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MODULAR WEIGHTED INEQUALITIES FOR PARTIAL SUMS OF FOURIER-VILENKIN SERIES

M. KHABAZI

Abstract. Some conditions for the convergence and boundedness of partial sums of Fouurier-Vilenkin series in weighted Orlicz Classes are derived.

Fourier-Vilenkin series is the generalization of Fourier-Walsh series. Let $(p_i)_{i\geq 0}$ be any sequense of natural numbers, such that $p_i \geq 2$, $i = 0, 1, \dots$ By Z_{p_i} we denote the cyclic group of order p_i, by G–the direct product of these groups: $G = \prod\limits_{i=0}^{\infty} Z_{p_i}$ and by μ–the Haar measure normalized $\mu(G) = 1$. The functions

$$\phi_k(x) = \exp\left(2\pi i \frac{x_k}{p_k}\right), \quad x = (x_k) \in G, \quad k = 0, 1, \dots$$

represent an orthonormal system on G. It can be completed by the following process: let $m_0 = 1$, $m_k = p_0 p_1 \dots p_{k-1}$; every nonnegative integer number n can be represented by the unique way as a finite sum, $n = \sum\limits_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$. Define the functions χ_n $(n = 0, 1, \dots)$:

$$\chi_n(x) = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}(x).$$

$\{\chi_n\}$ form the complete orthonormal system on G, known as a mulitiplicative system or Vilenkin system. For the details see [1],[2].

The group G can be identified with the interval $(0, 1)$, putting to each $\{x_i\} \in G$ into correspondence the point $\sum\limits_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0, 1)$. If we will not regard the countable set of p_i-rational points, this mapping is one-to-one, onto and measure-preserving.

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Wo-Sang Young [3] defined the Muckenhoupt classes for the group G. Let $\{G_k\}$ be the sequence of subgroups of G defined by

$$G_0 = G, \quad G_k = \prod_{i=0}^{k-1}\{0\} \times \prod_{i=k}^{\infty} Z_p, \quad k = 1, 2, \dots$$

On the interval $(0,1)$, cosets of G_k are intervals of the form $\left(\frac{i}{m_k}, \frac{i+1}{m_k}\right)$, $i = 0, 1, \dots, m_{k-1}$. For $k = 0, 1, \dots,$ $i = 1, \dots p_k$, let I_{ik} be the set in G_k corresponding to the interval $\left(0, \frac{i}{m_{k+1}}\right)$. Let F denote the collection of all translates of I_{ik} in G, for all $k = 0, 1, \dots,$ $i = 1, \dots, p_k$.

A weight function w (a.e., positive integrable function on G) belongs to the class $A_p(G)$ $(1 \le p < \infty)$ if

$$\sup_{I \in F} \left(\frac{1}{\mu I} \int\limits_I w^{-\frac{1}{p-1}} d\mu\right)^{p-1}, \quad 1 < p < \infty \tag{1}$$

and

$$\frac{1}{\mu I} \int\limits_I w(x)dx \le c \operatorname*{ess\,inf}_{y \in I} w(y), \quad p = 1, \tag{2}$$

where c is independent of $I \in F$.

The $A_p(G)$ classes have the main properties of classical Muckenhoupt classes: if $w \in A_p(G)$, then $w \in A_q(G)$ for every $q > p$ and when $p > 1$, there exists an $\varepsilon > 0$, such that $w \in A_{p-\varepsilon}(G)$. Also, if $w \in A_p(G)$ $(p > 1)$, then $w^{-\frac{1}{p-1}} \in A_{p'}(G)$ and w, $w^{-\frac{1}{p-1}} \in L^1(G)$ (p' is defined by the equality $\frac{1}{p} + \frac{1}{p'} = 1$).

Let $S_n f$ denote the n-th partial sum of the Fourier-Vilenkin series of a function f. We will assume that $S_n f \equiv \infty$ for $n = 1, 2, \dots$ when $f \notin L^1$. If w is a weight on G, by $L^p_w(G)$ we denote the class of all measurable functions f, such that $\int_G |f(x)|^p w(x) d\mu(x) < \infty$. The following theorem belongs to Wo-Sang Young:

Theorem A. *Let w be a weight on G and $1 < p < \infty$. The following statements are equivalent:*

(i) *There is a constant c, independent of $f \in L^p_w(G)$, such that*

$$\int\limits_G |S_n f|^p w d\mu \le c \int\limits_G |f|^p w d\mu, \quad n = 1, 2, \dots.$$

(ii) *For every $f \in L^p_w(G)$*

$$\lim_{n \to \infty} \int\limits_G |f - S_n f|^p w d\mu = 0.$$

(iii) $w \in A_p$.

Our goal was to investigate the same problem for the weighted Orlicz classes. We need some definitions to formulate our results.

Let ϕ denote the set of all functions $\varphi : \mathbf{R}^1 \to \mathbf{R}^1$ which are nonnegative, even, and increasing on $(0,\infty)$ such that $\varphi(0+) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$. If w is a weight on G, by $\varphi_w(L)$ we denote the class of all measurable functions f, such that $\int_G \varphi(f(x))w(x)d\mu(x) < \infty$.

A function ω is called a Young function on $[0,\infty)$ if ω is convex, $\omega(0) = 0$, and $\omega(\infty) = \infty$. A function φ is called quasiconvex if there exist a Young function ω and a constant $c > 1$ such that $\omega(t) \le \varphi(t) \le \omega(ct)$, $t \ge 0$. If these inequalities hold for $t > t_0 > 0$ we say that φ is quasiconvex in a neighborhood of ∞. The concept of quasiconvexity, as well as the fundamental definition of the number $p(\varphi)$, which follows, was introduced by V. Kokilashvili and thoroughly investigated by him and his colleagues (see e.g. [4], [5], [6]).

By definition the function φ satisfies Δ_2 condition ($\varphi \in \Delta_2$) if there exist numbers $c > 0$ and $t_0 > 0$ such that $\varphi(2t) \le c\varphi(t)$, when $t > t_0$. If this inequality holds for every $t > 0$ then they say that φ satisfies the global Δ_2 condition ($\varphi \in \overline{\Delta}_2$).

For any quasiconvex function φ let us define numbers $p(\varphi)$ and $q(\varphi)$ as

$$\frac{1}{p(\varphi)} = \inf\{\beta : \beta > 0, \quad \varphi^\beta \quad \text{is quasiconvex}\}$$

$$\frac{1}{q(\varphi)} = \inf\{\beta : \beta > 0, \varphi^\beta \quad \text{is quasiconvex in a neighborhood of} \quad \infty\}.$$

To each quasiconvex function φ corresponds the complementary function $\widetilde{\varphi}$, defined by the equality $\widetilde{\varphi}(t) = \sup_{s\ge 0}(st - \varphi(s))$. It is easy to check that $\widetilde{\varphi}$ is Young function and $\widetilde{\widetilde{\varphi}} \le \varphi$.

Now we can formulate our results. We suppose that $\sup\{p_i\} < \infty$.

Theorem 1. *Let w be a weight and $\varphi \in \phi$. The following statements are equivalent:*

(i) *There is a constant c, independent of $f \in \varphi_w(L)$, such that*

$$\int_G \varphi(S_n f)wd\mu \le c\int_G \varphi(f)wd\mu, \quad n = 1, 2, \dots. \tag{3}$$

(ii) *There exists a number α, $0 < \alpha < 1$, such that φ^α is quasiconvex, φ satisfies global Δ_2 condition and $w \in A_{p(\varphi)}(G)$.*

To prove this theorem we need some lemmas.

Lemma 1. *Let $\varphi \in \phi$. The following statements are equivalent:*

(i) *φ is quasiconvex.*

(ii) *There exists a constant $c_1 > 0$, such that*

$$\frac{\varphi(t_1)}{t_1} \le c_1 \frac{\varphi(c_1 t_2)}{t_2}$$

when $t_1 < t_2$.

(iii) *There exists a constant* $c_2 > 0$, *such that*

$$\varphi(t) \le c_2\widetilde{\widetilde{\varphi}}(t), \quad t > 0.$$

(iv) *There exists a constant* $c_3 > 0$, *such that*

$$\varphi\bigg(\frac{1}{|I|}\int\limits_I f(x)dx\bigg) \le \frac{c_3}{|I|}\int\limits_I \varphi(c_3 f(x))dx.$$

Lemma 2. *If* $\varphi \in \phi$ *is quasiconvex and satisfies global* Δ_2 *condition, then there exist a constant* $c > 0$, *such that*

$$\varphi\Big(\frac{\widetilde{\varphi}(t)}{t}\Big) \le c\widetilde{\varphi}(t), \quad t > 0.$$

Lemma 3. *Let* $\varphi \in \phi$. *Then the following conditions are equivalent:*

(i) φ^α *is quasiconvex for some* α, $0 < \alpha < 1$.

(ii) φ *is quasiconvex and* $\widetilde{\varphi} \in \overline{\Delta}_2$.

(iii) *There exists a constant* $c > 0$, *such that*

$$\int\limits_0^t \frac{\varphi(s)}{s^2}ds \le c\frac{\varphi(ct)}{t},$$

for every $t > 0$.

Let $\varphi \in \phi$, w_i, $i = 1, 2, 3, 4$ be weights on G and $f_I = \frac{1}{\mu I}\int_I f d\mu$ for $I \in F$. Let us suppose that there exists a positive number c such that for every nonnegative measurable function f and $I \in F$ the following inequality holds:

$$\int\limits_I \varphi(f_I w_1)w_2 d\mu \le c\int\limits_I \varphi(cfw_3)w_4 d\mu. \tag{4}$$

Lemma 4. *The following statements are true:*

(i) *if* $w_1 \equiv w_3 \equiv 1$ *and* $w_2 \equiv w_4 \equiv w$, *the* (4) *condition holds then and only then when* φ *is quasiconvex and* $w \in A_{p(\varphi)}$.

(ii) $w_1 \equiv w_3 \equiv w$ *and* $w_2 \equiv w_4 \equiv 1$, *the* (4) *condition holds then and only then when* φ *is quasiconvex,* $w^{p(\varphi)} \in A_{p(\varphi)}(G)$ *and* $w^{-p(\widetilde{\varphi})} \in A_{p(\widetilde{\varphi})}(G)$.

(iii) *if* $w_1 \equiv w_3 \equiv w$ *and* $w_2 \equiv w_4 \equiv \frac{1}{w}$, *the* (4) *condition holds then and only then when* φ *is quasiconvex and* $w \in A_{p(\widetilde{\varphi})}$.

The following interpolation theorem belongs to V. Kokilashvili and A. Gogatishvili [7]:

Lemma 5. *Let (M, S, ν) and (M_1, S_1, ν_1) be measure spaces, $T : L^0(M) \to L^0(M_1)$-semilinear operator, $\varphi \in \phi$-quasiconvex function and $1 \le r < p(\varphi) \le p'(\widetilde{\varphi}) < s < \infty$ and in the every $\lambda > 0$ and $f \in L^r(\nu) + L^s(\nu)$*

$$\int\limits_{\{x \in M_1 : |Tf(x)| > \lambda\}} \le c_1 \lambda^{-r} \int\limits_M |f(x)|^r d\nu,$$

$$\int\limits_{\{x \in M_1 : |Tf(x)| > \lambda\}} \le c_2 \lambda^{-s} \int\limits_M |f(x)|^s d\nu,$$

and in the case $s = \infty$

$$\|Tf\|_\infty \le c_2 \|f\|_\infty,$$

then there exists a positive constant c_3, independent of T, such that

$$\int\limits_{M_1} \varphi(Tf) d\nu_1 \le c_3 \int\limits_M \varphi(f) d\nu, \quad f \in \varphi(L, M).$$

These lemmas and its proofs can be found in [6], [7].

Proof of Theorem 1. (i)⇒(ii). As φ is quasiconvex, by Lemma 1 $\widetilde{\widetilde{\varphi}} \sim \varphi$, and as $\varphi \in \overline{\Delta}_2$, by Lemma 3 $\widetilde{\varphi}^\beta$ is quasiconvex for some β, $0 < \beta < 1$. In this case $p'(\widetilde{\varphi}) < \infty$. Let $p'(\widetilde{\varphi}) < s < \infty$ and $r < p(\varphi)$ be such a number that $w \in A_r$. By lemma 5, where $M = M_1$ and $d\nu = d\nu_1 = w d\mu$ and also by Theorem A we obtain (ii).

(ii)⇒(i). As $\sup\{p_i\} < \infty$, is enough to show that (1) holds for the intervals $\left(\frac{i}{m_k}, \frac{i+1}{m_k}\right)$, $k = 0, 1, \dots, m_k - 1$. Let I be one of those intervals, $f \in L^1(G)$, $f \ge 0$ and $\operatorname{supp} f \subset I$. As it is known

$$S_{m_k} f(x) = \frac{1}{\mu I} \int\limits_I f d\mu, \quad x \in I.$$

Then, by (3) we get

$$\varphi\left(\frac{1}{\mu I} \int\limits_I f d\mu\right) \le \frac{c}{wI} \int\limits_I \varphi(f) w d\mu.$$

By Lemma 4 this means that φ is quasiconvex and $w \in A_{p(\varphi)}(G)$.

Now we are going to show that $\varphi, \widetilde{\varphi} \in \overline{\Delta}_2$.

Lemma 6. *Let $E \subset G$ be any set of positive measure and there exists a constant $c > 0$, such that for any measurable function f, with* $\operatorname{supp} f \subset E$,

$$\int\limits_E \varphi(S_n f) d\mu \le c \int\limits_E \varphi(cf) d\mu, \quad n = 1, 2, \dots. \tag{5}$$

Then φ satisfies $\overline{\Delta}_2$ condition.

Proof. Suppose that $x=0$ is a density point of E. We are going to construct a bounded and measurable function f, such that $\|f\|_\infty \le 1$, $\operatorname{supp} f \subset E$ and $\limsup_{n\to\infty} |S_n f(0)| = \infty$.

Define n_s numbers in the following manner:

$$n_s = 1 + m_2 + \cdots + m_{2s}, \quad s = 0, 1, \dots$$

It can be easily seen that $n_3 < \frac{4}{3} m_{2s}$. Let (s_k) be a sequense of natural numbers, which we will define later, and β be a real number, $0<\beta<1$. Let D_n denote the Dirichlet kernel for the multiplicative system and define the function f,

$$f(x) = \begin{cases} \frac{|D_{n_{s_k}}(x)|}{D_{n_{s_k}}(x)}, & x \in E \cap (G_{2s_{k-1}} \backslash G_{2s_k}), \\ 0, & x \in G \backslash \cap_k (G_{2s_{k-1}} \backslash G_{2s_k}), \end{cases}$$

where the ratio is considered to be 1, if $D_{n_{s_k}}(x) = 0$, We want to estimate $J_k = S_{n_{s_k}} f(0)$, representing it as a sum of the following terms:

$$J_k = \int_G f(x) D_{n_{s_k}}(x) d\mu(x) = \int_{G_{2s_k}} + \int_{G_{2s_{k-1}} \backslash G_{2s_k}} + \int_{G \backslash G_{2s_{k-1}}} = J_k' + J_k'' + J_k'''.$$

For $|D_{n_{s_k}}(x)| \le n_{s_k} < \frac{4}{3} m_{2s_k}$,

$$|J_k'| \le \max |f(t)| m_{2s_k} \frac{4}{m_{2s_k}} \le \frac{4}{3}. \tag{6}$$

It is known that ([1], [2])

$$D_{n_{s_k}} = D_{m_{2s_k}} + \chi_{m_{2s_k}} D_{n_{s_k - 1}} \tag{7}$$

so

$$|J_k'''| = \left| \int_{G \backslash G_{2s_{k-1}}} f(x) D_{n_{s_k}}(x) d\mu(x) \right| =$$

$$= \left| \int_{G \backslash G_{2s_{k-1}}} f(x) (D_{m_{2s_k}}(x) + \chi_{m_{2s_k}}(x) D_{n_{s_k - 1}}(x)) d\mu(x) \right|,$$

but $D_{m_{2s_k}}(x) = 0$ when $x \in G_{2s_k}$ ([3] and)

$$|J_k'''| = \left| \int_{G \backslash G_{2s_{k-1}}} f(x) \chi_{m_{2s_k}}(x) D_{n_{s_{k-1}}}(x) d\mu(x) \right|.$$

We will construct the equense (s_k) by induction. Put $s_1 = 2$ and suppose that $s_1, \dots, s_{k-1}$ are already constructed. Then f is defined on $G_{2s_{k-1}}$ and $f(x) D_{n_{s_k - 1}}(x)$ is bounded there. As the Fourier-Vilenkin coefficients of a bounded function tend to zero, we can shose such a big s_k, that $s_k > s_{k-1}$ and

$$|J_k'''| < 1. \tag{8}$$

Now we will estimate J_k'':

$$J_k'' = \int\limits_{E\cap(G_{2s_{k-1}}\setminus G_{2s_k})} |D_{n_{s_k}}(x)|d\mu(x) \geq \int\limits_{E\cap G_{2s_{k-1}}} |D_{n_{s_k}}(x)|d\mu(x) -$$

$$- \int\limits_{G_{2s_k}} |D_{n_{s_k}}(x)|d\mu(x) \geq \int\limits_{E\cap G_{2s_{k-1}}} |D_{n_{s_k}}(x)|d\mu(x) - \frac{4}{3}. \tag{9}$$

We can chose s_k numbers so, that

$$\mu E \cap G_{2s_{k-1}} > \frac{\beta}{m_{2s_{k-1}}} \tag{10}$$

From (7) easily follows

$$|D_{n_{s_k}}(x)| > \frac{c}{x}, \quad x \in (0,1),$$

and taking into consideration that the function $\frac{1}{x}$ is decreasing on $(0,1)$, from (9) and (10) we get

$$J_k'' \geq c \int\limits_{E\cap G_{2s_{k-1}}} \frac{d\mu(x)}{x} - \frac{4}{3} \geq c \int\limits_{\left(\frac{1-\beta}{m_{2s_{k-1}}}, \frac{1}{m_{2s_{k-1}}}\right)} \frac{dt}{t} - \frac{4}{3} = c \ln \frac{1}{1-\beta} - \frac{4}{3},$$

and, as β was arbitrarily taken in (0,1), we have $\limsup\limits_{k\to\infty} J_k = \infty$, which means, that

$$\limsup_{n\to\infty} S_n f(0) = \infty.$$

From this follows that there exists a number $n \in N$, such that $|S_n f(0)| > 2c$. Then there exists a neighbourhood I_0 of zero, such that $|S_n f(0)| > 2c$ for every $x \in I_0$. Now let t be any positive number and a function g is defined by the equality: $g(x) = \frac{t}{c} f(x)$. It is obvious that $|S_n g(x)| > 2t$ when $x \in I_0$. Applying (5) for g, we get

$$\varphi(2t)\mu I_0 \cap E \leq c \int\limits_E \varphi(cg(x))d\mu(x) \leq c\varphi(t)\mu E.$$

Thus,

$$\varphi(2t) \leq \frac{c\mu E}{\mu I_0 \cap E}\varphi(t),$$

which means that φ satisfy global Δ_2 condition. In the case when $x = 0$ is not a point of density of E, but $x = x_0$ is it, the proof is the same. □

Let us continue the proof of theorem. suppose that k is such a number that the set $E=\left\{x:\frac{1}{k}\le w(x)\le k\right\}$ has a positive measure. Let $f\in\varphi_w(L)$ and $\operatorname{supp} f\subset E$. Then from (3) follows

$$\int\limits_E \varphi(S_nf(x))d\mu(x)\le c_1\int\limits_E \varphi(f(x))d\mu(x),\quad n=1,2,\dots,\tag{11}$$

where $c_1=ck^2$. By lemma 6 then $\varphi\in\overline{\Delta}_2$. According to Lemma 2, there exists a $c_2>0$, such that

$$\varphi\Big(\frac{\widetilde{\varphi}(t)}{t}\Big)\le c_2\widetilde{\varphi}(t),\quad t>0.\tag{12}$$

Using Young's inequality, (12) and (11) we get

$$\int\limits_E \widetilde{\varphi}(S_nf(x))d\mu(x)=\int\limits_E \frac{\widetilde{\varphi}(S_nf(x))}{S_nf(x)}S_nf(x)d\mu(x)=$$
$$=\int\limits_E S_n\Big(\frac{\widetilde{\varphi}(S_nf)}{S_nf}\chi_E\Big)(x)f(x)d\mu(x)\le$$
$$\le\frac{1}{2c_1c_2}\int\limits_E\varphi\Big(S_n\Big(\frac{\widetilde{\varphi}(S_nf)}{S_nf}\chi_E\Big)(x)\Big)d\mu(x)+\frac{1}{2c_1c_2}\int\limits_E\widetilde{\varphi}(2c_1c_2f(x))d\mu(x)\le$$
$$\le\frac{1}{2c_2}\int\limits_E\varphi\Big(\frac{\widetilde{\varphi}(S_nf(x))}{S_nf(x)}\Big)d\mu(x)+\frac{1}{2c_1c_2}\int\limits_E\widetilde{\varphi}(2c_1c_2f(x))d\mu(x)\le$$
$$\le\frac{1}{2}\int\limits_E\widetilde{\varphi}(S_nf(x))d\mu(x)+\frac{1}{2c_1c_2}\int\limits_E\widetilde{\varphi}(2c_1c_2f(x))d\mu(x),$$

and

$$\int\limits_E\widetilde{\varphi}(S_nf(x))d\mu(x)\le\frac{1}{c_1c_2}\int\limits_E\widetilde{\varphi}(2c_1c_2f(x))d\mu(x).$$

Applying once more Lemma 6 we conclude that $\widetilde{\varphi}$ satisfies the global Δ_2 condition. As we have already shown that φ is quasiconvex, by Lemma 3 φ^α is quasiconvex for some α, $0<\alpha<1$. Theorem 1 is proved.

The following theorems are the modifications of Theorem 1.

Theorem 2. *Let w be a weight and $\varphi\in\phi$. The following statements are equivalent:*

(i) *There is a constant c, independent of $f\in\varphi_w(L)$, such that for any measurable function f*

$$\int\limits_G\varphi(s_nfw)d\mu\le c\int\limits_G\varphi(fw)d\mu,\quad n=1,2,\dots.\tag{13}$$

(ii) *There exists a number* α, $0<\alpha<1$, *such that* φ^{α} *is quasiconvex,* φ *satisfies global* Δ_2 *condition and,* $w^{p(\varphi)} \in A_{p(\varphi)}(G)$ *and* $w^{-p(\widetilde{\varphi})} \in A_{p(\widetilde{\varphi})}(G)$.

Proof. (ii)$\Rightarrow$ (i). From the conditions of the theorem follows that $p(\varphi)>1$ and $p(\widetilde{\varphi})>1$. Then there exists an $\varepsilon>0$, such that $w^{p(\varphi)-\varepsilon} \in A_{p(\varphi)-\varepsilon}$ and $w^{-(p(\widetilde{\varphi})-\varepsilon} \in A_{p(\widetilde{\varphi})-\varepsilon}$. Therefore $w^{1-(p(\varphi)-\varepsilon)'} \in A_{(p(\varphi)-\varepsilon)'}$. Define the operators T_n by the following manner:

$$T_n f = w S_n\Big(\frac{f}{w}\Big), \quad n=1,2,\dots .$$

By virtue of Theorem A,

$$\int_G |T_n f|^{p(\varphi)-\varepsilon} d\mu \le c_1 \int_G |f|^{p(\varphi)-\varepsilon} d\mu,$$
$$\int_G |T_n f|^{(p(\widetilde{\varphi})-\varepsilon)'} d\mu \le c_2 \int_G |f|^{(p(\varphi)-\varepsilon)'} d\mu.$$

As $p(\varphi)-\varepsilon < p(\varphi) < p'(\widetilde{\varphi}) < (p(\widetilde{\varphi})-\varepsilon)'$, by lemma 5 we get

$$\int_G \varphi(T_n f) d\mu \le c_3 \int_G \varphi(f) d\mu.$$

Changing f by fw we obtain (i).

(i)$\Rightarrow$(ii). Let $i \in F$, $f \ge 0$ and $\operatorname{supp} f \subset I$. As $S_{m_k} f(x) = \frac{1}{\mu I}\int_I f d\mu$, $x\in I$, from (13) we have

$$\int_I \varphi(f_I w) d\mu \le c \int_I \varphi(fw) d\mu.$$

By Lemma 4 this means that φ is quasiconvex, $w^{p(\varphi)} \in A_{p(\varphi)}$ and $w^{-p(\widetilde{\varphi})} \in A_{p(\widetilde{\varphi})}$. Let k be a positive number, such that $E=\Big\{x : \frac{1}{k} \le w(x) \le k\Big\}$ has a positive measure. If $\operatorname{supp} f \subset E$, then from (13) follows

$$\int_E \varphi(S_n f) d\mu \le c \int_E \varphi(k^2 f) d\mu.$$

As we saw while proving Theorem 1, from here follows that $\varphi, \widetilde{\varphi} \in \Delta_2$. As φ is quasiconvex, by lemma 3 φ^{α} is quasiconvex for some α, $0<\alpha<1$. □

Theorem 3. *Let* w *be a weight and* $\varphi \in \phi$. *The following statements are equivalent:*

(i) *There is a constant* c, *independent of* $f \in \varphi_w(L)$, *such that for any measurable function* f

$$\int_G \varphi\Big(\frac{S_n f}{w}\Big) w d\mu \le c \int_G \varphi\Big(\frac{f}{w}\Big) w d\mu, \quad n=1,2,\dots . \tag{14}$$

(ii) *There exists a number* α, $0<\alpha<1$, *such that* φ^α *is quasiconvex,* φ *satisfies global* Δ_2 *condition and* $w \in A_{p(\widetilde{\varphi})}(G)$.

Proof. (ii)⇒(i). From the condition $w \in A_{p(\widetilde{\varphi})}(G)$ follows that $w \in A_{p(\widetilde{\varphi})-\varepsilon}(G)$ for some $\varepsilon > 0$. Then $w^{1-(p(\widetilde{\varphi})-\varepsilon)'} \in A_{p(\widetilde{\varphi})-\varepsilon)'}$. As φ is quasiconvex, $p(\widetilde{\widetilde{\varphi}}) = p(\varphi)$ and $p(\widetilde{\varphi}) < p'(\varphi) < (p(\varphi)-\varepsilon)'$, thus $w \in A_{p(\widetilde{\varphi})-\varepsilon}$. From this we get that $w^{1-(p(\varphi)-\varepsilon)} \in A_{p(\varphi)-\varepsilon)}$. Let us consider the operator

$$T_n f = \frac{1}{w} S_n(fw), \quad n = 1, 2, \ldots .$$

By virtue of Theorem A,

$$\int_G |T_n f|^{p(\varphi)-\varepsilon} d\mu \le c_1 \int_G |f|^{p(\varphi)-\varepsilon} d\mu,$$

$$\int_G |T_n f|^{(p(\varphi)-\varepsilon)'} d\mu \le c_2 \int_G |f|^{(p(\varphi)-\varepsilon)'} d\mu.$$

As $p(\varphi)-\varepsilon < p(\varphi) < p'(\widetilde{\varphi}) < (p(\widetilde{\varphi})-\varepsilon)'$, by lemma 5 we get

$$\int_G \varphi(T_n f) d\mu \le c_3 \int_G \varphi(f) d\mu.$$

Changing f by $\frac{f}{w}$ we obtain (i).

(i)⇒(ii). Let $I \in F$, $f \ge 0$ and $\operatorname{supp} f \subset I$. As $S_{m_k} f(x) = \frac{1}{\mu I}\int_I f d\mu$, $x \in I$, from (14) we have

$$\int_I \varphi(f_I w^{-1}) d\mu \le c \int_I \varphi(f w^{-1}) d\mu.$$

By Lemma 4 this means that φ is quasiconvex and $w \in A_{p(\widetilde{\varphi})}$. The rest of the proof coincides with the one of Theorem 2.

Finally we must note that analogous theorems for various classical operators were proved by V. Kokilashvili, A. Gogatishvili and M. Krbec in [4], [5–7]. □

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Author's address;
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi, 380093
Georgia