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TO THE EXISTENCE OF PROJECTIVE ABSOLUTELY NONMEASURABLE FUNCTIONS

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Abstract. It is shown that under some appropriate set-theoretical assumptions there exists an absolutely nonmeasurable function acting from $[0,1]$ into $[0,1]$, whose graph is a projective subset of $[0,1]^2$.

რეზიუმე. გარკვეული დამატებითი სიმრავლურ-თეორიული აქსიომების გამოყენებით ნაჩვენებია, რომ არსებობს აბსოლუტურად არაზომადი ფუნქცია $[0,1]$ სეგმენტიდან თავისთავში, რომლის გრაფიკი არის $[0,1]^2$ კვადრატის პროექციული ქვესიმრავლე.

The general measure extension problem requires to extend a nonzero σ-finite diffused (i.e., vanishing at all singletons) measure μ given on an uncountable base (ground) set E to a maximally wide class of subsets of E. As is well known, this problem does not admit a satisfactory solution because in many cases there is no maximal extensions of μ. On the other hand, for each set $X \subset E$, there exists a measure μ' on E extending μ and such that X becomes measurable with respect to μ'. By using induction it trivially follows from the above fact that, for any σ-finite measure μ on E and for any finite family $\{X_1, X_2, \dots, X_n\}$ of subsets of E, there exists a measure μ' on E extending μ and satisfying the relation

$$\{X_1, X_2, \dots, X_n\} \subset \mathrm{dom}(\mu').$$

Thus, one may say that there is no subset of E which is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$ of all nonzero σ-finite diffused measures on E.

Remark 1. If $\{X_j : j \in J\}$ is an arbitrary (in particular, uncountable) disjoint family of subsets of a base set E and μ is any σ-finite measure on E, then there always exists a measure μ' on E extending μ and satisfying the relation

$$\{X_j : j \in J\} \subset \mathrm{dom}(\mu').$$

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For a proof of this important result, see [1] or [6]. The disjointness of the family $\{X_j : j \in J\}$ is essential here, because without this assumption it may happen that there are some countable families of subsets of E which do not admit any nonzero σ-finite diffused measure (a more detailed explanation will be given below).

In the works [2], [3], [6], [16] some other aspects of the measure extension problem are considered in those cases when additional mathematical structures enter the scene and turn out to be compatible with measures, e.g., certain topological and algebraic structures.

For real-valued functions defined on E the situation is somewhat different. Indeed, denoting by $\mathbf{R}$ the real line, taking $E = \mathbf{R}$ and assuming Martin's Axiom, it can be shown that there are functions $f : \mathbf{R} \to \mathbf{R}$ which are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$ (see, for instance, [6]). The latter phrase means that f is nonmeasurable with respect to every measure $\mu \in \mathcal{M}(\mathbf{R})$. Such an f may be treated as an extremely nonmeasurable function, so f seems to be of much more pathological nature than, e.g., Lebesgue nonmeasurable real-valued functions or those real-valued functions which are nonmeasurable with respect to concrete nonzero σ-finite diffused measures on $\mathbf{R}$.

In order to characterize absolutely nonmeasurable functions, we need the notion of an absolute null subset of $\mathbf{R}$. Recall that a set $X \subset \mathbf{R}$ is absolute null (or universal measure zero) if, for every σ-finite diffused Borel measure μ on E, one has $\mu^*(X) = 0$, where μ^* denotes the outer measure associated with μ. Various properties of absolute null sets are discussed in [5], [6], [9], [10], [11]. The existence of uncountable absolute null subsets of $\mathbf{R}$ can be proved within **ZFC** set theory. There are several interesting constructions of such subsets, which essentially differ from each other (see, e.g., [5], [9], [10], [14], [17]).

One characterization of absolutely nonmeasurable real-valued functions, in terms of absolute null subsets of $\mathbf{R}$, looks as follows.

Lemma 1. *Let E be a ground set and let $f : E \to \mathbf{R}$ be a function. Then these two assertions are equivalent:*

(a) *f is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$;*

(b) *the range* $\mathrm{ran}(f)$ *of f is an absolute null subset of $\mathbf{R}$ and the sets $f^{-1}(t)$ are at most countable for all points $t \in \mathbf{R}$.*

For a proof of Lemma 1 (which is not difficult), see e.g., [5] or [6].

The absolute nonmeasurability of a function $f : E \to \mathbf{R}$ is closely connected with the existence of absolutely nonmeasurable countable families of subsets of a ground set E.

Let ω denote, as usual, the least infinite ordinal (cardinal) number.

We shall say that a countable family $\{Z_n : n < \omega\}$ of subsets of E is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$ if the following two conditions are fulfilled:

(i) all singletons in E belong to the σ-algebra $\sigma(\{Z_n : n < \omega\})$ generated by $\{Z_n : n < \omega\}$;

(ii) there exists no measure from $\mathcal{M}(E)$ whose domain coincides with the σ-algebra $\sigma(\{Z_n : n < \omega\})$.

It is easy to see that condition (i) is equivalent to the condition:

(i′) the family $\{Z_n : n < \omega\}$ separates points of E, i.e., for any two distinct points $x \in E$ and $y \in E$, there exists $n < \omega$ such that

$$\mathrm{card}(\{x, y\} \cap Z_n) = 1.$$

Notice also that condition (i) (or, equivalently, condition (i′)) implies the inequality $\mathrm{card}(E) \leq \mathbf{c}$, where $\mathbf{c}$ stands for the cardinality of the continuum.

Theorem 1. *Let E be a ground set. The following two assertions are valid.*

(1) *If there exists a function $f : E \to \mathbf{R}$ which is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$, then there exists a countable family $\{X_k : k < \omega\}$ of subsets of E which is absolutely nonmeasurable with respect to $\mathcal{M}(E)$;*

(2) *if there exists a countable family $\{X_k : k < \omega\}$ of subsets of E which is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$, then there exists an injective $\sigma(\{X_k : k < \omega\})$-measurable function $f : E \to \mathbf{R}$ which is absolutely nonmeasurable with respect to $\mathcal{M}(E)$.*

Proof. In order to establish the validity of (1), consider an arbitrary function $f : E \to \mathbf{R}$ which is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$. By virtue of (b) of Lemma 1, we have the inequality

$$\mathrm{card}(E) \leq \mathbf{c},$$

so E can be identified with some subset of $\mathbf{R}$. Take any countable family $\{\triangle_n : n < \omega\}$ of nonempty open intervals in $\mathbf{R}$ which collectively form a base of the standard topology of $\mathbf{R}$. Clearly, $\{\triangle_n : n < \omega\}$ separates points of $\mathbf{R}$. It directly follows from this observation that there exists a countable family $\{Y_n : n < \omega\}$ of subsets of E which separates points of E. Let us put

$$\{X_k : k < \omega\} = \{f^{-1}(\triangle_n) : n < \omega\} \cup \{Y_n : n < \omega\}.$$

Then it is not hard to check that the family $\{X_k : k < \omega\}$ turns out to be absolutely nonmeasurable with respect to $\mathcal{M}(E)$, i.e., assertion (1) holds true.

In order to establish the validity of (2), consider an arbitrary countable family $\{X_k : k < \omega\}$ of subsets of E, which is absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$. Denoting by $\{0, 1\}^\omega$ the Cantor discontinuum and using Marczewski's method of characteristic functions of sequences of sets, we may define a mapping

$$\phi : E \to \{0,1\}^{\omega}$$

by putting:

$$\phi(x) = \{i_k(x) : k < \omega\} \ (x \in E),$$

where $i_k(x) = 1$ if $x \in X_k$, and $i_k(x) = 0$ if $x \notin X_k$. It is easy to verify that the introduced in this manner mapping ϕ is injective and $\sigma(\{X_k : k < \omega\})$-measurable. Further, denote by ψ some Borel isomorphism acting from $\{0,1\}^{\omega}$ onto $\mathbf{R}$. Then the composition

$$f = \psi \circ \phi$$

is injective, $\sigma(\{X_k : k < \omega\})$-measurable and absolutely nonmeasurable with respect to the class $\mathcal{M}(E)$, i.e., assertion (2) holds true.

Theorem 1 has thus been proved. □

It is natural to ask whether there exist definable (in some natural sense) functions $f : \mathbf{R} \to \mathbf{R}$ which are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$. According to Theorem 1, this question is equivalent to the question whether there exists a definable countable family $\{Z_n : n < \omega\}$ of subsets of $\mathbf{R}$ absolutely nonmeasurable with respect to the same class $\mathcal{M}(\mathbf{R})$.

Since the notion of "definability" is not precisely determined and admits variations, we will restrict our further consideration to a more concrete version of definable sets and functions, namely, to those functions whose graphs are projective subsets of the plane $\mathbf{R}^2$. According to well-known set-theoretical results, there are models of **ZFC** theory, in which all projective subsets of $\mathbf{R}$ turn out to be of good descriptive structure and, in particular, turn out to be Lebesgue measurable. For instance, the so-called Axiom of Projective Determinacy (**PD**) implies all important regularity properties of the projective sets, including their measurability in the Lebesgue sense (see, e.g., [4], [5], [12]). This circumstance directly indicates that the existence of definable absolutely nonmeasurable functions on $\mathbf{R}$ can be detected only in special models of **ZFC** theory. The main goal of this note is to demonstrate the existence of such functions with the aid of some (natural) additional set-theoretical hypotheses.

For this purpose, we need several auxiliary notions and facts from classical measure theory and real analysis. First, let us fix the notation.

$\mathbf{Q}$ denotes the set of all rational numbers. Recall that this set is a countable everywhere dense subgroup of the additive group $(\mathbf{R}, +)$.

ω_1 is the least uncountable ordinal (cardinal) number.

λ stands for the ordinary Lebesgue measure on $\mathbf{R}$.

As usual, sets belonging to the Luzin–Sierpiński projective hierarchy are denoted by the symbols Σ^1_n and Π^1_n, according to projective levels of those sets. The standard monographs and text-books of classical descriptive set theory are [5] and [12] (see also [4]). It needless to say that projective sets are regarded as one of possible realizations of the notion of definability for point sets lying in Euclidean spaces or, more generally, in Polish topological spaces. However, it is well known that among those sets one can encounter some kind of pathology from the measure-theoretical and topological viewpoints.

Example 1. Recall that a Vitali set is an arbitrary selector of the quotient set $\mathbf{R}/\mathbf{Q}$. As was shown by Vitali [15], all Vitali sets are nonmeasurable with respect to any translation invariant measure on $\mathbf{R}$ extending the Lebesgue measure λ. Also, all Vitali sets do not possess the Baire property (see, e.g., [5], [6], [8], [11], [13]). Assuming that there exists a well-ordering of $\mathbf{R}$ whose graph is a Σ^1_2-subset of the plane $\mathbf{R}^2$, it can readily be demonstrated that there exists a Vitali set in $\mathbf{R}$ which is a Σ^1_2-subset of $\mathbf{R}$. Consequently, under the above assumption, there exist projective sets which are Lebesgue nonmeasurable and do not have the Baire property. Recall that this assumption holds true in certain models of **ZFC** theory, e.g., in Gödel's Constructible Universe $\mathbf{L}$ (see [4]).

As was announced earlier, our goal is to strengthen Example 1 in terms of absolutely nonmeasurable real-valued functions defined on $\mathbf{R}$, whose graphs are projective subsets of $\mathbf{R}^2$.

Lemma 2. *Suppose that there exists a well-ordering $\preceq$ of $\mathbf{R}$ (or, equivalently, of $[0,1]$) whose graph is a projective subset of $\mathbf{R}^2$ (of $[0,1]^2$). Then every projective set in $\mathbf{R}^2$ (in $[0,1]^2$) admits a uniformization by a projective set.*

Proof. The argument is easy and can be carried out in the standard manner. Take any projective set $Z \subset [0,1]^2$ and consider its projection $\mathrm{pr}_1(Z)$ on the axis of abscissae. For each point $x \in \mathrm{pr}_1(Z)$, the corresponding section

$$Z(x) = \{y \in [0,1] : (x,y) \in Z\}$$

is nonempty. Let us put

$$y(x) = \inf_{\preceq}\{y \in [0,1] : (x,y) \in Z\}.$$

A straightforward verification shows that the set

$$\{(x, y(x)) : x \in \mathrm{pr}_1(Z)\} \subset Z$$

is a projective subset of $[0,1]^2$ and, simultaneously, is the graph of a certain function acting from $\mathrm{pr}_1(Z)$ into $[0,1]$. □

Below, we will need the notion of a Luzin subset of $\mathbf{R}$. Recall that $X \subset \mathbf{R}$ is a Luzin set if X is uncountable and its intersection with every first category subset of $\mathbf{R}$ is at most countable. Various properties of Luzin sets are envisaged in [8], [9], [10], [11], [13]. One of them is formulated in the next auxiliary proposition.

Lemma 3. *Every Luzin subset of* $\mathbf{R}$ *is an absolute null set.*

This lemma is well known, so its proof is omitted here (cf. [5], [6], [9], [10], [11], [13]).

Example 2. If X is a Luzin subset of $\mathbf{R}$ and $Y \subset \mathbf{R}$ is a homeomorphic image of X, then one cannot assert that Y is a Luzin set. Moreover, such a Y can be a nowhere dense subset of $\mathbf{R}$. On the other hand, it is easy to see that if $f : \mathbf{R} \to \mathbf{R}$ is a bijection such that both f and f^{-1} preserve the σ-ideal of all first category sets in $\mathbf{R}$, then f transforms the class of all Luzin subsets of $\mathbf{R}$ onto itself.

We now are ready to formulate and prove the main result of this paper.

Theorem 2. *Suppose that there exists a well-ordering* $\preceq$ *of* $[0,1]$ *for which the following two conditions are fulfilled:*
$(*)$ $\preceq$ *is isomorphic to the natural well-ordering of* ω_1*;*
$(**)$ *the graph of* $\preceq$ *is a projective subset of* $[0,1]^2$.
Then there exists a function

$$\phi : [0,1] \to [0,1]$$

whose graph is a projective subset of $[0,1]^2$ *and which is absolutely nonmeasurable with respect to the class* $\mathcal{M}([0,1])$ *of all nonzero* σ*-finite diffused measures on* $[0,1]$.

Proof. We argue step by step as follows.

(a) First, denote by E the compact metric space consisting of all nonempty closed subsets of $[0,1]$, and consider its subspace E' consisting of all nonempty nowhere dense closed subsets of $[0,1]$. It can readily be checked that E' is of type G_δ in E, so E' can be treated as a Polish topological space (notice, by the way, that E' is everywhere dense in E).

(b) Further, identify the Baire canonical space ω^ω with the set I of all irrational numbers in $[0,1]$ and introduce a continuous surjection

$$\Phi : I \to E'.$$

Then, for each point $x \in [0,1]$, consider the set

$$Z(x) = \{y : x \preceq y \ \&\ y \notin \cup\{\Phi(i) : i \preceq x\}\}$$

and define the subset Z of the plane $\mathbf{R}^2$ by putting

$$Z = \cup\{\{x\} \times Z(x) : x \in [0,1]\}.$$

A straightforward verification shows that Z is a projective subset of the square $[0,1]^2$ and the equality $\mathrm{pr}_1(Z) = [0,1]$ holds true.

(c) Taking into account Lemma 2, we establish the existence of a function

$$\phi : [0,1] \to [0,1]$$

whose graph is a projective subset of $[0,1]^2$ and is contained in Z. Then we check that the range $\mathrm{ran}(\phi)$ of ϕ is a Luzin set in $[0,1]$ which simultaneously is a projective subset of $[0,1]$ and

$$\mathrm{card}(\phi^{-1}(t)) \leq \omega$$

for every point $t \in [0,1]$. Keeping in mind Lemma 1, we may conclude that ϕ is absolutely nonmeasurable with respect to the class $\mathcal{M}([0,1])$ of all nonzero σ-finite diffused measures on $[0,1]$.

According to Theorem 1, it follows from the above result that there is a countable family $\{T_j : j \in J\}$ of projective subsets of $[0,1]$ such that no nonzero σ-finite diffused measure μ on $[0,1]$ satisfies the relation

$$\{T_j : j \in J\} \subset \mathrm{dom}(\mu).$$

Indeed, for this purpose it suffices to take some countable base $\{U_n : n < \omega\}$ of open sets in $[0,1]$ and some countable base $\{V_n : n < \omega\}$ of open sets in the space $\mathrm{ran}(\phi) \subset [0,1]$. Denoting

$$\{T_j : j \in J\} = \{U_n : n < \omega\} \ \cup \ \{\phi^{-1}(V_n) : n < \omega\},$$

we see that the family $\{T_j : j \in J\}$ is as required. This completes the proof of Theorem 2. $\square$

Remark 2. The situation described by the conditions $(*)$ and $(**)$ of Theorem 2 is realizable in certain models of **ZFC** theory, e.g., in Gödel's Constructible Universe **L**.

Remark 3. In one of Ulam's letters to Gödel, it was underlined that it is impossible (at least, in some models of set theory) to introduce a diffused probability measure defined for all projective subsets of the unit interval $[0,1]$. In this connection, Ulam referred to his famous $(\omega \times \omega_1)$-matrix and especially underlined the fact that all members of that matrix can be taken to be projective subsets of $[0,1]$. Theorem 2 shows that certain projective Luzin subsets of $[0,1]$ also suffice for obtaining the same result. Moreover, it follows from the proof of Theorem 2 that an absolutely nonmeasurable countable family of projective subsets of $[0,1]$ can be assumed to have the property that all members of the family belong to a concrete projective class Σ^1_n. Of course, this Σ^1_n depends on the projective class corresponding to an initial well-ordering $\preceq$.

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