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SOME UNSOLVED PROBLEMS IN MEASURE THEORY

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Abstract. One of the central problems in measure theory is concerned with proper extensions of measures. This problem has various aspects: purely set theoretical, algebraic, topological. The present article is devoted to some open questions which are closely connected with the general measure extension problem and with the existence of sets nonmeasurable with respect to nonzero σ-finite invariant (quasi-invariant) measures.

რეზიუმე. ზომის თეორიის ერთ-ერთი ცენტრალური პრობლემა ეხება ზომების საკუთრივი გაგრძელებების არსებობას. ამ პრობლემას აქვს სხვადასხვა ასპექტი: წმინდა სიმრავლურ-თეორიული, ალგებრული, ტოპოლოგიური. სტატია ეძღვნება ზოგიერთ ღია საკითხს, რომლებიც უშუალოდ არიან დაკავშირებული ზომის გაგრძელების ზოგად ამოცანასთან და არანულოვანი სიგმა-სასრული ინვარიანტული (კვაზი-ინვარიანტული) ზომების მიმართ არაზომადი სიმრავლეების არსებობასთან.

There are many problems in real analysis and measure theory, which are not solved so far and which are attractive for researchers working in the above-mentioned classical disciplines of mathematics.

Of course, the significance of those problems is quite different: some of them are interesting for a limited circle of specialists, some are topical or, at least, deserve to be discussed and investigated, while others are of great importance and stimulate the further development of corresponding branches of mathematical analysis.

Here we would like to present a short list of problems and questions in measure theory, which still remain unsolved, although most of them were posed many years ago and may be regarded as old ones (cf. [24], [26], [32]). In our opinion, these problems and questions are interesting from the measure-theoretical point of view. Also, they have nontrivial connections with certain topics of abstract set theory, group theory, and general topology.

For the reader's convenience, in our further presentation we sometimes recall the corresponding notions before formulating problems. Also, we

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systematically provide the text with comments and examples which are related to the problems and questions discussed in this article.

ZFC abbreviates the standard Zermelo-Fraenkel set theory (see, e.g., [21]).

The symbol ω denotes the least infinite ordinal (cardinal) number.

The symbol ω_1 denotes the least uncountable ordinal (cardinal) number.

$\mathbf{R}$ stands, as usual, for the set of all real numbers (i.e., $\mathbf{R}$ is the real line).

$\mathbf{c}$ denotes the cardinality of the continuum (i.e., $\mathbf{c} = \mathrm{card}(\mathbf{R}) = 2^\omega$).

$\mathbf{Q}$ stands for the set of all rational numbers.

Let E be a ground set and let G be a group of transformations of E. So we may consider the pair (E, G) which is usually called a space equipped (endowed) with a transformation group.

Suppose that a nonzero σ-finite measure μ is defined on a σ-algebra of subsets of E. As a rule, this σ-algebra will be denoted by $\mathrm{dom}(\mu)$ (domain of μ). The σ-ideal generated by the family of all μ-measure zero sets will be denoted by $\mathcal{I}(\mu)$.

A measure μ on E is called continuous (or diffused) if $\{x\} \in \mathrm{dom}(\mu)$ and $\mu(\{x\}) = 0$ for each $x \in E$.

A cardinal number α is called measurable in the Ulam sense (or, sometimes, real-valued measurable) if there exists a nonzero σ-finite diffused measure μ whose domain coincides with the family of all subsets of α (i.e., $\mathrm{dom}(\mu) = \mathcal{P}(\alpha)$).

It is known that the existence of real-valued cardinal numbers cannot be established within contemporary **ZFC** set theory (see [21], [61], [68]).

Having a nonzero σ-finite measure μ on E, we can associate with μ the Hilbert space $L_2(\mu)$ consisting of all real-valued square integrable functions on E (of course, identifying μ-equivalent functions). The topological weight of $L_2(\mu)$ is a certain cardinal characteristic (invariant) of μ. In the sequel, we refer to this cardinal invariant as to the weight of μ. In the literature the terms "character of μ" or "separability character of μ" are usually exploited instead of the weight of μ (see, for instance, [15], [18], [22], [47]).

If the space $L_2(\mu)$ is separable, then μ is called a separable measure.

Accordingly, if $L_2(\mu)$ is nonseparable, then μ is called a nonseparable measure.

Notice that if, for a given σ-finite measure μ, the σ-algebra $\mathrm{dom}(\mu)$ is countably generated, then μ turns out to be separable. The converse assertion is not true, in general.

Let E be a ground set, G be a group of transformations of E, and let μ be a σ-finite measure defined on some σ-algebra of subsets of E.

μ is called G-quasi-invariant if both $\mathrm{dom}(\mu)$ and $\mathcal{I}(\mu)$ are G-invariant classes of subsets of E (see [3], [10], [17], [33], [74]).

A G-quasi-invariant measure μ is called G-invariant if $\mu(g(X)) = \mu(X)$ for all transformations $g \in G$ and all sets $X \in \mathrm{dom}(\mu)$ (see [3], [10], [17], [33], [74]).

Example 1. Let n be a natural number and let $\mathbf{R}^n$ denote the Euclidean n-dimensional space. This space is endowed with the following two classical groups of transformations:

T_n = the group of all translations of $\mathbf{R}^n$;

Is_n = the group of all isometric transformations of $\mathbf{R}^n$.

The standard Lebesgue measure λ_n on $\mathbf{R}^n$ is Is_n-invariant (consequently, also T_n-invariant). Notice that if $n \geq 1$, then there exist many measures on $\mathbf{R}^n$ which extend λ_n, are T_n-invariant but are not Is_n-invariant (see, e.g., [24], [26], [56]).

If $n = 1$, then, for the sake of brevity, we shall write λ instead of λ_1.

Example 2. Let $(G, \cdot)$ be a σ-compact locally compact topological group (identified with the group of all its left translations). As is well known, there exists a nonzero σ-finite G-invariant Borel measure μ on G, the so-called (left) Haar measure (see [17], [18], [50], [58]). This measure is unique in the following sense: every σ-finite G-invariant Borel measure ν on G is proportional to μ, i.e. the equality $\nu = t \cdot \mu$ holds for some real number $t \geq 0$ which depends on ν, i.e, $t = t(\nu)$.

Example 3. Let (E, G) be a space endowed with a transformation group and let μ be a σ-finite G-invariant measure on E. For every μ-measurable function $\phi : E \to]0, +\infty[$, we may put

$$\nu(X) = \int_X \phi(x) d\mu(x) \qquad (X \in \mathrm{dom}(\mu)).$$

So we get the σ-finite measure ν which is defined on the σ-algebra $\mathrm{dom}(\mu)$, and it can easily be seen that ν is a σ-finite G-quasi-invariant measure (actually, μ and ν are equivalent measures in the sense that $\mathcal{I}(\mu) = \mathcal{I}(\nu)$).

Since the notion of a quasi-invariant measure and the notion of an invariant measure are closely connected with each other, some problems listed below will be formulated simultaneously for both kinds of measures.

The concept of metrical transitivity of a quasi-invariant measure is classical and well known in the theory of dynamical systems and ergodic theory (see, e.g., [10], [16], [74]).

A G-quasi-invariant measure μ on E is called G-metrically transitive (or G-ergodic) if, for any μ-measurable set X with $\mu(X) > 0$, there exists a countable family $\{g_i : i \in I\}$ of transformations from G such that

$$\mu(E \setminus \cup\{g_i(X) : i \in I\}) = 0.$$

Example 4. The (left) Haar measure μ on a σ-compact locally compact topological group $(G,\cdot)$ is G-metrically transitive (in particular, the Lebesgue measure λ_n, considered as a T_n-invariant measure, is T_n-metrically transitive). More generally, if H is an everywhere dense subgroup of G, then μ treated as a left H-invariant measure is H-metrically transitive as well (cf. [30], [73]).

Having an abstract space (E,G) equipped with a transformation group, we may introduce the following two natural classes of measures:

$M_1(E,G)$ = the class of all those nonzero σ-finite diffused measures on E which are G-invariant;

$M_2(E,G)$ = the class of all those nonzero σ-finite diffused measures on E which are G-quasi-invariant.

Clearly, we have the inclusion $M_1(E,G) \subset M_2(E,G)$. The first problem concerning these two classes of measures is about proper extensions of members of $M_1(E,G)$ (respectively, of $M_2(E,G)$). It looks as follows.

Problem 1. Find necessary and sufficient conditions, in terms of (E,G), under which for every measure μ belonging to the class $M_1(E,G)$ (respectively, to the class $M_2(E,G)$) there exists a measure μ' also belonging to $M_1(E,G)$ (respectively, to $M_2(E,G)$) and strictly extending μ.

More or less trivial examples show that certain conditions on (E,G) are necessary for the existence of a required extension μ'.

Actually, if G is a small group of transformations of E, then Problem 1 becomes purely set-theoretical and its solution depends on additional hypotheses. To illustrate the said above, suppose for a moment that G contains only the identical transformation of E. In this case, if card(E) is nonmeasurable in the Ulam sense, then a required extension μ' does always exist, and if card(E) is measurable in the Ulam sense, then μ' may not exist.

Also, it is worth mentioning in connection with Problem 1 that, having a G-quasi-invariant (respectively, G-invariant) measure μ and considering various G-quasi-invariant (respectively, G-invariant) extensions of μ, one should be care of various "good" properties of μ. In other words, when extending μ, one may try to preserve "nice" properties of μ (e.g., its G-metrical transitivity, its separability, etc). Similar aspects of the general measure extension problem are touched upon in [39], [66], [67].

A special and important case of Problem 1 is when the ground set E is a group and G coincides with the group of all left translations of E (so we may identify E and G by a canonical isomorphism). Even in this particular case the reduced version of Problem 1 remains open. Let us formulate the corresponding question.

Problem 2. Let $(G,\cdot)$ be an uncountable group. Is it true that, for every measure μ belonging to the class $M_1(G,G)$ (respectively, to the class

$M_2(G,G)$), there exists a measure μ' also belonging to $M_1(G,G)$ (respectively, to $M_2(G,G)$) and strictly extending μ?

Let us make some remarks in connection with Problem 2. If μ is an arbitrary measure from the class $M_2(G,G)$, then the σ-algebra $\mathrm{dom}(\mu)$ differs from the family $\mathcal{P}(G)$. Moreover, a much stronger statement can be proved by using the fact that the least uncountable cardinal ω_1 is not measurable in the Ulam sense.

Theorem 1. *Let $(G,\cdot)$ be an uncountable group, μ be a measure from the class $M_2(G,G)$ and let X be a μ-measurable set with $\mu(X) > 0$. Then there exists a set $Y \subset X$ such that $Y \notin \mathrm{dom}(\mu)$.*

Theorem 1 was first established in [23] where this result was applied to the uniqueness property of σ-finite invariant measures (cf. also [13], [33], [65], [74]). Theorem 1 shows, in particular, that G-quasi-invariant extensions of μ a priori may exist. Indeed, as turns out, the answer to the question of Problem 2 is positive if one assumes the non-existence of cardinal numbers measurable in the Ulam sense (see [19], [33], [43], [56], [74]). Nevertheless, Problem 2 still remains open within **ZFC** set theory.

It is reasonable to consider two analogues of Problems 1 and 2 in terms of so-called absolutely negligible sets. First, let us give the precise definition of such sets.

Let (E,G) be an abstract space equipped with a transformation group.

A set $X \subset E$ is G-absolutely negligible if, for any measure μ from the class $M_1(E,G)$ (respectively, from the class $M_2(E,G)$), there exists a measure μ' from $M_1(E,G)$ (respectively, from $M_2(E,G)$) extending μ and such that $X \in \mathrm{dom}(\mu')$ and $\mu'(X) = 0$.

Various properties of absolutely negligible sets are discussed in [24], [26], [33], [43]. A purely algebraic (or, if one prefers, purely geometric) characterization of G-absolutely negligible sets in terms of the pair (E,G) can also be found in those works. Namely, we have the following result.

Theorem 2. *Let E be a space equipped with a transformation group G and let X be a subset of E. These two assertions are equivalent:*

(1) *for any countable family $\{g_i : i \in I\} \subset G$, there exists a countable family $\{h_j : j \in J\} \subset G$ such that*

$$\cap\{h_j(\cup\{g_i(X) : i \in I\}) : j \in J\} = \emptyset.$$

(2) *X is G-absolutely negligible.*

As one can see, condition (1) is purely geometric, because it does not appeal to the concept of measure.

It directly follows from the definition that the family of all G-absolutely negligible sets in E forms a G-invariant ideal of subsets of E which, in general, does not need to be a σ-ideal. Moreover, in certain situations it becomes possible to cover E by countably many G-absolutely negligible sets. So the next problem naturally arises.

Problem 3. Find necessary and sufficient conditions on a space (E, G), under which there exists a countable family $\{X_i : i \in I\}$ of subsets of E having the property that all X_i $(i \in I)$ are G-absolutely negligible and $\cup\{X_i : i \in I\} = E$.

Observe that if for a given space (E, G) the above-mentioned family $\{X_i : i \in I\}$ does exist, then Problem 1 is positively solvable for (E, G). Indeed, in such a case, having any measure μ from $M_2(E, G)$ (respectively, from $M_1(G, G)$), we get that at least one set X_i must be nonmeasurable with respect to μ. So, by using this X_i, we may strictly extend μ to a measure μ' belonging to $M_2(E, G)$ (respectively, belonging to $M_1(E, G)$).

In light of the just stated, the next special form of Problem 3 is of interest, too.

Problem 4. Let $(G, \cdot)$ be an uncountable group (identified with the group of all left translations of G). Does there exist a countable family $\{X_i : i \in I\}$ having the property that all sets X_i $(i \in I)$ are G-absolutely negligible and $\cup\{X_i : i \in I\} = G$?

It makes sense to point out here that in some particular cases Problem 4 admits a positive solution. To show this, let us recall the widely known notion of a solvable group.

Let $(G, \cdot)$ be a group. G is said to be a solvable group if there exists a finite k-sequence

$$G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_{k-1} \supset G_k$$

of subgroups of G such that:

(a) $G_0 = G$ and $G_k = \{e\}$, where e denotes the neutral element of G;

(b) for every natural index $i \in [1, k]$, the group G_i is a normal subgroup of G_{i-1} and the quotient-group G_{i-1}/G_i is commutative.

The following statement is true (see, e.g., [33], [35], [43]).

Theorem 3. *If $(G, \cdot)$ is an uncountable solvable group, then G admits a countable covering, all members of which are G-absolutely negligible sets.*

Moreover, the technique of absolutely negligible sets turned out to be useful even in those cases where a basic transformation group is not, in general, solvable.

For example, it was demonstrated that if E coincides with the Euclidean n-dimensional space $\mathbf{R}^n$, where $n \geq 1$, and G coincides with the group Is_n of all isometric transformations of $\mathbf{R}^n$, then there exists a countable covering of E with G-absolutely negligible sets. So, in this classical case, we have a positive solution of Problem 1 (for more details, see [5], [6], [9], [24], [26], [55], [71], [72], [74]).

Recall that, for $n \geq 3$, the group Is_n is very far from being solvable. This fact implies that, for $n \geq 3$, there are various paradoxical decompositions of bounded subsets of $\mathbf{R}^n$ with nonempty interiors (see the extensive monograph [69] in which many topics related to the famous Banach-Tarski paradox are envisaged).

From the above-mentioned result on proper extensions of Is_n-invariant (respectively, Is_n-quasi-invariant) measures it directly follows that if μ is any Is_n-invariant extension of the standard Lebesgue measure λ_n on $\mathbf{R}^n$, then there always exists an Is_n-invariant measure μ' on $\mathbf{R}^n$ strictly extending μ.

Furthermore, it turns out that there are even nonseparable Is_n-invariant measures μ on $\mathbf{R}^n$ which extend λ_n. Now, we would like to say a few words about such extensions.

As earlier, let T_n stand for the group of all translations of $\mathbf{R}^n$. The two methods of constructing nonseparable T_n-invariant extensions of λ_n can be distinguished. The first of them is due to Kakutani and Oxtoby [22], the second method is due to Kodaira and Kakutani [47].

The method of Kakutani and Oxtoby admits a natural generalization to a certain class of spaces (E, G), which includes some types of uncountable σ-compact locally compact topological groups (see [18]; cf. also [15], [20], [42], [54]). Besides, a nonseparable T_n-invariant measure μ on $\mathbf{R}^n$ which extends λ_n and is obtained by this method is such that its weight is equal to $2^{\mathbf{c}}$, i.e., attains maximum. However, there is a weak side of the method, because μ is not T_n-metrically transitive (T_n-ergodic). Thus, the metrical transitivity of λ_n is lost by μ.

The second method, due to Kodaira and Kakutani, has a certain advantage. Namely, the nonseparable T_n-invariant measure μ on $\mathbf{R}^n$ which extends λ_n and is obtained by their method preserves the property of T_n-metrical transitivity of λ_n (surprisingly, this important fact is not stated in [47]). But the construction of μ only yields that the weight of μ is equal to $\mathbf{c}$, i.e., is not maximal.

In this context, the following question arises.

Problem 5. Let $n \geq 1$ be a natural number. Does there exist an Is_n-invariant measure on $\mathbf{R}^n$ which extends the Lebesgue measure λ_n, which is also Is_n-metrically transitive and whose weight is equal to $2^{\mathbf{c}}$?

In [40] a nonseparable complete measure ν on $\mathbf{R}$ was constructed by using the Continuum Hypothesis. This ν extends the Lebesgue measure λ, the weight of ν is equal to $\mathbf{c}$ and the σ-ideal $\mathcal{I}(\nu)$ of all ν-measure zero sets coincides with the σ-ideal $\mathcal{I}(\lambda)$ of all λ-measure zero sets. In other words, the measure ν being a nonseparable extension of the Lebesgue measure λ does not expand the σ-ideal $\mathcal{I}(\lambda)$ of all Lebesgue measure zero sets.

In this connection, the next problem seems to be of some interest.

Problem 6. Investigate (possibly, with the aid of additional set-theoretic assumptions such as the Continuum Hypothesis or Martin's Axiom) whether there exists a measure θ on $\mathbf{R}$ satisfying the following relations:

(a) θ extends λ;

(b) the weight of θ is equal to $2^{\mathbf{c}}$;

(c) the σ-ideal $\mathcal{I}(\theta)$ of all θ-measure zero sets coincides with the σ-ideal $\mathcal{I}(\lambda)$ of all λ-measure zero sets.

As has been said earlier, G-absolutely negligible sets in a space (E, G) are very good from the point of view of the measure extension problem. But, in many cases, there are subsets of a ground space E which are extremely bad from the measure-theoretical view-point. Recall the definition of the latter subsets of E (see [24], [26], [33], [35], [43]).

A set $X \subset E$ is G-absolutely nonmeasurable if, for every measure μ belonging to the class $M_2(E, G)$, we have $X \notin \mathrm{dom}(\mu)$.

In other words, $X \subset E$ is G-absolutely nonmeasurable if X is simultaneously nonmeasurable with respect to all measures from $M_2(E, G)$.

Various properties of G-absolutely nonmeasurable sets are discussed in [24], [26], [33], [35] and [43]. One of the simplest properties is indicated in the following example.

Example 5. If X is a G-absolutely nonmeasurable subset of E, then there exists a countable family $\{g_i : i \in I\}$ of transformations from G such that $\cup\{g_i(X) : i \in I\} = E$.

However, the property described in the above example is very far from being sufficient to assert that X is G-absolutely nonmeasurable in E.

Example 6. In [11] a function $\phi : \mathbf{R} \to \mathbf{R}$ is constructed such that the graph $\mathrm{Gr}(\phi)$ of ϕ has the following property: there exists a countable family $\{g_i : i \in I\} \subset Is_2$ for which the equality

$$\cup\{g_i(\mathrm{Gr}(\phi)) : i \in I\} = \mathbf{R}^2$$

holds true (for some related results, see also [12] and [61]). It can easily be verified that $\mathrm{Gr}(\phi)$ is an Is_2-absolutely nonmeasurable subset of the plane $\mathbf{R}^2$.

The next problem remains open, too.

Problem 7. In terms of (E, G), find a characterization of G-absolutely nonmeasurable subsets of E.

Let us formulate one more problem also concerning absolutely nonmeasurable sets and closely connected with the previous one.

Problem 8. Find necessary and sufficient conditions on (E, G), under which there exists at least one G-absolutely nonmeasurable subset of E.

The following question is a special important case of Problem 8.

Problem 9. Let $(G, \cdot)$ be an uncountable group (identified with the group of all left translations of G). Does there exist a G-absolutely nonmeasurable subset of G?

Again, it should be indicated that, for some rather wide classes of uncountable groups, the answer to this question turns out to be positive. In particular, we have the following result.

Theorem 4. *If $(G, \cdot)$ is an uncountable solvable group, then there are G-absolutely nonmeasurable sets in G.*

A detailed proof of this statement can be found in [35] (see also [25], [33], [43]). As a consequence, we obtain that there are G-absolutely nonmeasurable sets in any uncountable commutative group $(G, +)$. So, if a natural number n is strictly positive, then there exist T_n-absolutely nonmeasurable sets in the Euclidean space $\mathbf{R}^n$ (cf. Example 6).

From the said above one can conclude that, for a large class of uncountable groups $(G, \cdot)$, there are G-absolutely negligible sets and G-absolutely nonmeasurable sets in G.

Now, we would like to recall one old result of Sierpiński [60], according to which there exist two λ-measure zero sets $X \subset \mathbf{R}$ and $Y \subset \mathbf{R}$ such that their algebraic sum

$$X + Y = \{x + y : x \in X,\ y \in Y\}$$

is nonmeasurable with respect to λ.

Notice that somewhat similar result is also known for the algebraic sum of Borel subsets of $\mathbf{R}$, namely, there exist two Borel sets $X' \subset \mathbf{R}$ and $Y' \subset \mathbf{R}$ whose algebraic sum

$$X' + Y' = \{x' + y' : x' \in X,\ y' \in Y'\}$$

is not Borel (see [4], [14], [59], [64]). Obviously, $X' + Y'$ is an analytic (i.e., Suslin) subset of $\mathbf{R}$.

Sierpiński's above-mentioned result initiated a series of publications in recent years (see, e.g., [4], [7], [8], [34], [36], [41], [44], [45], [46]). The next problem is also motivated by the same result.

Problem 10. Let $(G, +)$ be an uncountable commutative group. Do there exist two G-absolutely negligible sets $A \subset G$ and $B \subset G$ such that their algebraic sum

$$A + B = \{a + b : a \in A,\ b \in B\}$$

is a G-absolutely nonmeasurable set in G?

In this direction, the following statement has been established.

Theorem 5. *Let $(E, +)$ be a vector space over* $\mathbf{Q}$ *and let* $\mathrm{card}(E) \geq \mathbf{c}$. *Then there are two E-absolutely negligible sets $A \subset E$ and $B \subset E$ such that their algebraic sum $A + B$ is an E-absolutely nonmeasurable set in E.*

In particular, if $n \geq 1$, then there are two $\mathbf{R}^n$-absolutely negligible subsets of $\mathbf{R}^n$ whose algebraic sum is an $\mathbf{R}^n$-absolutely nonmeasurable set in $\mathbf{R}^n$.

For the proof of Theorem 5, see [36] or [43]. One direct consequence of this theorem should be indicated here. Namely, we have:

Under the Continuum Hypothesis, every uncountable vector space E over $\mathbf{Q}$ contains two E-absolutely negligible sets A and B whose algebraic sum $A + B$ turns out to be E-absolutely nonmeasurable.

We do not know whether it is possible to obtain the above consequence within **ZFC** set theory.

The properties of G-absolutely negligible subsets and G-absolutely nonmeasurable subsets of E substantially depend on the structural properties of G. It makes sense to distinguish especially the two classical cases where a ground space E coincides with the Euclidean n-dimensional space $\mathbf{R}^n$ and G coincides either with the group T_n of all translations of $\mathbf{R}^n$ or with the group Is_n of all isometric transformations of $\mathbf{R}^n$.

Problem 11. Let $n \geq 2$ be a natural number. Does there exist a T_n-absolutely negligible set in $\mathbf{R}^n$ which simultaneously is Is_n-absolutely nonmeasurable?

Even for $n = 2$ the answer to this question remains unknown. Notice that for $n = 1$ the class of all T_n-absolutely negligible sets is identical with the class of all Is_n-absolutely negligible sets, so in this situation there does not exist a T_n-absolutely negligible set which is Is_n-absolutely nonmeasurable.

As is widely known, a lot of analogies may be observed between the two fundamental concepts of real analysis: Lebesgue measurability and Baire property. An excellent reference is Oxtoby's small text-book [53] (see also Morgan's much more extensive monograph [49] in which a unified approach to these two concepts is developed). A simple fact concerning analogies between the Lebesgue measure and Baire property can be expressed in the following two statements:

(i) if $n \geq 1$, then there exists a λ_n-measure zero subset of $\mathbf{R}^n$ which does not possess the Baire property;

(ii) if $n \geq 1$, then there exists a first category subset of $\mathbf{R}^n$ which is not λ_n-measurable.

The assertions (i) and (ii) readily follow from the existence of a partition $\{X, Y\}$ of $\mathbf{R}^n$ such that X is of λ_n-measure zero and Y is of first category in $\mathbf{R}^n$ (see [49], [53]).

In connection with (ii), we would like to formulate the next question.

Problem 12. Let $n \geq 1$ be a natural number. Does there exist a T_n-absolutely nonmeasurable subset of $\mathbf{R}^n$ which has the Baire property?

Recall that a set $X \subset \mathbf{R}^n$ is almost Is_n-invariant (with respect to the Lebesgue measure λ_n) if, for any transformation $g \in Is_n$, we have

$$\lambda_n(g(X) \triangle X) = 0.$$

By using the Continuum Hypothesis (or Martin's Axiom), it can be demonstrated that if $n \geq 1$, then there exists a partition $\{X, Y\}$ of $\mathbf{R}^n$, where both sets X and Y are almost Is_n-invariant and none of them is of λ_n-measure zero (see, e.g., [24], [26], [33], [35], [43], [49], [53], [56]). It is unknown whether the analogous result holds within the standard **ZFC** set theory.

Problem 13. Let $n \geq 1$ be a natural number. Does there exist (within **ZFC** set theory) a partition $\{X, Y\}$ of $\mathbf{R}^n$ such that both sets X and Y are almost Is_n-invariant and none of them is of λ_n-measure zero?

It may happen that this question cannot be resolved without the aid of additional set-theoretical hypotheses.

Now, let us recall the notion of a Sierpiński-Zygmund function (see [6], [48], [62]). Any function of this kind is extremely discontinuous, so is utterly bad from the purely topological point of view.

Let $f : \mathbf{R} \to \mathbf{R}$ be a function. We say that f is a Sierpiński-Zygmund function if, for any set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restriction of f to X is not continuous (i.e., is discontinuous).

The existence of such functions can be established only by using uncountable forms of the Axiom of Choice. Various interesting and unexpected properties of Sierpiński-Zygmund functions are discussed, e.g., in [2], [6], [37], [38], [43], [51], [57].

Let $M(\mathbf{R})$ denote the class of the completions of all nonzero σ-finite continuous Borel measures on $\mathbf{R}$. It is not hard to show that every Sierpiński-Zygmund function f is absolutely nonmeasurable with respect to $M(\mathbf{R})$, i.e., f is nonmeasurable with respect to every measure from $M(\mathbf{R})$. In particular, f is nonmeasurable with respect to λ.

On the other hand, it was demonstrated in [38] that there exists a translation invariant measure μ on $\mathbf{R}$ which extends the Lebesgue measure λ and for which some Sierpiński-Zygmund function $\psi : \mathbf{R} \to \mathbf{R}$ turns out to be μ-measurable. This function ψ is not additive, i.e., is not an endomorphism of the additive group $(\mathbf{R}, +)$ into itself. So the following question arises.

Problem 14. Investigate whether there exists a Sierpiński-Zygmund function $f : \mathbf{R} \to \mathbf{R}$ satisfying these two relations:

(1) f is additive (in other words, f is a solution of the Cauchy functional equation);

(2) f is measurable with respect to some translation invariant measure on $\mathbf{R}$ which extends λ.

Notice that if in the formulation of Problem 14 we replace the translation invariance by the translation quasi-invariance, then such a function f does exist (for more details, see [38], [43]).

Now, let us introduce a notion somewhat similar to the concept of an absolutely negligible set.

Let (E, G) be again a space equipped with a transformation group and let X be a subset of E.

We say that X is G-negligible in E if these two conditions are fulfilled:

(a) there exists a nonzero σ-finite diffused G-invariant (G-quasi-invariant) measure μ_0 on E such that $X \in \mathrm{dom}(\mu_0)$ and $\mu_0(X) = 0$;

(b) if μ is an arbitrary σ-finite diffused G-invariant (G-quasi-invariant) measure on E such that $X \in \mathrm{dom}(\mu)$, then $\mu(X) = 0$.

It can easily be seen that every G-absolutely negligible set is G-negligible as well. The converse assertion fails to be true, in general, as simple examples show.

Since we have a suitable characterization of G-absolutely negligible sets (see Theorem 2), it makes sense to formulate the next problem.

Problem 15. Let (E, G) be a space equipped with a transformation group. In terms of the pair (E, G), give a characterization of all G-negligible subsets of E.

Let us consider a particular case, where a base set E is a commutative group and G coincides with the group of all translations of E (naturally, the latter is identified with E).

So, let $(G, +)$ be a commutative group and let X be a subset of G such that there exists an uncountable family $\{g_i : i \in I\}$ of elements of G for which the corresponding family of sets $\{g_i + X : i \in I\}$ is disjoint. Then we may assert that the set X is G-negligible. This fact is implied by the following auxiliary proposition which is not difficult to prove and is useful in many situations.

Theorem 6. *Let $(G, +)$ be a commutative group, Z be a subset of G and suppose that these two conditions are satisfied:*

(1) $\cup\{h_j + Z : j \in J\} = G$ *for some family* $\{h_j : j \in J\}$ *of elements of* G*;*

(2) *there exists a family* $\{g_i : i \in I\}$ *of elements of* G *such that the corresponding family of sets* $\{g_i + Z : i \in I\}$ *is disjoint.*

Then the inequality $\mathrm{card}(J) \geq \mathrm{card}(I)$ *holds true.*

It directly follows from this theorem that if a subset X of a commutative group $(G, +)$ is such that, for some uncountable family $\{g_i : i \in I\} \subset G$, the family of sets $\{g_i + X : i \in I\}$ is disjoint, then

$$\mathrm{card}(G \setminus \cup\{h_j + X : j \in J\}) > \omega$$

whenever $\{h_j : j \in J\}$ is a countable family of elements of G. Keeping in mind this circumstance, it is easy to define a probability diffused G-invariant measure μ_0 on G such that $X \in \mathrm{dom}(\mu_0)$ and $\mu_0(X) = 0$. Also, in view of the disjointness of the uncountable family of sets $\{g_i + X : i \in I\}$, we readily infer that $\mu(X) = 0$ for any σ-finite G-quasi-invariant measure μ on G such that $X \in \mathrm{dom}(\mu)$.

The following simple example serves as an illustration of the stated above.

Example 7. Let $f : \mathbf{R} \to \mathbf{R}$ be a function and let $\mathrm{Gr}(f)$ denote the graph of f. Then $\mathrm{Gr}(f)$ is a T_2-negligible subset of the plane $\mathbf{R}^2$, but the same $\mathrm{Gr}(f)$ does not need to be a T_2-absolutely negligible set (see [24], [26], [33], [35]). Here the symbol T_2 stands, as earlier, for the group of all translations of the plane $\mathbf{R}^2$.

Example 8. It can be proved, by assuming the Continuum Hypothesis, that there exists a subset Y of the space $\mathbf{R}^n$ $(n \geq 1)$ satisfying the following two conditions:

(a) $\cup\{h_j + Y : j \in J\} = \mathbf{R}^n$ for some countable family $\{h_j : j \in J\}$ of elements of $\mathbf{R}^n$;

(b) there exists an uncountable family $\{g_i : i \in I\}$ of elements of $\mathbf{R}^n$ such that

$$\mathrm{card}((g_i + Y) \cap (g_{i'} + Y)) \leq \omega$$

for any two distinct indices $i \in I$ and $i' \in I$.

For the proof, see [24] or [35] (in these works the more general case of a commutative group $(G, +)$ with $\mathrm{card}(G) = \omega_1$ is considered). Clearly, the above-mentioned set Y turns out to be T_n-absolutely nonmeasurable.

If (E, G) is a space endowed with a transformation group, then, for every ordinal number α, we may introduce the class $\mathcal{K}_\alpha$ of subsets of E. We first put:

$X \in \mathcal{K}_0$ if and only if there exists an uncountable family $\{g_i : i \in I\}$ of elements from G such that the family $\{g_i(X) : i \in I\}$ is disjoint.

Suppose now that, for an ordinal α, all the classes $\mathcal{K}_\beta$ $(\beta < \alpha)$ have already been defined. Then we put:

$X \in \mathcal{K}_\alpha$ if and only if there exists an uncountable family $\{g_i : i \in I\}$ of elements from G such that, for any two distinct indices $i \in I$ and $i' \in I$, the relation

$$g_i(X) \cap g_{i'}(X) \in \cup\{\mathcal{K}_\beta : \beta < \alpha\}$$

holds true.

By using the method of transfinite induction, it is easy to prove that, for any ordinal α, all sets X from the class $\mathcal{K}_\alpha$ have the following property: if

μ is an arbitrary σ-finite G-quasi-invariant measure on E and $X \in \mathrm{dom}(\mu)$, then $\mu(X) = 0$.

Example 9. Let $E = \mathbf{R}^n$ and let $G = T_n$, where $n \geq 1$. It can be shown that there exists a G-absolutely nonmeasurable subset of E belonging to the class $\mathcal{K}_1$ (cf. Example 8; see also [24], [35]).

Some interesting problems arise in connection with absolutely nonmeasurable additive functions, i.e., absolutely nonmeasurable homomorphisms of commutative groups.

Let $(G, +)$ be an uncountable commutative group and let $(H, +)$ be an uncountable commutative Polish group. We say that a homomorphism $\phi : G \to H$ is absolutely nonmeasurable if, for any nonzero σ-finite translation quasi-invariant measure μ on G, this ϕ is nonmeasurable with respect to μ.

Let $\mathbf{T}$ denote the additive group of the one-dimensional unit torus (actually, $\mathbf{T}$ stands for the circle group). The following statement is true.

Theorem 7. *Let $(G, +)$ be a commutative group and let G_0 be the torsion subgroup of G. These two conditions are equivalent:*

(1) *the quotient group G/G_0 is uncountable;*

(2) *there exists a homomorphism from G into $\mathbf{R}$ (into $\mathbf{T}$) which is absolutely nonmeasurable with respect to the class $M_2(G, G)$.*

In view of Theorem 7, the next example is relevant.

Example 10. Let $C = \{0, 1\}^\omega$ denote the Cantor space regarded as a commutative compact metrizable group with respect to the standard product topology and group operation modulo 2. By using the Continuum Hypothesis (Martin's axiom), it can be demonstrated that C contains a Luzin subset (a generalized Luzin subset) L which simultaneously is a subgroup of C. So, under these additional axioms, there exist universal measure zero subgroups of C which are equinumerous with C. Let now $(G, +)$ be an arbitrary 2-divisible commutative group (e.g., $G = \mathbf{R}$ or $G = \mathbf{T}$). Then it is clear that any homomorphism $\phi : G \to C$ is trivial and, consequently, there exist no absolutely nonmeasurable homomorphisms acting from G into C (although condition (1) of Theorem 7 may be satisfied for G). At the same time, one can see that the identical embedding of L into C is an absolutely nonmeasurable group monomorphism.

In view of Theorem 7 and Example 10, the following problem arises.

Problem 16. Let $(G, +)$ be an uncountable commutative group and let $(H, +)$ be an uncountable commutative Polish topological group. Find necessary and sufficient conditions for the existence of an absolutely nonmeasurable homomorphism of $(G, +)$ into $(H, +)$.

Obviously, the analogous problem can be formulated for uncountable non-commutative groups.

We now turn our attention to Borel measures in an infinite-dimensional Hausdorff topological vector space E.

A long time ago, it was demonstrated by various authors that, as a rule, there are no nonzero σ-finite Borel measures on E invariant (or quasi-invariant) with respect to a vector subspace of E of second category (in this connection, see especially [63] and the references therein; cf. also [27], [29]). Further, it was proved that if all cardinal numbers are nonmeasurable in the Ulam sense, then no metrizable nonseparable topological group G admits a nonzero σ-finite Borel measure quasi-invariant with respect to an everywhere dense subgroup of G. The latter fact substantially exploits the metrizability of G. For non-metrizable topological groups (and for non-metrizable topological vector spaces) the situation is radically different, as the next example shows.

Example 11. Let α be any infinite cardinal number. Consider the product space $\mathbf{R}^\alpha$ endowed with the Tychonoff topology. Let $\mathbf{R}^{(\alpha)}$ stand for all those elements of $\mathbf{R}^\alpha$ which have finite supports, i.e.,

$$\mathbf{R}^{(\alpha)} = \{(x_i)_{i\in\alpha} \in \mathbf{R}^\alpha : \mathrm{card}(\{i \in \alpha : x_i \neq 0\}) < \omega\}.$$

Obviously, $\mathbf{R}^{(\alpha)}$ is an everywhere dense vector subspace of the topological vector space $\mathbf{R}^\alpha$. It was shown (see, e.g., [31]) that there exists a probability Borel measure μ on $\mathbf{R}^\alpha$ which is $\mathbf{R}^{(\alpha)}$-quasi-invariant. Notice that if $\alpha > \mathbf{c}$, then $\mathbf{R}^\alpha$ is not separable and if $\alpha > \omega$, then this μ is not a Radon measure.

The said above leads to the next problem (see [32]).

Problem 17. Give a characterization of all those Hausdorff topological vector spaces E which possess the following property: there exists a nonzero σ-finite Borel measure on E quasi-invariant with respect to some everywhere dense vector subspace of E.

It should be noticed that, for certain natural classes of Hausdorff topological vector spaces, we have a complete solution of the formulated problem.

Example 12. If E is an arbitrary Banach space, then these two assertions are equivalent:

(a) E is separable;

(b) there exists a nonzero σ-finite Borel measure on E quasi-invariant with respect to an everywhere dense vector subspace of E.

The proof of the equivalence of (a) and (b) may be found in [32].

Let H be an infinite-dimensional separable Hilbert space. Some constructions are known for obtaining nonzero σ-finite Borel measures in H which are invariant with respect to an everywhere dense vector subspace of

H (see, for instance, [28], [70]). Those constructions are also applicable to other classical separable Banach spaces (e.g., for l_p where $p \geq 1$). Similarly to Problem 17, one can formulate the problem of a characterization of all those Hausdorff topological vector spaces E which possess the property that there exists a nonzero σ-finite Borel measure on E invariant with respect to an everywhere dense vector subspace of E.

Notice at the end of this article that there are several constructions of translation invariant Borel measures in some infinite-dimensional topological vector spaces E, which take nonzero finite value on certain canonical Borel subsets of E (see, for instance, [1], [52]). Of course, a weak side of those measures is that they are not σ-finite, so classical results of measure theory (such as Fubini's theorem, Radon-Nikodym theorem, etc.) are not applicable to them.

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