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# THE CAUCHY PROBLEM FOR ONE-DIMENSIONAL WAVE EQUATIONS WITH NONLINEAR DISSIPATIVE AND DAMPING TERMS

## 1. The Statement of the Problems

For one-dimensional wave equations with nonlinear dissipative term [1], [2, p. 57],

$$Lu := u_{tt} - u_{xx} + g(x,t,u)u_t = f(x,t), \tag{1.1}$$

in the half-plane $\Omega := \{(x,t) : x \in \mathbb{R},\ t > 0\}$ let us consider the Cauchy problem with the following initial conditions:

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $f, g, \varphi, \psi$ are given, and $u$ are unknown real functions.

Let $P_0 := P_0(x_0, t_0)$ be an arbitrary point of the domain $\Omega$ and $D_{P_0} := \{(x,t) : t + x_0 - t_0 < x < -t + x_0 + t_0,\ t > 0\}$ be a triangular domain bounded by characteristic segments $\gamma_{1,P_0} : x = t + x_0 - t_0,\ 0 \le t \le t_0$ and $\gamma_{2,P_0} : x = -t + x_0 + t_0,\ 0 \le t \le t_0$ of the equation (1.1), and the segment $\gamma_{P_0} : t = 0,\ x_0 - t_0 \le x \le x_0 + t_0$.

First, we consider the Cauchy problem for equation (1.1) in the finite domain $D_{P_0}$: find a solution $u = u(x,t),\ (x,t) \in D_{P_0}$, of the equation (1.1) by the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \gamma_{P_0}, \tag{1.3}$$

where $\varphi, \psi$ are the given real functions on $\gamma_{P_0}$.

**Definition 1.1.** Let $f \in C(\overline{D}_{P_0}),\ g \in C(\overline{D}_{P_0} \times \mathbb{R}),\ \varphi \in C^1(\gamma_{P_0})$ and $\psi \in C(\gamma_{P_0})$. We say that the function $u$ is a strong generalized solution of the problem (1.1), (1.3) of the class $C^1$ in the domain $D_{P_0}$, if $u \in C^1(\overline{D}_{P_0})$ and there exists a sequence of the functions $u_n \in C^2(\overline{D}_{P_0})$, such that $u_n \to u,\ Lu_n \to f,\ u_n(\cdot,0) \to \varphi$ and $u_{nt}(\cdot,0) \to \psi$ for $n \to \infty$ in the spaces $C^1(\overline{D}_{P_0}),\ C(\overline{D}_{P_0}),\ C^1(\gamma_{P_0})$ and $C(\gamma_{P_0})$, respectively.

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*Remark* 1.1. It is obvious that a classical solution of the problem (1.1), (1.3) of the class $C^2(\overline{D}_{P_0})$ is a strong generalized solution of this problem of the class $C^1$ in the domain $D_{P_0}$. Conversely, if a strong generalized solution of the problem (1.1), (1.3) of the class $C^1$ in the domain $D_{P_0}$ belongs to the space $C^2(\overline{D}_{P_0})$, then it will be also a classical solution of this problem.

**Definition 1.2.** Let $f \in C(\overline{\Omega})$, $g \in C(\overline{\Omega} \times \mathbb{R})$, $\varphi \in C^1(\mathbb{R})$, $\psi \in C(\mathbb{R})$. We say that the problem (1.1), (1.3) is globally solvable in the class $C^1$, if for any point $P_0 \in \Omega$ the problem has a strong generalized solution of the class $C^1$ in the domain $D_{P_0}$ in the sense of Definition 1.1.

**Definition 1.3.** Let $f \in C(\overline{\Omega})$, $g \in C(\overline{\Omega} \times \mathbb{R})$, $\varphi \in C^1(\mathbb{R})$, $\psi \in C(\mathbb{R})$. We say that the function $u \in C^1(\overline{\Omega})$ is a global strong generalized solution of the problem (1.1), (1.2) of the class $C^1$, if for any point $P_0 \in \Omega$ it is a strong generalized solution of the problem (1.1), (1.3) of the class $C^1$ in the domain $D_{P_0}$ in the sense of Definition 1.1.

*Remark* 1.2. Note that in the case when the theorem of the existence and uniqueness of a strong generalized solution of the problem (1.1), (1.3) of the class $C^1$ in the domain $D_{P_0}$ is valid for any $P_0 \in \Omega$, then there follows the existence of the unique global strong generalized solution of the problem (1.1), (1.2) of the class $C^1$ in the sense of Definition 1.3.

## 2. A Priori Estimates for the Strong Generalized Solution of the Problem (1.1), (1.3) in the Classes $\mathbf{C(\overline{D}_{P_0})}$ and $\mathbf{C^1(\overline{D}_{P_0})}$

Consider the conditions

$$g(x,t,s) \geq -M,\ (x,t,s) \in \overline{\Omega} \times \mathbb{R},\ M := \text{const} > 0 \tag{2.1}$$

and

$$f \in C(\overline{\Omega}), \quad g \in C(\overline{\Omega} \times \mathbb{R}), \quad \varphi \in C^1(\mathbb{R}), \quad \psi \in C(\mathbb{R}). \tag{2.2}$$

**Lemma 2.1.** *Let the conditions* (2.1), (2.2) *be fulfilled and* $P_0$ *be an arbitrary point of the domain* $\Omega$. *Then if* $u$ *is a strong generalized solution of the problem* (1.1), (1.3) *of the class* $C^1$ *in the domain* $D_{P_0}$, *then the following a priori estimate is valid:*

$$\|u\|_{C(\overline{D}_{P_0})} \leq c_0\big(\|f\|_{C(\overline{D}_{P_0})} + \|\varphi\|_{C^1(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})}\big),$$

*with positive constant* $c_0 = c_0(t_0, M)$ *not depending on* $u$ *and* $f, \varphi, \psi$, *where* $\|\varphi\|_{C^1(\gamma_{P_0})} := \max\big\{\|\varphi\|_{C(\gamma_{P_0})}, \|\varphi'\|_{C(\gamma_{P_0})}\big\}$.

**Lemma 2.2.** *In the conditions of Lemma* 2.1, *for a strong generalized solution of the problem* (1.1), (1.3) *of the class* $C^1$ *in the domain* $D_{P_0}$ *the following estimate is valid:*

$$\|u\|_{C^1(\overline{D}_{P_0})} \leq c_1,$$

*with a positive constant* $c_1 = c_1(P_0, c_0, \|f\|_{C(\overline{D}_{P_0})}, \|\varphi\|_{C^1(\gamma_{P_0})}, \|\psi\|_{C(\gamma_{P_0})})$, *where* $\|u\|_{C^1(\overline{D}_{P_0})} := \max\big\{\|u\|_{C(\overline{D}_{P_0})}, \|u_x\|_{C(\overline{D}_{P_0})}, \|u_t\|_{C(\overline{D}_{P_0})}\big\}$.

## 3. The Solvability of the Problem (1.1), (1.3)

Let together with (2.2) be fulfilled the following conditions:

$$\begin{gathered}\varphi_\infty := \max\big\{\sup_{x\in\mathbb{R}}|\varphi(x)| < +\infty,\ \sup_{x\in\mathbb{R}}|\varphi'(x)| < +\infty\big\},\\ \psi_\infty := \sup_{x\in\mathbb{R}}|\psi(x)| < +\infty,\end{gathered} \tag{3.1}$$

$$f_\infty := \sup_{(x,t)\in\overline{\Omega}}|f(x,t)| < +\infty \tag{3.2}$$

and

$$|g(x,t,s)| \le m(r), \quad |g(x,t,s_2) - g(x,t,s_1)| \le c(r)|s_2 - s_1| \tag{3.3}$$

for all $(x,t)\in\overline{\Omega}$, $|s|, |s_1|, |s_2| \le r$, where $m(r)$ and $c(r)$ are some nonnegative continuous functions of argument $r \ge 0$.

**Theorem 3.1.** *Let the functions $f, g, \varphi$ and $\psi$ satisfy the conditions* (2.2), (3.1)–(3.3). *Then there exists a positive number $t_* := t_*(f,g,\varphi,\psi)$, such that for $t_0 \le t_*$ the problem* (1.1), (1.3) *will have at least one strong generalized solution $u$ of the class $C^1$ in the domain $D_{P_0}$.*

*Remark* 3.1. In the role of the value $t_*$ we can take, for example,

$$\begin{gathered}t_* := \min\left\{\frac{1}{r_* m(r_*) + r_* + f_\infty}, \frac{1}{2}, \frac{1}{2\big(r_* c(r_*) + m(r_*)\big)}\right\},\\ r_* := 1 + \varphi_\infty + \psi_\infty.\end{gathered}$$

**Theorem 3.2.** *If the conditions* (2.1), (2.2), (3.1)–(3.3) *are fulfilled, then the problem* (1.1), (1.3) *is globally solvable in the class $C^1$ in the sense of Definition* 1.2, *i.e. for any $P_0 \in \Omega$ this problem has a strong generalized solution of the class $C^1$ in the domain $D_{P_0}$.*

*Remark* 3.2. One may present the examples of functions $g = g(x,t,s)$, satisfying the conditions of Theorem 3.2, or, the same, the conditions (2.1), $g \in C(\overline{\Omega}\times\mathbb{R})$ and (3.3). Such function is

$$g(x,t,s) = \sum_{k=1}^{n} \alpha_k(x,t)|s|^{\beta_k},$$

where $\alpha_k \in C(\overline{\Omega})$, $k = 1,\dots,n$; $\alpha_1(x,t) \ge \text{const} > 0$, $|\alpha_i(x,t)| \le \text{const}$ for $(x,t) \in \overline{\Omega}$ and $\beta_1 > \beta_i \ge 1$, $i = 2,\dots,n$; and also the function $g(x,t,s) = \alpha(x,t)g_0(s)$, where $\alpha \in C(\overline{\Omega})$, $|\alpha(x,t)| \le \text{const}$ for $(x,t) \in \overline{\Omega}$, $g_0 \in Lip_{loc}(\mathbb{R})$ and $\liminf_{|s|\to+\infty} g_0(s) > -\infty$, satisfy the conditions of Theorem 3.2.

## 4. The Uniqueness and the Existence Theorems

**Theorem 4.1.** *Let the conditions* (2.2) *and* (3.3) *be fulfilled. Then for any fixed point $P_0 \in \Omega$ the problem* (1.1), (1.3) *cannot have more than one strong generalized solution of the class $C^1$ in the domain $D_{P_0}$.*

**Theorem 4.2.** *If the conditions* (2.1)*,* (2.2)*,* (3.1)*–*(3.3) *are fulfilled, then the problem* (1.1)*,* (1.2) *has a unique global strong generalized solution of the class* $C^1$ *in the sense of Definition* 1.3*.*

*Remark* 4.1. For the conditions of Theorem 4.1 there exists the positive number $T_* := T_*(f, g, \varphi, \psi) > 0$, such that the problem (1.1), (1.2) in the strip $\Omega_1 := \mathbb{R} \times (0, T_*)$ has a unique strong generalized solution $u$ of the class $C^1$ in the domain $\Omega_1$, in the sense that for any point $P_0 \in \Omega_1$ the function $u\big|_{D_{P_0}}$ represents a strong generalized solution of the problem (1.1), (1.3) of the class $C^1$ in the domain $D_{P_0}$ in the sense of Definition 1.1.

*Remark* 4.2. Theorems 3.1, 3.2 and 4.2 remain valid without requirements of the conditions (3.1), (3.2), having only the condition of smoothness $\varphi \in C^1(\mathbb{R})$, $\psi \in C(\mathbb{R})$, $f \in C(\overline{\Omega})$.

## 5. The Case of Absence of a Global Solution of the Problem (1.1), (1.3)

*Remark* 5.1. Violation of the condition (2.1) may, generally speaking, cause an absence of the global solvability of the problem (1.1), (1.3) in the sense of Definition 1.2. Indeed, let $g(x, t, s) = -|s|^\alpha s$, $(x, t) \in \overline{\Omega}$, $s \in \mathbb{R}$ and the exponent of nonlinearity $\alpha > -1$. It is shown that for certain conditions on the functions $f \in C(\overline{\Omega})$, $\varphi \in C^1(\mathbb{R})$, $\psi \in C(\mathbb{R})$ for any fixed $x_0 \in \mathbb{R}$ there exists the number $t^* := t^*(x_0; f, \varphi, \psi) > 0$, such that for $t_0 \in (0, t^*)$ the problem (1.1), (1.3) has a strong generalized solution of the class $C^1$ in the domain $D_{P_0}$, while for $t_0 > t^*$ it does not have such solution in this domain.

Consider the function $\chi^0 := \chi^0(x, t)$, such that (see, e.g., [3], pp. 10–12, [4], [5])

$$\chi^0 \in C^2(\overline{D}_{(0,1)}), \quad \chi^0 + \chi^0_t \leq 0, \quad \chi^0\big|_{D_{(0,1)}} > 0, \quad \chi^0\big|_{\gamma_{i,(0,1)}} = 0, \quad i = 1, 2, \tag{5.1}$$

$$\int\limits_{D_{(0,1)}} \frac{|\Box \chi^0|^{p'}}{|\chi^0|^{p'-1}} dxdt < +\infty, \quad p' = \frac{\alpha + 2}{\alpha + 1}. \tag{5.2}$$

It is easy to verify that for the function $\chi^0$, satisfying the conditions (5.1) and (5.2), one may consider the function

$$\chi^0 = \chi^*(x, t) := [(1 - t)^2 - x^2]^n, \quad (x, t) \in \overline{D}_{(0,1)}, \tag{5.3}$$

for a sufficiently large natural number $n$.

Now, putting $\chi_{P_0}(x,t) = \chi^0\big(\frac{x-x_0}{t_0}, \frac{t}{t_0}\big)$. For fixed functions $f, \varphi, \psi$ and number $x_0 \in \mathbb{R}$, consider the function

$$\zeta(t_0) := \int\limits_{D_{P_0}} f\chi_{P_0}\,dxdt + \int\limits_{x_0-t_0}^{x_0+t_0} \Big[\psi(x)\chi_{P_0}(x,0) - \varphi(x)\frac{\partial\chi_{P_0}(x,0)}{\partial t}\Big]dx -$$
$$- \frac{1}{\alpha+2}\int\limits_{x_0-t_0}^{x_0+t_0} |\varphi(x)|^{\alpha+2}\chi_{P_0}(x,0)\,dx.$$

**Theorem 5.1.** *Let $g(x,t,s) = -|s|^\alpha s$, $(x,t) \in \overline{\Omega}$, $s \in \mathbb{R}$, $\alpha > -1$, $f \in C(\overline{\Omega})$, $\varphi \in C^1(\mathbb{R})$, $\psi \in C(\mathbb{R})$, and the function $u \in C^1(\overline{D}_{P_0})$ be a strong generalized solution of the problem* (1.1), (1.3) *of the class $C^1$ in the domain $D_{P_0}$. If*

$$\liminf_{t_0\to+\infty} \zeta(t_0) > 0, \tag{5.4}$$

*then there exists the positive number $t^0 := t^0(x_0; f, \varphi, \psi) > 0$, such that for $t_0 > t^0$ the problem* (1.1), (1.3) *cannot have a strong generalized solution of the class $C^1$ in the domain $D_{P_0}$.*

*Remark* 5.2. According to Remark 5.1, let us denote by $t^* := t^*(x_0; f, \varphi, \psi)$ the upper edge of those $t_0 > 0$, for which the problem (1.1), (1.3) is solvable in the domain $D_{P_0}$. From Theorems 3.1 and 5.1 it follows that $0 < t^* \le t^0$, and the problem (1.1), (1.3) is solvable in the domain $D_{P_0}$ for $t_0 < t^*$ and does not have solution for $t_0 > t^*$.

*Remark* 5.3. It is easy to verify that if $f \ge 0$, $\varphi \equiv 0$, $\psi \ge 0$ and one of the following conditions

$$1)\ f(x,t) \ge c,\ (x,t) \in \overline{\Omega};\quad 2)\ \psi(x) \ge c,\ x \in \mathbb{R},$$

is fulfilled, where $c := \text{const} > 0$, and for function $\chi_{P_0}$ we take $\chi_{P_0}(x,t) = \chi^*\big(\frac{x-x_0}{t_0}, \frac{t}{t_0}\big)$, where $\chi^*$ is defined by the equality (5.3), then the condition (5.4) will be fulfilled, and therefore in this case the problem (1.1), (1.3) for sufficiently large $t_0$ will not have a strong generalized solution $u$ of the class $C^1$ in the domain $D_{P_0}$.

## 6. The Cauchy Problem for One-Dimensional Wave Equation with Nonlinear Damping Term

In this section we use the obtained in the previous sections results for proving the existence of a unique global classical solution of the Cauchy problem for wave equations with nonlinear damping term of the type

$$u_{tt} - u_{xx} + h(u_t) = f(x,t). \tag{6.1}$$

**Theorem 6.1.** *Let the conditions*

$$h \in C^2(\mathbb{R}),\quad f \in C^1(\overline{\Omega}),\quad \varphi \in C^2(\mathbb{R}),\quad \psi \in C^1(\mathbb{R})$$

*be fulfilled and*

$$h'(s) \geq -M, \quad s \in \mathbb{R}, \quad M := \text{const} > 0. \tag{6.2}$$

*Then there exists a unique global classical solution* $u \in C^2(\overline{\Omega})$ *of the problem* (6.1), (1.2).

Note that the condition (6.2) is fulfilled for functions with exponential growth, e.g. $h(s) = e^{s^{2k+1}}$, $s \in \mathbb{R}$, $k = 0, 1, \dots$.

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