Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 340 (2008) 1033–1045
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On existence and nonexistence of global solutions of Cauchy–Goursat problem for nonlinear wave equations ☆

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Received 12 June 2007
Available online 25 September 2007
Submitted by M. Nakao

Abstract

We consider the Cauchy–Goursat initial characteristic problem for nonlinear wave equations with power nonlinearity. Depending on the power of nonlinearity and the parameter in an equation we investigate the problem on existence and nonexistence of global solutions of the Cauchy–Goursat problem. The question on local solvability of the problem is also considered.

Keywords: Nonlinear wave equations; Cauchy–Goursat problem; Global and local solvability; Blow-up

1. Statement of a problem

In the plane of independent variables x and t consider nonlinear wave equation of the following form:

$$L_\lambda u := u_{tt} - u_{xx} + \lambda |u|^\alpha u = f(x,t), \tag{1}$$

where λ and α are given real constants, and $\lambda\alpha \neq 0$, $\alpha > -1$; f is given, while u—unknown real functions.

Denote by $D_T := \{(x,t)\colon\ 0 < x < t,\ 0 < t < T\}$, $T \leqslant \infty$ triangle domain, bounded by characteristic segment $\gamma_{1,T}\colon\ x = t, 0 \leqslant t \leqslant T$, segments $\gamma_{2,T}\colon\ x = 0, 0 \leqslant t \leqslant T$ and $\gamma_{3,T}\colon\ t = T, 0 \leqslant x \leqslant T$.

For Eq. (1) in domain D_T consider the Cauchy–Goursat problem on determination of solution $u(x,t)$ by initial-characteristic conditions [1, p. 228]

$$u_x|_{\gamma_{2,T}} = 0, \qquad u|_{\gamma_{1,T}} = 0. \tag{2}$$

Certain papers have been devoted to the questions of existence and nonexistence of global solutions of nonlinear hyperbolic equations for different problems (such as initial, mixed and nonlocal problems) [2–11]. In linear case, i.e., for $\lambda\alpha = 0$, problem (1), (2) is posed correctly and we have global solvability in corresponding functional spaces [1,12].

We show that for certain assumption on the power of nonlinearity α and parameter λ problem (1), (2) in some cases is globally solvable, while in other cases it has not global solution, though, as it will be shown below, this problem is locally solvable.

☆ The present work was supported by the INTAS grant No. 05-1000008-7921.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.09.030

Definition 1. Let $f \in C(\overline{D}_T)$. Function u is called a strong generalized solution of problem (1), (2) of the class C in domain D_T, if $u \in C(\overline{D}_T)$ and there exists such the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C(\overline{D}_T)$ for $n \to \infty$, where $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T) := \{u \in C^2(\overline{D}_T)\colon\ u_x|_{\gamma_{2,T}} = 0,\ u|_{\gamma_{1,T}} = 0\}$, $\Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Remark 1. It is clear that a classical solution of problem (1), (2) in space $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a strong generalized solution of this problem of the class C in domain D_T. In turn, if a strong generalized solution of problem (1), (2) of the class C in domain D_T belongs to the space $C^2(\overline{D}_T)$, then it also is a classical solution of the problem.

Definition 2. Let $f \in C(\overline{D}_\infty)$. We say that problem (1), (2) is globally solvable in the class C, if for any finite $T > 0$ the problem has a strong generalized solution of the class C in domain D_T.

2. A priori estimate of the solution of problem (1), (2)

Lemma 1. *Let $-1 < \alpha < 0$ and in the case when $\alpha > 0$ let us additionally require that $\lambda > 0$. Then for a strong generalized solution of problem* (1), (2) *of the class C in domain D_T it is valid the following a priori estimate*:

$$\|u\|_{C(\overline{D}_T)} \leqslant c_1 \|f\|_{C(\overline{D}_T)} + c_2 \tag{3}$$

with positive constants $c_i(T, \alpha, \lambda)$, $i = 1, 2$, not dependent on u and f.

Proof. First consider the case when $\alpha > 0$ and $\lambda > 0$. Let u be a strong generalized solution of problem (1), (2) of the class C in domain D_T. Then due to Definition 1 there exists the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|L_\lambda u_n - f\|_{C(\overline{D}_T)} = 0, \tag{4}$$

and therefore

$$\lim_{n\to\infty} \big\| \lambda |u_n|^\alpha u_n - \lambda |u|^\alpha u \big\|_{C(\overline{D}_T)} = 0. \tag{5}$$

Consider function $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$, as a solution of the following problem:

$$L_\lambda u_n = f_n, \tag{6}$$

$$\frac{\partial u_n}{\partial x}\bigg|_{\gamma_{2,T}} = 0, \qquad u_n|_{\gamma_{1,T}} = 0. \tag{7}$$

Here

$$f_n := L_\lambda u_n. \tag{8}$$

Multiplying the both sides of equality (6) by $\frac{\partial u_n}{\partial t}$ and integrating in domain $D_\tau := \{(x,t) \in D_T\colon\ 0 < t < \tau\}$, $0 < \tau \leqslant T$ we receive

$$\frac{1}{2} \int_{D_\tau} \frac{\partial}{\partial t} \left(\frac{\partial u_n}{\partial t} \right)^2 dx\,dt - \int_{D_\tau} \frac{\partial^2 u_n}{\partial x^2} \frac{\partial u_n}{\partial t}\,dx\,dt + \frac{\lambda}{\alpha+2} \int_{D_\tau} \frac{\partial}{\partial t} |u_n|^{\alpha+2}\,dx\,dt = \int_{D_\tau} f_n \frac{\partial u_n}{\partial t}\,dx\,dt. \tag{9}$$

Assume that $\Omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \leqslant T$. Then by virtue of (7), integrating by parts the left side of equality (9), we have

$$\int_{D_\tau} f_n \frac{\partial u_n}{\partial t}\,dx\,dt = \int_{\gamma_{1,\tau}} \frac{1}{2\nu_t} \left[\left(\frac{\partial u_n}{\partial x} \nu_t - \frac{\partial u_n}{\partial t} \nu_x \right)^2 + \left(\frac{\partial u_n}{\partial t} \right)^2 \left(\nu_t^2 - \nu_x^2 \right) \right] ds$$
$$+ \frac{1}{2} \int_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial t} \right)^2 + \left(\frac{\partial u_n}{\partial x} \right)^2 \right] dx + \frac{\lambda}{\alpha+2} \int_{\Omega_\tau} |u_n|^{\alpha+2}\,dx, \tag{10}$$

where $\nu := (\nu_x, \nu_t)$ is unit vector of outer normal to ∂D_τ and $\gamma_{1,\tau} := \gamma_{1,T} \cap \{t \leqslant \tau\}$.

Taking into account that the operator $\nu_t \frac{\partial}{\partial x} - \nu_x \frac{\partial}{\partial t}$ is an interior differential operator on $\gamma_{1,T}$, due to the second condition in (7) we have

$$\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x\right)\Big|_{\gamma_{1,\tau}} = 0. \tag{11}$$

Further, it is clear that

$$\left(\nu_t^2 - \nu_x^2\right)\big|_{\gamma_{1,\tau}} = 0. \tag{12}$$

Therefore, by virtue of (11), (12) from (10) we get

$$w_n(\tau) := \int\limits_{\Omega_\tau} \left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right] dx \leqslant 2\int\limits_{D_\tau} f_n \frac{\partial u_n}{\partial t}\, dx\, dt. \tag{13}$$

Taking into account inequality

$$2 f_n \frac{\partial u_n}{\partial t} \leqslant \varepsilon\left(\frac{\partial u_n}{\partial t}\right)^2 + \frac{1}{\varepsilon} f_n^2,$$

which is valid for any $\varepsilon := \text{const} > 0$, from (13) we have

$$w_n(\tau) \leqslant \varepsilon \int\limits_0^\tau w_n(\sigma)\, d\sigma + \frac{1}{\varepsilon}\|f_n\|^2_{L_2(D_\tau)}, \quad 0 < \tau \leqslant T.$$

Whence, having the fact that the value $\|f_n\|^2_{L_2(D_\tau)}$, as a function of τ is nondecreasing, by the Gronwall lemma [13, p. 13] we receive

$$w_n(\tau) \leqslant \frac{1}{\varepsilon}\|f_n\|^2_{L_2(D_\tau)} \exp(\tau\varepsilon).$$

Thus, by taking into account that $\inf_{\varepsilon>0} \frac{\exp(\tau\varepsilon)}{\varepsilon} = e\tau$, which is achieved for $\varepsilon = \frac{1}{\tau}$, we obtain

$$w_n(\tau) \leqslant e\tau \|f_n\|^2_{L_2(D_\tau)}, \quad 0 < \tau \leqslant T. \tag{14}$$

If $(x,t) \in \overline{D}_T$, then by virtue of the second condition in (7) the following equality is valid:

$$u_n(x,t) = u_n(x,t) - u_n(t,t) = \int\limits_t^x \frac{\partial u_n(\sigma,t)}{\partial x}\, d\sigma,$$

thus due to (14) we have

$$\begin{aligned}\left|u_n(x,t)\right|^2 &\leqslant \int\limits_x^t d\sigma \int\limits_x^t \left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \leqslant (t-x)\int\limits_{\Omega_t}\left[\frac{\partial u_n(\sigma,t)}{\partial x}\right]^2 d\sigma \leqslant (t-x) w_n(t) \\ &\leqslant t w_n(t) \leqslant e t^2 \|f_n\|^2_{L_2(D_t)} \leqslant e t^2 \|f_n\|^2_{C(\overline{D}_t)} \operatorname{mes} D_t \leqslant \frac{1}{2} e t^4 \|f_n\|^2_{C(\overline{D}_T)}.\end{aligned} \tag{15}$$

From (15) it follows that

$$\|u_n\|_{C(\overline{D}_T)} \leqslant \sqrt{\frac{e}{2}}\, T^2 \|f_n\|_{C(\overline{D}_T)}.$$

According to (4)–(8), by passing in the last inequality to limit for $n \to \infty$, we receive

$$\|u\|_{C(\overline{D}_T)} \leqslant \sqrt{\frac{e}{2}}\, T^2 \|f\|_{C(\overline{D}_T)}. \tag{16}$$

From (16) follows estimate (3) in case when $\alpha > 0$ and $\lambda > 0$.

Now consider the case when $-1<\alpha<0$ for any λ. When $-1<\alpha<0$, i.e., when $1<\alpha+2<2$, using well-known inequality

$$ab \leqslant \frac{a^p}{p}+\frac{b^q}{q} \quad \left(a=|u_n|^{\alpha+2},\ b=1,\ p=\frac{2}{\alpha+2}>1,\ q=-\frac{2}{\alpha}>1,\ \frac{1}{p}+\frac{1}{q}=1\right)$$

we get

$$\int\limits_{\Omega_\tau} |u_n|^{\alpha+2}\,dx \leqslant \int\limits_{\Omega_\tau}\left[\frac{\alpha+2}{2}|u_n|^2-\frac{\alpha}{2}\right]dx=\frac{\alpha+2}{2}\int\limits_{\Omega_\tau}|u_n|^2\,dx+\frac{|\alpha|\tau}{2}.$$

From equality (10), by virtue of (11), (12) and the last inequality it follows that

$$\omega_n(\tau)\leqslant|\lambda|\int\limits_{\Omega_\tau}|u_n|^2\,dx+\frac{|\lambda\alpha|\tau}{\alpha+2}+2\int\limits_{D_\tau}f_n\frac{\partial u_n}{\partial t}\,dx\,dt. \tag{17}$$

In accordance with the theory of trace there holds estimate [14, pp. 77, 86]

$$\|u_n\|_{L_2(\Omega_\tau)}\leqslant\sqrt{\tau}\|u_n\|_{\mathring{W}_2^1(D_\tau,\gamma_{1,\tau})},\quad 0<\tau\leqslant T, \tag{18}$$

where $\mathring{W}_2^1(D_\tau,\gamma_{1,\tau}):=\{u\in W_2^1(D_\tau)\colon\ u|_{\gamma_{1,\tau}}=0\}$, $W_2^1(D_\tau)$ is well-known Sobolev space, and

$$\|u_n\|^2_{\mathring{W}_2^1(D_\tau,\gamma_{1,\tau})}:=\int\limits_{D_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2+\left(\frac{\partial u_n}{\partial x}\right)^2\right]dx\,dt.$$

Since $2f_n\frac{\partial u_n}{\partial t}\leqslant f_n^2+(\frac{\partial u_n}{\partial t})^2$, then due to (17), (18) we have

$$w_n(\tau)\leqslant|\lambda|\tau\int\limits_{D_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2+\left(\frac{\partial u_n}{\partial x}\right)^2\right]dx\,dt+\int\limits_{D_\tau}\left(\frac{\partial u_n}{\partial t}\right)^2dx\,dt+\int\limits_{D_\tau}f_n^2\,dx\,dt+\frac{|\lambda\alpha|\tau}{\alpha+2}.$$

Whence according to the form of function $w_n(\tau)$ we get

$$w_n(\tau)\leqslant\big(|\lambda|\tau+1\big)\int\limits_0^\tau w_n(\sigma)\,d\sigma+\|f_n\|^2_{L_2(D_\tau)}+\frac{|\lambda\alpha|\tau}{\alpha+2},$$

thus by Gronwall's lemma [13, p. 13] we obtain

$$w_n(\tau)\leqslant\left[\|f_n\|^2_{L_2(D_T)}+\frac{|\lambda\alpha|T}{\alpha+2}\right]\exp\big(|\lambda|T\tau+\tau\big).$$

Analogously to that as (16) was received from (15), from last inequality we receive

$$\begin{aligned}\big|u_n(x,t)\big|^2\leqslant t\,w_n(t)&\leqslant T\left[\|f_n\|^2_{C(\overline{D}_T)}\operatorname{mes}D_T+\frac{|\lambda\alpha|T}{\alpha+2}\right]\exp\big(|\lambda|T^2+T\big)\\&=T\left[\frac{T^2}{2}\|f_n\|^2_{C(\overline{D}_T)}+\frac{|\lambda\alpha|T}{\alpha+2}\right]\exp\big(|\lambda|T^2+T\big).\end{aligned}$$

From here follows that

$$\|u_n\|_{C(\overline{D}_T)}\leqslant\left[\sqrt{\frac{T}{2}}T\|f_n\|_{C(\overline{D}_T)}+\sqrt{\frac{|\lambda\alpha|}{\alpha+2}}T\right]\exp\left\{\frac{1}{2}\big(|\lambda|T^2+T\big)\right\},$$

whence due to (4)–(8), as a result of passing to limit when $n\to\infty$ we get estimate

$$\|u\|_{C(\overline{D}_T)}\leqslant\sqrt{\frac{T}{2}}T\exp\left\{\frac{1}{2}\big(|\lambda|T^2+T\big)\right\}\|f\|_{C(\overline{D}_T)}+\sqrt{\frac{|\lambda\alpha|}{\alpha+2}}T\exp\left\{\frac{1}{2}\big(|\lambda|T^2+T\big)\right\}. \tag{19}$$

This completely proves estimate (3). $\square$

Remark 2. From (16) and (19) follows that constants c_1 and c_2 in estimate (3) are equal

$$(1) \qquad c_1 = \sqrt{\frac{e}{2}} T^2, \qquad c_2 = 0, \quad \text{for } \alpha > 0,\ \lambda > 0; \tag{20}$$

$$(2) \qquad c_1 = \sqrt{\frac{T}{2}} T \exp\left\{\frac{1}{2}\bigl(|\lambda| T^2 + T\bigr)\right\}, \qquad c_2 = \sqrt{\frac{|\lambda\alpha|}{\alpha+2}} T \exp\left\{\frac{1}{2}\bigl(|\lambda| T^2 + T\bigr)\right\}, \tag{21}$$

for $-1 < \alpha < 0,\ \lambda \in (-\infty, 0) \cup (0, +\infty)$.

3. Equivalent reduction of problem (1), (2) to Volterra type nonlinear integral equation

Let $P_0 := P_0(x_0, t_0)$ be an arbitrary point in domain D_T. Denote by G_{x_0,t_0} a quadrangle with vertices $O(0,0)$, $P_0(x_0,t_0)$ and also P_1, P_3, which lay on data supports $\gamma_{2,T}$ and $\gamma_{1,T}$, respectively, i.e., $P_1 := P_1(0, t_0 - x_0)$, $P_3 := P_3(\frac{x_0+t_0}{2}, \frac{x_0+t_0}{2})$; and by Ω_{x_0,t_0}—a triangle domain with vertices P_1, O and P_2—belonging to characteristics $\gamma_{1,T}$, i.e., $P_2 := P_2(\frac{t_0-x_0}{2}, \frac{t_0-x_0}{2})$.

Let $u \in C^2(\overline{D}_T)$ be a classical solution of problem (1), (2). By integration of Eq. (1) in domain G_{x_0,t_0}, using homogeneous boundary conditions (2) and returning to initial variables x, t it is easy to see that

$$\begin{aligned} &u(x,t) + \frac{\lambda}{2} \int\limits_{G_{x,t}} |u|^\alpha u \, dx'\, dt' + \frac{\lambda}{2} \int\limits_{\Omega_{x,t}} |u|^\alpha u \, dx'\, dt' \\ &\quad = \frac{1}{2} \int\limits_{G_{x,t}} f(x',t')\, dx'\, dt' + \frac{1}{2} \int\limits_{\Omega_{x,t}} f(x',t')\, dx'\, dt', \quad (x,t) \in \overline{D}_T. \end{aligned} \tag{22}$$

Remark 3. Equality (22) can be considered as a nonlinear Volterra type integral equation, which can be rewritten as follows

$$u(x,t) + \lambda\bigl(\Box^{-1}|u|^\alpha u\bigr)(x,t) = F(x,t), \quad (x,t) \in \overline{D}_T. \tag{23}$$

Here $\Box := L_0 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ and $\Box^{-1}$ is a linear operator acting by formula

$$\bigl(\Box^{-1} v\bigr)(x,t) := \frac{1}{2} \int\limits_{G_{x,t}} v(x',t')\, dx'\, dt' + \frac{1}{2} \int\limits_{\Omega_{x,t}} v(x',t')\, dx'\, dt', \quad (x,t) \in \overline{D}_T, \tag{24}$$

and

$$F(x,t) := \bigl(\Box^{-1} f\bigr)(x,t), \quad (x,t) \in \overline{D}_T. \tag{25}$$

Lemma 2. *Function $u \in C(\overline{D}_T)$ is a strong generalized solution of problem* (1), (2) *of the class C in domain D_T if and only if, when it is a continuous solution of nonlinear integral equation* (23).

Proof. Indeed, let $u \in C(\overline{D}_T)$ be the solution of Eq. (23). Since $f \in C(\overline{D}_T)$ and space $C^2(\overline{D}_T)$ is dense in $C(\overline{D}_T)$ [15, p. 37], then there exists the sequence of functions $f_n \in C^2(\overline{D}_T)$, such that $f_n \to f$ in space $C(\overline{D}_T)$ for $n \to \infty$. Analogously, since $u \in C(\overline{D}_T)$, then there exists the sequence of functions $w_n \in C^2(\overline{D}_T)$, such that $w_n \to u$ in space $C(\overline{D}_T)$ for $n \to \infty$. Let $u_n := -\lambda(\Box^{-1}|w_n|^\alpha w_n) + \Box^{-1} f_n$, $n = 1, 2, \ldots$. It is easy to verify that $u_n \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$, but since $\Box^{-1}$ is a linear continuous operator acting in space $C(\overline{D}_T)$, and besides $\lim_{n\to\infty} \|w_n - u\|_{C(\overline{D}_T)} = 0$, $\lim_{n\to\infty} \|f_n - f\|_{C(\overline{D}_T)} = 0$, we have $u_n \to -\lambda(\Box^{-1}|u|^\alpha u) + \Box^{-1} f$ in space $C(\overline{D}_T)$ for $n \to \infty$. On the other hand from Eq. (23) it follows that $-\lambda(\Box^{-1}|u|^\alpha u) + \Box^{-1} f = u$. Therefore $\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0$. But $\Box u_n = -\lambda|w_n|^\alpha w_n + f_n$, whence by virtue of $\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \lim_{n\to\infty} \|w_n - u\|_{C(\overline{D}_T)} = 0$ and $\lim_{n\to\infty} \|f_n - f\|_{C(\overline{D}_T)} = 0$, we receive $L_\lambda u_n = \Box u_n + \lambda|u_n|^\alpha u_n = -\lambda|w_n|^\alpha w_n + f_n + \lambda|u_n|^\alpha u_n = -\lambda[|w_n|^\alpha w_n - |u|^\alpha u] + \lambda[|u_n|^\alpha u_n - |u|^\alpha u] + f_n \to f$ in space $C(\overline{D}_T)$ for $n \to \infty$. The only if part of the lemma is obvious. $\Box$

4. The case of global solvability of problem (1), (2) in the class of continuous functions

As it was said above operator $\square^{-1}$ from (24) is a linear continuous operator acting in space $C(\overline{D}_T)$.

Now let us show that this operator acts as a linear and continuous one from space $C(\overline{D}_T)$ into the space of differentiable functions $C^1(\overline{D}_T)$. For this purpose by linear nondegenerative transformation of independent variables $t = \xi + \tau$ and $x = \xi - \tau$ let us pass to a plane of variables ξ, τ. As a result triangular domain D_T will transform into triangle D'_T with vertices $O, N'_1(T,0), N'_2(\frac{T}{2}, \frac{T}{2})$; quadrangle $G_{x,t}$ from previous paragraph will transform into quadrangle $G'_{x,t}$ with vertices $P'(\frac{t+x}{2}, \frac{t-x}{2})$, $P'_1(\frac{t-x}{2}, \frac{t-x}{2})$, O, $P'_3(\frac{t+x}{2}, 0)$, i.e., into variables ξ, τ in quadrangle $G'_{\xi,\tau} (= G'_{x,t})$ with vertices $P'(\xi,\tau)$, $P'_1(\tau,\tau)$, O and $P'_3(\xi, 0)$, while triangular domain $\Omega_{x,t}$ will transform into triangle $\Omega'_{x,t}$ with vertices $P'_1(\frac{t-x}{2}, \frac{t-x}{2})$, O, $P'_2(\frac{t-x}{2}, 0)$, i.e., into variables ξ, τ in triangle $\Omega'_{\xi,\tau}$ $(= \Omega'_{x,t})$ with vertices $P'_1(\tau,\tau)$, O and $P'_2(\tau, 0)$.

The operator $\square^{-1}$ from (24) will transform into operator $(\square^{-1})'$, acting in space $C(\overline{D}'_T)$ by formula

$$\big((\square^{-1})'w\big)(\xi,\tau) = \int\limits_{G'_{\xi,\tau}} w(\xi',\tau')\,d\xi'\,d\tau' + \int\limits_{\Omega'_{\xi,\tau}} w(\xi',\tau')\,d\xi'\,d\tau'$$
$$= \int_0^\tau d\tau' \int_{\tau'}^{\xi} w(\xi',\tau')\,d\xi' + \int_0^\tau d\tau' \int_{\tau'}^{\tau} w(\xi',\tau')\,d\xi', \quad (\xi,\tau) \in \overline{D}'_T. \tag{26}$$

If $w \in C(\overline{D}'_T)$, then from (26) directly follows that

$$\frac{\partial}{\partial \xi}\big((\square^{-1})'w\big)(\xi,\tau) = \int_0^\tau w(\xi,\tau')\,d\tau', \quad (\xi,\tau) \in \overline{D}'_T, \tag{27}$$

$$\frac{\partial}{\partial \tau}\big((\square^{-1})'w\big)(\xi,\tau) = \int_\tau^\xi w(\xi',\tau)\,d\xi' + \int_0^\tau w(\tau,\tau')\,d\tau', \quad (\xi,\tau) \in \overline{D}'_T. \tag{28}$$

Now, taking into account that for $(\xi,\tau) \in \overline{D}'_T$ it is valid $0 \leqslant \xi \leqslant T$ and $0 \leqslant \tau \leqslant \frac{T}{2}$, then by virtue of (26)–(28) we get

$$\big\|(\square^{-1})'w\big\|_{C(\overline{D}'_T)} + \left\|\frac{\partial}{\partial \xi}(\square^{-1})'w\right\|_{C(\overline{D}'_T)} + \left\|\frac{\partial}{\partial \tau}(\square^{-1})'w\right\|_{C(\overline{D}'_T)}$$
$$\leqslant \xi\tau\|w\|_{C(\overline{D}'_T)} + \tau\|w\|_{C(\overline{D}'_T)} + (\xi-\tau)\|w\|_{C(\overline{D}'_T)} + \tau\|w\|_{C(\overline{D}'_T)} \leqslant 2^{-1}(T^2+3T)\|w\|_{C(\overline{D}'_T)},$$

i.e.,

$$\big\|(\square^{-1})'\big\|_{C(\overline{D}'_T)\to C^1(\overline{D}'_T)} \leqslant 2^{-1}\big(T^2+3T\big), \tag{29}$$

this concludes the proof.

Further, since space $C^1(\overline{D}'_T)$ is compactly embedded into space $C(\overline{D}'_T)$ [16, p. 135], then due to (29) operator $(\square^{-1})' : C(\overline{D}'_T) \to C(\overline{D}'_T)$ is a linear and compact operator. Thus, returning from variables ξ and τ to variables x and t, for operator $\square^{-1}$ from (24) we receive the validity of the following statement.

Lemma 3. *Operator $\square^{-1} : C(\overline{D}_T) \to C(\overline{D}_T)$ acting by formula* (24) *is a linear compact operator, moreover this operator transforms space $C(\overline{D}_T)$ into space $C^1(\overline{D}_T)$.*

Equation (23), taking into account (25), can be rewritten in the form

$$u = Au := \square^{-1}\big(-\lambda|u|^\alpha u + f\big), \tag{30}$$

where operator $A : C(\overline{D}_T) \to C(\overline{D}_T)$ is continuous and compact, since nonlinear operator $K : C(\overline{D}_T) \to C(\overline{D}_T)$, acting according to formula $Ku := -\lambda|u|^\alpha u + f$ for $\alpha > -1$, is bounded and continuous, while linear operator

$\Box^{-1} : C(\overline{D}_T) \to C(\overline{D}_T)$ due to Lemma 3 is compact. At the same time, by virtue of Lemmas 1 and 2, equalities (20) and (21), for any parameter $\tau \in [0,1]$ and any solution $u \in C(\overline{D}_T)$ equation $u = \tau A u$, it is valid a priori estimate $\|u\|_{C(\overline{D}_T)} \leqslant \tilde{c}_1 \|f\|_{C(\overline{D}_T)} + \tilde{c}_2$ with positive constants $\tilde{c}_1$ and $\tilde{c}_2$, not dependent on u, τ and f. Therefore, according to Leray–Schauder theorem [17, p. 375], Eq. (30) in conditions of Lemma 1 has at least one solution $u \in C(\overline{D}_T)$. In that way, by virtue of Lemma 2, we have proved the following

Theorem 1. *Let $-1 < \alpha < 0$ and in the case when $\alpha > 0$ parameter $\lambda > 0$. Then problem* (1), (2) *is globally solvable of the class C in the sense of Definition* 2, *i.e., if $f \in C(\overline{D}_\infty)$, then for any $T > 0$ problem* (1), (2) *has a strong generalized solution of the class C in domain D_T.*

5. Smoothness and uniqueness of the solution of problem (1), (2). The existence of a global solution in D_∞

From equality (30), due to Lemmas 2 and 3, it follows the following

Lemma 4. *Let u be a strong generalized solution of problem* (1), (2) *of the class C in domain D_T in the sense of Definition* 1. *If $\alpha > 0$ and $f \in C^1(\overline{D}_T)$, then $u \in C^2(\overline{D}_T)$.*

Indeed, in this case due to Lemma 2 function u represents a continuous solution for integral equation (23), therefore for Eq. (30), and moreover, $|u|^\alpha u \in C(\overline{D}_T)$. According to Lemma 3 and Eq. (30) we conclude that $u \in C^1(\overline{D}_T)$. Besides, in conditions of Lemma 4, $|u|^\alpha u \in C^1(\overline{D}_T)$ is true and again according to Lemma 3 and Eq. (30), we conclude that $u \in C^2(\overline{D}_T)$, and thus u is a classical solution of problem (1), (2).

Lemma 5. *For $\alpha > 0$ problem* (1), (2) *cannot have more than one strong generalized solution of the class C in domain D_T.*

Proof. Indeed, suppose that problem (1), (2) has two possible different strong generalized solutions u_1 and u_2 of the class C in domain D_T. According to Definition 1 there exists the sequence of functions $u_{in} \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$, $i = 1, 2$, such that

$$\lim_{n\to\infty} \|u_{in} - u_i\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|L_\lambda u_{in} - f\|_{C(\overline{D}_T)} = 0, \quad i = 1, 2. \tag{31}$$

Denote by $\omega_{nm} := u_{2n} - u_{1m}$. It is easy to see that function $\omega_{nm} \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$ satisfies the following identities:

$$\Box \omega_{nm} + g_{nm} \omega_{nm} = f_{nm}, \tag{32}$$

$$\frac{\partial \omega_{nm}}{\partial x}\bigg|_{\gamma_{2,T}} = 0, \qquad \omega_{nm}|_{\gamma_{1,T}} = 0. \tag{33}$$

Here

$$g_{nm} := \lambda(1+\alpha) \int_0^1 \bigl|u_{1m} + t(u_{2n} - u_{1m})\bigr|^\alpha \, dt, \tag{34}$$

$$f_{nm} := L_\lambda u_{2n} - L_\lambda u_{1m}, \tag{35}$$

where we used obvious equality $\varphi(x_2) - \varphi(x_1) = (x_2 - x_1) \int_0^1 \varphi'(x_1 + t(x_2 - x_1))\, dt$ for function $\varphi(x) := |x|^\alpha x$ when $x_2 = u_{2n}$, $x_1 = u_{1m}$, $\alpha > 0$. Due to the first equality from (31) there exists the number $M := \mathrm{const} > 0$, not dependent on indices i and n, such that $\|u_{in}\|_{C(\overline{D}_T)} \leqslant M$, whence by virtue of (34) it follows that

$$\|g_{n,m}\|_{C(\overline{D}_T)} \leqslant |\lambda|(1+\alpha) M^\alpha \quad \forall n, m. \tag{36}$$

According to (35) and the second equality from (31) it follows that

$$\lim_{n,m\to\infty} \|f_{nm}\|_{C(\overline{D}_T)} = 0. \tag{37}$$

Multiplying the both sides of equality (32) by $\frac{\partial\omega_{nm}}{\partial t}$ and integrating the received equality in domain $D_\tau := \{(x,t) \in D_T\colon\ 0 < t < \tau\}$, $0 < \tau \leqslant T$, due to equalities (33), as it was in receiving of inequality (13) from (6), (7), we shall have

$$\omega_{nm}(\tau) := \int\limits_{\Omega_\tau} \left[\left(\frac{\partial\omega_{nm}}{\partial t} \right)^2 + \left(\frac{\partial\omega_{nm}}{\partial x} \right)^2 \right] dx \leqslant 2 \int\limits_{D_\tau} (f_{nm} - g_{nm}\omega_{nm}) \frac{\partial\omega_{nm}}{\partial t}\, dx\, dt, \tag{38}$$

where $\Omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \leqslant T$.

Due to estimate (36) and the inequality of Cauchy we shall have

$$\begin{aligned}
&2 \int\limits_{D_\tau} (f_{nm} - g_{nm}\omega_{nm}) \frac{\partial\omega_{nm}}{\partial t}\, dx\, dt \\
&\quad\leqslant \int\limits_{D_\tau} \left(\frac{\partial\omega_{nm}}{\partial t} \right)^2 dx\, dt + \int\limits_{D_\tau} (f_{nm} - g_{nm}\omega_{nm})^2\, dx\, dt \\
&\quad\leqslant \int\limits_{D_\tau} \left(\frac{\partial\omega_{nm}}{\partial t} \right)^2 dx\, dt + 2 \int\limits_{D_\tau} f_{nm}^2\, dx\, dt + 2 \int\limits_{D_\tau} g_{nm}^2 \omega_{nm}^2\, dx\, dt \\
&\quad\leqslant \int\limits_{D_\tau} \left(\frac{\partial\omega_{nm}}{\partial t} \right)^2 dx\, dt + 2 \int\limits_{D_\tau} f_{nm}^2\, dx\, dt + 2\lambda^2(1+\alpha)^2 M^{2\alpha} \int\limits_{D_\tau} \omega_{nm}^2\, dx\, dt.
\end{aligned} \tag{39}$$

Further, from equality $\omega_{nm}(x,t) = \int_x^t \frac{\partial\omega_{nm}(x,\tau)}{\partial t}\, d\tau$, $(x,t) \in \overline{D}_T$, which follows from the second equality of (33), using standard considerations we receive inequality [14, p. 63]

$$\int\limits_{D_\tau} \omega_{nm}^2\, dx\, dt \leqslant \tau^2 \int\limits_{D_\tau} \left(\frac{\partial\omega_{nm}}{\partial t} \right)^2 dx\, dt. \tag{40}$$

From inequality (38), by virtue of (39) and (40), it follows that

$$\begin{aligned}
w_{nm}(\tau) &\leqslant \left(1 + 2\lambda^2(1+\alpha)^2 M^{2\alpha} \tau^2\right) \int\limits_{D_\tau} \left(\frac{\partial\omega_{nm}}{\partial t} \right)^2 dx\, dt + 2 \int\limits_{D_\tau} f_{nm}^2\, dx\, dt \\
&\leqslant \left(1 + 2\lambda^2(1+\alpha)^2 M^{2\alpha} T^2\right) \int\limits_0^\tau w_{nm}(\sigma)\, d\sigma + 2 \int\limits_{D_T} f_{nm}^2\, dx\, dt.
\end{aligned}$$

Whence by the lemma of Gronwall [13, p. 13] we receive that

$$w_{nm}(\tau) \leqslant c \| f_{nm} \|_{L_2(D_T)}^2, \quad 0 < \tau \leqslant T, \tag{41}$$

where $c := 2\exp(T + 2\lambda^2(1+\alpha)^2 M^{2\alpha} T^3)$.

Conducting the same considerations, as those used for receiving of inequality (15), taking into account obvious inequality

$$\| f_{nm} \|_{L_2(D_T)}^2 \leqslant \| f_{nm} \|_{C(\overline{D}_T)}^2 \operatorname{mes} D_T,$$

and also due to (41) we have

$$\begin{aligned}
\left|\omega_{nm}(x,t)\right|^2 &\leqslant t w_{nm}(t) \leqslant T c \operatorname{mes} D_T \| f_{nm} \|_{C(\overline{D}_T)}^2 \\
&= 2^{-1} c T^3 \| f_{nm} \|_{C(\overline{D}_T)}^2, \quad (x,t) \in \overline{D}_T.
\end{aligned}$$

From this inequality it follows that

$$\| \omega_{nm} \|_{C(\overline{D}_T)} \leqslant T \sqrt{2^{-1} c T}\, \| f_{nm} \|_{C(\overline{D}_T)}. \tag{42}$$

Recalling the definition of function ω_{nm}, according to the first inequality from (31) we have

$$\lim_{n,m\to\infty} \|\omega_{nm}\|_{C(\overline{D}_T)} = \|u_2 - u_1\|_{C(\overline{D}_T)}.$$

Due this equality and (37), passing in inequality (42) to limit for $n, m \to \infty$ we receive $\|u_2 - u_1\|_{C(\overline{D}_T)} = 0$, i.e., $u_1 = u_2$, which proves Lemma 5.

Here arises naturally the question on what happens in the sense of the theorem of uniqueness for $-1 < \alpha < 0$. In this case, as the example below shows clearly, at additional condition $\lambda < 0$ problem (1), (2) for $f \equiv 0$ has also other solutions in addition to trivial one $u \equiv 0$.

Indeed, as simple verification can confirm, the following function:

$$u(x,t) := \beta\big(t^2 - x^2\big)^\gamma \quad \text{if } |\beta| = \big(-4\lambda^{-1}\alpha^{-2}\big)^{\frac{1}{\alpha}},\ \gamma = -\alpha^{-1}$$

satisfies this condition. □

Theorem 2. *Let $\alpha > 0$ and $\lambda > 0$. Then for any $f \in C^1(\overline{D}_\infty)$ problem* (1), (2) *has unique global classical solution $u \in \mathring{C}^2(\overline{D}_\infty, \Gamma_\infty)$ in domain D_∞.*

Proof. If $\alpha > 0$, $\lambda > 0$ and $f \in C^1(\overline{D}_\infty)$, then according to Theorem 1 and Lemmas 4 and 5 in domain D_T for $T = n$ there exists unique classical solution $u_n \in \mathring{C}^2(\overline{D}_n, \Gamma_n)$ of problem (1), (2). Since u_{n+1} represents also a classical solution of problem (1), (2) in domain D_n, then by virtue of Lemma 5 we have $u_{n+1}|_{D_n} = u_n$. Therefore function u, constructed in domain D_∞ by rule $u(x,t) = u_n(x,t)$ at $n = [t] + 1$, where $[t]$ is an integer part of number t, while point $(x,t) \in D_\infty$, will be a unique classical solution of problem (1), (2) in domain D_∞ of class $\mathring{C}^2(\overline{D}_\infty, \Gamma_\infty)$. Theorem 2 is proved completely. □

6. The case of nonexistence of a global solution of problem (1), (2)

Below, we consider the case when parameter $\lambda < 0$ in Eq. (1), while a power of nonlinearity $\alpha > 0$.

Lemma 6. *Let u be a strong generalized solution of problem* (1), (2) *of the class C in domain D_T in the sense of Definition* 1. *Then it is valid the following integral equality*:

$$\int\limits_{D_T} u\Box\varphi\,dx\,dt = -\lambda \int\limits_{D_T} |u|^\alpha u\varphi\,dx\,dt + \int\limits_{D_T} f\varphi\,dx\,dt \tag{43}$$

for any function φ, such that

$$\varphi \in C^2(\overline{D}_T), \qquad \varphi|_{t=T} = 0, \qquad \varphi_t|_{t=T} = 0, \qquad \varphi_x|_{\gamma_{2,T}} = 0. \tag{44}$$

Proof. According to the definition of strong generalized solution u of problem (1), (2) of the class C in domain D_T, function $u \in C(\overline{D}_T)$ and there exists the sequence of functions $u_n \in \mathring{C}^2(\overline{D}_T, \Gamma_T)$, such that the equalities (4) are valid.

Suppose that $f_n := L_\lambda u_n$. Multiplying the both sides of equality $L_\lambda u_n = f_n$ by function φ let us integrate the received equality in domain D_T. As a result of integration by parts of the left side of this equality, due to (44) and boundary conditions (2) we receive

$$\int\limits_{D_T} u_n\Box\varphi\,dx\,dt = -\lambda \int\limits_{D_T} |u_n|^\alpha u_n\varphi\,dx\,dt + \int\limits_{D_T} f_n\varphi\,dx\,dt.$$

By passing to limit in this equality for $n \to \infty$, according to (4) we receive equality (43). Thus Lemma 6 is proved. □

Lemma 7. *Let $\lambda < 0$ and $\alpha > 0$, and function $u \in C(\overline{D}_T)$ be a strong generalized solution of problem* (1), (2) *of the class C in domain D_T. If $f \geqslant 0$ in domain D_T, then $u \geqslant 0$ in domain D_T.*

Proof. According to Lemma 2 and equalities (23)–(25) function u is a solution of the following Volterra type integral equation:

$$u(x,t)=\int\limits_{G_{x,t}} K(x',t')u(x',t')\,dx'\,dt'+\int\limits_{\Omega_{x,t}} K(x',t')u(x',t')\,dx'\,dt'+F(x,t),\quad (x,t)\in\overline{D}_T. \tag{45}$$

Here $K(x,t):=-\frac{\lambda}{2}|u(x,t)|^{\alpha}\in C(\overline{D}_T)$, and function $F(x,t)$ is given by equality (25). By virtue of suppositions made in Lemma 7 we have

$$K(x,t)\geqslant 0,\qquad F(x,t)\geqslant 0\quad \forall (x,t)\in\overline{D}_T. \tag{46}$$

Assuming that function $K(x,t)$ is given, let us consider Volterra type linear integral equation

$$v(x,t)=\int\limits_{G_{x,t}} K(x',t')v(x',t')\,dx'\,dt'+\int\limits_{\Omega_{x,t}} K(x',t')v(x',t')\,dx'\,dt'+F(x,t),\quad (x,t)\in\overline{D}_T, \tag{47}$$

in the class $C(\overline{D}_T)$ with respect to unknown function $v(x,t)$. As it is known [18], Eq. (47) in the class $C(\overline{D}_T)$ has unique continuous solution $v(x,t)$, which can be obtained by use of the method of consecutive approximations

$$v_0(x,t)=0,$$
$$v_{n+1}(x,t)=\int\limits_{G_{x,t}} K(x',t')v_n(x',t')\,dx'\,dt'+\int\limits_{\Omega_{x,t}} K(x',t')v_n(x',t')\,dx'\,dt'+F(x,t),\quad n\geqslant 1,\ (x,t)\in\overline{D}_T.$$

From these equalities according to (46) we have $v_n(x,t)\geqslant 0$ in $\overline{D}_T$ for all $n=0,1,\dots$. On the other hand, $v_n\to v$ in the class $C(\overline{D}_T)$ for $n\to\infty$. Therefore, limit function $v\geqslant 0$ in domain D_T. We have just note, that by virtue of equality (45) function u is also a solution of Eq. (47), and therefore due to the uniqueness of solution of this equation we finally receive $u=v\geqslant 0$ in domain D_T. Lemma 7 is proved. □

For $\lambda<0$, according to the last lemma, equality (43) can by rewritten in the form

$$\int\limits_{D_T} |u|\Box\varphi\,dx\,dt=|\lambda|\int\limits_{D_T} |u|^{\alpha+1}\varphi\,dx\,dt+\int\limits_{D_T} f\varphi\,dx\,dt. \tag{48}$$

Let us introduce into consideration function $\varphi^0:=\varphi^0(x,t)$ such that

$$\varphi^0\in C^2(\overline{D}_\infty),\qquad \varphi^0\big|_{D_{T=1}}>0,\qquad \varphi^0_x\big|_{\gamma_{2,\infty}}=0,\qquad \varphi^0\big|_{t\geqslant 1}=0 \tag{49}$$

and

$$\varkappa_0:=\int\limits_{D_{T=1}}\frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx\,dt<+\infty,\qquad p'=1+\frac{1}{\alpha}. \tag{50}$$

It is easy to verify that in the role of function φ^0, satisfying conditions (49) and (50), one may use function

$$\varphi^0(x,t):=\begin{cases} x^n(1-t)^m, & (x,t)\in D_{T=1},\\ 0, & t\geqslant 1,\end{cases}$$

for sufficiently large positive numbers n and m.

Suppose that $\varphi_T(x,t):=\varphi^0(\frac{x}{T},\frac{t}{T})$, $T>0$. Due to (49) it is easy to see that

$$\varphi_T\in C^2(\overline{D}_T),\qquad \varphi_T|_{D_T}>0,\qquad \varphi_T|_{t=T}=0,\qquad \frac{\partial\varphi_T}{\partial t}\bigg|_{t=T}=0,\qquad \frac{\partial\varphi_T}{\partial x}\bigg|_{\gamma_{2,T}}=0. \tag{51}$$

Supposing that function f is fixed, let us introduce into consideration a function of one variable T,

$$\zeta(T):=\int\limits_{D_T} f\varphi_T\,dx\,dt,\qquad T>0. \tag{52}$$

The following theorem on the nonexistence of a global solution of problem (1), (2) is valid.

Theorem 3. *Let $\lambda < 0$, $\alpha > 0$, $f \in C(\overline{D}_\infty)$ and $f \geqslant 0$ in domain D_∞. If*

$$\liminf_{T\to+\infty} \zeta(T) > 0, \tag{53}$$

then there exists positive number $T_0 := T_0(f)$, such that for $T > T_0$ problem (1), (2) *cannot have strong generalized solution u of the class C in domain D_T.*

Proof. Suppose, that in conditions of this theorem there exists strong generalized solution u of problem (1), (2) of the class C in domain D_T. Then according to Lemmas 6 and 7 equality (48) holds, where due to (51) in the role of function φ can be taken function $\varphi = \varphi_T$, i.e.,

$$\int\limits_{D_T} |u|\Box\varphi_T\,dx\,dt = |\lambda| \int\limits_{D_T} |u|^p \varphi_T\,dx\,dt + \int\limits_{D_T} f\varphi_T\,dx\,dt, \quad p := \alpha + 1.$$

Taking into account (52) this equality can be rewritten in the form

$$|\lambda| \int\limits_{D_T} |u|^p \varphi_T\,dx\,dt = \int\limits_{D_T} |u|\Box\varphi_T\,dx\,dt - \zeta(T). \tag{54}$$

If in Young inequality with parameter $\varepsilon > 0$,

$$ab \leqslant \frac{\varepsilon}{p}a^p + \frac{1}{p'\varepsilon^{p'-1}}b^{p'}; \quad a, b \geqslant 0,\ \frac{1}{p} + \frac{1}{p'} = 1,\ p := \alpha + 1 > 1,$$

we shall take $a = |u|\varphi_T^{\frac{1}{p}}$, $b = \frac{|\Box\varphi_T|}{\varphi_T^{\frac{1}{p}}}$, then since $\frac{p'}{p} = p' - 1$ we obtain

$$|u\Box\varphi_T| = |u|\varphi_T^{\frac{1}{p}} \frac{|\Box\varphi_T|}{\varphi_T^{\frac{1}{p}}} \leqslant \frac{\varepsilon}{p}|u|^p\varphi_T + \frac{1}{p'\varepsilon^{p'-1}} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}.$$

According last inequality from (54) we have

$$\left(|\lambda| - \frac{\varepsilon}{p}\right) \int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \leqslant \frac{1}{p'\varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \zeta(T),$$

whence for $\varepsilon < |\lambda|p$ we receive

$$\int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \leqslant \frac{p}{(|\lambda|p - \varepsilon)p'\varepsilon^{p'-1}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p}{|\lambda|p - \varepsilon}\zeta(T).$$

Since $p' = \frac{p}{p-1}$, $p = \frac{p'}{p'-1}$ and $\min_{0<\varepsilon<|\lambda|p} \frac{p}{(|\lambda|p-\varepsilon)p'\varepsilon^{p'-1}} = \frac{1}{|\lambda|^{p'}}$, which is achieved for $\varepsilon = |\lambda|$, it follows, that

$$\int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \leqslant \frac{1}{|\lambda|^{p'}} \int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p'}{|\lambda|}\zeta(T). \tag{55}$$

Since $\varphi_T(x,t) := \varphi^0(\frac{x}{T}, \frac{t}{T})$, then due to (49), (50), after changing variables $x = Tx'$, $t = Tt'$, it is easy to verify, that

$$\int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt = T^{-2(p'-1)} \int\limits_{D_{T=1}} \frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx'\,dt' = T^{-2(p'-1)}\varkappa_0.$$

According to (51) and the last inequality from (55) we receive

$$0 \leqslant \int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \leqslant \frac{1}{|\lambda|^{p'}}T^{-2(p'-1)}\varkappa_0 - \frac{p'}{|\lambda|}\zeta(T). \tag{56}$$

Since $p' = \frac{p}{p-1} > 1$, then $-2(p'-1) < 0$ and due to (50) we have

$$\lim_{T\to\infty} \frac{1}{|\lambda|^{p'}} T^{-2(p'-1)} \varkappa_0 = 0.$$

Therefore, by virtue of (53) there exists positive number $T_0 := T_0(f)$, such that for $T > T_0$ the right-hand side of inequality (56) will be negative, whereas the left-hand side of this inequality is nonnegative. This means that if there exists strong generalized solution u of problem (1), (2) of the class C in domain D_T, then necessarily $T \leqslant T_0$, which proves Theorem 3. □

Remark 4. It is easy to verify that if $f \in C(\overline{D}_\infty)$ and $f(x,t) \geqslant ct^{-m}$ for $t \geqslant 1$, where $c := \text{const} > 0$, $0 \leqslant m := \text{const} \leqslant 2$, then condition (53) will be fulfilled, and so for $\lambda < 0$, $\alpha > 0$ problem (1), (2) for sufficiently large T will not have strong generalized solution u of the class C in domain D_T.

Indeed, let us introduce in (52) the transformation of independent variables x and t by formula $x = Tx'$, $t = Tt'$, after some estimates we have

$$\begin{aligned}\zeta(T) &= T^2 \int\limits_{D_1} f(Tx', Tt')\varphi^0(x', t')\,dx'\,dt' \\ &\geqslant cT^{2-m} \int\limits_{D_1\cap\{t'\geqslant T^{-1}\}} t'^{-m}\varphi^0(x',t')\,dx'\,dt' + T^2 \int\limits_{D_1\cap\{t'<T^{-1}\}} f(Tx', Tt')\varphi^0(x',t')\,dx'\,dt'\end{aligned}$$

in supposition that $T > 1$. Further, let $T_1 > 1$ be any fixed number. Then from the last inequality for function ζ we have

$$\begin{aligned}\zeta(T) &\geqslant cT^{2-m} \int\limits_{D_1\cap\{t'\geqslant T^{-1}\}} t'^{-m}\varphi^0(x',t')\,dx'\,dt' \\ &\geqslant cT^{2-m} \int\limits_{D_1\cap\{t'\geqslant T_1^{-1}\}} t'^{-m}\varphi^0(x',t')\,dx'\,dt',\end{aligned}$$

if $T \geqslant T_1 > 1$. From the latter inequality immediately follows the validity of (53).

7. Local solvability of problem (1), (2) in the case when $\lambda < 0$ and $\alpha > 0$

Theorem 4. *Let $\lambda < 0$ and $\alpha > 0$, function $f \in C(\overline{D}_\infty)$, $f \not\equiv 0$. Then there exists positive number $T_* := T_*(f)$, such that for $T \leqslant T_*$ problem* (1), (2) *in domain D_T will have strong generalized solution u in the class C.*

Proof. In Section 4 problem (1), (2) in space $C(\overline{D}_T)$ equivalently was reduced to the functional equation (30), where operator $A : C(\overline{D}_T) \to C(\overline{D}_T)$ is a linear and compact. Therefore for the solvability of Eq. (30), according to the theorem of Schauder, it will suffice to show that operator A maps certain ball $B_R := \{v \in C(\overline{D}_T)\colon \|v\|_{C(\overline{D}_T)} \leqslant R\}$ of radius $R > 0$, which is closed and convex set in Banach space $C(\overline{D}_T)$, into itself. Let us show that this takes place for sufficiently small T.

Indeed, according to (24) and (30) for $\|u\|_{C(\overline{D}_T)} \leqslant R$ we have

$$\begin{aligned}\|Au\|_{C(\overline{D}_T)} &\leqslant \|\Box^{-1}\|_{C(\overline{D}_T)\to C(\overline{D}_T)}\big[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}\big] \\ &\leqslant 2^{-1} \sup_{(x,t)\in\overline{D}_T} (\operatorname{mes} G_{x,t} + \operatorname{mes} \Omega_{x,t})\big[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}\big] \\ &\leqslant \operatorname{mes} D_T\big[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}\big] \\ &= 2^{-1}T^2\big[|\lambda|\|u\|^{\alpha+1}_{C(\overline{D}_T)} + \|f\|_{C(\overline{D}_T)}\big] \leqslant 2^{-1}T^2\big[|\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_T)}\big].\end{aligned}$$

Let us fix arbitrarily positive number T_2. Then by virtue of the last estimate for $0 < T \leqslant T_2$, we shall have

$$\|Au\|_{C(\overline{D}_T)} \leqslant 2^{-1}T^2\big[|\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_{T_2})}\big].$$

In turn, from this estimation it follows that if

$$T_*^2 := \min\left\{T_2^2, \frac{2R}{|\lambda|R^{\alpha+1} + \|f\|_{C(\overline{D}_{T_2})}}\right\},$$

then $\|Au\|_{C(\overline{D}_T)} \leqslant R$ for $\|u\|_{C(\overline{D}_T)} \leqslant R$, $0 < T \leqslant T_*$. Theorem 4 is proved completely. □

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