

Galois theory of simplicial complexes

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Abstract

We examine basic notions of categorical Galois theory for the adjunction between Π_0 and the inclusion *as discrete*, in the case of simplicial complexes. Covering morphisms are characterized as the morphisms satisfying the *unique simplex lifting* property, and are classified by means of the fundamental groupoid, for which we give an explicit “Galois-theoretic” description. The class of covering morphisms is a part of a factorization system similar to the (purely inseparable, separable) factorization system in classical Galois theory, which however fails to be the (monotone, light) factorization.

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Introduction

Out of many good books in algebraic topology, let us pick up Brown [4], Gabriel and Zisman [8], Quillen [15], and Spanier [17]. We observe:

- The Galois theory of covering spaces, i.e., the classification of covering spaces (of a “good” space) via the fundamental groupoid and its actions, is presented in [4] and [8]. It can also be deduced from the results of [17], where however only connected coverings and their canonical projections and automorphisms are considered (rather than the whole *category of coverings*).

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- The passage from covering spaces to the actions of the fundamental groupoid, in [4] and [8], uses *coverings of groupoids*; these are the same as *discrete fibrations*, called “coverings” by *analogy* with the topological case.
- Gabriel and Zisman [8] also develop what we would call the Galois theory of covering morphisms of *simplicial sets*. This theory agrees with the topological one via the classical adjunction between *geometric realization* and the *singular complex* functor; here again, covering morphisms are defined by analogy with the topological ones.
- Spanier [17] does not make use of any combinatorial notion of covering, but constructs directly the so-called *edge-path groupoid* of a *simplicial complex* (as recalled in Remark 3.3).
- According to [15,8], the most important combinatorial objects for homotopy theory are simplicial sets; however, simplicial complexes are a simpler way of codifying “good spaces”, favored in [4,17].

Now, all the above mentioned notions of a covering are shown (see [3] and references there) to be special cases of the categorical notion, derived from *categorical Galois theory* (CGT for short). Therefore, we became interested in the “missing case” of simplicial complexes, for which no theory of covering morphisms has apparently been developed, arriving to the following conclusions:

- CGT can be applied to the category **SmC** of simplicial complexes, as to any category of *abstract families* (see [3,5]). Note that **SmC** is not a topos and hence does not fit in the framework of Barr and Diaconescu [2]; still, it is not far from a topos, of course (see Section 1 below).
- CGT produces a reasonable notion of covering morphism (Section 2) of simplicial complexes. Moreover, the classical edge-path groupoid of a simplicial complex is equivalent (see Section 3) to the fundamental groupoid which we deduce here from CGT, and hence classifies such covering morphisms.
- The class of covering morphisms in **SmC** is closed under composition. Moreover, it is a part of a factorization system (Section 3), which is *not* the (monotone, light) factorization in the sense of [6] (Section 4).

We have tried to make the paper reasonably self-contained, assuming however some familiarity with Chapters 5 and 6 of [3]. A more detailed presentation of descent theory, also containing a result used here, can be found in the expository paper [13]. Factorization systems in categories are treated in many books and papers; everything we need (and much more!) is recalled in [6]. The notation $F \dashv G$ means, as usual, that the functor F is left adjoint to G .

1. Simplicial complexes

A *simplicial complex* is a pair $A = (A, S(A))$, where A is a set and $S(A)$ is a down-closed set of finite non-empty parts of A , containing all one-element subsets. The elements of $S(A)$ are called *linked subsets*, or *simplexes* of A .

Simplicial complexes are related with the topos $\mathbf{Set}^{!\Delta^{\text{op}}}$ of symmetric simplicial sets, or presheaves on the category $!\Delta$ of non-zero finite cardinals (and all mappings between them), equivalent to the category of non-empty finite sets (see [9]). We will use the following well-known properties:

Proposition 1.1. (a) \mathbf{SmC} is a full reflexive subcategory in the topos $\mathbf{Set}^{!\Delta^{\text{op}}}$ of symmetric simplicial sets. An object A in \mathbf{SmC} , regarded as a functor $!\Delta^{\text{op}} \rightarrow \mathbf{Set}$, has $A(\{0, \dots, n\}) = A_n = \{(a_0, \dots, a_n) \in A^{n+1} \mid \{a_0, \dots, a_n\} \in S(A)\}$.

(b) An object A in $\mathbf{Set}^{!\Delta^{\text{op}}}$ is (isomorphic to) a simplicial complex if and only if all canonical maps $A_n \rightarrow A^{n+1}$ (sending a simplex to the family of its vertices) are injective.

(c) \mathbf{SmC} is closed under subobjects and products in $\mathbf{Set}^{!\Delta^{\text{op}}}$. Therefore, \mathbf{SmC} is regular and a morphism $p : E \rightarrow B$ in \mathbf{SmC} is a regular epimorphism if and only if it is a regular epimorphism in $\mathbf{Set}^{!\Delta^{\text{op}}}$, i.e., if and only if it is surjective on linked subsets.

(d) The connected components of a simplicial complex A are precisely the equivalence classes under the smallest equivalence relation containing all pairs (a_0, a_1) with $\{a_0, a_1\} \in S(A)$; moreover, every simplicial complex is isomorphic to the sum of its components. Thus, \mathbf{SmC} is a connected locally connected category, in the sense that it is equivalent to the category of families of its connected objects and its terminal object is connected. (Categories of families will be described at the beginning of the next section.)

From (b) we easily obtain

Corollary 1.2. Let

$$\begin{array}{ccc}
 E \times_B A & \xrightarrow{\text{pr}_2} & A \\
 \text{pr}_1 \downarrow & & \downarrow \alpha \\
 E & \xrightarrow{p} & B
 \end{array} \tag{1}$$

be a pullback diagram in $\mathbf{Set}^{!\Delta^{\text{op}}}$; if p is a regular epimorphism in \mathbf{SmC} and $E \times_B A$ is in \mathbf{SmC} , then A also is in \mathbf{SmC} (up to an isomorphism).

Since $\mathbf{Set}^{!\Delta^{\text{op}}}$ is an exact category, together with Proposition 1.1(d) and [13, Proposition 3.2], this gives³

Corollary 1.3. A morphism in \mathbf{SmC} is an effective descent morphism if and only if it is a regular epimorphism.

An object A in \mathbf{SmC} is said to be a *simplex* if $A \in S(A)$; such an object is necessarily finite and is also called an *n-simplex* if it has exactly $n + 1$ elements. From Corollary 1.3 and the characterization of regular epimorphisms in Proposition 1.1(b) (or using simplicial sets) we obtain

³ This observation has been extended to more general relational structures by A.H. Roque [16], a Ph.D student of the second named author.

Corollary 1.4. (a) *A connected object in \mathbf{SmC} is projective with respect to the class of effective descent morphisms (= regular epimorphisms) in \mathbf{SmC} if and only if it is a simplex.*

(b) *For every object B in \mathbf{SmC} , there is an effective descent morphism $p: E \rightarrow B$ with E projective. Moreover, there is a canonical choice for such a morphism: take E to be the coproduct of all linked subsets in B considered as simplexes, and $p: E \rightarrow B$ induced by the inclusion maps.*

An obvious but surprising conclusion of Corollary 1.4(a) is

Corollary 1.5. *Every quotient of a connected projective object in \mathbf{SmC} is projective.*

2. Covering maps of simplicial complexes

We are going to apply categorical Galois theory to the adjunction

$$\mathbf{SmC} \begin{array}{c} \xrightarrow{\Pi_0} \\ \xleftarrow{D} \end{array} \mathbf{Set} \quad (\Pi_0 \dashv D), \tag{2}$$

where $\Pi_0(A)$ is the set of connected components of a simplicial complex A , while D embeds \mathbf{Set} into \mathbf{SmC} regarding sets as *discrete simplicial complexes* (i.e., simplicial complexes with no n -simplexes for $n \neq 1$).

Now, we already noted, in Proposition 1.1(d), that \mathbf{SmC} is equivalent to the category $\mathbf{Fam}(\mathbf{A})$ of families on the category \mathbf{A} of *connected simplicial complexes*. In this light, the previous adjunction appears to be a special case of the basic adjunction of the Galois theory of abstract families

$$\mathbf{Fam}(\mathbf{A}) \begin{array}{c} \xrightarrow{I} \\ \xleftarrow{H} \end{array} \mathbf{Fam}(\mathbf{1}) = \mathbf{Set} \quad (I \dashv H) \tag{3}$$

where \mathbf{A} is an arbitrary category with terminal object and $\mathbf{Fam}(\mathbf{A})$ has pullbacks. We recall from [5,3] that I sends a family to its set of indices, or—equivalently—to the set of its connected components. Thus, an object of $\mathbf{Fam}(\mathbf{A})$ can (and will) be written as a family $A = (A_i)_{i \in I(A)}$, while a morphism $\alpha: A \rightarrow B$ consists of a mapping $I(\alpha): I(A) \rightarrow I(B)$ together with a family $(\alpha_i: A_i \rightarrow B_{I(\alpha)(i)})_{i \in I(A)}$ of morphisms of \mathbf{A} .

We recall

Proposition 2.1. *The following conditions on a morphism $\alpha: A \rightarrow B$ in $\mathbf{Fam}(\mathbf{A})$ are equivalent:*

(a) *α is a trivial covering, i.e., the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ \alpha \downarrow & & \downarrow HI(\alpha) \\ B & \xrightarrow{\eta_B} & HI(B) \end{array} \tag{4}$$

in which η is the unit of the adjunction (3), is a pullback;

- (b) each α_i in the presentation of α as the family $(\alpha_i : A_i \rightarrow B_{I(\alpha)(i)})_{i \in I(A)}$ of morphisms in \mathbf{A} , is an isomorphism.

Corollary 2.2. *The following conditions on a morphism $\alpha : A \rightarrow B$ in \mathbf{SmC} are equivalent:*

- (a) α is a trivial covering, i.e., the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & D\Pi_0(A) \\
 \beta \downarrow & & \downarrow \Pi_0(\beta) \\
 B & \xrightarrow{\eta_B} & D\Pi_0(B)
 \end{array} \tag{5}$$

is a pullback;

- (b) for every connected component C in A , $\alpha(C)$ is a connected component in B , and α induces an isomorphism $C \cong \alpha(C)$.

After that we are ready to describe all coverings:

Theorem 2.3. *Suppose $\mathbf{C} = \text{Fam}(\mathbf{A})$ satisfies the following conditions:*

- (i) it is a regular category in which every regular epimorphism is an effective descent morphism;
- (ii) it has enough projectives (with respect to the class of regular epimorphisms), i.e., every object in \mathbf{C} is a quotient of a projective object;
- (iii) every quotient of a connected projective object in it is itself projective.

Then the conditions (a)–(d) below on a morphism $\alpha : A \rightarrow B$ in \mathbf{C} are equivalent to each other and imply (e):

- (a) α is a covering, i.e., there exists a pullback diagram (1), in which p is an effective descent morphism, and $pr_1 : E \times_B A \rightarrow E$ is a trivial covering;
- (b) for every morphism $p : E \rightarrow B$ with projective E , the morphism $pr_1 : E \times_B A \rightarrow E$ is a trivial covering;
- (c) for every morphism $p : E \rightarrow B$ with E connected projective, the morphism $pr_1 : E \times_B A \rightarrow E$ is a trivial covering;
- (d) for every connected projective subobject E in B , the morphism from the inverse image of E to E induced by α is a trivial covering;
- (e) (The unique connected projective subobject lifting property) for every connected projective subobject E in B and every connected subobject C in A whose image is contained in E , there exists a unique connected projective subobject D in A containing C such that α induces an isomorphism $D \cong E$.

Proof. (a) \Rightarrow (b) follows from the fact that the class of trivial coverings is pullback stable. (b) \Rightarrow (a) follows from (ii) and the fact that every regular epimorphism in \mathbf{C} is an effective descent morphism. The implications (b) \Rightarrow (c) \Rightarrow (d) are trivial. (c) \Rightarrow (b) can easily be

shown using the fact that \mathbf{C} is a category of families. (d) \Rightarrow (c) follows from (iii) and the regularity of \mathbf{C} . (d) \Rightarrow (e) can easily be shown using Proposition 2.1. \square

Corollary and Remark 2.4. *Theorem 2.3 applies to $\mathbf{C} = \mathbf{SmC}$, and moreover, in this case the condition (e) clearly implies (d), and hence all the other conditions. Note also that it can be equivalently reformulated as the following unique simplex lifting property: for every simplex E in B and every point a in A whose image is contained in E , there exists a unique simplex D in A containing a such that α induces an isomorphism $D \cong E$.*

Corollary 2.5. *The class of covering morphisms in \mathbf{SmC} is the second component of a factorization system.*

Proof. As follows from Corollary 1.4 and the results of [12] (see point 2⁰ in Introduction and Theorem 4.2 there), the category of coverings of an arbitrary object B in \mathbf{SmC} is reflective in $(\mathbf{SmC} \downarrow B)$. After that, since the class of covering morphisms is pullback stable, we only need to check that it is closed under composition—which however can easily be deduced from Theorem 2.3. \square

3. The fundamental groupoid

Let B be a fixed simplicial complex, and $p : E \rightarrow B$ be what we called the canonical effective descent morphism (with projective E) in Corollary 1.4(b). In other words:

- E is the set of pairs (b, s) with $b \in s \in S(B)$;
- a subset $\{(b_0, s_0), \dots, (b_n, s_n)\}$ in E is linked if and only if $s_0 = \dots = s_n$;
- p is defined by $p(b, s) = b$.

As follows from the fundamental theorem of categorical Galois theory [11] (see also [3]) and Corollary 1.4(b), there is a canonical category equivalence

$$\text{Cov}(B) \simeq \mathbf{Set}^{\text{Gal}(E,p)}, \tag{6}$$

where $\text{Cov}(B)$ is the category of coverings of B , and the precategory

$$\text{Gal}(E, p) = \left(\Pi_0(E \times_B E \times_B E) \rightrightarrows \Pi_0(E \times_B E) \leftarrow \Pi_0(E) \right) \tag{7}$$

is the Π_0 -image of the kernel pair of p considered as an internal precategory in \mathbf{SmC} . The precategory $\text{Gal}(E, p)$ has an obvious explicit description, as soon as we make

Observation 3.1. Since two elements in E belong to the same connected component if and only if they are linked, it is easy to see that $\Pi_0(E \times_B \dots \times_B E)$ can be identified with the set of all sequences (of the appropriate length) of elements in $S(B)$ having non-empty intersection.

Let

$$L : \mathbf{Precat} \rightarrow \mathbf{Cat} \tag{8}$$

be the left adjoint of the inclusion functor from the category of precategories to the category of categories. We define now the *fundamental groupoid* $\Pi_1(B)$ of the simplicial complex B as $\Pi_1(B) = L(\text{Gal}(E, p))$, where E and p are as above. Since for every precategory P , and every category \mathbf{S} we have $\mathbf{S}^P \approx \mathbf{S}^{L(P)}$, (6) gives

$$\text{Cov}(B) \simeq \mathbf{Set}^{\Pi_1(B)}. \tag{9}$$

From the obvious description of L and Observation 3.1 we obtain:

Theorem 3.2. *The fundamental groupoid $\Pi_1(B)$ can be described as follows:*

- (a) *the objects in $\Pi_1(B)$ are all simplexes in B ;*
- (b) *a morphism $s \rightarrow s'$ in $\Pi_1(B)$ is the equivalence class of a finite sequence (s_0, \dots, s_n) of simplexes in B with $s_0 = s$, $s_{i-1} \cap s_i$ non-empty for each $i = 1, \dots, n$, and $s_n = s'$, where*
- (c) *the sequences compose by concatenation (as in the free category), and their equivalence is defined as the smallest congruence under which (s_0, s_1, s_2) is congruent to (s_0, s_2) whenever $s_0 \cap s_1 \cap s_2$ is not empty; the composition of equivalence classes is induced by the composition of sequences.*

Remark 3.3. (a) In the groupoid $\Pi_1(B)$, an object (i.e., a simplex of B) is isomorphic to each of its points, and we can equivalently use the full subgroupoid of points. The latter is plainly isomorphic to the classical *edge-path groupoid* of B [17], which can be constructed as follows:

- (i) the objects are the points of B ,
- (ii) a morphism $[b_0, \dots, b_n]: b_0 \rightarrow b_n$ is the equivalence class of a finite sequence of points of B , where each subset $\{b_{i-1}, b_i\}$ is linked,
- (iii) such sequences compose by concatenation, and their equivalence is the smallest congruence under which $[b, b', b''] = [b, b'']$ whenever the subset $\{b, b', b''\}$ is linked.

(b) One can find in [10] an equivalent construction, based on an intrinsic homotopy theory for simplicial complexes that also deals with their higher homotopy groups. Higher fundamental groupoids Π_n of *symmetric* simplicial sets have been studied in [9], together with higher fundamental *categories* for simplicial sets.

(c) Similar geometrical constructions, one of which uses simply connected open subsets in the same way as we use simplexes (and goes back at least to Artin and Mazur [1]), are discussed by Kennison [14].

4. Stabilization fails

Let (\mathbf{E}, \mathbf{M}) be the reflective factorization system (see [7] or [6]) in $\mathbf{C} = \text{Fam}(\mathbf{A})$ associated with the adjunction (3); accordingly \mathbf{E} consists of all morphisms e in \mathbf{C} , for which $I(e)$ is a bijection, and \mathbf{M} is the class of trivial coverings. Restricting ourselves to the case $\mathbf{C} = \mathbf{SmC}$, we could try to compare (\mathbf{E}, \mathbf{M}) with the new factorization system

$(\mathbf{E}^*, \mathbf{M}^*)$ obtained from Corollary 2.5. Since \mathbf{M}^* , the class of covering morphisms in \mathbf{SmC} , is what was described in [6] as the *localization* of \mathbf{M} , the results of [6] suggest that \mathbf{E}^* might be the *stabilization* of \mathbf{E} . However, this is not the case, as follows from [6, Theorem 6.9] and Proposition 4.1 and Example 4.2 below:

Proposition 4.1. *Suppose there exists a pullback diagram*

$$\begin{array}{ccc} 0 & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & E \end{array} \quad (10)$$

in $\mathbf{C} = \mathbf{Fam}(\mathbf{A})$, with 0 denoting the initial object (= the empty family), U and V non-initial, and E connected projective. Then the reflective factorization system in \mathbf{C} associated with the adjunction (3) is not locally stable in the sense of [6].

Proof. Let me be the (\mathbf{E}, \mathbf{M}) -factorization of the morphism $V \rightarrow E$ in (7). Since E is projective, me is locally stable if and only if it is stable. Therefore it suffices to show that it is not stable. Indeed, we have:

- (i) since V is not initial, so is the codomain of e , which is the same as the domain of m ;
- (ii) since E is connected, and m is a trivial covering with non-initial domain, m is a split epimorphism;
- (iii) therefore the pullback m' of m along $U \rightarrow E$ also is a split epimorphism;
- (iv) since U is not initial, we conclude that the domain of m' is not initial;
- (v) the pullback $m'e'$ of the factorization me along $U \rightarrow E$ has therefore e' with non-initial codomain but the domain initial (since (7) is a pullback);
- (vi) therefore $I(e')$ is not a bijection, i.e., e' is not in \mathbf{E} , as desired. \square

Example 4.2. It is a triviality to find a pullback of the form (7) in \mathbf{SmC} ; for instance, take $U = V = 1$, $E = \text{codiscrete } 2$ (= 2-simplex), and use the two maps from 1 to 2. Thus the reflective factorization system in \mathbf{SmC} associated with the adjunction (2) is not locally stable, or, equivalently, the adjunction (2) does not yield the (monotone, light) factorization system.

Remark 4.3. (a) Since products of connected objects in \mathbf{SmC} are connected, it is easy to show that the adjunction (2) *has stable units* in the sense of [7]. Hence \mathbf{SmC} provides one more example where the stable-unit-property does not imply the existence of the (monotone, light) factorization system.

(b) Proposition 4.1 and Example 4.2 also show that the class \mathbf{E}^* in \mathbf{SmC} is not pullback stable.

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