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Quantization of the AdS_3 superparticle on $OSP(1|2)^2/SL(2,\mathbb{R})$

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Abstract

We analyze AdS_3 superparticle dynamics on the coset $OSP(1|2) \times OSP(1|2)/SL(2,\mathbb{R})$. The system is quantized in canonical coordinates obtained by gauge invariant Hamiltonian reduction. The left and right Noether charges of a massive particle are parametrized by coadjoint orbits of a timelike element of $\mathfrak{osp}(1|2)$. Each chiral sector is described by two bosonic and two fermionic canonical coordinates corresponding to a superparticle with superpotential $W = q - m/q$, where m is the particle mass. Canonical quantization then provides a quantum realization of $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$. For the massless particle the chiral charges lie on the coadjoint orbit of a nilpotent element of $\mathfrak{osp}(1|2)$ and each of them depends only on one real fermion, which demonstrates the underlying κ-symmetry. These remaining left and right fermionic variables form a canonical pair and the system is described by four bosonic and two fermionic canonical coordinates. Due to conformal invariance of the massless particle, the $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ extends to the corresponding superconformal algebra $\mathfrak{osp}(2|4)$. Its 19 charges are given by all real quadratic combinations of the canonical coordinates, which trivializes their quantization.

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1. Introduction

For more than a decade the existence of integrability in the AdS/CFT correspondence has excited astonishing insights into non-perturbative aspects of both conformal field theories (CFT) as well as string theories in Anti-de Sitter space (AdS) [1–3]. In particular, unraveled first for the duality between $\mathcal{N}=4$ super Yang–Mills theory and the $AdS_5 \times S^5$ superstring, the conjectured quantum integrability has allowed for a solution of the spectral problem through the mirror Thermodynamic Bethe Ansatz (TBA) [4–7] as well as the Quantum Spectral Curve [8], which ostensibly amounts to quantization of the system.

However, in spite of this progress it is worth noting that our understanding of quantization of the $AdS_5 \times S^5$ superstrings from first principles is still limited. The spectrum of the 1/2-BPS subsector, *viz.* of the corresponding supergravity, is well-known [9,10] and, using the results of [11,12], it was shown to match with quantization of the massless $AdS_5 \times S^5$ superparticle [13], see also [14] as well as the recent work on the supertwistor formulation [15]. In fact, it seems favorable to attain a rigorous understanding of the massless superparticle before attempting to quantize the superstring.

Another well-studied sector is the class of *heavy*, respectively, *long* string states captured by semi-classical string solutions. As in the seminal works [16–20], here one relies on some of the global $\mathfrak{psu}(2,2|4)$ charges to diverge in the 't Hooft coupling as $\sqrt{\lambda}$, resulting in a similar scaling of the string energy, $E \propto \sqrt{\lambda}$. Fluctuations around such configurations can then be quantized perturbatively. For instance, fluctuations around the point particle of diverging S^5 momentum are described by the BMN string [16] and the corresponding quantum corrections were calculated in [21–24], which allowed to construct the scattering S-matrix in this limit [25–27].

For *light*, respectively, *short* string states with finite $\mathfrak{psu}(2,2|4)$ charges, however, such a perturbative description formally breaks down and it has been a renowned problem to obtain the spectrum beyond the leading order [17], $E \propto \lambda^{1/4}$. The difficulties seem to be caused by the particular scaling of the string zero-modes [28], *viz.* the particle-like degrees of freedom of the center-of-mass. At the same time, this points out that for short strings the customary uniform light cone gauge [29,30] might not be the most appropriate gauge choice.

Therefore, restricting to bosonic $AdS_5 \times S^5$ and employed static gauge [31], in [32] a semi-classical string solution has been constructed generalizing the pulsating string [33,34] by allowing for unconstrained zero-modes. Apart from showing classical integrability and invariance under the isometries $SO(2,4) \times SO(6)$, the energy of the lowest excitation of this so called single-mode string proved to match with integrability based results for the Konishi anomalous dimension up the first quantum corrections, the order $\lambda^{-1/4}$. For this the crucial step has been to reformulate the system as a massive $AdS_5 \times S^5$ particle [35,36] with the mass term determined by the stringy non-zero-modes. Hence, in order to understand quantization of the AdS superstrings from first principles it seems favorable to study not only massless but also massive AdS superparticles.

Notably, the previous observation is equivalent to the statement that the single-mode string [32] is the $SO(2,4) \times SO(6)$ orbit of the pulsating string [33,34]. This suggests to construct (super)isometry group orbits of other semi-classical string solutions, which has the additional appeal that the Kirillov–Kostant–Souriau method of coadjoint orbits yields a quantization scheme in terms of the symmetry generators, which is manifestly gauge-independent. In [37] we followed this idea by constructing the isometry group orbits of the bosonic particle and spinning string in $AdS_3 \times S^3$, leading to a Holstein–Primakoff realization for the isometry algebra [38–40] in agreement with previous results. We then turned to superisometry group orbits by applying orbit method quantization to the AdS_2 superparticle on $OSP(1|2)/SO(1,1)$ [41], yielding a Holstein–Primakoff-like realization of the superisometries $\mathfrak{osp}(1|2)$. For the massless case however the κ-symmetry transformation left only one physical real fermion, rendering the model quantum inconsistent.

In this work we continue this program and apply superisometry group orbit quantization to the $\mathcal{N}=1$ superparticle on the AdS_3 superspace defined on the coset[1] $OSP(1|2)^2/SL(2,\mathbb{R})$. More specifically, we will investigate the action showing κ-symmetry in the massless case, as it constitutes a truncation of the Green–Schwarz superstring encountered in the AdS/CFT correspondence. Additionally, we will demonstrate that only for this κ-symmetric action there is a close relation to the superparticle on the supergroup $OSP(1|2)$, a statement which carries over to general cosets of the form G^2/H.

Let us note already that in comparison to [41] the present coset exactly doubles the number of fermionic degrees of freedom. Hence, by construction we are circumventing the problems encountered in the massless case of [41], as now κ-symmetry will leave us with two real fermions, enough to form one fermionic canonical pair. Therefore, this model amounts to what is arguably the simplest quantum consistent massless AdS superparticle.[2]

Indeed, for both the massive and the massless superparticle, by using the orbit method we will obtain not only physical canonical variables, which can be quantized in terms of bosonic and fermionic oscillators, but also conserved charges forming a Holstein–Primakoff-like quantum realization [41] of the superisometry algebra $\mathfrak{osp}_l(1|2)\oplus\mathfrak{osp}_r(1|2)$.

For the massive case we point out that both chiral subsectors are described by supersymmetric quantum mechanics with superpotential $W=q-\frac{2\mu-1/2}{q}$ [48].

For massless particles it is well-known that the action is invariant not only under the isometries but under the full conformal symmetries of the underlying space–time. For AdS_{N+1} this yields an extension of the isometry algebra $\mathfrak{so}(2,N)$ to the conformal algebra $\mathfrak{so}(2,N+1)$ [49]. Correspondingly, for the massless superparticle at hand we find that the superisometries $\mathfrak{osp}_l(1|2)\oplus\mathfrak{osp}_r(1|2)$ extend to the superconformal algebra $\mathfrak{osp}(2|4)$.

This work has clearly been motivated by and is aimed towards a future application to semi-classical string solutions of the $AdS_5\times S^5$ superstring. However, there is actually a whole plethora of semi-symmetric AdS supercoset [50] which might serve as backgrounds for integrable sigma-models encountered in the AdS/CFT correspondence. In particular, initiated by [51–57] there has been remarkable progress on the AdS_3/CFT_2 correspondence on $AdS_3\times S^3\times T^4$ and $AdS_3\times S^3\times S^3\times S^1$, see also the more recent works [58–63] as well as the review [64].[3] The AdS_3 superparticle under investigation is naturally viewed as a truncation of these string theories. Similarly, this work might also prove relevant for supersymmetric versions of the non-critical AdS_3 string [71], see also [72] as well as the work [73] on the $OSP(1|2)$ WZNW model, and even of the AdS_3 higher spin theory [74,75].

Particle dynamics in AdS_3 (super)space have also been investigated in a series of other works. The dynamical sectors of the bosonic AdS_3 particle were investigated in [76], where for critical spin $J=J_{12}=m$ the systems reduces to a particle on AdS_2. Higher derivative actions for the

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[1] Here and in the following, we abbreviate the direct product of supergroups G as $G\times G=G^2$.

[2] Contestants to this title might be the AdS_2 superparticle actions on $SU(1,1|1)/(SO(1,1)\times U(1))$ or $SU(1,1|1)/SO(1,1)$, see for example [42,43] and the more recent works [44–47], as well as a non-κ-symmetric version of the AdS_2 superparticles on $OSP(1|2)/SO(1,1)$.

[3] Further studies on AdS_3 superstrings, especially in the RNS description, include [65–68], see also the more recent works [69,70].

AdS_3 superparticle on SU(1, 1|1) were derived in [77], see also [78], and similar techniques have been applied to multi-particle dynamics, see [79,80] and references therein, which are relevant for the duality between black holes and superconformal Calogero models [81,82].

The paper is organized as follows. In Section 2 we study the bosonic AdS_3 particle on $SL(2,\mathbb{R})$. After establishing the isometry between AdS_3 and $SL(2,\mathbb{R})$ and the AdS_3 conformal algebra we discuss the massive and massless particle dynamics. In Section 3 we then turn to the AdS_3 superparticle on $OSP(1|2)^2/SL(2,\mathbb{R})$. Here, we first discuss the coset construction to then study the massive and massless case. A conclusion and outlook are given in Section 4. Some technical details of the calculations are collected in three appendices.

2. The bosonic AdS_3 particle

2.1. *Isometry between* AdS_3 *and* $SL(2,\mathbb{R})$

Let us consider the space $\mathbb{R}^{2,2}$ with coordinates X^A, $A = 0', 0, 1, 2$, and the metric tensor $\eta_{AB} = \text{diag}(-1, -1, 1, 1)$. The hyperboloid embedded in $\mathbb{R}^{2,2}$,

$$X^A X_A + 1 = 0, \tag{2.1}$$

is identified with AdS_3 and its map to the $SL(2,\mathbb{R})$ group manifold is given by

$$g = \begin{pmatrix} X^{0'} + X^2 & X^1 + X^0 \\ X^1 - X^0 & X^{0'} - X^2 \end{pmatrix}. \tag{2.2}$$

This group element and its inverse can be written as

$$g = X^{0'}\,\mathbf{I} + X^\mu\,\mathbf{t}_\mu, \qquad g^{-1} = X^{0'}\,\mathbf{I} - X^\mu\,\mathbf{t}_\mu, \tag{2.3}$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{t}_\mu$ for $\mu = 0, 1, 2$ form a basis of $\mathfrak{sl}(2,\mathbb{R})$,

$$\mathbf{t}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{t}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{t}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

The commutation relations of the basis vectors is given by

$$[\mathbf{t}_\mu, \mathbf{t}_\nu] = 2\epsilon_{\mu\nu}{}^{\rho}\,\mathbf{t}_\rho \tag{2.5}$$

where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita tensor, $\epsilon_{012} = 1$. Here, rising and lowering of indices is provided by the metric tensor $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$, which corresponds to the inner product in $\mathfrak{sl}(2,\mathbb{R})$ defined by $\langle \mathbf{t}_\mu\,\mathbf{t}_\nu \rangle \equiv \frac{1}{2}\,\text{tr}(\mathbf{t}_\mu\,\mathbf{t}_\nu) = \eta_{\mu\nu}$. With the help of (2.3) and (2.1) one then obtains the isometry between $SL(2,\mathbb{R})$ and AdS_3,

$$\langle (\text{d}g\,g^{-1})\,(\text{d}g\,g^{-1}) \rangle = \text{d}X^A \text{d}X_A. \tag{2.6}$$

The SO(2, 2) isometry group of AdS_3 is generated by the infinitesimal transformations

$$X^A \mapsto X^A + \alpha^A{}_B\,X^B, \qquad \alpha^{AB} = -\alpha^{BA}. \tag{2.7}$$

In $SL(2,\mathbb{R})$, these correspond to left-right multiplication of the group element

$$g \mapsto g + \alpha_l^\nu\,\mathbf{t}_\nu\,g + \alpha_r^\nu\,g\,\mathbf{t}_\nu, \tag{2.8}$$

with $\alpha_l^\nu = \frac{1}{2}\alpha^{0'\nu} + \frac{1}{4}\epsilon^\nu{}_{\rho\mu}\,\alpha^{\rho\mu}$ and $\alpha_r^\nu = \frac{1}{2}\alpha^{0'\nu} - \frac{1}{4}\epsilon^\nu{}_{\rho\mu}\,\alpha^{\rho\mu}$ (cf. Appendix A), hence establishing the algebra isomorphism $\mathfrak{so}(2,2) = \mathfrak{sl}_l(2,\mathbb{R}) \oplus \mathfrak{sl}_r(2,\mathbb{R})$.

The infinitesimal conformal transformation of AdS_3 read [49]

$$X^A \mapsto X^A + \varepsilon^B (\delta_B{}^A + X_B X^A), \tag{2.9}$$

leading to a rescaling of the metric (2.6) by the factor $1 + 2\varepsilon^B X_B$. Correspondingly, these extend the isometry algebra $\mathfrak{so}(2,2)$ to the AdS_3 conformal algebra $\mathfrak{so}(2,3)$. For the $SL(2,\mathbb{R})$ group element (2.3) the transformations (2.9) take the form (see also (A.4))

$$g \mapsto g + \varepsilon^{0'} \mathbf{I} + \varepsilon^\mu \mathbf{t}_\mu + (\varepsilon^A X_A) g. \tag{2.10}$$

2.2. *Particle dynamics on* $SL(2,\mathbb{R})$

The dynamics of a particle in $SL(2,\mathbb{R})$ can be described by the action

$$S = \int d\tau \left(\frac{1}{2\xi} \left\langle \dot g\, g^{-1}\, \dot g\, g^{-1} \right\rangle - \frac{\xi\, m^2}{2} \right). \tag{2.11}$$

Here, τ is an evolution parameter, ξ plays the role of a worldline einbein and m is a particle mass. The isometry transformations (2.8) yield the Noether charges

$$L = \frac{\dot g\, g^{-1}}{\xi}, \qquad R = \frac{g^{-1} \dot g}{\xi}, \tag{2.12}$$

which are related by $L = gRg^{-1}$ and therefore have the same length, $\langle L\, L \rangle = \langle R\, R \rangle$.

In the first order formalism the action (2.11) is equivalent to

$$S = \int d\tau \left(\left\langle L\, \dot g\, g^{-1} \right\rangle - \frac{\xi}{2} \left(\langle L\, L \rangle + m^2 \right) \right), \tag{2.13}$$

which leads to the Hamilton equations

$$\dot g g^{-1} = \xi L, \qquad \dot L = 0, \tag{2.14}$$

and the mass-shell condition

$$\langle L\, L \rangle + m^2 = 0. \tag{2.15}$$

We use the Faddeev–Jackiw method that reduces the system to the physical degrees of freedom. The reduction schemes for the massive and the massless cases are different.

2.3. *Massive particle on* $SL(2,\mathbb{R})$

We first analyze the massive case, which corresponds to timelike L and R. Due to the mass-shell condition (2.15) they are on the adjoint orbit of the $\mathfrak{sl}(2,\mathbb{R})$ element $m\mathbf{t}_0$ and one can use the parametrization

$$L = m\, g_l\, \mathbf{t}_0\, g_l^{-1}, \qquad R = m\, g_r^{-1}\, \mathbf{t}_0\, g_r, \qquad g = g_l\, g_r. \tag{2.16}$$

The presymplectic form $\Theta = \langle L\, dg\, g^{-1} \rangle$ then splits into the sum of left and right parts

$$\Theta = m \langle \mathbf{t}_0\, g_l^{-1}\, dg_l \rangle + m \langle \mathbf{t}_0\, dg_r\, g_r^{-1} \rangle. \tag{2.17}$$

Defining the nilpotent generators $\mathbf{t}_\pm = \frac{1}{2}(\mathbf{t}_1 \pm \mathbf{t}_0)$, we use the Iwasawa decomposition

$$g_l = e^{\gamma_l\, \mathbf{t}_+}\, e^{\alpha_l\, \mathbf{t}_2}\, e^{\theta_l\, \mathbf{t}_0}, \qquad g_r = e^{\theta_r\, \mathbf{t}_0}\, e^{\alpha_r\, \mathbf{t}_2}\, e^{\gamma_r\, \mathbf{t}_+}, \tag{2.18}$$

see also Appendix A. Plugging this parametrization into (2.17), we find the presymplectic form

$$\Theta = -m\,\mathrm{d}\theta_l - \frac{m}{2}\,e^{-2\alpha_l}\,\mathrm{d}\gamma_l - m\,\mathrm{d}\theta_r - \frac{m}{2}\,e^{2\alpha_r}\,\mathrm{d}\gamma_r\,, \tag{2.19}$$

and the Noether charges (2.16)

$$L = m\begin{pmatrix} -\gamma_l\,e^{-2\alpha_l} & \gamma_l^2\,e^{-2\alpha_l} + e^{2\alpha_l} \\ -e^{-2\alpha_l} & \gamma_l\,e^{-2\alpha_l}\end{pmatrix}, \qquad R = m\begin{pmatrix} \gamma_r\,e^{2\alpha_r} & \gamma_r^2\,e^{2\alpha_r} + e^{-2\alpha_r} \\ -e^{2\alpha_r} & -\gamma_r\,e^{2\alpha_r}\end{pmatrix}, \tag{2.20}$$

hence rendering θ_l and θ_r unphysical. Introducing the canonical coordinates by

$$p_l = \sqrt{m}\,\gamma_l\,e^{-\alpha_l}, \quad q_l = \sqrt{m}\,e^{-\alpha_l}, \qquad p_r = \sqrt{m}\,\gamma_r\,e^{\alpha_r}, \quad q_r = \sqrt{m}\,e^{\alpha_r}, \tag{2.21}$$

from (2.19) and (2.20) we find $\mathrm{d}\Theta = \mathrm{d}p_l \wedge \mathrm{d}q_l + \mathrm{d}p_r \wedge \mathrm{d}q_r$ and

$$L = \begin{pmatrix} -p_l q_l & p_l^2 + m^2\,q_l^{-2} \\ -q_l^2 & p_l q_l\end{pmatrix}, \qquad R = \begin{pmatrix} p_r q_r & p_r^2 + m^2\,q_r^{-2} \\ -q_r^2 & -p_r q_r\end{pmatrix}. \tag{2.22}$$

The dynamical integrals $L_\mu = \langle\,\mathbf{t}_\mu\,L\,\rangle$ and $R_\mu = \langle\,\mathbf{t}_\mu\,R\,\rangle$ then take the form

$$\begin{aligned} L^0 &= \frac{1}{2}(p_l^2 + q_l^2) + \frac{m^2}{2q_l^2}, \qquad L_1 = \frac{1}{2}(p_l^2 - q_l^2) + \frac{m^2}{2q_l^2}, \qquad L_2 = -p_l q_l\,, \\ R^0 &= \frac{1}{2}(p_r^2 + q_r^2) + \frac{m^2}{2q_r^2}, \qquad R_1 = \frac{1}{2}(p_r^2 - q_r^2) + \frac{m^2}{2q_r^2}, \qquad R_2 = p_r q_r\,, \end{aligned} \tag{2.23}$$

and their Poisson brackets form the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.5)

$$\{L_\mu, L_\nu\} = -2\epsilon_{\mu\nu}{}^{\rho}\,L_\rho\,, \qquad \{R_\mu, R_\nu\} = 2\epsilon_{\mu\nu}{}^{\rho}\,R_\rho\,, \tag{2.24}$$

which reflects the isometry invariance on the mass-shell.

The time translation parameter in (2.7) is ${\alpha^{0'}}_0$ and due to (2.8) the energy reads

$$E = \frac{1}{2}(L^0 + R^0). \tag{2.25}$$

Now we describe quantization of the system (2.23)–(2.24). Since the canonical coordinates (2.21) are given on the half-planes ($q_l > 0$, $q_r > 0$), it is natural to quantize the system in the coordinate representation. Thus, only the charges L_2 and R_2 exhibit ambiguous operator ordering. A quantum realization of the algebra (2.24) is provided by the Weyl ordering and the energy spectrum is obtained from the analysis of the eigenvalue problem for the operator

$$H = \frac{1}{4}(-\partial_q^2 + q^2 + m^2/q^2). \tag{2.26}$$

Due to the commutation relations of $\mathfrak{sl}(2,\mathbb{R})$, the operators

$$J^\pm = \frac{1}{4}(-\partial_q^2 - q^2 + m^2/q^2) \pm \frac{1}{2}(q\,\partial_q + 1/2) \tag{2.27}$$

are rising and lowering for H, i.e. $[H, J^\pm] = \pm J^\pm$. H then has the harmonic oscillator spectrum with some minimal eigenvalue μ and the ground state wave function $\Psi_0(q)$ satisfies the equations $H\Psi_0 = \mu\,\Psi_0$ and $J^-\Psi_0 = 0$.

Derivation of the eigenfunctions is simplified due to the relation

$$J^\pm = H - \frac{q^2}{2} \pm \frac{1}{2}(q\,\partial_q + 1/2), \tag{2.28}$$

which leads to a first order differential equation for $\Psi_0(q)$. Up to a normalization constant, one simply obtains

$$\Psi_0(q) \propto q^{2\mu-\frac{1}{2}}\, e^{-\frac{1}{2}q^2} \tag{2.29}$$

and the minimal eigenvalue μ is related to the mass parameter by[4]

$$\mu = \frac{1 \pm \sqrt{m^2+1/4}}{2}. \tag{2.30}$$

Note that the ground state wave function (2.29) is normalizable for $\mu > 0$, which corresponds to the unitarity bound in AdS_3 [83].

Higher level eigenstates are obtained by acting with the rising operator J^+ (2.28), yielding

$$\Psi_n \propto P_n(q^2)\, q^{2\mu-\frac{1}{2}}\, e^{-\frac{1}{2}q^2}, \tag{2.31}$$

with the recursive relations $P_{n+1}(x) = (2\mu + n - x)P_n(x) + x P_n'(x)$, such that after a suitable normalization $P_n(x)$ become generalized Laguerre polynomials.

The left and right copies of the generators (2.26)–(2.27) form a representation of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, which is unitary equivalent to the Holstein–Primakoff type representation obtained in [39], see also [37,84]. The Holstein–Primakoff representation of $\mathfrak{sl}(2,\mathbb{R})$ reads

$$E = \mu + b^\dagger b, \quad B^\dagger = b^\dagger \sqrt{2\mu + b^\dagger b}, \quad B = \sqrt{2\mu + b^\dagger b}\; b, \tag{2.32}$$

where $b^\dagger$ and b are the standard creation–annihilation operators with canonical commutation $[b, b^\dagger] = 1$ and to recover (2.26)–(2.27) we employ the unitary map $E \mapsto H$, $B^\dagger \mapsto -J^+$ and $B \mapsto -J^-$. The corresponding canonical transformation can be found in Appendix B.

Furthermore, note that the operators (2.26)–(2.27) can be written as

$$H = \mu + A^+A^-, \qquad J^+ = A^+(A^- - q), \qquad J^- = (A^+ - q)A^-, \tag{2.33}$$

where

$$A^+ = \frac{1}{2}\left(q - \frac{2\mu - \frac{1}{2}}{q} - \partial_q\right), \qquad A^- = \frac{1}{2}\left(q - \frac{2\mu - \frac{1}{2}}{q} + \partial_q\right). \tag{2.34}$$

This form of the Hamiltonian prepares the system for a supersymmetric extension [48], with superpotential $W = q - \frac{2\mu - 1/2}{q}$.

2.4. *Massless particle on* $\mathrm{SL}(2,\mathbb{R})$

The massless case corresponds to lightlike L and R. These are on the orbit of the nilpotent element, say $\mathbf{t}_+$, implying the parametrization

$$L = g_l\, \mathbf{t}_+\, g_l^{-1}, \qquad R = g_r^{-1}\, \mathbf{t}_+\, g_r, \qquad g = g_l\, g_r. \tag{2.35}$$

Analogously to (2.17), the presymplectic one-form becomes

$$\Theta = \langle \mathbf{t}_+\, g_l^{-1}\, \mathrm{d}g_l \rangle + \langle \mathbf{t}_+\, \mathrm{d}g_r\, g_r^{-1} \rangle. \tag{2.36}$$

[4] Two possible signs in (2.30) correspond to two different self-adjoint extensions of the operator (2.26) valid for $m^2 \in [-1/4, < 3/4)$ [40], see also the massless case below.

With the help of the Iwasawa decompositions

$$g_l = e^{\theta_l\,\mathbf{t}_0}\, e^{\alpha_l\,\mathbf{t}_2}\, e^{\gamma_l\,\mathbf{t}_+}, \qquad g_r = e^{\gamma_r\,\mathbf{t}_+}\, e^{\alpha_r\,\mathbf{t}_2}\, e^{\theta_r\,\mathbf{t}_0}, \tag{2.37}$$

we find the one-form (see Appendix A for details of calculation)

$$\Theta = -\frac{1}{2}\, e^{2\alpha_l}\, d\theta_l - \frac{1}{2}\, e^{-2\alpha_r}\, d\theta_r, \tag{2.38}$$

and the Noether charges (2.35)

$$\begin{aligned} L &= \begin{pmatrix} e^{2\alpha_l}\cos\theta_l\,\sin\theta_l & e^{2\alpha_l}\cos^2\theta_l \\ -e^{2\alpha_l}\sin^2\theta_l & -e^{2\alpha_l}\cos\theta_l\sin\theta_l \end{pmatrix}, \\ R &= \begin{pmatrix} -e^{-2\alpha_r}\cos\theta_r\sin\theta_r & e^{-2\alpha_r}\cos^2\theta_r \\ -e^{-2\alpha_r}\sin^2\theta_r & e^{-2\alpha_r}\cos\theta_r\sin\theta_r \end{pmatrix}, \end{aligned} \tag{2.39}$$

yielding that γ_l and γ_r are unphysical.

The parameters θ_l and θ_r are cyclic variables and global canonical coordinates read

$$p_l = e^{\alpha_l}\cos\theta_l, \quad q_l = -e^{\alpha_l}\sin\theta_l, \qquad p_r = -e^{-\alpha_r}\cos\theta_r, \quad q_r = e^{-\alpha_r}\sin\theta_r. \tag{2.40}$$

This provides $d\Theta = dp_l \wedge dq_l + dp_r \wedge dq_r$ and the Noether charges in (2.35) take the form

$$L = \begin{pmatrix} -p_l q_l & p_l^2 \\ -q_l^2 & p_l q_l \end{pmatrix}, \qquad R = \begin{pmatrix} p_r q_r & p_r^2 \\ -q_r^2 & -p_r q_r \end{pmatrix}. \tag{2.41}$$

Then, similarly to (2.23), one gets the dynamical integrals

$$\begin{aligned} L^0 &= \frac{1}{2}(p_l^2 + q_l^2), \qquad L_1 = \frac{1}{2}(p_l^2 - q_l^2), \qquad L_2 = -p_l q_l, \\ R^0 &= \frac{1}{2}(p_r^2 + q_r^2), \qquad R_1 = \frac{1}{2}(p_r^2 - q_r^2), \qquad R_2 = p_r q_r, \end{aligned} \tag{2.42}$$

which form the same Poisson brackets algebra (2.24). Formally, these dynamical integrals are obtained from (2.23) at $m = 0$. However, it has to be noticed that the canonical coordinates in (2.42) are given on the full planes without the origin, whereas in the massive case they are defined on the half-planes ($q_l > 0$, $q_r > 0$).

In the massless case there are additional Noether charges C_A related to the conformal transformations (2.10), which yield $C_{0'} = \langle g^{-1} L \rangle$, and $C_\mu = \langle g^{-1} L\, \mathbf{t}_\mu \rangle$. These dynamical integrals can be combined in the matrix $C = g^{-1} L = g_r^{-1} \mathbf{t}_\pm\, g_l^{-1}$ and, using the canonical coordinates (2.40), one finds (see equation (A.7) in Appendix A)

$$C = \begin{pmatrix} q_l p_r & -p_l p_r \\ -q_l q_r & p_l q_r \end{pmatrix}. \tag{2.43}$$

Note that the conservation of $C = g^{-1} L$ follows from the equations of motion (2.14) and from the nilpotency condition $L^2 = 0$, valid for the massless case.

As a result, we obtain ten dynamical integrals given by quadratic combinations of four canonical variables $\{p_l, q_l, p_r, q_r\}$. The Poisson brackets of these functions obviously form $\mathfrak{sp}(4)$,

$$\mathfrak{sp}(4) = \text{span}\left\{ p_l^2,\ q_l^2,\ p_l\, q_l,\ \ p_l\, p_r,\ p_l q_r,\ q_l\, p_r,\ q_l q_r,\ \ p_r^2,\ q_r^2,\ p_r\, q_r \right\}. \tag{2.44}$$

This algebra is isomorphic to $\mathfrak{so}(2,3)$, which corresponds to the conformal symmetry of AdS_3.

Usually, the standard form of the $\mathfrak{so}(2,3)$ algebra is depicted as

$$\{J_{AB}, J_{CD}\} = \eta_{AC}\, J_{BD} + \eta_{BD}\, J_{AC} - \eta_{AD}\, J_{BC} - \eta_{BC}\, J_{AD}, \tag{2.45}$$

with $A, B \ldots = 0', 0, 1, 2, 3$ and $\eta_{AB} = \mathrm{diag}(-1, -1, 1, 1, 1)$ being the metric tensor of $\mathbb{R}^{2,3}$. On the basis of (2.7)–(2.10) one obtains

$$J_{\mu 0'} = \frac{1}{2}(L_\mu + R_\mu), \quad J_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho}(L_\rho - R_\rho), \quad J_{30'} = -C_{0'}, \quad J_{3\mu} = C_\mu, \tag{2.46}$$

where L_μ, R_μ are given by (2.42) and $C_{0'}, C_\mu$, with $\mu = 0, 1, 2$, are obtained from (2.43). The canonical Poisson brackets $\{p_l, q_l\} = \{p_r, q_r\} = 1$ indeed provide the algebra (2.45).

Quantum realization of these commutation relations is obtained by the Weyl ordering. Using the creation–annihilation operators, $a_l^\pm = \frac{1}{\sqrt{2}}(p_l \pm i q_l)$ and $a_r^\pm = \frac{1}{\sqrt{2}}(p_l \pm i q_l)$, one obtains the energy operator

$$E = \frac{1}{2}(a_l^+ a_l^- + a_r^+ a_r^-) + \frac{1}{2}, \tag{2.47}$$

with eigenstates $|n_l, n_r\rangle$. The operators of the right sector $H_r = \frac{1}{2}(a_r^+ a_r^- + \frac{1}{2})$, $J_r^\pm = \frac{1}{2} a_r^{\pm\,2}$ realize the $\mathfrak{sl}(2,\mathbb{R})$ algebra which contains two unitary irreducible representations, with minimal eigenvalues of H_r equal to $1/4$ and to $3/4$. The first is realized on the even level eigenstates ($n_r = 2k$) and the second on the odd ones ($n_r = 2k + 1$). The operators of the left sector $H_l = \frac{1}{2}(a_l^+ a_l^- + \frac{1}{2})$, $J_l^\pm = \frac{1}{2} a_l^{\pm\,2}$ give a similar representation of $\mathfrak{sl}(2,\mathbb{R})$. In addition, one has four $\mathfrak{sp}(4)$ generators $a_l^- a_r^-$, $a_l^- a_r^+$, $a_l^+ a_r^-$, $a_l^+ a_r^+$. Since the symmetry generators are quadratic in creation–annihilation operators, they preserve the parity of $n_l + n_r$. Thus, the constructed representation of $\mathfrak{sp}(4)$ splits in two irreducible representations, with even and odd $n_l + n_r$, respectively.

Note that the representation of $\mathfrak{sl}(2,\mathbb{R})$ given by (2.26)–(2.27) at $m = 0$ describes the unitary irreducible representations either with $\mu = 1/4$ or with $\mu = 3/4$ (see (2.30)). They correspond to the Neumann and Dirichlet boundary conditions of the oscillator eigenfunctions at $q = 0$, respectively. Therefore, in the limit $m \to 0$ one does not get the $\mathfrak{sp}(4)$ symmetry and the case $m = 0$ has to be treated separately.

3. The AdS_3 superparticle

3.1. Coset construction

In the context of the AdS/CFT correspondence, a particularly interesting class of AdS string theories are the ones exhibiting classical integrability. Typically, these are formulated as sigma models on semi-symmetric spaces [50], that is supercosets G/H with G containing the AdS_{N+1} isometry group $SO(2, N)$ and its stabilizer H containing $SO(1, N)$.

The case of AdS_3 is somewhat special as here the cosets of interest take the form G^2/H with H the bosonic part of the diagonal subgroup of G^2, which is isomorphic to the bosonic subgroup of G. Especially, in case of the AdS_3/CFT_2, see e.g. [51], the relevant coset is $\mathfrak{D}(2, 1; \alpha)^2/SO(1, 2) \times SO(3)^2$, with the special cases $PSU(1, 1|2)^2/SO(1, 2) \times SO(3)$ for $\alpha = 0$ or $\alpha = 1$ as well as $OSP(4|2)^2/SO(1, 2) \times SO(3)^2$ for $\alpha = 1/2$. In this work we will instead study the simpler coset $OSP(1|2)^2/SL(2; \mathbb{R})$, which also has this feature.

But first, let us discuss the general case of a coset of the form G^2/H, where H does not necessarily have to correspond to the bosonic subgroup of G. The group element $g \in G^2$ is given as the pair $g = (u, v)$ with $u \in G$ and $v \in G$ and the action of stabilizer subgroup $H \subset G$ on G^2 is defined by $(u, v) \mapsto (hu, hv)$, where $h \in H$. The Lie algebras of G and H are denoted by $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and we introduce the orthogonal completion of $\mathfrak{h}$ in $\mathfrak{g}$, which is denoted by $\mathfrak{h}_\perp$. The metric tensor on $\mathfrak{h}$ is defined by a normalized Killing form $\rho_{ab} = \langle \mathbf{t}_a \mathbf{t}_b \rangle$ of basis vectors $\mathbf{t}_a \in \mathfrak{h}$, whereas the basis of $\mathfrak{h}_\perp$ is denoted by $\mathbf{s}_\alpha$. It is easy to check that the quadratic form $\rho^{ab}\langle \mathbf{t}_a\, v \rangle\langle \mathbf{t}_b\, v \rangle$ with $v \in \mathfrak{g}$ is invariant under the transformations $v \mapsto hvh^{-1}$ for any $h \in H$.

The superparticle action is then given in the coset scheme by

$$S = \int \mathrm{d}\tau \left[\frac{\langle \mathbf{t}_a(\dot{u}\, u^{-1} - \dot{v}\, v^{-1}) \rangle \langle \mathbf{t}^a(\dot{u}\, u^{-1} - \dot{v}\, v^{-1}) \rangle}{2e} - \frac{em^2}{2} \right], \tag{3.1}$$

and it is invariant under the gauge transformations $u(\tau) \mapsto h(\tau)u(\tau)$, $v(\tau) \mapsto h(\tau)v(\tau)$, with $h(\tau) \in H$. The Faddeev–Jackiw method provides the following first order action

$$\begin{aligned} S = \int \mathrm{d}\tau \Big[&\langle L_u\, \dot{u}\, u^{-1} \rangle + \langle L_v\, \dot{v}\, v^{-1} \rangle - \frac{e}{2}\Big(\langle L_u\, L_u \rangle + m^2\Big) + \\ &\lambda^a \langle \mathbf{t}_a (L_u + L_v) \rangle + \xi_u^\alpha \langle \mathbf{s}_\alpha\, L_u \rangle + \xi_v^\alpha \langle \mathbf{s}_\alpha\, L_v \rangle \Big], \end{aligned} \tag{3.2}$$

where e, λ^a, ξ_u^α and ξ_v^α are Lagrange multipliers and one obtains the constraints

$$\langle L_u\, L_u \rangle + m^2 = 0, \qquad L_u \in \mathfrak{h}, \qquad L_v = -L_u. \tag{3.3}$$

The system is then described by the 1-form and the Noether charges

$$\Theta = \langle L_u (\mathrm{d}u\, u^{-1} - \mathrm{d}v\, v^{-1}) \rangle, \qquad R_u = u^{-1} L_u\, u, \qquad R_v = -v^{-1} L_u\, v. \tag{3.4}$$

Introducing gauge invariant variables $g = v^{-1}u$ and $L = v^{-1} L_u\, v$, from (3.4) we find

$$\Theta = \langle L\, \mathrm{d}g\, g^{-1} \rangle, \qquad R_u = g^{-1} L\, g, \qquad R_v = -L. \tag{3.5}$$

It is interesting to note that, in comparison, the superparticle action on the (super)group manifold G (2.13) would yield G orbits of some element L_u of $\mathfrak{g}$ instead of an element of its subalgebra $\mathfrak{h}$. Hence, the action (3.1) on the coset G^2/H corresponds to a subclass of orbits of the action (2.13) on the group manifold G.[5]

As mentioned above, we will be interested in the AdS$_3$ superparticle corresponding to $G = \mathrm{OSP}(1|2)$ and $H = \mathrm{SL}(2, \mathbb{R})$. Then $L_u \in \mathrm{sl}(2, \mathbb{R})$ and L is on its OSP(1|2) orbit. The bosonic case is given by $G = H = \mathrm{SL}(2, \mathbb{R})$, for which the reduction scheme describes a particle on $\mathrm{SL}(2, \mathbb{R})$ considered just in the previous section.

3.2. *Massive particle on* OSP(1|2)

First we introduce necessary notations and normalization in the $\mathfrak{osp}(1|2)$ algebra.

The standard basis of $\mathfrak{osp}(1|2)$ is given by the matrices

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{T}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{T}_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

[5] The actions coincide, if in fact the mass-shell condition (2.15) requires L_u to be an element of $\mathfrak{h}$. This appears to happen for $\mathfrak{h}$ being the bosonic subalgebra of $\mathfrak{g}$, as is the case for $\mathrm{OSP}(1|2)^2/\mathrm{SL}(2, \mathbb{R})$, and the mass parameter taken to be pure body, $m \in \mathbb{R}$.

$$\mathbf{S}_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{S}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.7}$$

and they satisfy the commutation relations

$$\begin{aligned} &[\mathbf{T},\, \mathbf{T}_\pm] = \pm 2\mathbf{T}_\pm, \qquad [\mathbf{T}_+,\, \mathbf{T}_-] = \mathbf{T}, \\ &[\mathbf{T},\, \mathbf{S}_\pm] = \pm \mathbf{S}_\pm, \qquad [\mathbf{T}_\pm,\, \mathbf{S}_\mp] = -\mathbf{S}_\pm, \qquad [\mathbf{T}_\pm,\, \mathbf{S}_\pm] = 0, \\ &[\mathbf{S}_+,\, \mathbf{S}_-]_+ = \mathbf{T}, \qquad [\mathbf{S}_\pm,\, \mathbf{S}_\pm]_+ = \pm 2\mathbf{T}_\pm. \end{aligned} \tag{3.8}$$

The normalized supertrace $\langle \mathbf{a}\,\mathbf{b}\rangle = \frac{1}{2}\big((\mathbf{a}\,\mathbf{b})_{11} + (\mathbf{a}\,\mathbf{b})_{22} - (\mathbf{a}\,\mathbf{b})_{33}\big)$ provides an inner product on $\mathfrak{osp}(1|2)$ with the following nonzero components

$$\langle \mathbf{T}\,\mathbf{T}\rangle = 1, \qquad \langle \mathbf{T}_+\,\mathbf{T}_-\rangle = \frac{1}{2}, \qquad \langle \mathbf{S}_+\,\mathbf{S}_-\rangle = -\langle \mathbf{S}_-\,\mathbf{S}_+\rangle = 1. \tag{3.9}$$

With this, we start from the action (3.1), where u and v are group elements in OSP(1|2), the basis elements $\mathbf{t}_a$ correspond to the bosonic generators (3.6) and $\langle\,\cdot\,\rangle$ denotes the normalized supertrace. In the first order formalism one again gets the action (3.2) where now L lies on the OSP(1|2) orbit of an element of the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{R})$. As for the purely bosonic particle in the last section, all that is left is to analyze the presymplectic form $\Theta = \langle L\, \mathrm{d}g\, g^{-1}\rangle$ and the Noether charges L and $R = g^{-1}\, L\, g$ on the constrained surface $\langle L\, L\rangle + m^2 = 0$.

In the massive case L and R are on the adjoint orbit of $m\,\mathbf{T}_0$, where $\mathbf{T}_0 = \mathbf{T}_+ - \mathbf{T}_-$ is a unit timelike element of $\mathfrak{osp}(1|2)$. Taking a parametrization similar to (2.16),

$$L = m\, g_l\, \mathbf{T}_0\, g_l^{-1}, \qquad R = m\, g_r^{-1}\, \mathbf{T}_0\, g_r, \qquad g = g_l\, g_r, \tag{3.10}$$

splits the presymplectic form again into the left and right parts

$$\Theta = \Theta_l + \Theta_r, \qquad \Theta_l = m\langle\, \mathbf{T}_0\, g_l^{-1}\, \mathrm{d}g_l\,\rangle, \qquad \Theta_r = m\langle\, \mathbf{T}_0\, \mathrm{d}g_r\, g_r^{-1}\,\rangle. \tag{3.11}$$

For g_l and g_r let us take the parametrization

$$g_l = e^{\gamma_l\, \mathbf{T}_+}\, e^{\alpha_l\, \mathbf{T}}\, e^{\zeta_l\, \mathbf{S}_+}\, e^{\eta_l\, \mathbf{S}_-}\, e^{\theta_l\, \mathbf{T}_0}, \qquad g_r = e^{\theta_r\, \mathbf{T}_0}\, e^{\eta_r\, \mathbf{S}_-}\, e^{\zeta_r\, \mathbf{S}_+}\, e^{\alpha_r\, \mathbf{T}}\, e^{\gamma_r\, \mathbf{T}_+}, \tag{3.12}$$

where $\eta_{l,r}$ and $\zeta_{l,r}$ are fermionic, i.e. Grassmann odd, parameters while the bosonic parameters $\theta_{l,r}$, $\alpha_{l,r}$ and $\gamma_{l,r}$ correspond to Iwasawa type decomposition (2.18). Technical details of the parametrization (3.12) are deferred to Appendix C, where we also present some useful formulas.

Calculations of the Noether charges $L = m g_l\, \mathbf{T}_0\, g_l^{-1}$ and $R = m g_r^{-1}\, \mathbf{T}_0\, g_r$ (3.10) as well as of the presymplectic forms $\Theta_l = m\langle\, \mathbf{T}_0\, g_l^{-1}\, \mathrm{d}g_l\,\rangle$ and $\Theta_r = m\langle\, \mathbf{T}_0\, \mathrm{d}g_r\, g_r^{-1}\,\rangle$ (3.11) then yields

$$\begin{aligned} L &= m \begin{pmatrix} -\gamma_l\, e^{-2\alpha_l} & \gamma_l^2\, e^{-2\alpha_l} + e^{2\alpha_l} - 2\, e^{2\alpha_l}\, \eta_l \zeta_l & \gamma_l e^{-\alpha_l}\, \zeta_l + e^{\alpha_l}\, \eta_l \\ -e^{-2\alpha_l} & \gamma_l\, e^{-2\alpha_l} & e^{-\alpha_l}\, \zeta_l \\ -e^{-\alpha_l}\, \zeta_l & \gamma_l e^{-\alpha_l}\, \zeta_l + e^{\alpha_l}\, \eta_l & 0 \end{pmatrix}, \\ R &= m \begin{pmatrix} \gamma_r\, e^{2\alpha_r} & \gamma_r^2\, e^{2\alpha_r} + e^{-2\alpha_r} - 2\, e^{-2\alpha_r}\, \eta_r \zeta_r & \gamma_r e^{\alpha_r}\, \zeta_r - e^{-\alpha_r}\, \eta_r \\ -e^{2\alpha_r} & -\gamma_r\, e^{2\alpha_r} & -e^{\alpha_r}\, \zeta_r \\ e^{\alpha_r}\, \zeta_r & \gamma_r e^{\alpha_r}\, \zeta_r - e^{-\alpha_r}\, \eta_r & 0 \end{pmatrix}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \Theta_l &= \frac{m}{2}(\eta_l \mathrm{d}\eta_l + \zeta_l \mathrm{d}\zeta_l - e^{-2\alpha_l}\, \mathrm{d}\gamma_l - 2\mathrm{d}\theta_l), \\ \Theta_r &= -\frac{m}{2}(\eta_r \mathrm{d}\eta_r + \zeta_r \mathrm{d}\zeta_r + e^{2\alpha_r}\, \mathrm{d}\gamma_r + 2\mathrm{d}\theta_r). \end{aligned} \tag{3.14}$$

Similarly to the bosonic case we introduce the variables

$$\begin{aligned} p_l &= \sqrt{m}\,\gamma_l\, e^{-\alpha_l}, \quad q_l = \sqrt{m}\, e^{-\alpha_l}, \qquad p_r = \sqrt{m}\,\gamma_r\, e^{\alpha_r}, \quad q_r = \sqrt{m}\, e^{\alpha_r}, \\ \psi_l &= \sqrt{m}\,\zeta_l\, e^{-i\,\frac{\pi}{4}}, \quad \chi_l = \sqrt{m}\,\eta_l\, e^{-i\,\frac{\pi}{4}}, \quad \psi_r = \sqrt{m}\,\zeta_r\, e^{i\,\frac{\pi}{4}}, \quad \chi_r = \sqrt{m}\,\eta_r\, e^{i\,\frac{\pi}{4}}, \end{aligned} \tag{3.15}$$

and obtain the canonical symplectic form

$$\Omega = \mathrm{d}\Theta = \Omega_l + \Omega_r, \qquad \Omega_l = \mathrm{d}\Theta_l = \frac{i}{2}(\mathrm{d}\psi_l \wedge \mathrm{d}\psi_l + \mathrm{d}\chi_l \wedge \mathrm{d}\chi_l) + \mathrm{d}p_l \wedge \mathrm{d}q_l, \tag{3.16}$$

and similarly for $\Omega_r = \mathrm{d}\Theta_r$. Suppressing indices, let us gather phase space variables of the left, respectively, right sector into (2|2) vectors $\rho^a = (p, q, \psi, \chi)$, hence $\Omega = \frac{1}{2}\mathrm{d}\rho^a \omega_{ab} \mathrm{d}\rho^b$ with

$$\omega_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \qquad \omega^{ab} = (\omega_{ab})^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = -\omega_{ab}. \tag{3.17}$$

Up to an overall sign, this then determines the Poisson bracket of two functions $\mathcal{A}$ and $\mathcal{B}$ on phase space to take the form

$$\{\mathcal{A}, \mathcal{B}\} = -\mathcal{A}\,\frac{\overleftarrow{\partial}}{\partial\rho^a}\omega^{ab}\frac{\overrightarrow{\partial}}{\partial\rho^b}\mathcal{B} = \mathcal{A}\left(\overleftarrow{\partial_p}\,\overrightarrow{\partial_q} - \overleftarrow{\partial_q}\,\overrightarrow{\partial_p} + i\,\overleftarrow{\partial_\psi}\,\overrightarrow{\partial_\psi} + i\,\overleftarrow{\partial_\chi}\,\overrightarrow{\partial_\chi}\right)\mathcal{B}. \tag{3.18}$$

In particular, this yields the non-vanishing Poisson brackets

$$\{p_l, q_l\} = 1, \quad \{\psi_l, \psi_l\} = \{\chi_l, \chi_l\} = i, \qquad \{p_r, q_r\} = 1, \quad \{\psi_r, \psi_r\} = \{\chi_r, \chi_r\} = i, \tag{3.19}$$

and for example $\{i\chi\,\psi, \psi\} = -\chi$ and $\{i\chi\,\psi, \chi\} = \psi$. The odd variables $\psi_{l,r}$ and $\chi_{l,r}$ are real and we construct the standard fermionic creation–annihilation variables by[6]

$$f_l^{\pm} = \frac{\psi_l \pm i\,\chi_l}{\sqrt{2}}, \qquad f_r^{\pm} = \frac{\psi_r \pm i\,\chi_r}{\sqrt{2}}, \tag{3.20}$$

hence $\{f_l^{\pm}, f_l^{\mp}\} = \{f_r^{\pm}, f_r^{\mp}\} = i$ and all other vanishing. Note that $i\chi\psi = f^+ f^-$ is also real.

In terms of the canonical variables (3.15) the Noether charges become

$$\begin{aligned} L &= \begin{pmatrix} -p_l\, q_l & p_l^2 + m^2 q_l^{-2} - 2im\, q_l^{-2}\,\chi_l\psi_l & (p_l\psi_l + m q_l^{-1}\,\chi_l)e^{i\,\frac{\pi}{4}} \\ -q_l^2 & p_l\, q_l & q_l\psi_l\, e^{i\,\frac{\pi}{4}} \\ -q_l\psi_l\, e^{i\,\frac{\pi}{4}} & (p_l\psi_l + m q_l^{-1}\,\chi_l)e^{i\,\frac{\pi}{4}} & 0 \end{pmatrix}, \\ R &= \begin{pmatrix} p_r\, q_r & p_r^2 + m^2 q_r^{-2} + 2im\, q_r^{-2}\,\chi_r\psi_r & (p_r\psi_r + m q_r^{-1}\,\chi_r)e^{-i\,\frac{\pi}{4}} \\ -q_r^2 & -p_r\, q_r & -q_r\psi_r\, e^{-i\,\frac{\pi}{4}} \\ q_r\psi_r\, e^{-i\,\frac{\pi}{4}} & (p_l\psi_r + m q_r^{-1}\,\chi_r)e^{-i\,\frac{\pi}{4}} & 0 \end{pmatrix}. \end{aligned} \tag{3.21}$$

Introducing the dynamical integrals related to the Noether charges

$$\begin{aligned} L_2 &= \langle \mathbf{T}\, L\rangle, \quad L_\pm = \langle \mathbf{T}_\pm\, L\rangle, \qquad R_2 = \langle \mathbf{T}\, R\rangle, \quad R_\pm = \langle \mathbf{T}_\pm\, R\rangle, \\ l_\pm &= \langle \mathbf{S}_\pm\, L\rangle\, e^{-i\,\frac{\pi}{4}}, \qquad\qquad r_\pm = \langle \mathbf{S}_\pm\, R\rangle\, e^{i\,\frac{\pi}{4}} \end{aligned} \tag{3.22}$$

6 We use the down ± indices for 'chiral' components and the upper ± indices for complex coordinates.

from (3.21) we find

$$\begin{aligned}
&L_2=-p_l\,q_l\,, \qquad L_+=-\frac{1}{2}\,q_l^2\,, \qquad R_2=p_r\,q_r\,, \quad R_+=-\frac{1}{2}\,q_r^2\,,\\
&L_-=\frac{1}{2}\left(p_l^2+\frac{m^2}{q_l^2}\right)+i\,\frac{m}{q_l^2}\,\chi_l\,\psi_l\,,\; R_-=\frac{1}{2}\left(p_r^2+\frac{m^2}{q_r^2}\right)+i\,\frac{m}{q_r^2}\,\chi_r\,\psi_r\,,\\
&l_+=q_l\psi_l\,, \quad l_-=\frac{m}{q_l}\,\chi_l-p_l\,\psi_l\,, \quad r_+=q_r\psi_r\,, \qquad r_-=\frac{m}{q_r}\,\chi_r-p_r\,\psi_r\,.
\end{aligned} \tag{3.23}$$

The Poisson brackets of the right functions form the algebra

$$\begin{aligned}
&\{R_2,R_\pm\}=\pm 2R_\pm\,, \qquad \{R_+,R_-\}=R_2\,,\\
&\{R_2,r_\pm\}=\pm r_\pm\,, \qquad \{R_\pm,r_\mp\}=-r_\pm\,, \qquad \{R_\pm,r_\pm\}=0\,,\\
&\{r_+,r_-\}=-i\,R_2\,, \qquad \{r_\pm,r_\pm\}=\mp 2i\,R_\pm\,,
\end{aligned} \tag{3.24}$$

which is equivalent to the commutation relations of the basis elements (3.8) with the replacements $\mathbf{S}_\pm \mapsto \mathbf{S}_\pm\, e^{-i\,\frac{\pi}{4}}$. The Poisson brackets of the left functions form the same algebra up to a sign, as in (2.24). Therefore, due to similarity of the left and right sectors, in the following let us focus on the right sector and drop the corresponding index r.

To pass to the quantum theory we apply the usual canonical quantization rule

$$[q,p]=i\,, \qquad [\chi,\chi]_+=[\psi,\psi]_+=1\,, \qquad [\chi,\psi]_+=0. \tag{3.25}$$

The quantum version of the symmetry generators are then obtained from the classical expressions (3.23). As in the purely bosonic case, see above (2.26), by this only R_2 (and L_2) exhibit ambiguous operator ordering. Choosing again the Weyl ordering and the coordinate representation, we get $R_2=-i(q\,\partial_p+1/2)$.

Computation of the commutation relations then yields

$$\begin{aligned}
&[R_2,R_\pm]=\mp 2i\,R_\pm\,, \qquad [R_+,R_-]=-i\,R_2\,,\\
&[R_2,r_\pm]=\mp i\,r_\pm\,, \qquad [R_\pm,r_\mp]=i\,r_\pm\,, \qquad [R_\pm,r_\pm]=0\,,\\
&[r_+,r_-]_+=-R_2\,, \qquad [r_\pm,r_\pm]_+=\mp 2R_\pm\,,
\end{aligned} \tag{3.26}$$

which is the quantum version of (3.24) in compliance with the rule $\{\mathcal{A},\mathcal{B}\}\mapsto i[\mathcal{A},\mathcal{B}]_\pm$.

In terms of the fermionic creation and annihilation operators (f^+,f^-) introduced in (3.20) one gets $i\,\chi\,\psi=f^+f^- - 1/2$ and the (right) energy operator $H=\frac{1}{2}(R_- - R_+)$ becomes

$$H=\frac{1}{4}\left(-\partial_q^2+q^2+\frac{m^2}{q^2}+\frac{m}{q^2}(2f^+f^- -1)\right). \tag{3.27}$$

The canonical anti-commutation relations in (3.25) are equivalent to

$$[f^-,f^-]_+=[f^+,f^+]_+=0, \qquad [f^-,f^+]_+=1, \tag{3.28}$$

which is realized in the space of two component spinors and one gets

$$f^-=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \qquad f^+=\begin{pmatrix}0&0\\1&0\end{pmatrix}, \qquad H=\begin{pmatrix}H_0&0\\0&H_1\end{pmatrix}, \tag{3.29}$$

with

$$H_0=\frac{1}{4}\left(-\partial_q^2+q^2+\frac{m^2-m}{q^2}\right), \qquad H_1=\frac{1}{4}\left(-\partial_q^2+q^2+\frac{m^2+m}{q^2}\right). \tag{3.30}$$

The Hamiltonians H_0 and H_1 have the oscillator spectrum with minimal eigenvalues $\mu_0 = \frac{2m+1}{4}$ and $\mu_1 = \frac{2m+3}{4}$, respectively, and they are represented in the form of supersymmetric quantum mechanics [48]

$$H_0 = A^+A^- + \frac{2m+1}{4}, \qquad H_1 = A^-A^+ + \frac{2m-1}{4}, \tag{3.31}$$

with

$$A^+ = \frac{1}{2}\left(q - \frac{m}{q} - \partial_q\right), \qquad A^- = \frac{1}{2}\left(q - \frac{m}{q} + \partial_q\right). \tag{3.32}$$

Introducing the rising-lowering operators for the Hamiltonian H

$$J^\pm = \frac{1}{2}(R_+ + R_- \pm i R_2), \qquad j^\pm = \frac{1}{\sqrt{2}}(r_+ \pm i r_-), \tag{3.33}$$

one gets the following form of the $\mathfrak{osp}(1|2)$ algebra (3.26)

$$\begin{aligned} &[H, J^\pm] = \pm J^\pm, \quad [H, j^\pm] = \pm\frac{1}{2}\, j^\pm, \quad [j^\pm, j^\pm]_+ = -2J^\pm, \quad [j^+, j^-]_+ = 2H, \\ &[J^-, J^+] = 2H, \quad [J^\pm, j^\mp] = \pm j^\pm, \quad [J^\pm, j^\pm] = 0. \end{aligned} \tag{3.34}$$

This representation of $\mathfrak{osp}(1|2)$ is unitary equivalent to the Holstein–Primakoff type representation given by [41]

$$E = \mu + b^\dagger b + \frac{f^\dagger f}{2}, \quad B = \sqrt{2\mu + b^\dagger b + f^\dagger f}\; b, \quad F = \sqrt{2\mu + b^\dagger b}\; f + f^\dagger\, b, \tag{3.35}$$

together with $B^\dagger$ and $F^\dagger$. Here, $\mu = \frac{2m+1}{4}$, b and $b^\dagger$ as well as f and $f^\dagger$ are the standard bosonic and fermionic creation–annihilation operators, $[b, b^\dagger] = 1$ and $[f, f^\dagger]_+ = 1$, and the unitary map to (3.35) is provided by $\{E, B, B^\dagger, F, F^\dagger\} \leftrightarrow \{H, -J^-, -J^+, j^-, j^+\}$.

Let us establish the corresponding canonical transformation at the classical level. Using (3.20) and (3.23), the dynamical integrals H, $J^\pm$, $j^\pm$ can be written as

$$H = \frac{1}{4}\left(p^2 + q^2 + \tilde{m}^2/q^2\right), \qquad J^\pm = \frac{1}{4}\left(p^2 - q^2 + \tilde{m}^2/q^2\right) \pm \frac{i}{2}\, pq \tag{3.36}$$

$$j^\pm = \frac{1}{2}(q + m/q \mp ip)f^\pm + \frac{1}{2}(q - m/q \mp ip)f^\mp, \tag{3.37}$$

with $\tilde{m} = m + f^+f^-$ and one gets $j^+j^- = mf^+f^-$. Similarly, $F^*F = mf^*f$, as it follows from the classical form of (3.35)

$$E = \frac{\hat{m}}{2} + b^*b, \quad B = \sqrt{\hat{m} + b^*b}\; b, \quad F = \sqrt{m + b^*b}\; f + f^*\, b, \tag{3.38}$$

where $\hat{m} = m + f^*f$. Both sets of generators then have the same Casimir

$$H^2 - J^+J^- - \frac{1}{2}\, j^+j^- = \frac{m^2}{4}, \qquad E^2 - B^*B - \frac{1}{2}F^*F = \frac{m^2}{4}. \tag{3.39}$$

We use the equations

$$H = E, \quad J^+ = -B^*, \quad J^- = -B, \quad j^+ = F^*, \quad j^- = F, \tag{3.40}$$

to find the canonical map between the variables $(p, q, f^\pm)$ and (b^*, b, f^*, f).

Note that the odd part of (3.40) implies $f^+f^- = f^*f$, hence $\tilde{m} = \hat{m}$. Then, from (3.36)

$$q^2 \pm ipq = 2(H - J^{\mp}), \tag{3.41}$$

and by the bosonic part of (3.40) one finds

$$q = \sqrt{2E + B^* + B}, \qquad pq = i(B^* - B). \tag{3.42}$$

Using again (3.41) and the odd part of (3.40), we obtain

$$f^+ = \frac{E + B + m/2}{\sqrt{(E + m/2)(2E + B^* + B)}}\, f^*, \tag{3.43}$$

and f^- is its complex conjugated.

Since $f^*f^* = 0$, one can replace $\hat{m}$ by m in the expressions of E, B^*, B standing in the right hand side of (3.43).[7] After this replacement, the bosonic factor in (3.43) gets unit norm, which helps to check that the transformation (3.42)–(3.43) from (b^*, b, f^*, f) to $(p, q, f^\pm)$ is indeed canonical

$$\mathrm{d}p \wedge \mathrm{d}q + i\mathrm{d}f^+ \wedge \mathrm{d}f^- = i\mathrm{d}b^* \wedge \mathrm{d}b + i\mathrm{d}f^* \wedge \mathrm{d}f. \tag{3.44}$$

One can repeat the same for the right part of the system and obtain a parametrization of all dynamical integrals in terms of bosonic and fermionic oscillator variables.

3.3. *Massless particle on* OSP(1|2)

In the massless case L and R are on the adjoint orbit of the nilpotent element $\mathbf{T}_+$

$$L = g_l\, \mathbf{T}_+\, g_l^{-1}, \qquad R = g_r^{-1}\, \mathbf{T}_+\, g_r. \tag{3.45}$$

Here we use the parametrization (see Appendix C)

$$g_l = e^{\theta_l\, \mathbf{T}_0}\, e^{\alpha_l\, \mathbf{T}}\, e^{\zeta_l\, \mathbf{S}_-}\, e^{\eta_l\, \mathbf{S}_+}\, e^{\gamma_l\, \mathbf{T}_+}, \qquad g_r = e^{\gamma_r\, \mathbf{T}_+}\, e^{\eta_r\, \mathbf{S}_+}\, e^{\zeta_r\, \mathbf{S}_-}\, e^{\alpha_r\, \mathbf{T}}\, e^{\theta_r\, \mathbf{T}_0}, \tag{3.46}$$

which yields the Noether charges

$$L = \begin{pmatrix} e^{2\alpha_l}\cos\theta_l\, \sin\theta_l & e^{2\alpha_l}\cos^2\theta_l & e^{\alpha_l}\cos\theta_l\; \zeta_l \\ -e^{2\alpha_l}\sin^2\theta_l & -e^{2\alpha_l}\cos\theta_l\, \sin\theta_l & -e^{\alpha_l}\sin\theta_l\; \zeta_l \\ e^{\alpha_l}\sin\theta_l\; \zeta_l & e^{\alpha_l}\cos\theta_l\; \zeta_l & 0 \end{pmatrix}, \tag{3.47}$$

$$R = \begin{pmatrix} -e^{-2\alpha_r}\cos\theta_r\, \sin\theta_r & e^{-2\alpha_r}\cos^2\theta_r & -e^{-\alpha_r}\cos\theta_r\; \zeta_r \\ -e^{-2\alpha_r}\sin^2\theta_r & e^{-2\alpha_r}\cos\theta_r\, \sin\theta_r & -e^{-\alpha_r}\sin\theta_r\; \zeta_r \\ e^{-\alpha_r}\sin\theta_r\; \zeta_r & -e^{-\alpha_r}\cos\theta_r\; \zeta_r & 0 \end{pmatrix}. \tag{3.48}$$

The presymplectic form is again given as the sum of the left and right parts $\Theta = \Theta_l + \Theta_r$, with $\Theta_l = \langle \mathbf{T}_+\, g_l^{-1}\, \mathrm{d}g_l \rangle$ and $\Theta_r = \langle \mathbf{T}_+\, \mathrm{d}g_r\, g_r^{-1} \rangle$. Using again (3.46), one finds

$$\Theta_l = \frac{1}{2}(\zeta_l\, \mathrm{d}\zeta_l - e^{2\alpha_l}\mathrm{d}\theta_l), \qquad \Theta_r = -\frac{1}{2}(\zeta_r\, \mathrm{d}\zeta_r + e^{-2\alpha_r}\mathrm{d}\theta_r). \tag{3.49}$$

The Noether charges and the symplectic form do not depend on the odd variables $(\eta_l,\, \eta_r)$, which reflects the κ-symmetry of the massless case.

[7] By the same reason we use m instead of $\hat{m}$ in the odd element of (3.38).

Similarly to (2.40), canonical variables here are introduced by

$$p_l - i\, q_l = e^{\alpha_l} e^{i\,\theta_l}, \quad \psi_l = \zeta_l\, e^{-i\,\frac{\pi}{4}}, \qquad p_r - i\, q_r = -e^{-\alpha_r} e^{i\,\theta_r}, \quad \psi_r = \zeta_r\, e^{i\,\frac{\pi}{4}}, \tag{3.50}$$

and one obtains

$$L = \begin{pmatrix} -p_l\, q_l & p_l^2 & p_l\, \zeta_l \\ -q_l^2 & p_l\, q_l & q_l\, \zeta_l \\ -q_l\, \zeta_l & p_l\, \zeta_l & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} p_r\, q_r & p_r^2 & p_r\, \zeta_r \\ -q_r^2 & -p_r\, q_r & -q_r\, \zeta_r \\ q_r\, \zeta_r & p_r\, \zeta_r & 0 \end{pmatrix}, \tag{3.51}$$

$$\mathrm{d}\Theta = \frac{i}{2}(\mathrm{d}\psi_l \wedge \mathrm{d}\psi_l + \mathrm{d}\psi_r \wedge \mathrm{d}\psi_r) + \mathrm{d}p_l \wedge \mathrm{d}q_l + \mathrm{d}p_r \wedge \mathrm{d}q_r. \tag{3.52}$$

Hence, following (3.22) we obtain ten dynamical integrals corresponding to the isometry group $\mathrm{OSP}_l(1|2) \oplus \mathrm{OSP}_r(1|2)$, which take the simple form

$$p_l^2, \;\; q_l^2, \;\; p_l\, q_l, \;\; p_l\, \psi_l, \;\; q_l \psi_l, \qquad p_r^2, \;\; q_r^2, \;\; p_r\, q_r, \;\; p_r \psi_r, \;\; q_r\, \psi_r. \tag{3.53}$$

Due to the masslessness, the symmetry algebra extends by nine additional dynamical integrals,

$$p_l\, p_r, \quad p_l q_r, \quad q_l\, p_r, \quad q_l q_r, \qquad p_l \psi_r, \;\; q_l \psi_r, \;\; \psi_l\, p_r, \;\; \psi_l q_r, \qquad i\, \psi_l \psi_r. \tag{3.54}$$

The leftmost four correspond to the $\mathrm{SL}(2,\mathbb{R})$ conformal transformations, while the remaining five follow from closure of the algebra. Altogether, the 19 functions in (3.53) and (3.54) comprise all possible real quadratic combinations of phase space variables and form the algebra $\mathfrak{osp}(2|4)$. This result matches with the supertwistor representation for the massless superparticle in three dimensional flat space, see for example [85], because at least locally conformal theories do not distinguish between flat and AdS backgrounds.

Quantization of the model is then straightforward. Similarly to (2.47), for the bosonic variables we define creation–annihilation operators $a_l^\pm = \frac{1}{\sqrt{2}}(p_l \pm i q_l)$ and $a_r^\pm = \frac{1}{\sqrt{2}}(p_r \pm i q_r)$. Concerning the fermionic variables, it is crucial that after κ-symmetry we are still left with two real fermions, ψ_l in the left and ψ_r in the right sector, which is just enough to form one fermionic oscillator $\psi = \frac{1}{\sqrt{2}}(\psi_l + i\, \psi_r)$.[8] Following the canonical quantization rule $\{\mathcal{A}, \mathcal{B}\} \mapsto i[\mathcal{A}, \mathcal{B}]_\pm$ and adopting the ordering in (2.47) then yields a quantum realization of $\mathfrak{osp}(2|4)$.

Finally, we would like to note that at the classical level there is another attractive extension of the $\mathfrak{osp}_l(1|2) \oplus \mathfrak{osp}_r(1|2)$ algebra. Recall that the isometry algebra of the bosonic AdS_2 particle on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ consists out of only one $\mathfrak{sl}(2,\mathbb{R})$, see also [41]. For the massless case, this symmetry extends to the corresponding conformal symmetry of AdS_2, which is the infinite dimensional Virasoro algebra $\mathfrak{Vir}$, viz. the Witt algebra at the classical level. Moreover, it is well known that $\mathfrak{osp}(1|2)$ is a subalgebra of the super Virasoro algebra in the NS sector $\mathfrak{sVir}_{\mathrm{NS}}$. Therefore, it is tempting to extend the $\mathfrak{osp}_l(1|2) \oplus \mathfrak{osp}_r(1|2)$ algebra of the present massless AdS_3 superparticle to a double copy of the classical super Virasoro algebra in the NS–NS sector, $\mathfrak{sVir}_{\mathrm{NS},l} \oplus \mathfrak{sVir}_{\mathrm{NS},r}$.

Focusing again on the right sector and suppressing once more the index r, for this we again introduce the Hamiltonian $H = \langle R\mathbf{T}_0 \rangle$ (3.27) and raising–lowering functions (3.33). By (3.51), these become

[8] In comparison, recall that for the massless AdS_2 superparticle on $\mathrm{OSP}(1|2)/\mathrm{SO}(2)$ [41] κ-symmetry reduced the phase space to only one real fermion, which is insufficient for quantization of the model.

$$H=\frac{1}{4}(p^2+q^2), \qquad J^\pm=\frac{1}{4}(p\pm i\,q)^2, \qquad j^\pm=\frac{-i}{\sqrt{2}}(p\pm i\,q)\psi, \tag{3.55}$$

which resembles the $m\to 0$ limit of (3.36) and which fulfill the classical version of (3.34). Introducing the angle ϕ conjugate to H as

$$\phi=\arg\left(J^+\right), \qquad e^{i\,\phi}=\frac{J_+}{H}=\frac{(p+iq)^2}{p^2+q^2}, \qquad \{H,e^{i\,n\,\phi}\}=i\,n\,e^{i\,n\,\phi}, \tag{3.56}$$

we get $J^\pm=H\,e^{\pm i\,\phi}$ and $j^\pm=\sqrt{2\,J^\pm}\,\psi=\sqrt{2\,H}\,e^{\pm\frac{i}{2}\,\phi}$. From this we guess the charges

$$J_n=H\,e^{\pm i\,n\,\phi}, \qquad j_s=-i\sqrt{2\,J_{2\,s}}\,\psi=-i\sqrt{2\,H}\,e^{i\,s\,\phi}\,\psi, \tag{3.57}$$

for $n\in\mathbb{Z}$ and s being half-integer, $s\in\mathbb{Z}+\frac{1}{2}$, especially $J_0=H$, $J_{\pm 1}=J^\pm$ and $j_{\pm 1/2}=j^\pm$. These charges indeed fulfill the classical super Virasoro algebra in the NS sector,

$$\{J_m,J_n\}=-i(m-n)J_{m+n}, \quad \{J_m,j_s\}=-i\Big(\frac{m}{2}-s\Big)j_{m+s}, \quad \{j_s,j_t\}=-2i\,J_{s+t}. \tag{3.58}$$

We have not been able to quantize this classical representation of $\mathfrak{sVir}_{\mathrm{NS},l}\oplus\mathfrak{sVir}_{\mathrm{NS},r}$. This comes to no surprise as even quantization of the Virasoro algebra for the bosonic AdS_2 particle is still an open question.

Furthermore, it has to be pointed out that in contrast to the extension of $\mathfrak{osp}_l(1|2)\oplus\mathfrak{osp}_r(1|2)$ superisometries to the superconformal algebra $\mathfrak{osp}(2|4)$ the above extension to $\mathfrak{sVir}_{\mathrm{NS},l}\oplus\mathfrak{sVir}_{\mathrm{NS},r}$ does not actually correspond to a symmetry of the AdS_3 superparticle action. However, it is known that the $\mathrm{AdS}_3/\mathrm{CFT}_2$ on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{M}_4$ enjoys a *small*, respectively, *large* $\mathcal{N}=(4,4)$ superconformal algebra, see e.g. [65–68] as well as the more recent works [69,70], which have $\mathfrak{sVir}_{\mathrm{NS},l}\oplus\mathfrak{sVir}_{\mathrm{NS},r}$ subalgebras. Hence, the discussed extension might show relevant for effective descriptions of string and even higher spin states.

4. Conclusions

Quantization of the Green–Schwarz superstring on AdS superspaces from first principles is still an open problem. To attain a better understanding, the work [32] suggested to study orbit method quantization of semi-classical string solutions, where we have explored this idea in [37] and [41]. In this work, we continued this program and applied superisometry group orbit quantization to the κ-symmetric AdS_3 superparticle on the coset $\mathrm{OSP}(1|2)^2/\mathrm{SL}(2,\mathbb{R})$.

First, we reviewed how the method applies to bosonic AdS_3 on the group manifold $\mathrm{SL}(2,\mathbb{R})$. The massive particle is described by orbits of a temporal $\mathfrak{sl}(2,\mathbb{R})$ element, with its norm given by the mass m, while for the massless case one has to consider orbits of a lightlike $\mathfrak{sl}(2,\mathbb{R})$ element. For both cases the calculations split up into left and right chiral sectors and the physical phase space of each sector is two dimensional, being a half-plane for the massive case while a full plane without origin for the massless case. From the left and right Noether currents we read off the dynamical integrals, where at the classical level the massless charges can formally be viewed as the $m\to 0$ limit of the massive charges. These fulfill the isometry algebra $\mathfrak{sl}_l(2,\mathbb{R})\oplus\mathfrak{sl}_r(2,\mathbb{R})\cong\mathfrak{so}(2,2)$, which determined quantization of the system, yielding a quantum realization unitarily equivalent to the Holstein–Primakoff representation [39], see also [37,84]. For the massless case we then observed how the isometries extend to the AdS_3 conformal symmetries $\mathfrak{sp}(4)\cong\mathfrak{so}(2,3)$.

Next, we turned to the AdS_3 superparticle. For this we first discussed the superparticle action on cosets of the form G^2/H and pointed out that generally in comparison to the superparticle action on G it amounts to a subclass of orbits. In particular, focusing then on

$G^2/H = \mathrm{OSP}(1|2)^2/\mathrm{SL}(2,\mathbb{R})$, the massive and massless particle are described by OSP(1|2) orbits of timelike and lightlike elements of $\mathfrak{sl}(2,\mathbb{R})$, respectively. Again, the calculation split into left and right chiral sectors, where apart from two real bosons the physical phase space of each sector contains two real fermions in the massive case whilst only one real fermion for the massless case. As anticipated [41], the latter reflects the underlying $\varkappa$-symmetry for the massless superparticle and importantly the two remaining real fermions could be combined into a fermionic oscillator, which can be quantized. The dynamical integrals respected the $\mathfrak{osp}_l(1|2) \oplus \mathfrak{osp}_r(1|2)$ super isometry algebra and quantization amounted to two copies of Holstein–Primakoff type quantum representations of $\mathfrak{osp}(1|2)$ [41]. For the massive case we observed that each chiral sector corresponds to the superparticle with superpotential $W = q - \frac{2\mu-1/2}{q}$ [48]. For the massless case we demonstrated how the superisometries extend to the corresponding superconformal algebra $\mathfrak{osp}(2|4)$. We finally pointed out that, at least at the classical level, there is another interesting extension of the superisometry algebra $\mathfrak{osp}_l(1|2) \oplus \mathfrak{osp}_r(1|2)$ to a double copy of the super Virasoro algebra in the NS sector, $\mathfrak{sVir}_{\mathrm{NS},l} \oplus \mathfrak{sVir}_{\mathrm{NS},r}$.

Our work offers several future directions of research. As this article discusses orbit method quantization for what arguably amounts to the simplest quantum consistent massless AdS superparticle, a natural next step is to investigate AdS superparticles with a higher amount of supersymmetry, in particular the AdS_2 and AdS_3 superparticles build on the superalgebras $\mathfrak{su}(1,1|1)$, $\mathfrak{psu}(1,1|2)$, and $\mathfrak{d}(2,1;\alpha)$, see also [42–47] and [77–80] and references therein.

In a longer term we would like to utilize this quantization scheme to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superparticle, see also [13] and the recent work [15]. Hence, apart from increasing the amount of supersymmetry another intermediate goal is to raise the dimension of the AdS space. Indeed, the found charges forming a quantum realization of $\mathfrak{sl}_l(2,\mathbb{R}) \oplus \mathfrak{sl}_r(2,\mathbb{R})$, respectively, $\mathfrak{osp}_l(1|2) \oplus \mathfrak{osp}_r(1|2)$ can be rewritten in an $\mathfrak{so}(2,2)$ scheme. By this, the expressions become covariant under the $\mathfrak{so}(2) \subset \mathfrak{so}(2,2)$ corresponding to the rotations of the spatial directions of $\mathbb{R}^{2,2}$ embedding space. As we show in [86], generalization of the $\mathfrak{so}(2)$ to an $\mathfrak{so}(N)$ covariance then yields an ansatz for quantum prescription of the bosonic AdS_{N+1} particle, respectively, the $\mathcal{N} = 1$ AdS_{N+1} superparticle. A similar idea has been adopted in [80], where the dynamical realization on SU(1, 1|2) has been generalized to SU(1, 1|N).

Another direction is to apply the orbit method to honest string solution, *viz.* ones storing more information than only the particle degrees of freedom. As advocated previously [32], we hope that such orbits open a window into computation of the string spectrum from first principles, especially for short strings. In particular, in view of [37] we expect that the step from the superparticle on for example $\frac{\mathrm{PSU}(1,1|2)}{\mathrm{SO}(1,1)\times\mathrm{SO}(2)} \times \mathrm{T}^4$ to orbit quantization of more involved string solutions on this background should be manageable, thus yielding results for the spectral problem in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ [51–57].

Furthermore, we are also curious if our results may find application for the non-critical AdS_3 superstring, see also [73], and even of the AdS_3 higher spin theory [74,75]. Even for the non-critical string in bosonic AdS_3 [71,72] it seems promising to apply the orbit method as the model resembles the WZNW model on $\mathrm{SL}(2,\mathbb{R})$ [87].

Finally, lately there has been considerable interest in the so-called η-deformation [88–90], which amounts to a one-parameter integrable deformation of the of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. However, even the particle dynamics on this background are still an open problem, see also [91]. For the truncation to $(\mathrm{S}^2)_\eta$, corresponding to the Fateev sausage model, geodesic motion has been solved recently [92] but the non-closure of the orbits seems to stem a fundamental obstacle to quantization of this system. Also here the Kirillov–Kostant–Souriau method of coadjoint orbits might lead the way out, as its extension to quantum groups has been investigated [93].

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Appendix A. More on SL(2, $\mathbb{R}$) calculus

In this appendix we describe some technical details of SL(2, $\mathbb{R}$) calculations.

The basis vectors (2.4) satisfy the matrix relations

$$\mathbf{t}_\mu \mathbf{t}_\nu = \eta_{\mu\nu}\mathbf{I} + \epsilon_{\mu\nu}{}^{\rho}\mathbf{t}_\rho \tag{A.1}$$

and by (2.3) one obtains

$$\mathbf{t}_\nu\, g = X^{0'}\mathbf{t}_\nu + X_\nu\mathbf{I} - \epsilon_{\mu\nu}{}^{\rho}X^\mu\mathbf{t}_\rho, \qquad g\,\mathbf{t}_\nu = X^{0'}\mathbf{t}_\nu + X_\nu\mathbf{I} + \epsilon_{\mu\nu}{}^{\rho}X^\mu\mathbf{t}_\rho. \tag{A.2}$$

From (2.3) one also finds the infinitesimal transformations of g corresponding to (2.7)

$$g \mapsto g + \alpha^{0'\nu}(X^{0'}\mathbf{t}_\nu + X_\nu\mathbf{I}) + \alpha^{\rho\mu}X_\mu\mathbf{t}_\rho. \tag{A.3}$$

Introducing dual parameters β_ν by $\beta_\nu = -\frac{1}{2}\epsilon_{\nu\rho\mu}\alpha^{\rho\mu}$, one gets $\alpha^{\rho\mu} = \epsilon^{\rho\mu\nu}\beta_\nu$, due to $\epsilon^{\rho\mu\nu}\epsilon_{\nu\rho'\mu'} = \delta^{\rho}_{\mu'}\delta^{\mu}_{\rho'} - \delta^{\rho}_{\rho'}\delta^{\mu}_{\mu'}$, and (A.3) takes the form (2.8), because of (A.2).

Note that the infinitesimal conformal transformation (2.10) has the form

$$g \mapsto g + \varepsilon^A K^\mu_A \mathbf{t}_\mu\, g, \tag{A.4}$$

which exhibits that the transformed matrix in (2.10) is also an element of SL(2, $\mathbb{R}$). Indeed, comparing (2.10) and (A.4) for the terms containing $\varepsilon^{0'}$, we get $g^{-1} + X_{0'}\mathbf{I} = K^\mu_{0'}\mathbf{t}_\mu$ and inserting here g^{-1} from (2.3), we find $K^\mu_{0'} = -X^\mu$. Similarly, the terms containing ε^ν yield $\mathbf{t}_\nu\, g^{-1} + X_\nu\mathbf{I} = K^\mu_\nu\mathbf{t}_\mu$ and lead to $K^\mu_\nu = X^{0'}\delta^\mu_n - \epsilon^\mu{}_{\nu\rho}X^\rho$.

In practical calculations it is helpful to use the Chevalley basis of $\mathfrak{sl}(2,\mathbb{R})$

$$\mathbf{t}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{t}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{t}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{A.5}$$

which satisfies the commutation relations

$$[\mathbf{t}_2, \mathbf{t}_\pm] = \pm 2\mathbf{t}_\pm, \qquad [\mathbf{t}_+, \mathbf{t}_-] = \mathbf{t}_2, \tag{A.6}$$

and therefore $e^{\alpha\mathbf{t}_2}\,\mathbf{t}_\pm\, e^{-\alpha\mathbf{t}_2} = e^{\pm 2\alpha}\,\mathbf{t}_+$. The nonzero components of the normalized Killing form are $\langle\mathbf{t}_2\,\mathbf{t}_2\rangle = 1$, $\langle\mathbf{t}_+\,\mathbf{t}_-\rangle = \frac{1}{2}$. These equations simplify the calculation of the presymplectic forms (2.17), (2.36) and of the Noether charges (2.20), (2.39).

Using the parametrization (2.35), the Noether charge related to the conformal transformations $C = g^{-1}\,L$ takes the following form $C = g_r^{-1}\,\mathbf{t}_+\,g_l^{-1}$. The Iwasawa decomposition (2.37) then leads to

$$C = \begin{pmatrix} e^{\alpha_l}\,e^{-\alpha_r}\,\sin\theta_l\,\cos\theta_r & e^{\alpha_l}\,e^{-\alpha_r}\,\cos\theta_l\,\cos\theta_r \\ e^{\alpha_l}\,e^{-\alpha_r}\,\sin\theta_l\,\sin\theta_r & e^{\alpha_l}\,e^{-\alpha_r}\,\cos\theta_l\,\sin\theta_r \end{pmatrix}, \tag{A.7}$$

and in the canonical coordinates (2.40) on obtains (2.43). This yields $C_{0'} = \frac{1}{2}(q_l\,p_r + p_l q_r)$, $C_0 = \frac{1}{2}(p_l\,p_r - q_l q_r)$, $C_1 = -\frac{1}{2}(q_l\,p_r + q_l q_r)$, $C_2 = \frac{1}{2}(q_l\,p_r - p_l q_r)$.

Now we prove that the Iwasawa decomposition (2.18) for a given $g_l \in \mathrm{SL}(2,\mathbb{R})$ uniquely fixes the parameters $\gamma_l \in \mathbb{R}^1$, $a_l \in \mathbb{R}^1$, and $\theta_l \in S^1$. For simplicity we omit the index l. First note that for a given $g \in \mathrm{SL}(2,\mathbb{R})$ one can find the parameter γ such that the matrix $\tilde{g} = e^{-\gamma\mathbf{t}_+}\,g$ has rows orthogonal to each other. Indeed, for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{one has} \qquad \tilde{g} = e^{-\gamma\mathbf{t}_+}\,g = \begin{pmatrix} a-\gamma c & b-\gamma d \\ c & d \end{pmatrix}, \tag{A.8}$$

and requiring orthogonality of the rows uniquely fixes the parameter $\gamma = \frac{ac+bd}{c^2+d^2}$. Now, any such matrix $\tilde{g}$ can be parametrized as

$$\tilde{g} = e^{\alpha\mathbf{t}_2}\,e^{\theta\mathbf{t}_0} = \begin{pmatrix} e^{\alpha}\cos\theta & e^{\alpha}\sin\theta \\ -e^{-\alpha}\sin\theta & e^{-\alpha}\cos\theta \end{pmatrix}. \tag{A.9}$$

Especially, as the two orthogonal rows are nonzero two-vectors, the parameters α and θ are uniquely defined by one of the rows. The proof can easily be repeated for g_r in (2.18).

Appendix B. Canonical map to the Holstein–Primakoff realization

Here we describe a canonical map which relates the Holstein–Primakoff type realization of $\mathfrak{sl}(2,\mathbb{R})$ to the realization given by (2.26)–(2.27). This map provides a one to one correspondence between complex canonical coordinates (b^*, b) on a plane and canonical coordinates (p, q) given on the half-plane $q > 0$.

The Holstein–Primakoff realization classically is represented as (see (2.32))

$$E = b^*b + m/2, \quad B^* = \sqrt{m + b^*b}\,b^*, \quad B = \sqrt{m + b^*b}\,b, \tag{B.1}$$

where (b^*, b) are complex coordinates on $\mathbb{R}^2$ and the Poisson bracket $\{b, b^*\} = i$ provides the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$\{E, B^*\} = i\,B^*, \qquad \{E, B\} = -i\,B, \qquad \{B, B^*\} = 2i\,E. \tag{B.2}$$

The generators (2.26)–(2.27) classically can be written as

$$H = \frac{1}{4}\left(p^2 + q^2 + m^2/q^2\right), \qquad J^{\pm} = \frac{1}{4}\left(p^2 - q^2 + m^2/q^2\right) \pm \frac{i}{2}\,pq \tag{B.3}$$

and the canonical bracket $\{p, q\} = 1$ leads to the algebra

$$\{H, J^{\pm}\} = \pm i\,J^{\pm}, \qquad \{J^-, J^+\} = 2i\,H, \tag{B.4}$$

which is equivalent to (B.2) under the identifications $E = H$, $B^* = -J^+$, $B = -J^-$.

From these equations one finds $q=\sqrt{2E+B^*+B}$, $pq=i(B^*-B)$ and the inverse map is given by

$$b^*=-\frac{J^+}{\sqrt{H+m/2}}, \qquad b=-\frac{J^-}{\sqrt{H+m/2}}. \tag{B.5}$$

Then, from $\{p,q\}=1$ follows $\{b,b^*\}=i$ and vice versa, i.e. $\mathrm{d}p\wedge\mathrm{d}q=i\,\mathrm{d}b^*\wedge\mathrm{d}b$.

Appendix C. Parametrization of OSP(1|2)

In the last appendix we describe parameterizations of OSP(1|2) group elements and give some useful formulas for calculations of the Noether charges and the symplectic forms.

In the massive case one can start with $g_l=g_l^{(b)}\,g_l^{(f)}$, where $g_l^{(b)}$ and $g_l^{(f)}$ are purely bosonic and purely fermionic parts, respectively. For the bosonic part we use the Iwasawa decomposition $g_l^{(b)}=e^{\gamma_l\,\mathbf{T}_+}\,e^{\alpha_l\,\mathbf{T}}\,e^{\theta_l\,\mathbf{T}_0}$, as in (2.18), and the fermionic part we represent in a symmetric form $g_l^{(f)}=e^{\zeta_l\,\mathbf{S}_+ +\eta_l\,\mathbf{S}_-}$. Using then the relations

$$e^{\theta\,\mathbf{T}_0}\,\mathbf{S}_\pm\,e^{-\theta\,\mathbf{T}_0}=\cos\theta\,\mathbf{S}_\pm\pm\sin\theta\,\mathbf{S}_\mp, \tag{C.1}$$

$$e^{\zeta\,\mathbf{S}_+ +\eta\,\mathbf{S}_-}=e^{\frac{1}{2}\eta\zeta\mathbf{T}}e^{\zeta\,\mathbf{S}_+}\,e^{\eta\,\mathbf{S}_-}, \tag{C.2}$$

which follow from the $\mathfrak{osp}(1|2)$ algebra (3.8), we represent g_l as

$$g_l=e^{\gamma_l\,\mathbf{T}_+}\,e^{\tilde\alpha_l\,\mathbf{T}}\,e^{\tilde\zeta_l\,\mathbf{S}_+}\,e^{\tilde\eta_l\,\mathbf{S}_-}\,e^{\theta_l\,\mathbf{T}_0}, \tag{C.3}$$

where $\tilde\zeta_l=\cos\theta_l\,\zeta_l-\sin\theta_l\,\eta_l$, $\tilde\eta_l=\sin\theta_l\,\zeta_l+\cos\theta_l\,\eta_l$ and $\tilde\alpha_l=\alpha_l+\frac{1}{2}\,\tilde\eta_l\,\tilde\zeta_l$. Removing then 'tilde' in (C.3), we obtain g_l in (3.12). g_r is obtained in a similar way, starting with $g_r=g_r^{(f)}\,g_r^{(b)}$ and using the same steps as for g_l.

The calculation of the Noether charges and the presymplectic forms in the massive case (see (3.13)–(3.14)) is based on the following relations

$$\begin{aligned}
&e^{\zeta\,\mathbf{S}_+}\,\mathbf{T}_0\,e^{-\zeta\,\mathbf{S}_+}=\mathbf{T}_0-\zeta\,\mathbf{S}_-, && e^{\eta\,\mathbf{S}_-}\,\mathbf{T}_0\,e^{-\eta\,\mathbf{S}_-}=\mathbf{T}_0+\eta\,\mathbf{S}_+,\\
&e^{\zeta\,\mathbf{S}_+}\,\eta\,\mathbf{S}_+\,e^{-\zeta\,\mathbf{S}_+}=\eta\,\mathbf{S}_+ +2\,\zeta\,\eta\,\mathbf{T}_+, && e^{\gamma\,\mathbf{T}_+}\,\mathbf{T}_-\,e^{-\gamma\,\mathbf{T}_+}=\mathbf{T}_- +\gamma\,\mathbf{T}-\gamma^2\mathbf{T}_+,
\end{aligned} \tag{C.4}$$

which also follow from (3.8).

In the massless case we start again with $g_l=g_l^{(b)}\,g_l^{(f)}$, where $g_l^{(b)}=e^{\theta_l\,\mathbf{T}_0}\,e^{\alpha_l\,\mathbf{T}}\,e^{\gamma_l\,\mathbf{T}_+}$ and $g_l^{(f)}=e^{\zeta_l\,\mathbf{S}_- +\eta_l\,\mathbf{S}_+}$. Similarly to (C.1)–(C.2), here we use

$$e^{\gamma\,\mathbf{T}_+}\,\mathbf{S}_+\,e^{-\gamma\,\mathbf{T}_+}=\mathbf{S}_+, \qquad e^{\gamma\,\mathbf{T}_+}\,\mathbf{S}_-\,e^{-\gamma\,\mathbf{T}_+}=\mathbf{S}_- -\gamma\,\mathbf{S}_+, \tag{C.5}$$

$$e^{\zeta\,\mathbf{S}_- +\eta\,\mathbf{S}_+}=e^{\frac{1}{2}\eta\zeta\mathbf{T}}e^{\zeta\,\mathbf{S}_-}\,e^{\eta\,\mathbf{S}_+}, \tag{C.6}$$

which leads to

$$g_l=e^{\theta_l\,\mathbf{T}_0}\,e^{\tilde\alpha_l\,\mathbf{T}}\,e^{\zeta_l\,\mathbf{S}_-}\,e^{\tilde\eta_l\,\mathbf{S}_+}\,e^{\gamma_l\,\mathbf{T}_+}, \tag{C.7}$$

where $\tilde\eta_l=\eta_l-\gamma_l\,\zeta_l$ and $\tilde\alpha_l=\alpha_l+\frac{1}{2}\,\eta_l\zeta_l$. Removing again the 'tilde', we get g_l in (3.46). The parametrization of g_r is derived in a similar way.

Finally, we present some helpful formulas for the calculations in the massless case

$$\begin{aligned}
&e^{\eta\,\mathbf{S}_+}\,\mathbf{T}_+\,e^{-\eta\,\mathbf{S}_+}=\mathbf{T}_+, && e^{\zeta\,\mathbf{S}_-}\,\mathbf{T}_+\,e^{-\zeta\,\mathbf{S}_-}=\mathbf{T}_+ +\zeta\,\mathbf{S}_+,\\
&e^{\alpha\,\mathbf{T}}\,\mathbf{T}_\pm\,e^{-\alpha\,\mathbf{T}}=e^{\pm 2\alpha}\,\mathbf{T}_\pm, && e^{\alpha\,\mathbf{T}}\,\mathbf{S}_\pm\,e^{-\alpha\,\mathbf{T}}=e^{\pm\alpha}\,\mathbf{S}_\pm,
\end{aligned} \tag{C.8}$$

$$e^{\theta\,\mathbf{T}_0}\,\mathbf{T}_+\,e^{-\theta\,\mathbf{T}_0}=\cos^2\theta\,\mathbf{T}_+ -\sin^2\theta\,\mathbf{T}_- +\sin\theta\,\cos\theta\,\mathbf{T}. \tag{C.9}$$

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