Available online at www.sciencedirect.com

ScienceDirect

Transactions of A. Razmadze Mathematical Institute 170 (2016) 200–204

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

On the cardinal number of the family of all invariant extensions of a nonzero σ-finite invariant measure

Alexander Kharazishvili

A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University, 6 Tamarashvili st., Tbilisi 0177, Georgia

Available online 24 May 2016

Abstract

It is shown that, for any nonzero σ-finite translation invariant (translation quasi-invariant) measure μ on the real line $\mathbf{R}$, the cardinality of the family of all translation invariant (translation quasi-invariant) measures on $\mathbf{R}$ extending μ is greater than or equal to 2^{ω_1}, where ω_1 denotes the first uncountable cardinal number. Some related results are also considered.

Keywords: Invariant measure; Quasi-invariant measure; Metrical transitivity; Nonseparable extension of measure; Ulam's transfinite matrix

Let E be a base (ground) set and let G be a group of transformations of E. The pair (E, G) is usually called a space equipped with a transformation group.

A measure μ defined on some G-invariant σ-algebra of subsets of E is called quasi-invariant with respect to G (briefly, G-quasi-invariant) if, for any μ-measurable set X and for any transformation g from G, the relation

$$\mu(X) = 0 \quad \Leftrightarrow \quad \mu(g(X)) = 0$$

holds true. Moreover, if the equality $\mu(g(X)) = \mu(X)$ is valid for any μ-measurable X and for any g from G, then μ is called an invariant measure with respect to G (briefly, G-invariant measure).

According to these definitions, the triplet of the form (E, G, μ) determines the structure of an invariant (quasi-invariant) measure on E.

Suppose that μ is a nonzero σ-finite G-invariant (G-quasi-invariant) measure on E. It is known that if a group G is uncountable and acts freely in E, then there always exist subsets of E nonmeasurable with respect to μ (see [1]; cf. also [2]). So the domain of μ differs from the family of all subsets of E, i.e., $\mathrm{dom}(\mu) \neq \mathcal{P}(E)$. In this connection, the natural question arises whether there exists a G-invariant (G-quasi-invariant) measure μ' on E strongly extending μ. This question was studied for various types of spaces (E, G, μ). Undoubtedly, the most interesting case for classical Real Analysis is when E coincides with the n-dimensional Euclidean space $\mathbf{R}^n$, a group G is a subgroup of the group of all isometric transformations of $\mathbf{R}^n$, and μ is a G-invariant extension of the standard n-dimensional Lebesgue measure λ_n on $\mathbf{R}^n$ (see, for instance, [3–8]).

E-mail address: kharaz2@yahoo.com.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.05.002

Another important case is when $E = \Gamma$, where Γ is an uncountable σ-compact locally compact topological group, Γ coincides with the group of all left (right) translations of Γ, and μ is a G-invariant extension of the left (right) Haar measure on Γ (cf. [9,5,10,11]).

A more general form of the above question is as follows. For a given space (E, G, μ), denote by $\mathcal{M}_G(\mu)$ the family of all measures on E extending μ and invariant (quasi-invariant) with respect to G. It is natural to try to evaluate the cardinality of $\mathcal{M}_G(\mu)$ in terms of card(E) and card(G). In the present paper, we will be dealing with this problem for the case when E coincides with the real line $\mathbf{R}$ and G is the group of all translations of $\mathbf{R}$. Notice that the method applied in our further considerations is primarily taken from [6].

Below, we will use the following standard notation:

$X \triangle Y =$ the symmetric difference of two sets X and Y;
$\omega =$ the least infinite cardinal (ordinal) number;
$\omega_1 =$ the least uncountable cardinal (ordinal) number;
$\mathbf{c} =$ the cardinality of the continuum.

Let μ be a measure defined on some σ-algebra of subsets of E (here μ is not assumed to be invariant or quasi-invariant under a nontrivial group of transformations of E). The Hilbert space of all square μ-integrable real-valued functions on E is usually denoted by the symbol $L_2(\mu)$. If $L_2(\mu)$ is a separable Hilbert space, then μ is called a separable measure. Otherwise, μ is called a nonseparable measure.

Treating the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers and keeping in mind the existence of a Hamel basis in $\mathbf{R}$, it is not difficult to show that the additive group $(\mathbf{R}, +)$ admits a representation in the form

$$\mathbf{R} = G + H \quad (G \cap H = \{0\}),$$

where G and H are some two subgroups of $(\mathbf{R}, +)$ and

$$\mathrm{card}(G) = \omega_1, \quad \mathrm{card}(H) \leq \mathbf{c}.$$

We denote by $\mathcal{I}$ the σ-ideal generated by all those subsets X of $\mathbf{R}$ which are representable in the form $X = Y + H$, where $Y \subset G$ and card$(Y) \leq \omega$.

It can readily be seen that $\mathcal{I}$ is a translation invariant σ-ideal of sets in $\mathbf{R}$.

We begin with the following auxiliary statement.

Lemma 1. *There exists a partition $\{X_\xi : \xi < \omega_1\}$ of $\mathbf{R}$ satisfying these two relations:*

(1) *for any ordinal $\xi < \omega_1$, the set X_ξ belongs to the σ-ideal $\mathcal{I}$;*
(2) *for each subset Ξ of ω_1 and for any $r \in \mathbf{R}$, the relation*

$$(\cup\{X_\xi : \xi \in \Xi\}) \triangle (r + \cup\{X_\xi : \xi \in \Xi\}) \in \mathcal{I}$$

holds true, i.e., the set $\cup\{X_\xi : \xi \in \Xi\}$ is $\mathcal{I}$-almost translation invariant in $\mathbf{R}$.

The proof of this lemma is given in [6].

By combining Lemma 1 with the well-known $(\omega \times \omega_1)$-matrix of Ulam (see, e.g., [12]), the next auxiliary statement can be deduced.

Lemma 2. *Let $\{X_\xi : \xi < \omega_1\}$ be a partition of $\mathbf{R}$ described in Lemma 1 and let μ be a nonzero σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$.*

There exists a disjoint family $\{\Xi_j : j \in J\}$ of subsets of ω_1 such that:

(1) card$(J) = \omega_1$*;*
(2) *for each index $j \in J$, the set $Z_j = \cup\{X_\xi : \xi \in \Xi_j\}$ is nonmeasurable with respect to μ (where $\{X_\xi : \xi < \omega_1\}$ is a partition of $\mathbf{R}$ described in Lemma 1);*
(3) $\mu_*(\cup\{Z_j : j \in J\}) = 0$ *(where the symbol μ_* denotes the inner measure associated with μ).*

Notice that the proof of Lemma 2 is similar to the argument presented in [6] (cf. also [7]).

Lemma 3. *Let μ be a σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$. There exists a measure μ' on $\mathbf{R}$ such that:*

(1) *μ' is translation invariant (translation quasi-invariant);*
(2) *μ' extends μ;*
(3) $\mathcal{I} \subset \mathrm{dom}(\mu')$.

Proof. If X is any set belonging to $\mathcal{I}$, then the equality $\mu_*(X) = 0$ is satisfied, because in $\mathbf{R}$ there are uncountably many pairwise disjoint translates of X. So we may apply Marczewski's standard method to μ and $\mathcal{I}$ for extending μ. Namely, introduce the σ-algebra $\mathcal{S}'$ of all those subsets Z of $\mathbf{R}$ which admit a representation

$$Z = (Y \cup X') \setminus X'' \quad (Y \in \mathrm{dom}(\mu), X' \in \mathcal{I}, X'' \in \mathcal{I})$$

and define on $\mathcal{S}'$ the functional μ' by the formula

$$\mu'(Z) = \mu(Y) \quad (Z \in \mathcal{S}').$$

It is not hard to verify that the definition of μ' is correct (i.e., the value $\mu'(Z)$ does not depend on a representation of Z in the above-mentioned form), and μ' satisfies the relations. (1), (2), and (3) of Lemma 3. □

The preceding lemmas enable us to establish the following statement.

Theorem 1. *Let μ be a nonzero σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$. Then the inequality $\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\omega_1}$ holds true. In particular, there are measures on $\mathbf{R}$ strictly extending μ and invariant (quasi-invariant) under the group of all translations of $\mathbf{R}$.*

Proof. Taking into account Lemma 3, we may assume without loss of generality that the measure μ is complete and $\mathcal{I} \subset \mathrm{dom}(\mu)$.

Let $\{Z_j : j \in J\}$ be the disjoint family of subsets of $\mathbf{R}$ described in Lemma 2. This family has the following properties:

(a) $\mathrm{card}(J) = \omega_1$ and the sets Z_j $(j \in J)$ are pairwise disjoint;
(b) every set Z_j $(j \in J)$ is nonmeasurable with respect to μ;
(c) for each set $J_0 \subset J$ and for every $r \in \mathbf{R}$, the equality

$$\mu((\cup\{Z_j : j \in J_0\}) \triangle (r + \cup\{Z_j : j \in J_0\})) = 0$$

is valid;
(d) $\mu_*(\cup\{Z_j : j \in J\}) = 0$.

Further, take a subset J_1 of J and associate to this J_1 the set

$$Z(J_1) = \cup\{Z_j : j \in J_1\}.$$

By virtue of (c) and (d), we get the relations:
(e) for every $r \in \mathbf{R}$, the set $Z(J_1)$ is μ-almost translation invariant, i.e.,

$$\mu(Z(J_1) \triangle (r + Z(J_1))) = 0;$$

(f) $\mu_*(Z(J_1)) = 0$.

Consequently, applying Marczewski's method of extending invariant and quasi-invariant measures (cf. the proof of Lemma 3), we obtain the measure μ_{J_1} on $\mathbf{R}$ which extends μ, is invariant (quasi-invariant) under the group of all translations of $\mathbf{R}$, and satisfies the equality $\mu_{J_1}(Z(J_1)) = 0$.

Now, let us establish that if J_1 and J_2 are any two distinct subsets of J, then the associated measures μ_{J_1} and μ_{J_2} differ from each other. Indeed, if $J_1 \neq J_2$, then either $J_1 \setminus J_2 \neq \emptyset$ or $J_2 \setminus J_1 \neq \emptyset$. We may suppose that $J_1 \setminus J_2 \neq \emptyset$, so there is an index $j \in J_1 \setminus J_2$. According to the definition of μ_{J_1}, the set Z_j turns out to be of μ_{J_1}-measure zero. On the other hand, the same set Z_j cannot be of μ_{J_2}-measure zero. To see this circumstance, suppose to the contrary that $\mu_{J_2}(Z_j) = 0$. Then, keeping in mind the construction of μ_{J_2}, we must have

$$Z_j = (T \cup T') \setminus T'',$$

where

$$\mu(T) = 0, \qquad T' \subset Z(J_2), \qquad T'' \subset Z(J_2).$$

However, it can easily be verified that the above relations imply the inclusion $Z_j \subset T$ and the equality $\mu(Z_j) = 0$. In particular, we obtain that Z_j is a μ-measurable set, which contradicts (b).

Thus, we have an injective mapping from the power set $\mathcal{P}(\omega_1)$ into the family of all those measures on **R** which extend μ and are translation invariant (translation quasi-invariant). The existence of such a mapping trivially yields the desired inequality $\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\omega_1}$, and the proof of Theorem 1 is finished. □

Remark 1. Consider the n-dimensional Euclidean space $\mathbf{R}^n$, where $n \geq 1$. Since there exists an isomorphism between the additive groups $(\mathbf{R}, +)$ and $(\mathbf{R}^n, +)$, the direct analogue of Theorem 1 is valid for the space $\mathbf{R}^n$ (and, more generally, for any uncountable vector space over the field **Q** of all rational numbers).

Remark 2. As an immediate consequence of Theorem 1, we get the relation

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\omega_1} \geq 2^{\omega} = \mathbf{c}.$$

This relation is a statement of **ZFC** set theory. Assuming the Continuum Hypothesis (**CH**), we directly come to the much stronger inequality

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) \geq 2^{\mathbf{c}}.$$

We do not know whether the latter inequality can be proved within the framework of **ZFC** theory.

Let the symbol λ $(= \lambda_1)$ denote the standard Lebesgue measure on the real line **R**. Kakutani and Oxtoby demonstrated in 1950 that there exist nonseparable measures on **R** belonging to the class $\mathcal{M}_{\mathbf{R}}(\lambda)$ (see [13]). Obviously, all those measures are strict extensions of λ. A radically different approach to the problem of the existence of nonseparable measures belonging to $\mathcal{M}_{\mathbf{R}}(\lambda)$ was given in the work by Kodaira and Kakutani (see again [13]).

The method of Kakutani and Oxtoby allows one to conclude that there exist at least $2^{2^{\mathbf{c}}}$ nonseparable measures on **R**, all of which extend λ and are translation invariant. Thus, for the concrete measure λ on **R**, the inequality of Theorem 1 can be essentially strengthened and, in fact, we have the following equality:

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\lambda)) = 2^{2^{\mathbf{c}}}.$$

In this context, the natural question arises whether the analogous equality

$$\mathrm{card}(\mathcal{M}_{\mathbf{R}}(\mu)) = 2^{2^{\mathbf{c}}}$$

is valid for any nonzero σ-finite translation invariant (translation quasi-invariant) measure μ on **R**. We do not know the answer to this question. Nevertheless, assuming the Continuum Hypothesis (**CH**), for a sufficiently wide class of measures μ on **R** it can be proved that the last equality holds true, too.

Let (E, G, μ) be a space equipped with a σ-finite G-invariant (G-quasi-invariant) measure μ. Recall that μ is metrically transitive (or ergodic) if, for any μ-measurable set X with $\mu(X) > 0$, there exists a countable family $\{g_k : k < \omega\}$ of transformations from G such that

$$\mu(E \setminus \cup\{g_k(X) : k < \omega\}) = 0.$$

It is well known that metrically transitive (ergodic) measures play an important role in many topics of mathematical analysis and probability theory.

Lemma 4. *Let (E, G) be a space equipped with a transformation group satisfying these two conditions:*

(1) $\mathrm{card}(E) = \omega_1$;
(2) *the group G acts freely and transitively in E.*

If μ is a nonzero σ-finite ergodic G-invariant (G-quasi-invariant) measure on E, then there exists a partition $\{X_\xi : \xi < \omega_1\}$ of E such that:

(i) *every set X_ξ $(\xi < \omega_1)$ is μ-thick in E, i.e., $\mu_*(E \setminus X_\xi) = 0$;*
(ii) *for any set $\Xi \subset \omega_1$ and for each transformation $g \in G$, the inequality*

$$\mathrm{card}((\cup\{X_\xi : \xi \in \Xi\}) \triangle (g(\cup\{X_\xi : \xi \in \Xi\}))) \leq \omega$$

is valid.

The proof of Lemma 4 is presented in [10].

Starting with the previous lemma and applying some modified version of the method of Kakutani and Oxtoby, we get the following statement.

Theorem 2. *Assume* **CH** *and let* (E, G) *be a space equipped with a transformation group, satisfying the conditions* (1) *and* (2) *of Lemma* 4.

Then, for every nonzero σ*-finite ergodic* G*-invariant (*G*-quasi-invariant) measure* μ *on* E*, the class* $\mathcal{M}_G(\mu)$ *contains at least* $2^{2^{\mathfrak{c}}}$ *nonseparable measures.*

As has already been mentioned, the proof of Theorem 2 is based on Lemma 4 and on the argument of Kakutani and Oxtoby [13] (cf. also [10]).

Remark 3. Marczewski's method of extending σ-finite invariant (quasi-invariant) measures does not substantially change the structure of an initial measure. On the other hand, the method of Kakutani and Oxtoby allows one to obtain nonseparable translation invariant extensions of λ on $\mathbf{R}$, starting with the separable measure λ (however, those extensions are not ergodic). Further modifications of this method were applied to the Haar measure on an uncountable σ-compact locally compact Polish topological group (see, for instance, [9]). Notice that various properties of invariant and quasi-invariant measures given on algebraic-topological structures are thoroughly discussed in [14].

Theorem 3. *Assume* **CH** *and let* (E, G) *be again a space equipped with a transformation group, satisfying the conditions* (1) *and* (2) *of Lemma* 4.

Then, for every nonzero σ*-finite ergodic* G*-invariant (*G*-quasi-invariant) measure* μ *on* E*, the class* $\mathcal{M}_G(\mu)$ *contains* $2^{2^{\mathfrak{c}}}$ *ergodic measures.*

The proof of Theorem 3 follows the method presented in [7] for a concrete space (E, G, μ). Namely, in [7] the role of (E, G, μ) is played by the triplet $(\mathbf{R}^n, D_n, \lambda_n)$, where $n \geq 1$ and D_n denotes the group of all isometric transformations of $\mathbf{R}^n$. Under **CH**, the argument given in [7] for $(\mathbf{R}^n, D_n, \lambda_n)$ works also for a space (E, G, μ) of Theorem 3.

Remark 4. Both Theorems 2 and 3 show that, supposing **CH**, the cardinality of the class $\mathcal{M}_G(\mu)$ is equal to the cardinality of the class of all measures on E (where a space (E, G) satisfies (1) and (2) of Lemma 4 and μ is a nonzero σ-finite ergodic G-invariant or G-quasi-invariant measure on E).

References

[1] A.B. Kharazishvili, Certain types of invariant measures, Dokl. Akad. Nauk SSSR 222 (3) (1975) 538–540 (in Russian).
[2] P. Erdös, R.D. Mauldin, The nonexistence of certain invariant measures, Proc. Amer. Math. Soc. 59 (2) (1976) 321–322.
[3] K. Ciesielski, A. Pelc, Extensions of invariant measures on Euclidean spaces, Fund. Math. 125 (1) (1985) 1–10.
[4] H. Friedman, A definable nonseparable invariant extension of Lebesgue measure, Illinois J. Math. 21 (1) (1977) 140–147.
[5] A. Hulanicki, Invariant extensions of the Lebesgue measure, Fund. Math. 51 (1962–1963) 111–115.
[6] A.B. Kharazishvili, Some applications of Hamel bases, Sakharth. SSR Mecn. Akad. Moambe 85 (1) (1977) 17–20 (in Russian).
[7] A.B. Kharazishvili, Invariant Extensions of the Lebesgue Measure, Tbilis. Gos. Univ., Tbilisi, 1983 (in Russian).
[8] Sh.S. Pkhakadze, The theory of Lebesgue measure, Proc. A. Razmadze Math. Inst. 25 (1958) 3–272 (in Russian).
[9] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis. Vol. I: Structure of Topological Groups, in: Integration Theory, Group Representations (Die Grundlehren der mathematischen Wissenschaften), Bd. 115, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin, Göttingen, Heidelberg, 1963.
[10] A.B. Kharazishvili, Metrical transitivity and nonseparable extensions of invariant measures, Taiwanese J. Math. 13 (3) (2009) 943–949.
[11] A. Pelc, Invariant measures and ideals on discrete groups, Dissertationes Math. (Rozprawy Mat.) 255 (1986).
[12] J.C. Oxtoby, Measure and Category. A Survey of the Analogies Between Topological and Measure Spaces, in: Graduate Texts in Mathematics, vol. 2, Springer-Verlag, New York, Berlin, 1971.
[13] S. Kakutani, in: Robert R. Kallman. (Ed.), Selected Papers. Vol. 2, in: Contemporary Mathematicians, Birkhäuser Boston, Inc., Boston, MA, 1986.
[14] P. Zakrzewski, Measures on algebraic-topological structures, in: Handbook of Measure Theory, Vols. I, II, North-Holland, Amsterdam, 2002, pp. 1091–1130.