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On negligible and absolutely nonmeasurable subsets of uncountable solvable groups

Alexander Kharazishvili

A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University, 6, Tamarashvili st., Tbilisi 0177, Georgia

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Abstract

It is proved that every uncountable solvable group contains two negligible sets whose union is an absolutely nonmeasurable subset of the same group.

Keywords: Solvable group; Invariant measure; Quasi-invariant measure; Negligible set; Absolutely nonmeasurable set

In this paper we will be dealing with measures invariant (or, more generally, quasi-invariant) under various transformation groups. We will be interested in the behavior of certain sets with respect to such measures. The notation and terminology used in the paper is primarily taken from [1] and [2]. All basic facts of modern measure theory can be found in [3]. An extensive survey devoted to measures given on different algebraic-topological structures is presented in [4].

Let E be a base (ground) set and let G be some group of transformations of E. In this case, the pair (E, G) is usually called a space equipped with a transformation group.

We shall say that a set $X \subset E$ is G-negligible (in E) if the following two conditions are fulfilled for X:

(a) there exists at least one nonzero σ-finite G-invariant (G-quasi-invariant) measure μ on E such that $X \in \mathrm{dom}(\mu)$;

(b) for every σ-finite G-invariant (G-quasi-invariant) measure ν on E such that $X \in \mathrm{dom}(\nu)$, the equality $\nu(X) = 0$ holds true.

We shall say that a set $Y \subset E$ is G-absolutely nonmeasurable (in E) if, for any nonzero σ-finite G-quasi-invariant measure θ on E, we have $X \notin \mathrm{dom}(\theta)$.

If $(G, \cdot)$ is a group, then we may consider G as a ground set E and take the group of all left translations of G as a group of transformations of E. Obviously, identifying G with the group of all left translations of G, we may speak of left G-invariant (left G-quasi-invariant) measures on E (=G) and, respectively, we may consider G-negligible and G-absolutely nonmeasurable subsets of G.

E-mail address: kharaz2@yahoo.com.
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Example 1. If $(G, \cdot)$ is an arbitrary uncountable solvable group, then there exists a G-absolutely nonmeasurable subset of G (in this connection, see e.g. [2] and references therein). At the same time, it is still unknown whether there exists a Γ-absolutely nonmeasurable set in any uncountable group $(\Gamma, \cdot)$.

The main goal of this paper is to show (for a certain class of spaces (E, G)) that there exist two G-negligible sets in E, the union of which turns out to be G-absolutely nonmeasurable in E. In particular, if E itself is an uncountable solvable group and G coincides with the group of all left translations of E, then the above-mentioned fact is valid for (E, G). Clearly, this yields some generalization of the statement formulated in Example 1.

It should be noticed that basic technical tools which lead us to the required result are motivated by the method of surjective homomorphisms (cf. [1,2,5]).

For our further purposes, we need several auxiliary propositions. The first of them is essentially contained in [2].

As usual, the symbol $\omega(=\omega_0)$ denotes the least infinite cardinal (ordinal) number and ω_1 denotes the least uncountable cardinal (ordinal) number.

Lemma 1. *Let a space (E, G) satisfy the following two relations:*

(1) $\mathrm{card}(E) = \omega_1$ *and the group G acts freely and transitively in E;*

(2) *there are two subgroups G_0 and G_1 of G such that*

$$\mathrm{card}(G_0) = \omega, \qquad \mathrm{card}(G_1) = \omega_1, \qquad G_0 \cap G_1 = \{\mathrm{Id}_E\},$$

where Id_E is the identity transformation of E.

Then there exist two G-negligible subsets T_1 and T_2 of E such that the set $T_1 \cup T_2$ is G-absolutely nonmeasurable in E.

Proof. We would like to recall one construction of a G-absolutely nonmeasurable subset of E (see [2], Chapter 11, Lemma 3). First, let us observe that relation (1) directly implies the equality

$$\mathrm{card}(G) = \omega_1.$$

So we may take an ω_1-sequence $\{\Gamma_\xi : \xi < \omega_1\}$ of subgroups of G, such that:

(a) $\Gamma_0 = G_0$;

(b) for all ordinals $\xi < \omega_1$, we have $\mathrm{card}(\Gamma_\xi) = \omega$;

(c) for each ordinal $\xi < \omega_1$, the set $\cup\{\Gamma_\zeta : \zeta < \xi\}$ is a proper subset of Γ_ξ (in particular, this ω_1-sequence of subgroups of G is strictly increasing by inclusion);

(d) $\cup\{\Gamma_\xi : \xi < \omega_1\} = G$.

Further, fix a point $y \in E$ and, for any ordinal number $\xi < \omega_1$, put

$$Y_\xi = \Gamma_\xi(y) \setminus \cup\{\Gamma_\zeta(y) : \zeta < \xi\}.$$

A straightforward verification shows that the family of sets $\{Y_\xi : \xi < \omega_1\}$ forms a partition of E and each Y_ξ is a Γ'_ξ-invariant subset of E, where the group Γ'_ξ is defined by the formula

$$\Gamma'_\xi = \cup\{\Gamma_\zeta : \zeta < \xi\}.$$

According to relation (c), the group Γ'_ξ is a proper subgroup of Γ_ξ. Also, by virtue of the free action of G in E, it is not hard to see that

$$\mathrm{card}(Y_\xi) = \omega \quad (\xi < \omega_1).$$

Now, for each ordinal number $\xi < \omega_1$, introduce the group

$$G_{1,\xi} = G_1 \cap \Gamma'_\xi.$$

Obviously, the ω_1-sequence $\{G_{1,\xi} : \xi < \omega_1\}$ of groups is increasing by inclusion and

$$\cup\{G_{1,\xi} : \xi < \omega_1\} = G_1.$$

Fix for a while an ordinal $\xi < \omega_1$ and consider the two partitions of Y_ξ into orbits associated with the groups G_0 and $G_{1,\xi}$, respectively. Taking into account the free action of G in E and the relation

$$G_0 \cap G_1 = \{\mathrm{Id}_E\},$$

we infer that the above-mentioned two partitions of Y_ξ are mutually transversal; in other words, any equivalence class of the first partition has at most one common point with any equivalence class of the second partition. Starting with this fact, we define by recursion an ω-sequence

$$\{x_{\xi,0}, x_{\xi,1}, \ldots, x_{\xi,k}, \ldots\}$$

of points from Y_ξ, such that:

(i) $G_0(\{x_{\xi,k} : k < \omega\}) = Y_\xi$;

(ii) for any two distinct natural numbers k and m, the point $x_{\xi,k}$ does not belong to the orbit $G_{1,\xi}(x_{\xi,m})$.

Indeed, let $\{Z_{\xi,k} : k < \omega\}$ denote an injective family of all those G_0-orbits which are contained in Y_ξ. Suppose that, for a natural number k, the elements

$$x_{\xi,0} \in Z_{\xi,0}, x_{\xi,1} \in Z_{\xi,1}, \ldots, x_{\xi,k-1} \in Z_{\xi,k-1}$$

have already been defined and that they lie in pairwise distinct $G_{1,\xi}$-orbits. Consider the set

$$P_k = G_{1,\xi}(x_{\xi,0}) \cup G_{1,\xi}(x_{\xi,1}) \cup \ldots \cup G_{1,\xi}(x_{\xi,k-1}).$$

Clearly, we have

$$\mathrm{card}(P_k \cap Z_{\xi,k}) \leq k, \qquad \mathrm{card}(Z_{\xi,k}) = \omega.$$

Consequently, there exists an element $x \in Z_{\xi,k} \setminus P_k$. So we can put $x_{\xi,k} = x$.

Therefore, for each ordinal $\xi < \omega_1$, we get the corresponding ω-sequence $\{x_{\xi,k} : k < \omega\}$ of points from Y_ξ, fulfilling conditions (i) and (ii).

Now, we define $X = \{x_{\xi,k} : \xi < \omega_1,\ k < \omega\}$ and verify that the set X is G-absolutely nonmeasurable in E.

Indeed, on the one hand, we may write

$$G_0(X) = \cup\{G_0(\{x_{\xi,k} : k < \omega\}) : \xi < \omega_1\} = \cup\{Y_\xi : \xi < \omega_1\} = E$$

and the above relation implies that if X is measurable with respect to some nonzero σ-finite G-quasi-invariant measure μ on E, then necessarily $\mu(X) > 0$.

On the other hand, let us take an arbitrary element $g \in G_1 \setminus \{\mathrm{Id}_E\}$. Then there exists an ordinal $\xi_0 < \omega_1$ for which $g \in G_{1,\xi_0}$. Further, for any $\xi < \omega_1$, let us denote

$$X_\xi = \{x_{\xi,k} : k < \omega\}.$$

Evidently, we have

$$(\forall \xi < \omega_1)(\mathrm{card}(X_\xi) = \omega).$$

Also, the equality

$$X = \cup\{X_\xi : \xi < \omega_1\}$$

implies the inclusion

$$g(X) \cap X \subset \cup\{g(X_\zeta) \cap X_\eta : \zeta < \omega_1,\ \eta < \omega_1\}.$$

If $\zeta < \omega_1$ and $\eta < \omega_1$ satisfy the relations $\xi_0 < \zeta$ and $\xi_0 < \eta$, then

$$g(X_\zeta) \cap X_\eta = \emptyset.$$

In addition to this, if $\zeta < \xi_0$ and $\eta > \xi_0$, or, respectively, $\zeta > \xi_0$ and $\eta < \xi_0$, then

$$g(X_\zeta) \cap X_\eta = g(X_\zeta \cap g^{-1}(X_\eta)) \subset g(Y_\zeta \cap Y_\eta) = \emptyset,$$

or, respectively,

$$g(X_\zeta) \cap X_\eta \subset Y_\zeta \cap Y_\eta = \emptyset.$$

We thus get the inclusion

$$g(X) \cap X \subset (\cup\{g(X_\zeta) : \zeta \leq \xi_0\}) \cup (\cup\{X_\eta : \eta \leq \xi_0\})$$

and, therefore,

$$\mathrm{card}(g(X) \cap X) \leq \omega.$$

Finally, suppose that g and h are any two distinct elements of G_1. Then

$$h^{-1} \circ g \neq \mathrm{Id}_E, \qquad h^{-1} \circ g \in G_1,$$

and, according to the fact established above, we may write

$$\mathrm{card}((h^{-1} \circ g)(X) \cap X) \leq \omega$$

which implies at once that

$$\mathrm{card}(g(X) \cap h(X)) \leq \omega.$$

The last inequality shows that if the set X is measurable with respect to some σ-finite G-quasi-invariant measure μ on E, then $\mu(X) = 0$. So we must have simultaneously $\mu(X) > 0$ and $\mu(X) = 0$. Obviously, this yields a contradiction and hence X is a G-absolutely nonmeasurable subset of E.

Now, let us return to the partition $\{Y_\xi : \xi < \omega_1\}$ of our ground set E and introduce the following two sets:

$$T_1 = \cup\{X \cap Y_\xi : \xi < \omega_1, \ \xi \text{ is an odd ordinal number}\},$$

$$T_2 = \cup\{X \cap Y_\xi : \xi < \omega_1, \ \xi \text{ is an even ordinal number}\}.$$

Clearly, $X = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Further, if $\{g_i : i \in I\}$ is an arbitrary countable family of elements of G, then

$$E \setminus \cup\{g_i(T_1) : i \in I\} \neq \emptyset, \qquad E \setminus \cup\{g_i(T_2) : i \in I\} \neq \emptyset.$$

It is not hard to infer from this property of the sets T_1 and T_2 that there exist two probability G-invariant measures μ_1 and μ_2 on E such that

$$T_1 \in \mathrm{dom}(\mu_1), \qquad T_2 \in \mathrm{dom}(\mu_2), \qquad \mu_1(T_1) = \mu_2(T_2) = 0.$$

Finally, keeping in mind the relations $T_1 \subset X$ and $T_2 \subset X$, we conclude that both T_1 and T_2 are G-negligible sets in E. Lemma 1 has thus been proved. □

Lemma 2. *Let $(G, \cdot)$ and $(H, \cdot)$ be two groups and let*

$$\phi : (G, \cdot) \to (H, \cdot)$$

be a surjective homomorphism. The following assertions are valid for any two sets $X \subset H$ and $Y \subset H$:

(1) *if X is an H-negligible subset of H, then $\phi^{-1}(X)$ is a G-negligible subset of G;*

(2) *if Y is an H-absolutely nonmeasurable subset of H, then $\phi^{-1}(Y)$ is a G-absolutely nonmeasurable subset of G.*

The proof of Lemma 2 is not difficult (see, e.g., [1] or [2]).

The next two propositions are purely algebraic and can be deduced from well-known theorems of the general theory of commutative groups (cf. [6,7]).

Lemma 3. *If $(H, +)$ is an uncountable commutative group, then there exist two subgroups H_0 and H_1 of $(H, +)$ such that:*

(1) $\mathrm{card}(H_0) = \omega$ *and* $\mathrm{card}(H_1) = \omega_1$*;*

(2) $H_0 \cap H_1 = \{0\}$*, where 0 stands for the neutral element of H.*

Lemma 4. *If $(G, +)$ is an uncountable commutative group, then there exists a surjective homomorphism*

$$\phi : (G, +) \to (H, +),$$

where $(H, +)$ is some commutative group of cardinality ω_1.

Lemma 5. *Let $(G, \cdot)$ be a group and let H be a normal subgroup of G such that* $\mathrm{card}(G/H) \leq \omega$. *The following two assertions are valid:*

(1) *if a set X is H-absolutely nonmeasurable in H, then X is also G-absolutely nonmeasurable in G;*

(2) *if a set Y is H-negligible in H, then Y is also G-negligible in G.*

The proof of Lemma 5 readily follows from the definitions of negligible and absolutely nonmeasurable sets.

Theorem 1. *If $(G, +)$ is an uncountable commutative group, then there exist two G-negligible subsets Y_1 and Y_2 in G such that their union $Y_1 \cup Y_2$ is G-absolutely nonmeasurable in G.*

Proof. According to Lemma 4, there exists a surjective homomorphism

$$\phi : (G, +) \to (H, +)$$

for some commutative group $(H, +)$ of cardinality ω_1. Applying Lemmas 1 and 3 to $(H, +)$, we obtain two H-negligible subsets X_1 and X_2 of H such that the set $X_1 \cup X_2$ is H-absolutely nonmeasurable in H. Let us denote

$$Y_1 = \phi^{-1}(X_1), \qquad Y_2 = \phi^{-1}(X_2).$$

By virtue of Lemma 2, both sets Y_1 and Y_2 are G-negligible in G. Also, in view of the same lemma, the set

$$Y_1 \cup Y_2 = \phi^{-1}(X_1) \cup \phi^{-1}(X_2) = \phi^{-1}(X_1 \cup X_2)$$

turns out to be G-absolutely nonmeasurable in G. This finishes the proof of Theorem 1. □

Theorem 2. *If $(G, \cdot)$ is an uncountable solvable group, then there exist two G-negligible sets Y_1 and Y_2 in G such that the set $Y_1 \cup Y_2$ is G-absolutely nonmeasurable in G.*

Proof. Since $(G, \cdot)$ is solvable, there exists a finite sequence

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_{n-1} \subset G_n = G$$

of subgroups of G satisfying these two relations:

(i) for each natural index $k \in [1, n]$, the group G_{k-1} is normal in the group G_k;

(ii) for each natural index $k \in [1, n]$, the quotient group G_k/G_{k-1} is commutative.

To demonstrate the validity of our assertion, we argue by induction on n.

If $n = 1$, then the uncountable group $G = G_n$ is commutative, and we may apply Theorem 1 to this G.

Suppose now that the assertion holds true for a natural number $n - 1 \geq 1$ and let us establish its validity for n.

For this purpose, consider the commutative quotient group $H = G_n/G_{n-1}$, where, as above, $G_n = G$. Here only two cases are possible.

(a) the group $H = G_n/G_{n-1}$ is uncountable.

In this case, we take the canonical surjective homomorphism

$$\phi : (G_n, \cdot) \to (H, +).$$

By virtue of Theorem 1, there are two H-negligible subsets X_1 and X_2 in H such that their union $X_1 \cup X_2$ is H-absolutely nonmeasurable in H. We put

$$Y_1 = \phi^{-1}(X_1), \qquad Y_2 = \phi^{-1}(X_2).$$

Then, keeping in mind Lemma 2, we see that both sets Y_1 and Y_2 are G-negligible in G, and we also deduce that the set

$$Y_1 \cup Y_2 = \phi^{-1}(X_1) \cup \phi^{-1}(X_2) = \phi^{-1}(X_1 \cup X_2)$$

turns out to be G-absolutely nonmeasurable in G.

(b) the group $H = G_n/G_{n-1}$ is countable.

In this case, in view of the uncountability of $G_n = G$, the group G_{n-1} is necessarily uncountable, and we can apply the inductive assumption to this G_{n-1}. So there are two G_{n-1}-negligible subsets Y_1 and Y_2 of G_{n-1} such that the set $Y_1 \cup Y_2$ is G_{n-1}-absolutely nonmeasurable in G_{n-1}. Lemma 5 now yields that, simultaneously, Y_1 and Y_2 are G-negligible subsets of G and their union $Y_1 \cup Y_2$ is a G-absolutely nonmeasurable set in G. Theorem 2 has thus been proved. □

Example 2. Let $(G, \cdot)$ be an arbitrary uncountable solvable group. It directly follows from Theorem 2 that there are two G-negligible sets Y_1 and Y_2 in G possessing the following property: for any nonzero σ-finite left G-quasi-invariant measure μ on G, at least one of the sets Y_1 and Y_2 is nonmeasurable with respect to μ.

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