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THE MEAN CONVERGENCE OF TRIGONOMETRIC FOURIER SERIES IN WEIGHTED ORLICZ CLASSES

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Abstract. Some conditions for the mean convergence of trigonometric Fourier series in weighted Orlicz classes are derived.

Let $S_n f$ $(n = 1, 2, \dots)$ denote the n-th partial sum of the trigonometric series of the function f. We will assume that $S_n f \equiv \infty$ for $n = 1, 2, \dots$ when $f \notin L^1$. If w is a weight on $(-\pi, \pi)$ (a.e., positive summable function), by $L^p_w(-\pi, \pi)$ we denote the class of all measurable functions f, such that $\int\limits_{-\pi}^{\pi} |f(x)|^p w(x)\, dx < \infty$. The following theorem of Hunt, Muckenhoupt and Wheeden [1] is well known:

Theorem A. *Let w be a weight on $(-\pi, \pi)$ and $1 < p < \infty$. The following statements are equivalent:*

(i) *For every $f \in L^p_w(-\pi, \pi)$*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \big|f(x) - S_n f(x)\big|^p w(x)\, dx = 0.$$

(ii) $w \in A_p$.

(For the definition of A_p see e.g., [1], [2]).

Our goal was to investigate the same problem for the weighted Orlicz classes. We need some definitions to formulate our results.

Let Φ denote the set of all functions $\varphi : \mathbf{R^1} \to \mathbf{R^1}$ which are nonnegative, even, and increasing on $(0, \infty)$ such that $\varphi(0+) = 0$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$. If w is a weight on $(-\pi, \pi)$, by $\varphi_w(L)$ we denote the class of all measurable functions f, such that $\int\limits_{-\pi}^{\pi} \varphi(f(x)) w(x) dx < \infty$.

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A function ω is called a Young function on $[0,\infty)$ if ω is convex, $\omega(0)=0$, and $\omega(\infty)=\infty$. A function φ is called quasiconvex if there exist a Young function ω and a constant $c>1$ such that $\omega(t)\le\varphi(t)\le\omega(ct)$, $t\ge 0$. If these inequalities hold for $t>t_0>0$ we say that φ is quasiconvex in a neighborhood of ∞. The concept of quasiconvexity, as well as the fundamental definition of the number $p(\varphi)$, which follows, was introduced by V. Kokilashvili and thoroughly investigated by him and his colleagues (see e.g., [3], [4], [5]).

By definition the function φ satisfies Δ_2 condition ($\varphi\in\Delta_2$) if there exist numbers $c>0$ and $t_0>0$ such that $\varphi(2t)\le c\varphi(t)$, when $t>t_0$. If this inequality holds for every t then they say that φ satisfies the global Δ_2 condition ($\varphi\in\overline{\Delta_2}$).

For any quasiconvex function φ let us define a number $p(\varphi)$ and $q(\varphi)$ as

$$\frac{1}{p(\varphi)}=\inf\big\{\beta : p>0,\ \varphi^{\beta}\ \text{ is quasiconvex}\big\},$$

$$\frac{1}{q(\varphi)}=\inf\big\{\beta : \beta>0,\ \ \varphi^{\beta}\ \text{ is quasiconvex in a neighborhood of }\infty\big\}. \quad (1)$$

Now we can formulate our results.

Theorem 1. *Let w be weight and $\varphi\in\Phi$. If φ satisfies Δ_2 condition, there exists a number α, $0<\alpha<1$, such that φ^{α} is quasiconvex in a neighborhood of ∞ and $w\in A_{q(\varphi)}$, where $q(\varphi)$ is defined by (1), then for every $f\in\varphi_w(L)$*

$$\lim_{n\to\infty}\int\limits_{-\pi}^{\pi}\varphi\big(f(x)-S_nf(x)\big)w(x)\,dx=0. \quad (2)$$

To prove this theorem we need some lemmas.

Lemma 1. *Let $\varphi\in\Phi$ satisfies Δ_2 condition, there exists a number α, $0<\alpha<1$, such that φ^{α} is quasiconvex in a neighborhood of ∞ and $\varepsilon>0$. Then there exist a $\psi\in\Phi$ and a number $x_0>0$ such that ψ satisfies global Δ_2 condition, ψ^{β} is quasiconvex for some β, $0<\beta<1$, $\varphi(x)=\psi(x)$, when $x>x_0$ and $p(\psi)>q(\varphi)-\varepsilon$.*

Proof. Let us suppose that $\varepsilon<1$, $\frac{1}{q(\varphi)}<a<\frac{1}{q(\varphi)-\varepsilon}$ and φ^{α} is quasiconvex in a neighborhood of ∞. It means that there exist a Young function ω and a number $x_1>0$, such that

$$\omega(x)\le\varphi^{\alpha}(x)\le\omega(cx),$$

when $x>x_1$. Since $\alpha>\frac{1}{p(\varphi)}$, $\lim\limits_{x\to\infty}\frac{\omega(x)}{x}=\lim\limits_{x\to\infty}\frac{\varphi^{\alpha}(x)}{x}=\infty$. Then ([6], Theorem 3.3), there exist $\gamma>1$ and $x_2>0$, such that the function

$$\omega_1(x)=\begin{cases}k|x|^{\gamma}, & |x|\le x_2\\ \omega(x), & |x|>x_2\end{cases}$$

is a Young function. Let us define the function ψ in the following way:

$$\psi(x)=\begin{cases}\omega_1^{\frac{1}{\alpha}}(x), & |x|\le \max(x_1,x_2)\\ \varphi(x), & |x|>\max(x_1,x_2).\end{cases}$$

It is not difficult to see that ψ satisfies global Δ_2 condition and ψ^α is quasiconvex. Also,

$$\frac{1}{p(\psi)}<\alpha<\frac{1}{q(\varphi)-\varepsilon},$$

from where follows that $p(\psi)>q(\varphi)-\varepsilon$. The lemma is proved. □

Lemma 2. *Let $\varphi\in\Phi$, there exists a number α, $0<\alpha<1$, such that φ^α is quasiconvex in a neighborhood of ∞ and $w\in A_{q(\varphi)}$. Then for every $f\in\varphi_w(L)$ and $\sigma>0$*

$$\lim_{n\to\infty} w\big\{x:\big|f(x)-S_n(x)\big|>\sigma\big\}=0. \tag{3}$$

Proof. It is easy to prove that if ψ is quasiconvex in a neighborhood of ∞, then there exist $t_0>0$ and $c>0$, such that

$$\frac{\psi(t_1)}{t_1}\le\frac{\psi(ct_2)}{t_2},$$

when $t_2>t_1>t_0$. Hence, if φ^α is quasiconvex in a neighborhood of ∞, then $\varphi_w(L)\subset L^p_w$, where $p=\frac{1}{\alpha}$. Let us take such $\varepsilon>0$ that $w\in A_{q(\varphi)-\varepsilon}$ and $q=q(\varphi)-\varepsilon>1$. If α is chosen so that $\frac{1}{\alpha}>q(\varphi)-\varepsilon$ and φ^α is quasiconvex in a neighborhood of ∞, then, by Theorem A

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\big|f(x)-S_nf(x)\big|^p w(x)\,dx=0.$$

From where immediately follows (3). □

Lemma 3. *Let $\varphi\in\Phi$ satisfies Δ_2 condition and w is a weight on $(-\pi,\pi)$. Then the class of all trigonometric polinomes is everywhere dense in $\varphi_w(L)$, i.e., for every $f\in\varphi_w(L)$ and $\varepsilon>0$ there exists a trigonometric polinome T, such that*

$$\int_{-\pi}^{\pi}\varphi\big(f(x)-T(x)\big)w(x)\,dx<\varepsilon.$$

Proof. Let $f\in\varphi_w(L)$. Since φ satisfies Δ_2 condition, $2f\in\varphi_w(L)$ and by the absolute continuity of Lebesgue integral there is $\delta_1>0$, such that

$$\int_e\varphi\big(2f(x)\big)w(x)\,dx<\varepsilon, \tag{4}$$

whenever $|e|<\delta_1$. Let a be a positive number, such that $|\{x:|f(x)|>a\}|<\delta_1$. Using once more the absolute continuity of Lebesgue integral, we can find a number $\delta_2>0$, such that

$$\int_e w(x)\,dx<\frac{\varepsilon}{\varphi(82)}, \tag{5}$$

whenever $|e|<\delta_2$. Let us define a function h,

$$h(x)={}_a f(x)=\begin{cases} f(x), & |f(x)|\le a\\ 0, & |f(x)|>a. \end{cases}$$

By Lusin's Theorem there exists a continuos function g, such that $\|g\|_\infty\le\|f\|_\infty\le a$ and

$$\big|\{x:h(x)\ne g(x)\}\big|<\delta_2. \tag{6}$$

If T is trigonometric polinome with $\|g-T\|_\infty<\varepsilon$, then taking in consideration (4), (5) and (6), we get

$$\begin{aligned}
&\int_{-\pi}^{\pi}\varphi\big(f(x)-T(x)\big)w(x)\,dx\le\int_{-\pi}^{\pi}\varphi\big(2\big(f(x)-h(x)\big)\big)w(x)\,dx+\\
&+\int_{-\pi}^{\pi}\varphi\big(2\big(h(x)-T(x)\big)\big)w(x)\,dx\le\int_{\{x:|f(x)|>a\}}^{\pi}\varphi\big(2\big(f(x)\big)\big)w(x)\,dx+\\
&+\int_{-\pi}^{\pi}\varphi\big(4\big(h(x)-g(x)\big)\big)(x)w(x)\,dx+\int_{-\pi}^{\pi}\varphi\big(4\big(g(x)-T(x)\big)\big)(x)w(x)\,dx\le\\
&\le\varepsilon+\varphi(8a)\int_{\{h\ne g\}}w(x)\,dx+\varphi(4\varepsilon)w(-\pi,\pi)<2\varepsilon+\varphi(4\varepsilon)w(-\pi,\pi),
\end{aligned}$$

and, as $\lim\limits_{\varepsilon\to0}\varphi(4\varepsilon)=0$, the lemma is proved. □

We will also use our result, obtained earlier in [7].

Theorem B. *Let w be a weight function and $\varphi\in\Phi$. The following conditions are equivalent:*

(i) *there is $c>0$, such that the inequalities*

$$\int_{-\pi}^{\pi}\varphi\big(S_nf(x)\big)w(x)\,dx<c\int_{-\pi}^{\pi}\varphi\big(f(x)\big)w(x)\,dx,\quad n=1,2,\dots.$$

(ii) *φ satisfies global Δ_2 condition, φ^α is quasiconvex for some α, $0<\alpha<1$, and $w\in A_{p(\varphi)}$.*

Proof of Theorem 1. Let ε be such that $w \in A_{q(\varphi)-\varepsilon}$ and ψ be the function defined by Lemma 1. Since $p(\psi) > q(\varphi) - \varepsilon$, $w \in A_{p(\psi)}$. If $\sigma > 0$, then

$$\int_{-\pi}^{\pi} \varphi\big(f(x) - S_n f(x)\big) w(x)\,dx = \int_{\{x:|f(x)-S_n f(x)|\le\sigma\}} \varphi\big(f(x)-S_n f(x)\big) w(x)\,dx + \\ + \int_{\{x:\sigma<|f(x)-S_n f(x)|\le x_0\}} \varphi\big(f(x) - S_n f(x)\big) w(x)\,dx + \\ + \int_{\{x:|f(x)-S_n f(x)|>x_0\}} \varphi\big(f(x)-S_n f(x)\big) w(x)\,dx = J_1 + J_2 + J_3.$$

We will estimate each of J_1, J_2 and J_3.

$$J_1 \le \varphi(\sigma) w(-\pi, \pi), \tag{7}$$

$$J_2 \le \varphi(x_0) w\big\{x : \big|f(x) - S_n f(x)\big| > \sigma\big\}, \tag{8}$$

$$J_3 = \int_{\{x:|f(x)-S_n f(x)|>0\}} \varphi\big(f(x) - S_n f(x)\big) w(x)\,dx \le \\ \le \int_{-\pi}^{\pi} \psi\big(f(x) - S_n f(x)\big) w(x)\,dx.$$

According to Lemma 3 there is a trigonometric polinome T, such that

$$\int_{-\pi}^{\pi} \psi\big(f(x) - T(x)\big) w(x)\,dx < \sigma.$$

If n is greater then the order of the polinome T,then by Theorem B,

$$J_3 \le c_1 \int_{-\pi}^{\pi} \psi\big(f(x) - T(x)\big) w(x)\,dx + c_1 \int_{-\pi}^{\pi} \psi\big(T(x) - S_n f(x)\big) w(x)\,dx \le \\ \le c_1\sigma + c_1 \int_{-\pi}^{\pi} \psi\big(S_n(f-T)(x)\big) w(x)\,dx \le \\ \le c_1\sigma + c_2 \int_{-\pi}^{\pi} \psi\big(f(x) - T(x)\big) w(x)\,dx \le c_3\sigma. \tag{9}$$

From (7), (8) and (9) follows

$$\int_{-\pi}^{\pi} \varphi\big(f(x) - S_n f(x)\big) w(x)\,dx \le w(-\pi, \pi)\varphi(\sigma) +$$

$$+\varphi(x_0)w\{x : |f(x) - S_n f(x)|\} > \sigma\} + c_3\sigma,$$

when n is great enough. Then, by Lemma 2,

$$\limsup_{n\to\infty} \int_{-\pi}^{\pi} \varphi\big(f(x) - S_n f(x)\big) w(x)\, dx \le w(-\pi, \pi)\varphi(\sigma) + c_3\sigma.$$

and sending σ to zero we get

$$\limsup_{n\to\infty} \int_{-\pi}^{\pi} \varphi\big(f(x) - S_n f(x)\big) w(x)\, dx = 0$$

and the proof is completed. $\square$

The following theorem is a partial reversion of Theorem 1.

Theorem 2. *If for any $f \in \varphi_w(L)$ the equality* (2) *holds, then φ satisfies Δ_2 condition and φ is quasiconvex in a neighborhood of ∞.*

The proof of this theorem is based on the following lemma, which at the same time generalizes a result of P. Oswald's [8].

Lemma 4. *If $E \subset (-\pi, \pi)$ has a positive Lebesgue measure and for every $f \in \varphi_w(L)$,* $\operatorname{supp} f \subset E$

$$\lim_{n\to\infty} \int_E \varphi\big(f(x) - S_n f(x)\big)\, dx = 0, \tag{10}$$

then φ satisfies Δ_2 condition and φ is quasiconvex in a neighborhood of ∞.

Proof. We will start with the proof of $\varphi \in \Delta_2$. We can suppose that $x = 0$ is the density point of E. Let $\varphi \notin \Delta_2$. Then there exists an increasing sequence $(t_k)_{k\ge1}$, such that

$$\varphi\Big(\frac{21}{20} t_k\Big) \ge 2^{2k} \varphi(t_k) \ \ (k = 1, 2, \dots), \ \ \varphi(t_1) > 1, \tag{11}$$

$$t_k > 2^k \sum_{i=1}^{k-1} t_i \quad (k = 2, 3, \dots). \tag{12}$$

Then we can find integer numbers $2 \le n_1 < n_2 < \dots$, which satisfy the inequalities

$$2^{-k-1} < 2^{-n_k} \varphi(t_k) \le 2^{-k} \quad (k = 1, 2, \dots). \tag{13}$$

Let us define the function f in the following way:

$$f(x) = \begin{cases} t_k, & x \in \Big(2^{-n_k},\, \frac{11}{10} 2^{-n_k}\Big) \cap E \\ -t_k, & x \in \Big(\frac{11}{10} 2^{-n_k},\, 2^{-n_k+1}\Big) \cap E \\ 0, & x \in (-\pi, \pi) \backslash \Big(2^{-n_k},\, \frac{11}{10} 2^{-n_k}\Big) \cap E. \end{cases} \tag{14}$$

It is obvious that $\operatorname{supp} f \subset E$ and

$$\int_{-\pi}^{\pi} \varphi\big(f(x)\big) w(x)\, dx \le \sum_{k=1}^{\infty} \varphi(t_k) 2^{-n_k} \le \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

One can easily check that the Dirichlet kernel $D_n(x) = \frac{\sin(n+\frac{1}{2})(x)}{2\sin\frac{1}{2}x}$ satisfies the inequalities:

$$\frac{2}{\pi}\Big(n+\frac{1}{2}\Big) \le D_n(x) \le \frac{\pi}{2}\Big(n+\frac{1}{2}\Big), \quad x \in \Big[-\frac{1}{n}, \frac{1}{n}\Big]. \tag{15}$$

Let us take $x \in [0, 2^{-n_k+1}]$ and estimate $S_{2^{n_k-2}} f(x)$.

$$S_{2^{n_k-2}} f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_{2^{n_k-2}}(x-t)\, dt = \frac{1}{\pi}\Bigg(\int_0^{2^{-n_k+1}} + \int_{2^{-n_k+1}}^{\pi} \Bigg) = \frac{1}{\pi}(I_1 + I_2).$$

Applying (15) we get

$$I_1 = \sum_{i=k}^{\infty} t_i \Bigg(\int_{2^{-n_i}}^{\frac{11}{10}2^{-n_i}} 1_E(t) D_{2^{n_k-2}}(t-x)\, dt - \int_{\frac{11}{10}2^{-n_i}}^{2^{-n_i+1}} 1_E(t) D_{2^{n_k-2}}(t-x)\, dt \Bigg) \le$$

$$\le \sum_{i=k}^{\infty} t_i \Big(\frac{\pi}{2}\Big(2^{n_k-2} + \frac{1}{2}\Big)\frac{2^{-n_k}}{10} - \frac{2}{\pi}\Big(2^{n_k-2}+\frac{1}{2}\Big)\Big|\Big(\frac{11}{10}2^{-n_i}, 2^{-n_i+1}\Big) \cap E\Big|\Big).$$

As $x=0$ is a denisity point of E, for a k_0 great enough when $i \ge k_0$ we have

$$\Big|\Big(\frac{11}{10}2^{-n_i}, 2^{-n_i+1}\Big) \cap E\Big| > \frac{8}{9}\Big|\Big(\frac{11}{10}2^{-n_i}, 2^{-n_i+1}\Big)\Big| = \frac{8}{10}2^{-n_i}.$$

Hence,

$$I_1 \le \sum_{i=k}^{\infty} 2^{-n_i} t_i \Big(2^{n_k-2} + \frac{1}{2}\Big)\Big(\frac{\pi}{20} - \frac{16}{10\pi}\Big) \le$$

$$\le 2^{-n_k} t_k, \Big(2^{n_k-2} + \frac{1}{2}\Big)\Big(\frac{\pi^2 - 32}{20\pi}\Big) < \frac{t_k}{4}\frac{\pi^2-32}{20\pi} < -\frac{t_k}{16}.$$

Now let us apply (12) and (13) to estimate I_2.

$$|I_2| \le \sum_{i=1}^{k-1} t_i \int_{2^{-n_i}}^{2^{-n_i+1}} \frac{dt}{2|\sin\frac{1}{2}(t-x)|} \le \sum_{i=1}^{k-1} t_i 2^{-n_i} 2^{n_i+2} < 2^{-k+2} t_k.$$

So, if $k \ge k_0$ is great enough, we have

$$S_{2^{n_k-2}} f(x) \le \frac{1}{\pi}\Big(-\frac{1}{16} + 2^{-k+2}\Big) t_k < -\frac{t_k}{20} \quad \big(x \in [0, 2^{-n_k+1}]\big).$$

Then, by (11) and (13)

$$\int_E \varphi\big(f(x) - S_{2^{n_k-2}} f(x)\big)\,dx \geq \int_{E\cap(2^{-n_k},\frac{11}{10}2^{-n_k})} \varphi\big(f(x) - S_{2^{n_k-2}} f(x)\big)\,dx \geq$$

$$\geq \varphi\Big(\frac{21}{20}t_k\Big)\Big|E\cap\Big(2^{-n_k},\frac{11}{10}\,2^{-n_k}\Big)\Big| \geq c\frac{2^{-n_k}}{10}\,2^{2k}\varphi(t_k) > \frac{c}{20}\,2^k, \quad (k\geq k_0).$$

Therefore,

$$\limsup_{n\to\infty}\int_E \varphi\big(f(x) - S_n f(x)\big)\,dx = \infty$$

and this is in the contradiction with the condition of the lemma. So, the statement $\varphi\in\Delta_2$ is proved.

No we will show that if φ is not quasiconvex in a neighborhood of ∞, then there exists an f, such that

$$\limsup_{n\to\infty}\int_E \varphi\big(f(x) - S_n f(x)\big)\,dx = \infty. \tag{16}$$

In fact it is enough to prove the existance of an f, for which

$$\limsup_{n\to\infty}\int_E \varphi\big(S_n f(x)\big)\,dx = \infty. \tag{17}$$

Indeed, as we have already shown, $\varphi\in\Delta_2$. So, there exists a number c, such that $\varphi(2u)\leq c\varphi(u)$ when $u\geq u_0$. Then,

$$\int_E \varphi\big(S_n f(x)\big)\,dx = \int_E \varphi\big(S_n f(x) - f(x) + f(x)\big)\,dx \leq$$

$$\leq \int_E \varphi\big(2\big(S_n f(x) - f(x)\big)\big)\,dx + \int_E \varphi\big(2f(x)\big)\,dx =$$

$$= \int_{E\cap(|S_nf-f|>u_0)} \varphi\big(2\big(S_n f(x)-f(x)\big)\big)\,dx + \int_{E\cap(|S_nf-f|\leq u_0)} \varphi\big(2\big(S_n f(x)-f(x)\big)\big)\,dx +$$

$$+ \int_{E\cap(|f|>u_0)} \varphi\big(2f(x)\big)\,dx + \int_{E\cap(|f|\leq u_0)} \varphi\big(2f(x)\big)\,dx \leq$$

$$\leq c\int_E \varphi\big(S_n f(x) - f(x)\big)\,dx + c\int_E \varphi\big(f(x)\big)\,dx + 2\varphi(2u_0),$$

and it is obvious that from (17) follows (16).

Once more suppose that $x=0$ is the density point of E. By the Oswald's lemma ([8], Lemma 4), there exists a sequence of positive functions f_k with the following properties:

$$\int_0^1 f_k(x)\,dx \geq 8\pi 2^k \sum_{i=1}^{k-1} \int_0^1 f_i(x)\,dx \quad (k = 2, 3, \dots), \tag{18}$$

$$\int_0^1 \varphi\big(f_k(x)\big)\,dx \geq 2 \int_0^1 \varphi\big(f_{k-1}(x)\big)\,dx > 2 \quad (k = 2, 3, \dots), \tag{19}$$

$$\varphi\bigg(\int_0^1 f_k(x)\,dx\bigg) \geq 2^{2k} \int_0^1 \varphi\big(f_k(x)\big)\,dx > 2, \quad (k = 1, 2, \dots). \tag{20}$$

Let us chose natural numbers n_k, $2 \leq n_k \leq n_{k+1} - 2$ $(k = 1, 2, \dots)$ so that

$$2^{-k-1} < 2^{-n_k} \int_0^1 \varphi\big(f_k(x)\big)\,dx \leq 2^{-k} \quad (k = 1, 2,) \tag{21}$$

and define the function f:

$$f(x) = \begin{cases} f_k\big(2^{n_k}(x - 2^{-n_k})\big), & x \in \big[2^{-n_k},\, 2^{-n_k+1}\big], \quad k = 1, 2, \dots \\ 0, & x \in (-\pi, \pi) \setminus \bigcup\limits_{k=1}^{\infty} \big[2^{-n_k},\, 2^{-n_k+1}\big]. \end{cases}$$

We will show that $f \in \varphi(L)$.

$$\int_{-\pi}^{\pi} \varphi\big(f(x)\big)\,dx = \sum_{k=1}^{\infty} \int_{2^{-n_k}}^{2^{-n_k+1}} \varphi\Big(f_k\big(2^{n_k}(x - 2^{-n_k})\big)\Big)\,dx =$$

$$= \sum_{k=1}^{\infty} 2^{-n_k} \int_0^1 \varphi\big(f_k(x)\big)\,dx \leq \sum_{k=1}^{\infty} 2^{-n_k} < \infty.$$

Applying (15) and (18) we get the following estimation when $x \in [0, 2^{-n_k+1}]$:

$$S_{2^{n_k-2}} f(x) \geq$$

$$\geq \frac{1}{\pi}\Bigg(\int_{2^{-n_k}}^{2^{-n_k+1}} f(t) D_{2^{n_k-2}}(t - x)\,dt - \sum_{i=1}^{k-1} \int_{2^{-n_i}}^{2^{-n_i+1}} f(t) |D_{2^{n_i-2}}(t - x)|\,dt \Bigg) \geq$$

$$\geq \frac{1}{\pi}\Bigg(\frac{2}{\pi}\Big(2^{n_k-2} + \frac{1}{2}\Big) \int_{2^{-n_k}}^{2^{-n_k+1}} f_k\big(2^{n_k}(t - 2^{-n_k})\big)\,dt -$$

$$- \sum_{i=1}^{k-1} 2^{n_i+2} \int_{2^{-n_i}}^{2^{-n_i+1}} f_i\big(2^{n_i}(t - 2^{-n_i})\big)\,dt \Bigg) =$$

$$= \frac{1}{\pi}\bigg(\frac{2}{\pi}\Big(2^{n_k-2}+\frac{1}{2}\Big)2^{-n_k}\int\limits_0^1 f_k(t)\,dt - \sum_{i=1}^{k-1} 2^{n_i+2}2^{-n_i}\int\limits_0^1 f_i(t)\,dt\bigg) \geq$$

$$\geq \frac{1}{\pi}\bigg(\frac{1}{2\pi}\int\limits_0^1 f_k(t)\,dt - 4\sum_{i=1}^{k-1}\int\limits_0^1 f_i(t)\,dt\bigg) \geq \frac{1}{4\pi^2}\int\limits_0^1 f_k(t)\,dt.$$

Then,

$$\int\limits_E \varphi\big(S_{2^{n_k-2}}f(x)\big)\,dx \geq \int\limits_{E\cap(0,2^{-n_k+1})} \varphi\big(S_{2^{n_k-2}}f(x)\big)\,dx \geq$$

$$\geq \varphi\bigg(\frac{1}{2\pi^2}\int\limits_0^1 f_k(x)\,dx\bigg)\big|E\cap\big(0,2^{-n_k+1}\big)\big| \geq \alpha 2^{-n_k}\varphi\bigg(\frac{1}{4\pi^2}\int\limits_0^1 f_k(x)\,dx\bigg),$$

when k is great enough (as $x=0$ is the density point of E).

According (18), $\int\limits_0^1 f_k(x)dx \to \infty$. Then, as $\varphi \in \Delta_2$,

$$\varphi\bigg(\frac{1}{4\pi^2}\int\limits_0^1 f_k(x)\,dx\bigg) \geq c_1\varphi\bigg(\int\limits_0^1 f_k(x)\,dx\bigg)$$

and by (19) and (20)

$$\int\limits_E \varphi\big(S_{2^{n_k-2}}f(x)\big)\,dx \geq \alpha c_1 2^{-n_k}\varphi\bigg(\int\limits_0^1 f_k(x)\,dx\bigg) \geq$$

$$\geq \alpha c_1 2^{2k}2^{-n_k}\int\limits_0^1 \varphi\big(f_k(x)\big)\,dx \geq c_2 2^k.$$

This shows that (17) is held and the proof of the lemma is completed. □

Proof of Theorem 2. Suppose that k is such a real number, that $E=\{x\colon \frac{1}{k} \leq w(x) \leq k\}$ has s positive measure. If $\varphi \notin \Delta_2$ or if φ is not quasiconvex in a neighborrhood of ∞, then by Lemma 4 there exists an $f \in \varphi(L)$, $\operatorname{supp} f \subset E$ and

$$\limsup_{n\to\infty}\int\limits_{-\pi}^{\pi} \varphi\big(f(x)-S_nf(x)\big)\,dx = \infty. \tag{22}$$

Let us show that $f \in \varphi_w(L)$:

$$\int\limits_{-\pi}^{\pi} \varphi\big(f(x)\big)w(x)\,dx = \int\limits_E \varphi\big(f(x)\big)w(x)\,dx \leq k\int\limits_E \varphi\big(f(x)\big)\,dx < \infty.$$

On the other hand, by (22)

$$\limsup_{n\to\infty}\int_{-\pi}^{\pi}\varphi\big(f(x)-S_nf(x)\big)w(x)\,dx \geq$$

$$\geq \frac{1}{k}\limsup_{n\to\infty}\int_{-\pi}^{\pi}\varphi\big(f(x)-S_nf(x)\big)\,dx=\infty,$$

what is in contradiction with (10). Theorem 2 is proved. □

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