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Vanishing of Hochschild, Cyclic and Periodic Homologies on the Category of Fredholm Modules

Let A be an involutive algebra over k, the field of complex or real numbers, and let $\mathscr{H}$ be a countably generated Hilbert space over k. A pair (ϕ, p) is said to be a separable Fredholm module over A if

- $\phi : A \to \mathscr{L}(\mathscr{H})$ is a $*$-homomorphism, where the latter algebra is the C*-algebra of bounded linear maps from $\mathscr{H}$ to itself;
- the closure of $\phi(A)$ in $\mathscr{L}(\mathscr{H})$ is a separable C*-algebra;
- p is a projection in $\mathscr{L}(\mathscr{H})$ such that
$$[\phi(a), p] \in \mathscr{K}(\mathscr{H})$$
for all $a \in A$, where $\mathscr{K}(\mathscr{H})$ is the ideal of compact operators in $\mathscr{L}(\mathscr{H})$.

One can construct the following category, denoted $\mathscr{F}_\sigma(A)$. It contains the separable Fredholm modules as objects, and a morphism $f : (\phi, p) \to (\phi', p')$ is a bounded linear map $f : \mathscr{H} \to \mathscr{H}'$ such that
$$fp = pf \text{ and } f\phi(a) - \phi'(a)f \in \mathscr{K}(\mathscr{H}, \mathscr{H}')$$
for all $a \in A$, where $\mathscr{K}(\mathscr{H}, \mathscr{H}')$ is the linear space of compact linear maps from $\mathscr{H}$ to $\mathscr{H}'$. One easily checks that $\mathscr{F}_\sigma(A)$ is a pseudo-abelian category.

Our objective in this article is to give a scheme of how to prove the following theorem.

Theorem. *Let A be an involutive algebra over the field k of complex or real numbers. Then*
$$HH_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A)) = 0,$$
$$HC_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A)) = 0,$$
$$HP_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A)) = 0,$$
where HH_*^{Mc}, HC_*^{Mc} and HP_*^{Mc} are McCarthy's Hochschild, cyclic and periodic homologies of additive categories with split short exact sequences [5].

One can prove this theorem step by step in the following way.

Step 1. The category $\mathscr{F}_\sigma(A)$ has the natural structure of a C*-category. Let $s : (\phi, p) \to (\phi', p')$ be an isometry, i. e. $s^*s = \mathrm{id}_{(\phi,\mathrm{p})}$. Using Morita invariance of usual Hochschild, cyclic and periodic homologies of k-algebras one has homomorphisms
$$\tau_{(\phi,p)}^{(\phi',p')} : HH_*(\mathrm{End}(\phi, p)) \to HH_*(\mathrm{End}(\phi', p'))$$
(resp., for HC_* and HP_*), which arises from a $*$-homomorphism $t_s : \mathrm{End}(\phi, \mathrm{p}) \to \mathrm{End}(\phi', \mathrm{p}')$ defined by the map $x \mapsto sxs^*$. The homomorphism τ does not depend on choice of s. Since the latter homology groups commute with directed colimits as well as the McCarthy's, one gets the following isomorphisms
$$HH_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A)) = \varinjlim \Big(HH_*(\mathrm{End}(\phi, \mathrm{p})); \tau_{(\phi,\mathrm{p})}^{(\phi',\mathrm{p}')}\Big)$$

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(resp. for $HC_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$ and $HP_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$).

Step 2. For any object $(\phi, 1)$ consider the C*-algebra $A_\phi = \overline{\phi(A)}$, i. e. the closure of $\phi(A)$ in $\mathscr{L}(\mathscr{H})$. The category $\mathscr{F}_\sigma(A_\phi)$ is a full subcategory of $\mathscr{F}_\sigma(A)$. Let us say $\phi \leq \phi'$ if $\ker(\phi') \subseteq \ker(\phi)$. An easy checking shows that $\mathscr{F}_\sigma(A) = \varinjlim \mathscr{F}_\sigma(A_\phi)$ and $HH_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A)) = \varinjlim HH_*(A_\phi)$ (resp. for $HC_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$ and $HP_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$). Thus it suffices to prove the theorem when A is a separable C*-algebra.

Step 3. Let $0 \to I \to B \to A \to 0$ be an exact sequence of separable C*-algebras such that the epimorphism has a completely positive and contractive section. Then the following sequence of homology groups

$$\cdots \to HH_{n+1}^{\mathrm{Mc}}(\mathscr{F}_\sigma(I)) \to HH_n^{\mathrm{Mc}}(\mathscr{F}_\sigma(A)) \to HH_n^{\mathrm{Mc}}(\mathscr{F}_\sigma(B)) \to HH_n^{\mathrm{Mc}}(\mathscr{F}_\sigma(I)) \to \cdots$$

is exact (resp. for $HC_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$ and $HP_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$). The proof of this statement is the same as the proof of exactness in [2]. But our approach is purely algebraic and it uses only Morita invariance of homology groups and the well known fact that C*-algebras are H-unital over k [6].

Step 4. Using Higson's homotopy invariance theorem [3], one can conclude that the functors $HH_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(-))$, $HC_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(-))$ and $HP_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(-))$ are homotopy invariant on the category of separable C*-algebras. Then after using the Cuntz-Bott periodicity theorem [1] one gets that the above functors have period 2 in the complex case and 8 in the real case. Using the same periodicity theorem one can express functorially all the $HH_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$ by $HH_0^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$.

Step 5. There is a natural transformation $\mu_* : HH_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A)) \to HC_*^{\mathrm{Mc}}(\mathscr{F}_\sigma(A))$ which is an isomorphism in dimension zero. Taking into account step 4, one easily checks that μ_* is an isomorphism in any dimension $* \geq 1$. Now, the Connes' periodicity theorem (see [4]) guarantees our theorem.

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