dots \cup A_k.$$

Since the whole space $\mathbf{R}^n$ can be covered by countably many translates of the set $A_1 \cup A_2 \cup \dots \cup A_k$, we readily deduce that there is no nonzero σ-finite translation-quasi-invariant measure μ on $\mathbf{R}^n$ satisfying the relation $\{C_i, C_j\} \subset \mathrm{dom}(\mu)$. Theorem 2 has thus been proved. □

In connection with the obtained results, the following open combinatorial problem seems to be of some interest.

Problem. Let $n \geq 1$, $k > 2$ and $0 < l < k$ be natural numbers. Prove (or disprove) that there is a family $\{A_1, A_2, \ldots, A_k\}$ of subsets of $\mathbf{R}^n$ satisfying the following conditions:

(a) for any l-element subfamily of $\{A_1, A_2, \ldots, A_k\}$, there exists a translation-invariant extension of λ_n such that all members from the subfamily are measurable with respect to this extension;

(b) for any $(l + 1)$-element subfamily of $\{A_1, A_2, \ldots, A_k\}$, there exists no nonzero σ-finite translation-quasi-invariant measure on $\mathbf{R}^n$ whose domain contains this subfamily.

Example 2. Let us consider the Euclidean plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ and let a set $X \subset \mathbf{R}^2$ be such that $\mathrm{card}(X \cap (\{t\} \times \mathbf{R})) < \omega$ for all $t \in \mathbf{R}$. Then, according to Lemma 4, X is $\mathbf{R}^2$-negligible in $\mathbf{R}^2$. At the same time, there exists a set $Z \subset \mathbf{R}^2$ which satisfies the relation $\mathrm{card}(Z \cap (\{t\} \times \mathbf{R})) \leq \omega$ for every $t \in \mathbf{R}$, but which is not $\mathbf{R}^2$-negligible in $\mathbf{R}^2$ (see, for instance, [4] or [6] where a much stronger result is presented).

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