Bulletin of TICMI
Vol. 16, No. 1, 2012, 34–42

Some Geometric Consequences of Ramsey's Combinatorial Theorem

Kharazishvili Alexander

A. Razmadze Mathematical Institute, University Street, 2, Tbilisi 0186, Georgia
I. Vekua Institute of Applied Mathematics, University Street, 2, Tbilisi 0186, Georgia
e-mail: kharaz2@yahoo.com

(*Received April 19, 2012; Accepted May 20, 2012*)

Some geometric applications of the finite and countably infinite versions of Ramsey's combinatorial theorem are discussed. In particular, the existence of those point sets is envisaged, all three-element subsets of which form triangles of a prescribed type.

Keywords: Ramsey's theorem, Obtuse-angled triangle, Isosceles triangle, Luzin set, Suslin line.

AMS Subject Classification: 52A15, 52B05, 52B10, 52B35, 52B45.

There are interesting geometric applications of Ramsey's famous combinatorial theorem [17]. Among them the best known and most impressive application was found many years ago by Erdös and Szekeres. Namely, it was demonstrated in their joint paper [8] that in the Euclidean space $\mathbf{R}^d$ $(d \geq 2)$ any finite set, whose all points are in general position and whose cardinality is sufficiently large, contains a prescribed number of points in convex position. An extensive survey about the above-mentioned Erdös-Szekeres result and its extensions is given in [14].

In this article we would like to present some other applications of Ramsey's theorem to questions of geometric flavor. Actually, the questions considered below are concerned with certain combinatorial properties of point sets, lying either in a finite-dimensional Euclidean space or in an infinite-dimensional Hilbert space.

First, let us recall the fairly standard notation which will be utilized throughout the article.

As a rule, the symbol $\mathbf{N}$ denotes the set of all natural numbers. The cardinality of $\mathbf{N}$ is denoted by ω (which is usually identified with $\mathbf{N}$).

$\mathbf{Z}$ is the set of all integers.

$\mathbf{Q}$ is the set of all rational numbers.

$\mathbf{R}$ is the real line and, for any natural number $d \geq 1$, the symbol $\mathbf{R}^d$ denotes the d-dimensional Euclidean space (consequently, $\mathbf{R} = \mathbf{R}^1$).

$\mathbf{c}$ is the cardinality of the continuum, i.e., $\mathbf{c} = \mathrm{card}(\mathbf{R}) = 2^\omega$.

ω_1 is the least uncountable ordinal (cardinal) number.

If X is an arbitrary set and k is a natural number, then the symbol $[X]^k$ denotes the family of all k-element subsets of X.

Let us formulate the two standard versions of Ramsey's theorem: finite and countably infinite.

Theorem 1: *Let n, m and k be three natural numbers such that $k \leq n$. There exists a natural number $r = r(n, m, k)$ having the following property:*
for any set X with $\mathrm{card}(X) \geq r$ and for any partition $\{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_m\}$ of $[X]^k$, there is a subset Y of X with $\mathrm{card}(Y) \geq n$ such that $[Y]^k$ is entirely contained in some member $\mathcal{A}_i$ of this partition.

Theorem 2: *Let m and k be two natural numbers, let X be an infinite set, and let $\{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_m\}$ be a partition of $[X]^k$. Then there is an infinite subset Y of X such that $[Y]^k$ is entirely contained in some member $\mathcal{A}_i$ of this partition.*

For the proofs of Theorems 1 and 2, see e.g. [10], [15], [17] or any other text-book of combinatorics. Notice that, by using some weak form of the Axiom of Choice, the finite version of Ramsey's statement (i.e., Theorem 1) can be deduced from the infinite version (i.e., from Theorem 2).

Now, let us consider several applications of the above-mentioned Theorems 1 and 2 to questions of somewhat geometric nature. We begin with a simple example concerning mutual positions of straight lines in the ordinary three-dimensional Euclidean space $\mathbf{R}^3$.

Example 1. Let n be a natural number. There exists a natural number r having the following property:

If $\mathcal{L}$ is a family of straight lines in $\mathbf{R}^3$ such that $\mathrm{card}(\mathcal{L}) \geq r$ and no two distinct lines from $\mathcal{L}$ are parallel, then there is a subfamily $\mathcal{L}'$ of $\mathcal{L}$ such that $\mathrm{card}(\mathcal{L}') \geq n$ and the disjunction of these two relations holds true:

(a) all lines from $\mathcal{L}'$ lie in one plane or all of them pass through one point;

(b) no two distinct lines from $\mathcal{L}'$ lie in a plane (or, in other words, any two distinct lines from $\mathcal{L}'$ are skew).

Indeed, take $m = 2$ and $k = 2$. Let $r = r(n, m, k)$ be as in the formulation of Theorem 1. Consider any family $\mathcal{L}$ of straight lines in $\mathbf{R}^3$ such that $\mathrm{card}(\mathcal{L}) \geq r$ and no two distinct lines from $\mathcal{L}$ are parallel. Define a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $[\mathcal{L}]^2$ as follows: if two distinct lines $l_1 \in \mathcal{L}$ and $l_2 \in \mathcal{L}$ have a common point, then put $\{l_1, l_2\} \in \mathcal{A}_1$, otherwise put $\{l_1, l_2\} \in \mathcal{A}_2$. According to Theorem 1, $\mathcal{L}$ contains some subfamily $\mathcal{L}'$ such that $\mathrm{card}(\mathcal{L}') \geq n$ and $[\mathcal{L}']^2$ is entirely contained either in $\mathcal{A}_1$ or in $\mathcal{A}_2$. It can easily be seen that this $\mathcal{L}'$ satisfies the disjunction of the relations (a) and (b).

Notice that the number r of this example can be roughly estimated from above. Namely, r can be taken to be equal to $\frac{(2(n-1))!}{((n-1)!)^2}$.

In a similar manner, utilizing Theorem 2, we obtain that if $\mathcal{M}$ is an infinite family of straight lines in the space $\mathbf{R}^3$ such that no two distinct lines from $\mathcal{M}$ are parallel, then there exists an infinite subfamily $\mathcal{M}'$ of $\mathcal{M}$ satisfying the disjunction of these two relations:

(c) all lines from $\mathcal{M}'$ lie in one plane or all of them pass through one point;

(d) any two distinct lines from $\mathcal{M}'$ are skew.

The next example is very similar to the Erdös-Szekeres result [8] mentioned at the beginning of this paper.

Example 2. Recall that a point set X in the plane $\mathbf{R}^2$ is in general position if no three distinct points of X are collinear (i.e., no three distinct points of X belong to a straight line). It is easy to see that among any five points in $\mathbf{R}^2$, which are in general position, there always exist three points which form an obtuse-angled triangle. Let us fix natural numbers $n \geq 5$, $m = 2$ and $k = 3$, and take the number $r = r(n, m, k)$ as in Theorem 1. Consider any set $X \subset \mathbf{R}^2$ of points in general position such that $\mathrm{card}(X) \geq r$. We may assert that there is a set $Y \subset X$ satisfying the following relations:

(a) $\mathrm{card}(Y) \geq n$;

(b) every three-element subset of Y forms an obtuse-angled triangle.

To see this, define the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $[X]^3$ as follows: if $V \in [X]^3$ is the set of vertices of an acute-angled or right-angled triangle, then put $V \in \mathcal{A}_1$, otherwise put $V \in \mathcal{A}_2$.

According to Theorem 1, there exists a set $Y \subset X$ with $\mathrm{card}(Y) \geq n$, all three-element subsets of which lie in exactly one member of $\{\mathcal{A}_1, \mathcal{A}_2\}$. Since $n \geq 5$, that member cannot be $\mathcal{A}_1$. So we get $[Y]^3 \subset \mathcal{A}_2$, which obviously means that all three-element subsets of Y form obtuse-angled triangles.

Now, let $X \subset \mathbf{R}^2$ be an infinite set of points in general position. Then there exists an infinite set $Y \subset X$ such that every $V \in [Y]^3$ forms an obtuse-angled triangle.

Indeed, define the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $[X]^3$ in the same manner as above: if $V \in [X]^3$ is the set of vertices of an acute-angled or right-angled triangle, then put $V \in \mathcal{A}_1$, otherwise put $V \in \mathcal{A}_2$. According to Theorem 2, there exists an infinite set $Y \subset X$ such that $[Y]^3 \subset \mathcal{A}_1$ or $[Y]^3 \subset \mathcal{A}_2$. But, as has already been shown, the relation $[Y]^3 \subset \mathcal{A}_1$ is impossible, so $[Y]^3 \subset \mathcal{A}_2$ and we obtain the desired result.

Remark 1: Example 2 admits a natural generalizations to the case of the d-dimensional Euclidean space $\mathbf{R}^d$, where $d \geq 2$. Let Z be a subset of $\mathbf{R}^d$ such that any three points from Z form either acute-angled or right-angled triangle. Then $\mathrm{card}(Z) \leq 2^d$ (see [1], [2], [4]). By starting with this fact and applying Theorem 1, it can be demonstrated that, for any natural number n, there exists a natural number r having the following property:

if $X \subset \mathbf{R}^d$ and $\mathrm{card}(X) \geq r$, then a set $Y \subset X$ can be found such that $\mathrm{card}(Y) \geq n$ and all three-element subsets of Y form obtuse-angled triangles.

Here, for the sake of convenience, the angle whose measure is equal to π is assumed to be obtuse.

In a similar way, applying Theorem 2, we readily get that if $X \subset \mathbf{R}^d$ is an infinite set, then there exists an infinite set $Y \subset X$ such that all three-element subsets of Y form obtuse-angled triangles.

In this connection, it should also be noticed that the uncountable version of Example 2 fails to be true (cf. Example 11 below).

Remark 2: In the Euclidean space $\mathbf{R}^d$, where $d \geq 2$, consider the curve given by the formula

$$t \to (t, t^2, ..., t^d) \qquad (t \in [0, 1]).$$

It is not hard to check that the range of this curve is a set in $\mathbf{R}^d$, all whose points are in general position (i.e., no $d+1$ of them lie in an affine hyperplane of $\mathbf{R}^d$) and every three-element subset of which forms an obtuse-angled triangle.

The above-mentioned curve plays an important role in the theory of convex polyhedra, because for $d \geq 4$ it provides various examples of so-called Carathéodory-Gale polyhedra (see, for instance, [9]).

Example 3. Let H be a Hilbert (more generally, pre-Hilbert) space over $\mathbf{R}$ and let a set $X \subset H$ be such that any three-point subset of X forms either right-angled or obtuse-angled triangle. Then one can assert that X is separable and, consequently, $\mathrm{card}(X) \leq \mathbf{c}$.

Indeed, suppose otherwise, i.e., the given set X is not separable. Then there exists a real $\varepsilon > 0$ such that X contains an uncountable ε-discrete subset Y. This means that $||y - y'|| \geq \varepsilon$ for any two distinct points y and y' from Y. Since Y is uncountable, we may assume without loss of generality that Y is also bounded in H. Furthermore, we may suppose that $\varepsilon = 1$ and the diameter of Y is strictly less than $2^{m/2}$, where $m > 0$ is some sufficiently great natural number.

Now, define the partition $\{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_m\}$ of $[Y]^2$ as follows: $\{y, y'\} \in \mathcal{A}_i$ if and only if $||y - y'|| \in [2^{(i-1)/2}, 2^{i/2}[$.

According to Theorem 2, there are an index $i \in \{1, 2, ..., m\}$ and an infinite set $Y' \subset Y$ such that $[Y']^2 \subset \mathcal{A}_i$. Now, it follows from the definition of $\mathcal{A}_i$ that any three-point subset of Y' is an acute-angled triangle, which is impossible. The obtained contradiction yields the desired result.

Remark 3: Let E be a metric space, $\varepsilon > 0$, and let $\{x, y, z\} \subset E$ be a triangle in E whose side lengths are a_1, a_2, a_3. We shall say that $\{x, y, z\}$ is an equilateral triangle with exactness to ε if the inequalities

$$1 - \varepsilon < a_i/a_j < 1 + \varepsilon$$

are valid for all indices $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3\}$.

Generalizing Example 3, one can show that if E is a nonseparable metric space, then for every $\varepsilon > 0$, there exists an infinite set $Z \subset E$ such that all three-element subsets of Z form equilateral triangles with exactness to ε.

Remark 4: Let H be an infinite-dimensional separable Hilbert space (over $\mathbf{R}$). The following assertions are true:

(a) there exists a set $X \subset H$ such that the cardinality of X is equal to $\mathbf{c}$ and any three distinct points of X form an acute-angled triangle;

(b) there exists a set $W \subset H$ homeomorphic to the line segment $[0, 1]$ such that any three distinct points of W form a right-angled triangle (this is the so-called Wiener curve);

(c) there exists a set $Y \subset H$ such that Y is homeomorphic to the line segment $[0, 1]$, all points of Y are in general and convex position, and any three distinct points of Y form an obtuse-angled triangle.

Now, we would like to recall one elementary geometric problem concerning equilateral triangles. Let n be a natural number and let X be a subset of $\mathbf{R}^2$ with $\mathrm{card}(X) = n^2$. Then there exists a set $Y \subset X$ with $\mathrm{card}(Y) \geq n$ such that no three distinct points of Y form an equilateral triangle. To demonstrate this, consider a maximal (with respect to inclusion) subset Y of X no three points of which form an equilateral triangle, and denote $k = \mathrm{card}(Y)$. According to the definition of Y, for any point $x \in X \setminus Y$, there are two points y and z in Y such that the triangle $[x, y, z]$ is equilateral. Since the number of all possible line segments $[y, z]$ does not exceed $k(k-1)/2$ and there are at most two equilateral triangles for which $[y, z]$ is a side, we easily infer the inequality

$$n^2 - k \leq 2(k(k-1)/2) = k^2 - k,$$

whence the inequality $n \leq k$ immediately follows, showing that Y is as required.

However, this very simple argument does not work in the case where X is a subset of $\mathbf{R}^3$. So the argument must be changed by another reasoning based on Theorem 1.

Example 4. Let $\triangle$ be a fixed triangle in the space $\mathbf{R}^3$ and let $n \geq 5$ be a natural number. There exists a natural number r satisfying the following condition:

for any set $X \subset \mathbf{R}^3$ with $\mathrm{card}(X) \geq r$, there is a set $Y \subset X$ with $\mathrm{card}(Y) \geq n$ no three-element subset of which forms a triangle similar to $\triangle$.

To demonstrate this fact, first observe that there is no five-point set in $\mathbf{R}^3$, all three-element subsets of which form triangles similar to $\triangle$. Keeping in mind this circumstance, take as earlier $m = 2$, $k = 3$, $r = r(n, m, k)$ and produce the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $[X]^3$ as follows: $V \in \mathcal{A}_1$ if V is the set of vertices of a triangle similar to $\triangle$, otherwise $V \in \mathcal{A}_2$. Applying Theorem 1 to this partition, we readily get the required result.

An analogous application of Theorem 2 yields that if a set $X \subset \mathbf{R}^3$ is infinite, then there exists an infinite set $Y \subset X$, no three-element subset of which forms a triangle similar to $\triangle$.

Remark 5 : The previous example also can be extended to the case of $\mathbf{R}^d$, where $d \geq 2$. Let some triangle $\triangle$ be given in $\mathbf{R}^d$ and let n be a natural number. There exists a natural number r satisfying the following condition:

for any set $X \subset \mathbf{R}^d$ with $\mathrm{card}(X) \geq r$, there is a set $Y \subset X$ with $\mathrm{card}(Y) \geq n$ such that no three-element subset of Y forms a triangle similar to $\triangle$.

To demonstrate this statement, it suffices to keep in mind the following two simple geometric facts:

(a) if $\triangle$ is not an equilateral triangle and a set $Z \subset \mathbf{R}^d$ is such that all three-element subsets of Z form triangles similar to $\triangle$, then $\mathrm{card}(Z) \leq 4$;

(b) if $\triangle$ is an equilateral triangle and a set $Z \subset \mathbf{R}^d$ is such that all three-element subsets of Z form triangles similar to $\triangle$, then $\mathrm{card}(Z) \leq d + 1$.

Now, applying Theorem 1, we obtain the desired result.

Analogously, applying Theorem 2, one can show that if a set $X \subset \mathbf{R}^d$ is infinite, then there exists an infinite set $Y \subset X$ such that no three-element subset of Y forms a triangle similar to $\triangle$.

Example 5. It is easy to see that the vertices of a regular pentagon in $\mathbf{R}^2$ with its center constitute a six-point set, all three-element subsets of which form isosceles triangles. Also, it is not difficult to show that there is no seven-point set in $\mathbf{R}^2$, all three-element subsets of which form isosceles triangles. Now, let $n \geq 7$ be a natural number. We may assert that there exists a natural number r satisfying the following condition:

for any set $X \subset \mathbf{R}^2$ with $\mathrm{card}(X) \geq r$, there is a set $Y \subset X$ with $\mathrm{card}(Y) \geq n$ no three-element subset of which forms an isosceles triangle.

To demonstrate this, let us take again $m = 2$, $k = 3$, $r = r(n, m, k)$ and produce the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $[X]^3$ as follows: $V \in \mathcal{A}_1$ if V is the set of vertices of an isosceles triangle, otherwise $V \in \mathcal{A}_2$. Applying Theorem 1 to this partition, we come to the required result.

In a similar way, using Theorem 2, we infer that if a set $X \subset \mathbf{R}^2$ is infinite, then there exists an infinite set $Y \subset X$ such that no three-element subset of Y forms an isosceles triangle.

Remark 6 : In the space $\mathbf{R}^3$ there exists a set X with $\mathrm{card}(X) = 8$ all three-element subsets of which form isosceles triangles, and 8 is the maximal cardinality of a set in $\mathbf{R}^3$ with the above-mentioned property (see, e.g., [3], [12]). In $\mathbf{R}^4$ it is not hard to indicate a point set of cardinality 11, having the same property. We do not know whether 11 is the maximum value for this property in $\mathbf{R}^4$.

Remark 7 : By using induction on d, it is possible to prove that there exists a natural number $i(d)$ satisfying the following condition:

if a set $X \subset \mathbf{R}^d$ is such that all three-element subsets of X form isosceles triangles, then $\mathrm{card}(X) \leq i(d)$.

Indeed, denote by $h(d)$ the maximum number of points in the unit sphere $\mathbf{S}_{d-1} \subset \mathbf{R}^d$, all nonzero distances between which are greater than or equal to 1. Then we have the inequality

$$i(d+1) \leq i(d) + 2h(d+1) + 2 \quad (d \geq 1).$$

By taking into account this inequality and Theorem 1, Example 5 can be trivially generalized to the case of $\mathbf{R}^d$.

Now, for any set X lying in $\mathbf{R}^d$ (or in a pre-Hilbert space H), let us introduce the notation:

$$D(X) = \{||x - x'|| : x \in X, x' \in X, x \neq x'\}.$$

Utilizing this notation, we may consider the next example.

Example 6. Let $n \geq 6$ be a natural number. There exists a natural number r satisfying the following condition:

if $X \subset \mathbf{R}^2$ and $\mathrm{card}(X) \geq r$, then $\mathrm{card}(D(X)) \geq n$.

To see this, take $n' = n + 1$, $m = 2$, $k = 3$ and $r = r(n', m, k)$. By virtue of Example 5, if $X \subset \mathbf{R}^2$ and $\mathrm{card}(X) \geq r$, then there exists a set $Y \subset X$ with $\mathrm{card}(Y) \geq n'$ such that no three-element subset of Y forms an isosceles triangle. Fix a point $y_0 \in Y$ and consider the real numbers

$$||y - y_0|| \quad (y \in Y \setminus \{y_0\}).$$

All these numbers are distinct, so $\mathrm{card}(D(Y)) \geq n' - 1 = n$, which yields the desired result.

Example 7. Let X be an infinite subset of the plane $\mathbf{R}^2$. It follows from Example 6 that the set $D(X)$ is infinite. Moreover, one may assert that if X is uncountable, then $D(X)$ is also uncountable. Indeed, suppose to the contrary that $\mathrm{card}(X) > \omega$ but $\mathrm{card}(D(X)) = \omega$. Fix a point $x_0 \in X$. Then all other points of X belong to the union of countably many circles. Consequently, one of those circles, say T, contains uncountably many points of X. But it is easy to see that the set $D(T \cap X)$ is uncountable, which contradicts our assumption.

Remark 8 : To obtain the corresponding analogues of Examples 6 and 7 for the space $\mathbf{R}^d$, let us introduce one auxiliary notion.

We shall say that a four-element set $\{x, y, z, t\} \subset \mathbf{R}^d$ is admissible if at least two numbers from $||x - y||$, $||x - z||$, $||x - t||$, $||y - z||$, $||y - t||$, $||z - t||$ are equal to each other.

It turns out that, for every natural number $d \geq 1$, there exists a natural number $p(d)$ having the following property: if $Z \subset \mathbf{R}^d$ is such that all four-element subsets of Z are admissible, then $\mathrm{card}(Z) \leq p(d)$.

The fact just formulated can be proved by induction on d. By virtue of this fact and Theorem 1, we readily deduce the next statement:

Let $d \geq 1$ and n be two natural numbers. There is a natural number r possessing the following property: if $X \subset \mathbf{R}^d$ is such that $\mathrm{card}(X) \geq r$, then there exists a set $Y \subset X$ with $\mathrm{card}(Y) \geq n$, all nonzero distances between points of which differ from each other.

Indeed, assume without loss of generality that $n > p(d)$ and put $m = 2$, $k = 4$, $r = r(n, m, k)$. Let $X \subset \mathbf{R}^d$ and $\mathrm{card}(X) \geq r$. Consider the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $[X]^4$ defined as follows: a set $V \in [X]^4$ belongs to $\mathcal{A}_1$ if and only if V is admissible. Applying to this partition Theorem 1 and taking into account the above-mentioned fact, we come to the required result.

Example 8. Let X be an infinite (respectively, uncountable) set in the space $\mathbf{R}^d$. Then there exists an infinite (respectively, uncountable) set $Y \subset X$ such that all nonzero distances between the points of Y differ from each other.

The proof can be obtained by induction on d.

Remark 9 : Consider a separable infinite-dimensional Hilbert space H and identify it with the standard space

$$\mathbf{l}_2 = \{t \in \mathbf{R}^{\mathbf{N}} : \sum \{(t(n))^2 : n \in \mathbf{N}\} < +\infty\}.$$

Let $\{N_j : j \in J\}$ be an almost disjoint family of infinite subsets of $\mathbf{N}$ such that $\mathrm{card}(J) = \mathbf{c}$. The almost disjointness of this family means that

$$\mathrm{card}(N_i \cap N_j) < \omega$$

whenever $i \in J$, $j \in J$ and $i \neq j$. For any index $j \in J$, let x_j be an element of $\mathbf{l}_2$ satisfying the following conditions:

(a) the norm of x_j is equal to 1;

(b) $x_j(n) \in \mathbf{Q}$ for all $n \in \mathbf{N}$;

(c) if $n \notin N_j$, then $x_j(n) = 0$.

Putting $X = \{x_j : j \in J\}$, it is easy to verify that $\mathrm{card}(X) = \mathbf{c}$ and $\mathrm{card}(D(X)) = \omega$. Thus, the analogue of Example 8 does not hold in H.

Example 9. Let $\mathcal{D}$ be an infinite (respectively, uncountable) family of line-segments on $\mathbf{R}$ (some of these segments may be degenerate, i.e. may be singletons). One can assert that there exists an infinite (respectively, uncountable) family $\mathcal{D}' \subset \mathcal{D}$ satisfying the disjunction of the following two assertions:

(a) all segments from $\mathcal{D}'$ are pairwise disjoint;

(b) all segments from $\mathcal{D}'$ have a common point.

To see this, first consider the case when the family $\mathcal{D}$ is countably infinite. In this case, define the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $[\mathcal{D}]^2$ as follows: $\{x, y\} \in \mathcal{A}_1$ if and only if $x \cap y = \emptyset$. According to Theorem 2, there exists an infinite $\mathcal{D}' \subset \mathcal{D}$ such that either $[\mathcal{D}']^2 \subset \mathcal{A}_1$ or $[\mathcal{D}']^2 \subset \mathcal{A}_2$. If $[\mathcal{D}']^2 \subset \mathcal{A}_1$, then $\mathcal{D}'$ satisfies (a). If $[\mathcal{D}']^2 \subset \mathcal{A}_2$, then any two segments from $\mathcal{D}'$ have nonempty intersection and it easily follows that any finite subfamily of $\mathcal{D}'$ has nonempty intersection, too. Taking into account the compactness of line segments, we get (b).

Now, consider the case when the family $\mathcal{D}$ is uncountable. We may assume, without loss of generality, that all segments from $\mathcal{D}$ are non-degenerate and, moreover, we may assume that the length of each segment from $\mathcal{D}$ is greater than δ, where δ is a fixed strictly positive real number. Let us put $T = \{n\delta : n \in \mathbf{Z}\}$. Obviously, the set T is countable and any member of $\mathcal{D}$ has a common point with T. So there exists a point $t \in T$ belonging to uncountably many segments from $\mathcal{D}$. Denoting by $\mathcal{D}'$ the family of all those segments from $\mathcal{D}$ which contain t, we conclude that $\mathcal{D}'$ satisfies (b).

Example 10. Recall that a linearly ordered set $(E, \leq)$ is Dedekind complete if, for any nonempty and bounded from above subset X of E, there exists $\sup(X)$. In such $(E, \leq)$ consider any infinite family $\mathcal{D}$ of segments. Then, similarly to the previous example, one can deduce from Theorem 2 that there exists an infinite family $\mathcal{D}' \subset \mathcal{D}$ satisfying the disjunction of the following two assertions:

(a) all segments from $\mathcal{D}'$ are pairwise disjoint;

(b) all segments from $\mathcal{D}'$ have a common point.

Unfortunately, the uncountable variant of the previous example is not provable for $(E, \leq)$ in contemporary set theory. Indeed, it is consistent with the standard axioms of set theory that there exists a linearly ordered set $(S, \leq)$ such that:

(i) $(S, \leq)$ is Dedekind complete, dense, nonseparable, and has neither least nor greatest elements;

(ii) S satisfies the countable chain condition, i.e., any disjoint family of nonempty open subintervals of S is at most countable.

Actually, S is the so-called Suslin line (see, e.g., [11]). Recall that the existence of S is valid in the Gödel Universe $\mathbf{L}$, where the Continuum Hypothesis $\mathbf{c} = \omega_1$ holds true, too. Notice that $\mathrm{card}(S) = \omega_1$, so S can be represented as an ω_1-sequence of points $\{s_\xi : \xi < \omega_1\}$. Now, by using the method of transfinite recursion up to ω_1, one can readily construct a family $\mathcal{D} = \{d_\xi : \xi < \omega_1\}$ of non-degenerate segments in S such that, for any ordinal $\xi < \omega_1$, the segment d_ξ does not intersect the closure of $\{s_\zeta : \zeta < \xi\}$ (it suffices to use the fact that S itself is nonseparable while the closure of $\{s_\zeta : \zeta < \xi\}$ is separable). It directly follows from the construction of $\mathcal{D}$ that every point of S belongs to at most countably many segments from $\mathcal{D}$, so $\mathcal{D}$ does not contain an uncountable subfamily $\mathcal{D}'$ satisfying the disjunction of the assertions (a) and (b).

Example 11. In 1914, by assuming the Continuum Hypothesis, Luzin constructed an uncountable set in $\mathbf{R}$ whose intersection with every nowhere dense subset of $\mathbf{R}$ is at most countable. Luzin's construction is considered in detail in the widely known text-book by Oxtoby [16] and some applications of Luzin's sets to certain questions of measure theory and general topology are also presented therein. By utilizing an argument similar to Luzin's one, it is possible to show that, under the same Continuum Hypothesis, there exists an uncountable set $L \subset \mathbf{R}^2$ of points in general position, such that every nowhere dense subset of $\mathbf{R}^2$ has at most countably many common points with L. These properties of L allow to infer that if L' is any uncountable subset of L, then:

(a) there are three points in L' which form an acute-angled triangle;

(b) there are three points in L' which form an obtuse-angled triangle.

We thus conclude that, within the standard **ZFC** set theory, there is no uncountable analogue of Example 2.

Example12. Let $(E, \leq)$ be a Dedekind complete linearly ordered set and let $\mathcal{L}$

be an uncountable family of segments in E. Then the disjunction of these two statements holds true:

(a) there exists an uncountable subfamily of $\mathcal{D}$, all segments from which are pairwise disjoint;

(b) there exists an infinite subfamily of $\mathcal{D}$, all segments from which have a common point.

Also, the disjunction of these two statements holds true:

(a') there exists an uncountable subfamily of $\mathcal{D}$, all segments from which have a common point;

(b') there exists an infinite subfamily of $\mathcal{D}$, all segments from which are pairwise disjoint.

To establish the validity of both indicated disjunctions, one needs to utilize a Ramsey type theorem due to Dushnik and Miller [5].

Example 13. According to the result of Erdos and Szekeres [8], every infinite set of points in general position in $\mathbf{R}^d$, where $d \geq 2$, contains an infinite convexly independent subset. It is natural to ask whether every uncountable set of points in general position in $\mathbf{R}^d$ contains an uncountable convexly independent subset. It turns out that the negative answer to this question does not contradict the axioms of contemporary **ZFC** set theory. More precisely, by assuming the Continuum Hypothesis, it can be demonstrated that there exists an uncountable set $X \subset \mathbf{R}^d$ of points in general position, such that no uncountable subset of X is convexly independent. For $d = 2$, a detailed proof of this fact is given in [13].

References

[1] M. Aigner, G.M. Ziegler, *Proofs from The Book*, Springer-Verlag, New York-Heidelberg, 2004
[2] V. G. Boltyanskii, I.Ts. Gokhberg, *Theorems and Problems from Combinatorial Geometry*, Izd. Nauka, Moscow, 1965 (Russian)
[3] H. T. Croft, 9*-point and* 7*-point configurations in* 3*-space*, Proc. London Math. Soc., (3), **12** (1962), 400-424
[4] L. Danzer, B. Grünbaum, *Über zwei Probleme bezüglich konvexer Körper von P. Erdös und von V.L. Klee*, Math. Zeitschrift, **79** (1962), 95-99
[5] B. Dushnik, E. W. Miller, *Partially ordered sets*, Amer. Journ. of Math., **63** (1941), 600-610
[6] P. Erdös, *Problems and results in combinatorial geometry. Discrete geometry and convexity*, Ann. New York Acad. Sci., **440** (1985), 1-11.
[7] P. Erdös, Z. Füredi, *The greatest angle among n points in the d-dimensional Euclidean space*, Annals of Discrete Mathematics, **17** (1983), 275-283
[8] P. Erdös, G. Szekeres, *A combinatorial problem in geometry*, Compos. Math., **2** (1935), 463-470
[9] B. Grünbaum, *Convex Polytopes*, Wiley Interscience, New York, 1967
[10] H. Hadwiger, H. Debrunner, V. Klee, *Combinatorial geometry in the plane*, New York, 1964
[11] T. Jech, *Set Theory*, Academic Press, New York-London, 1978
[12] A. B. Kharazishvili, *Selected Topics in Geometry of Euclidean spaces*, Izd. Tbil. Gos. Univ., Tbilisi, 1978 (Russian)
[13] A.B. Kharazishvili, *A note on convexly independent subsets of an infinite set of points*, Georgian Mathematical Journal, **9**, no. 2 (2002), 303 - 307
[14] W. Morris and V. Soltan, *The Erdös–Szekeres problem on points in convex position – a survey*, Bull. Amer. Math. Soc., **37**, n. 4 (2000), 437–458
[15] O. Ore, *Theory of Graphs*, Amer. Math. Soc., Rhode Island, 1962
[16] J. C. Oxtoby, *Measure and Category*, Springer-Verlag, New York, 1971
[17] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc., **30** (1930), 264–286