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MATRIX SPECTRAL FACTORIZATION AND WAVELETS

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ABSTRACT. In this paper, recently published results on matrix spectral factorization is reviewed, and their connection to wavelet matrices is revealed.

CONTENTS

1. Matrix Spectral Factorization

Spectral factorization is an important mathematical tool that plays a key role in the solution of various applied problems in control engineering and communications. Wiener's matrix spectral factorization theorem [9, 20] asserts that if

$$S(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1n}(t) \\ s_{21}(t) & s_{22}(t) & \cdots & s_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1}(t) & s_{n2}(t) & \cdots & s_{nn}(t) \end{pmatrix}, \tag{1}$$

$|t| = 1$, is a matrix function with integrable entries, $s_{ij}(t) \in L_1(\mathbb{T})$, which is positive definite for a.a. $t \in \mathbb{T}$, and such that the Paley–Wiener condition

$$\log\det S(t) \in L_1(\mathbb{T}) \tag{2}$$

is satisfied, then it admits a factorization

$$S(t) = S^+(t)S^-(t) = S^+(t)\big(S^+(t)\big)^*, \tag{3}$$

where $S^+(z)$ is an analytic on $\mathbb{D} = \mathbb{T}_+ := \{z \in \mathbb{C} : |z| < 1\}$ $m \times m$ matrix function with entries from the Hardy space

$$H_2 := \left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} \big|f(re^{i\theta})\big|^2 \, d\theta < \infty \right\}$$

and $S^-(z) = \widetilde{S^+}(z) = \big(S^+(1/\overline{z})\big)^*$, $|z| > 1$, is its adjoint. $S^\pm(t)$ are boundary values of $S^\pm(z)$, $S^\pm(t) = S^\pm(z)|_{z=t}$. It is assumed that Eq. (3) holds a.e. on $\mathbb{T}$. If the determinant of a spectral factor $S^+(z)$ is required to be an outer analytic function, then the factorization (3) is unique up to a constant

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unitary factor, and the unique spectral factor $S_c^+(z)$ which is positive definite at the origin, $S_c^+(0) > 0$, is called *canonical.* Together with the *left* factorization (3), one can consider the *right* factorization $S(t) = \mathcal{S}^-(t)\mathcal{S}^+(t)$ and those theories are mathematically equivalent as $\left(\mathcal{S}^+\right)^T(t) = (S^T)^+(t)$. In the scalar case, $n = 1$, a spectral factor can be explicitly written by the formula

$$S^+(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log S(e^{i\theta})\,d\theta\right). \tag{4}$$

However, there is no analog of this formula in the matrix case because, generally speaking, $e^{A+B} \neq e^A e^B$ for noncommutative matrices A and B. This is the main reason why the matrix spectral factorization is more demanding than the scalar spectral factorization. Wiener used the theory of multi-dimensional stochastic processes to prove the existence of the above spectral factorization, and Helson and Lowdenslager derived the theorem from the theory of invariant subspaces. A new approach to matrix spectral factorization problems proposed by the authors provides an analytic proof of Wiener's theorem (see [5]), which looks natural as the problem itself arises in the theory of complex functions. This proof is described in Sec. 3.

2. Polynomial Matrix Spectral Factorization

It is well known that if a matrix spectral density (1) is a Laurent polynomial

$$S(z) = \sum_{k=-N}^{N} S_k z^k \tag{5}$$

(S_k are $(n \times n)$-matrix coefficients) in Wiener's theorem, then the spectral factor is also a polynomial (of the same order N):

$$S^+(z) = \sum_{k=0}^{N} \sigma_k z^k \tag{6}$$

(then $S^-(z)$ in (3) is the adjoint matrix polynomial $\widetilde{S^+} = \sum_{k=0}^{N} \sigma_k^* z^{-k}$, and $\det S^+(z) \neq 0$ for $|z| < 1$). This theorem was first noted by Rosenblatt [16] applying the ideas of Wiener's proof, and since then many different proofs of the polynomial matrix spectral factorization theorem have appeared in the literature (see, e.g., [1, 8, 10]). A very short proof of this result is given in [6]; however, it uses Wiener's above existence theorem and some facts from the theory of the Hardy spaces. Recently, we have found a very simple and short proof [2] which uses elementary facts from linear algebra and complex analysis. In a few words but on a satisfactorily understandable level, this proof can be described as follows.

The existence of factorization $S(z) = S_0(z)\widetilde{S_0}(z)$ for some rational matrix function $S_0(z)$ is a simple consequence of the Gauss elimination method and the scalar spectral factorization, and we can remove all poles of entries of $S_0(z)$ inside $\mathbb{T}$ by multiplying on the right by unitary matrix functions $U(z) = \operatorname{diag}[1, \dots, u(z), \dots, 1]$, where $u(z) = (z - a)/(1 - \overline{a}z)$, $|a| < 1$, establishing a factorization $S(z) = S_0^+(z)\widetilde{S_0^+}(z)$ where $S_0^+(z)$ is rational and analytic (we mean its entries are analytic, as superscript "+" indicates) on $\mathbb{D}$. Now if $a \in \mathbb{D}$ and $\det S_0^+(a) = 0$, we can select a unitary matrix U which makes, say, in the first column of $S_0^+(a)U$ all 0s, and multiplication on the right by $U(z) = \operatorname{diag}[u(z), 1, \dots, 1]$, where $u(z) = (1-\overline{a}z)/(z-a)$, removes the singularity of the determinant at a. Repeating this process, if necessary, we obtain a factorization

$$S(z) = S^+(z)\widetilde{S^+}(z),$$

where $S^+(z)$ is rational, analytic and nonsingular inside $\mathbb{T}$ (consequently $\widetilde{S^+}(z)$ is such on $\mathbb{T}_- = \mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$). Then it can be shown that in fact $S^+(z)$ is polynomial. Indeed, $S^+(z)$ is analytic on $\mathbb{D}$ by assumption; $S^+(z)$ cannot have poles on $\mathbb{T}$ as $S(z) = S^+(z)\big(S^+(z)\big)^*$, for $|z| = 1$ and $S(z)$ is bounded on $\mathbb{T}$; and $z^{-N}S^+(z) = z^{-N}S(z)\big(S^-(z)\big)^{-1}$ is analytic on $\mathbb{T}_-$, which implies that $S^+(z)$ is a polynomial of order N. This proof can be extended to the spectral factorization of rank-deficient matrix polynomials [3].

Theorem. *Let $S(z)$ be an $n \times n$ Laurent matrix polynomial* (5) *of order N, which is nonnegative definite and of rank $r \le n$ for almost every $z \in \mathbb{T}$. Then there exists a unique* (*up to a $r \times r$ unitary matrix right multiplier*) *$n \times r$ matrix polynomial* (6) *of order N* (*σ_k are $n \times r$ matrix coefficients*), *which is of full rank k for each z inside $\mathbb{T}$, such that* (3) *holds.*

3. Matrix Spectral Factorization Algorithm

As numerous applied problems in multi-dimensional system theory are computationally reduced to matrix spectral factorization, a development of the method that approximately constructs a spectral factor S^+ for a given spectral density (1) has become a challenging problem. Since Wiener's original efforts [20] to create a sound computational algorithm of matrix spectral factorization, tens of different methods have appeared in the literature (see the survey papers [13, 17] and the references therein). Most of these algorithms impose extra restrictions on matrix spectral densities (e.g., to be real or rational or nonsingular on the boundary), while the Paley–Wiener necessary and sufficient condition (2) will do for the existence of spectral factorization. In [11], a new efficient algorithm of matrix spectral factorization is proposed that can be applied to compute an approximate spectral factor of any positive definite matrix function that satisfies (2). To describe this algorithm in a few words, it first performs the lower-upper triangular factorization with analytic entries on the diagonal, $S(t) = M(t)M^*(t)$,

$$M(t) = \begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0 \\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{n-1,1}(t) & \xi_{n-1,2}(t) & \cdots & f_{n-1}^+(t) & 0 \\ \xi_{n1}(t) & \xi_{n2}(t) & \cdots & \xi_{n,n-1}(t) & f_n^+(t) \end{pmatrix}, \tag{7}$$

where $\xi_{ij} \in L_2(\mathbb{T})$ and $f_i^+ \in H_2^O$ (H_2^O stands for the class of outer functions from H_2), and then carries out an approximate spectral factorization of $m \times m$ left-upper submatrices step by step, $m = 2, 3, \dots, n$, by multiplication on the right by unitary matrix functions of specific structures. As we make analytic each additional row on every such step and

$$\begin{pmatrix} s_{11}^+ & \cdots & s_{1,m-1}^+ & 0 \\ s_{21}^+ & \cdots & s_{2,m-1}^+ & 0 \\ \vdots & \ddots & \vdots & \vdots \\ s_{m-1,1}^+ & \cdots & s_{m-1,m-1}^+ & 0 \\ \zeta_1 & \cdots & \zeta_{m-1} & f^+ \end{pmatrix} = \begin{pmatrix} s_{11}^+ & \cdots & s_{1,m-1}^+ & 0 \\ s_{21}^+ & \cdots & s_{2,m-1}^+ & 0 \\ \vdots & \ddots & \vdots & \vdots \\ s_{m-1,1}^+ & \cdots & s_{m-1,m-1}^+ & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \cdot F,$$

where F is the matrix-function (8) below, the following theorem plays the decisive role in this process.

Theorem 1. *For an $(m \times m)$-matrix function $F(t)$ of the form*

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & f^+(t) \end{pmatrix}, \tag{8}$$

where $\zeta_j(t) \in L_2(\mathbb{T})$, $j = 1, 2, \dots, m-1$, and $f^+(t) \in H_2^0$, there exists a unitary matrix function $U(t)$ of the form

$$U(t) = \begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t) \\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t) \\ \widetilde{u_{m1}^+}(t) & \widetilde{u_{m2}^+}(t) & \cdots & \widetilde{u_{m,m-1}^+}(t) & \widetilde{u_{mm}^+}(t) \end{pmatrix}, \tag{9}$$

where $u_{ij}^+(t) \in L_\infty^+(\mathbb{T})$, with determinant 1, $\det U(t) = 1$, such that

$$F(t)U(t) \in L_2^+(\mathbb{T})$$

(L_p^+ is the set of boundary values of H_p).

This theorem allows us to represent a spectral factor as

$$S^+(t) = M(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\cdots\mathbf{U}_n(t). \tag{10}$$

In this equation

$$\mathbf{U}_m(t) = \begin{pmatrix} U(t) & 0 \\ 0 & I_{n-m} \end{pmatrix}, \quad m = 2, 3, \dots, n,$$

where $U(t)$ is the unitary matrix function determined according to Theorem 1 for a matrix function (8) the last row of which coincides with the first m entries in the mth row of $M(t)\mathbf{U}_2(t)\dots\mathbf{U}_{m-1}(t)$. The representation (10) actually gives an analytic proof of the existence of the matrix spectral factorization theorem considered in Sec. 1. A core of the numerical algorithm for approximate computation of S^+ is a constructive proof of Theorem 1 in the case where functions ζ_j and f^+ are (Laurent) polynomials (of order N), $\zeta_i \in \mathcal{P}_N^-$, $f^+ \in \mathcal{P}_N^+$. In this case, $u_{ij}^+ \in \mathcal{P}_N^+$ as well in (9), and they can be explicitly computed (see [11, Theorem 1]). Thus approximating (7) in L_2 by $\xi(t) \approx \sum_{k=-N}^{\infty} c_k(\xi)t^k$ and then constructing unitary matrix functions $\mathbf{U}_m$ in (10) explicitly, we get an approximate spectral factor $\widehat{S}^+ \approx S^+$. That $\widehat{S}^+ \to S^+$ as $N \to \infty$ follows from the general stability criterion given in the next section.

4. Stability of Spectral Factorization

In applications, a matrix spectral density S is frequently constructed empirically and therefore is replaced by its approximation $\widehat{S}$. It is therefore important to know how close S^+ remains to $\widehat{S}^+$ during the approximation. The following theorem provides this criterion [7, Theorem 1].

Theorem 2. *Let $S_k(t)$, $k = 1, 2, \dots$, be a sequence of positive definite $n \times n$ matrix functions with integrable entries such that*

$$\log\det S_k(t) \in L_1(\mathbb{T}), \quad k = 1, 2, \dots, \tag{11}$$

and let $(S_k)_c^+(t)$, $k = 1, 2, \dots$, *be the sequence of corresponding canonical spectral factors. If*

$$\|S_k(t) - S(t)\|_{L_1} \longrightarrow 0 \tag{12}$$

and

$$\int_0^{2\pi} \log\det S_k(e^{i\theta})\, d\theta \longrightarrow \int_0^{2\pi} \log\det S(e^{i\theta})\, d\theta, \tag{13}$$

then

$$\|(S_k)_c^+ - S_c^+\|_{H_2} \longrightarrow 0. \tag{14}$$

It is well known that, in general, (12) alone does not imply (14) even in the scalar case. On the other hand, if (14) holds, then

$$S_k^+(0) \longrightarrow S^+(0) \Longrightarrow \det S_k^+(0) \longrightarrow \det S^+(0),$$

and since Wiener's matrix spectral factorization theorem provides the scalar spectral factorization of the determinant

$$(\det S)(t) = (\det S)^+(t)(\det S)^-(t) = \det S^+(t) \det S^-(t)$$

and

$$(\det S_k)^+(0) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log\det S_k(e^{i\theta})\, d\theta\right)$$

(see (4)), we have that (13) is valid. Thus one can easily see that the condition (13) is necessary for the convergence (14) to hold. One more important practical consequence of Theorem 2 is the following fact: as was discussed in Section 2, if a sequence of the considered spectral densities are matrix polynomials of fixed order N, then the corresponding spectral factors are polynomials of the same order N. In this case (12) always implies (13) and consequently (14) follows directly from (12). Thus it should be expected that a "small" perturbation of the coefficients of S will not much affect the coefficients of S^+ even in the case where the determinant of $S(t)$ has zeros on the boundary. This fact was empirically observed during computer simulations of different numerical polynomial spectral factorization algorithms.

5. Compact Wavelet Matrices

It turns out that the unitary matrix functions constructed in Theorem 1 represent another way of describing wavelet matrices that we are going to discuss now. A formally infinite matrix $\mathcal{A}$ with m rows

$$\mathcal{A} = \begin{pmatrix} \cdots & a_{-1}^0 & a_0^0 & a_1^0 & a_2^0 & \cdots \\ \cdots & a_{-1}^1 & a_0^1 & a_1^1 & a_2^1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_{-1}^{m-1} & a_0^{m-1} & a_1^{m-1} & a_2^{m-1} & \cdots \end{pmatrix} \tag{15}$$

is called a *wavelet matrix* (of rank m) if its rows satisfy the so-called *shifted orthogonality condition*:

$$\sum_k a_{k+mp}^i \,\overline{a}_{k+mq}^j = \delta_{ij}\delta_{pq}. \tag{16}$$

In *polyphase representation* of matrix $\mathcal{A}$,

$$\mathbf{A}(z) = \sum_{k=-\infty}^{\infty} A_k z^k, \tag{17}$$

where $\mathcal{A} = (\dots, A_{-1}, A_0, A_1, A_2, \dots)$, is the partition of $\mathcal{A}$ into block $(m \times m)$-matrices $A_k = (a^i_{km+j})$, $0 \le i, j \le m-1$, the condition (16) is equivalent to

$$\mathbf{A}(z)\widetilde{\mathbf{A}}(z) = I_m \tag{18}$$

(it is easy to see that (16) can be written in the block matrix form $\sum_{k=-\infty}^{\infty} A_k A^*_{i-k} = \delta_{i0} I_m$). Such matrices play a central role in the theory of wavelets and multirate filter banks (see [15]) and they are known under different names such as lossless systems [19], perfect reconstruction m-filters [14], paraunitary m-channel filters [18], and so on. Our notion of wavelet matrix is weaker than usual. So as the orthogonal basis of $L^2(\mathbb{R})$ can be developed, also the *linear condition* $\mathbf{A}(1)\,\mathbf{1} = \sqrt{m}\, e_1$, where $\mathbf{1} = (1, 1, \dots, 1)^T$ and $e_1 = (1, 0, \dots, 0)$, must be satisfied. In our consideration, the linear condition is irrelevant. Instead, we require a more simple condition

$$\mathbf{A}(1) = I_m. \tag{19}$$

For any wavelet matrix $\mathcal{A}$, $\mathbf{A}(1)$ is unitary (see (18)), so that the conditions (16) and (19) will be satisfied for $\mathcal{A}_0 = \big(\mathbf{A}(1)\big)^{-1}\mathcal{A}$. Note also that for any wavelet matrix $\mathcal{A}_0$ satisfying (19) and unitary matrix U, the matrix $\mathcal{A} = U\mathcal{A}_0$ is also a wavelet matrix satisfying $\mathbf{A}(1) = U$. Thus the additional restriction (19) does not lose generality in the description of wavelet matrices. Observe that the polyphase representation of $U\mathcal{A}$ is $\sum_{k=-\infty}^{\infty} UA_k z^k$ (obviously, there is a one-to-one correspondence between matrices (15) and their polyphase representation (17) and they are naturally identified). We consider compact wavelet matrices, which means that only a finite number of coefficients in (15) are nonzero (the corresponding wavelet functions in $L^2(\mathbb{R})$ have then a compact support, and the corresponding filters in signal processing applications are physically realizable). Namely, compact wavelet matrices with polyphase representation

$$\mathbf{A}(z) = \sum_{k=0}^{N} A_k z^k, \tag{20}$$

where $A_N \neq 0$ (it is also always assumed that $A_0 \neq 0$), are called of *rank* m and *order* $N := \mathrm{ord}(\mathcal{A})$ (in some books they are called of *genus* $N+1$); we write $\mathcal{A} \in \mathcal{W}(m, N)$. The property (18) for matrix polynomial (20) means that $\mathbf{A}(z)$ is a paraunitary matrix function. Note that $\mathbf{A}(z)$ is the usual unitary matrix for each $z \in \mathbb{T}$. As the determinant of $\mathbf{A}(z) \in \mathcal{W}(m, N)$ is always monomial, i.e., $\det \mathbf{A}(z) = c\, z^d$, the positive integer d is defined to be the *degree* of $\mathcal{A}$, $d = \deg(\mathcal{A})$. It can be proved that $\deg(\mathcal{A}) \ge \mathrm{ord}(\mathcal{A})$ for any $\mathcal{A} \in \mathcal{W}(m, N)$ (see, e.g. [15] or [3, Lemma 2]) and $\deg(\mathcal{A}) = \mathrm{ord}(\mathcal{A})$ if and only if $\mathrm{rank}(A_0) = m - 1$ (see [4]). As $A_0 A_N^* = 0$ for each $\mathcal{A} \in \mathcal{W}(m, N)$ and $A_N \neq 0$, we have $\mathrm{rank}(A_0) < m$. Thus, $\deg(\mathcal{A}) = N$ whenever A_0 has the maximal possible rank. The set of wavelet matrices of rank m order N and degree d will be denoted by $\mathcal{W}(m, N, d)$. As $\mathcal{A}$ and $U\mathcal{A}$, where U is nonsingular, have the same rank, order, and degree, we will assume that in addition $\mathcal{A}$ satisfies (19), and the subset of such compact wavelet matrices will be denoted by $\mathcal{W}_0(m, N, d)$. If $\mathbf{A}(z) = Q + Pz$, where P and Q are complementary projector operators on $\mathbb{C}^m$ and $\mathrm{rank}(P) = d$ ($\mathrm{rank}(Q) = m - d$), then $\mathbf{A}(z) \in \mathcal{W}_0(m, 1, d)$, and every wavelet matrix of order 1 and degree d has such form (see [4, 12]). The matrix polynomials $\mathbf{V}(z) \in \mathcal{W}_0(m, 1, 1)$ are called *primitive* wavelet matrices and every $\mathbf{A}(z) \in \mathcal{W}_0(m, N, d)$ can be factorized as follows (see [15, Theorem 4.4.15]):

$$\mathbf{A}(z) = \prod_{j=1}^{d} \mathbf{V}_j(z), \quad \mathbf{V}_j(z) \in \mathcal{W}_0(m, 1, 1). \tag{21}$$

If $\mathbf{V}_j(z) = Q_j + P_j z$ and no consecutive operators P are orthogonal to each other, $P_j P_{j+1} \neq 0$, then $N = d$ and $\mathbf{A}(z) \in \mathcal{W}_0(m, N, N)$ in (21). This factorization is unique in this case (see [12]). Thus so far the only known way to construct $\mathcal{A} \in \mathcal{W}_0(m, N, N)$ for arbitrary N was the following procedure: choose column vectors $v_j \in \mathbb{C}^m$ with $\|v_j\| = 1$, $j = 1, 2, \ldots, N$, such that $v_j \not\perp v_{j+1}$, take corresponding projector operators $P_j = v_j v_j^*$ and primitive wavelet matrices $\mathbf{V}_j = P_j^\perp + P_j z$, and construct the product $\mathbf{A}(z) = \prod_{j=1}^{N} \mathbf{V}_j(z) \in \mathcal{W}_0(m, N, N)$. This scheme gives rise to the map (parametrization)

$$\underbrace{\mathbb{CP}^{m-1} \times \mathbb{CP}^{m-1} \times \cdots \times \mathbb{CP}^{m-1}}_{N} \supset \mathcal{B} \longleftrightarrow \mathcal{W}_0(m, N, N),$$

where $\mathbb{CP}^{m-1}$ is the projective space $\mathbb{C}^m/(\mathbb{C} \setminus \{0\})$ (the space of one-dimensional subspaces of $\mathbb{C}^m$), which is one-to-one and onto but defined only on the subset $\mathcal{B}$, where consecutive unit vectors are not orthogonal, i.e., some singular points are excluded from the set of parameters. In the next section, we give a more efficient way of construction and parametrization of $\mathcal{W}_0(m, N, N)$ depending on new ideas developed during spectral factorization.

6. Complete Parametrization of Compact Wavelet Matrices

We consider compact wavelet matrices $\mathcal{A}$ of rank m and order and degree N, $\mathcal{A} \in \mathcal{W}(m, N, N)$. Recall that we identify $\mathcal{A}$ with its polyphase representation (20). A linear condition we introduced was (19) and this subclass was denoted by $\mathcal{W}_0(m, N, N)$. We further require that the last row of A_N is not a zero vector and denote such subclass by $\mathcal{W}_1(m, N, N)$. This is done without loss of generality as $A_N \neq 0$ and we can interchange the rows of $\mathcal{A}$ if necessary. In particular, for any $\mathcal{A} \in \mathcal{W}(m, N, N)$ there exist constant unitary $m \times m$ matrices U_0 and U_1 such that $U_0 \mathcal{A} U_1 \in \mathcal{W}_1(m, N, N)$ (by multiplication on the right we assume that $\mathcal{A} U_1(z) = \mathbf{A}(z) U_1$). Indeed, we can interchange the rows of $\mathcal{A}$, if necessary, by multiplication by U_0 and then take $U_1 = (U_0 \mathbf{A}(1))^{-1}$. Let now $\mathcal{P}_N^+$ be the set of polynomials

$$\mathcal{P}_N^+ = \left\{ \sum_{k=0}^{N} c_k z^k : \ c_0, c_1, \ldots, c_N \in \mathbb{C} \right\}$$

and

$$\mathcal{P}_N^- = \left\{ \sum_{k=1}^{N} c_k z^{-k} : \ c_1, c_2, \ldots, c_N \in \mathbb{C} \right\}.$$

Observe that constant functions belong only to $\mathcal{P}_N^+$ according to our notation. Formally we assume that every $p \in \mathcal{P}_N^+$ has exactly $N+1$ coefficients (it might happen that $c_k = c_{k+1} = \cdots = c_N = 0$), so that we identify $\mathcal{P}_N^+$ with $\mathbb{C}^{N+1}$ and $\mathcal{P}_N^-$ with $\mathbb{C}^N$. Let $\mathcal{U}_1(m, N, N)$ be the set of paraunitary matrix polynomials $U(z)$, $U(z)\widetilde{U}(z) = I_m$, of the form (9) where each $u_{ij}^+(z) \in \mathcal{P}_N^+$, $U(1) = I_m$, $\det U(z) = 1$, and not all polynomials u_{mj}^+ (in the last row) are 0 in the origin ($\sum_{j=1}^{m} |u_{mj}(0)| > 0$). Then for each $\mathbf{A}(z) \in \mathcal{W}_1(m, N, N)$, we have $U(z) = \mathrm{diag}[1, \ldots, 1, z^{-N}]\mathbf{A}(z) \in \mathcal{U}_1(m, N, N)$ (the last row is multiplied by z^{-N}. Thus there is a simple one-to-one correspondence

$$\mathcal{W}_1(m, N, N) \longleftrightarrow \mathcal{U}_1(m, N, N),$$

and we parameterize the latter class using the extension of Theorem 1.

Theorem 3. *Let $N \geq 1$. For any Laurent matrix polynomial $F(z)$ of the form*

$$F(z) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(z) & \zeta_2(z) & \zeta_3(z) & \cdots & \zeta_{m-1}(z) & 1 \end{pmatrix}, \quad \zeta_j(z) \in \mathcal{P}_N^-, \tag{22}$$

there exists a unique

$$U(z) \in \mathcal{U}_1(m, N, N) \tag{23}$$

such that

$$F(z)U(z) \in \mathcal{P}_N^+. \tag{24}$$

Conversely, for each (23), *there exists a unique* (22) *such that* (24) *holds.*

For a given $F(z)$, the corresponding $U(z)$ can be efficiently computed using the constructive proof of Theorem 1 (see [11, Theorem 1]).

Let

$$\zeta_i(z) = \sum_{k=1}^{N} a_{ik} z^{-k}, \quad i = 1, 2, \ldots, m-1,$$

let Θ_i be the $(N+1) \times (N+1)$ Toeplitz-like matrix

$$\Theta_i = \begin{pmatrix} 0 & a_{i1} & a_{i2} & \cdots & a_{i,N-1} & a_{iN} \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{iN} & 0 \\ a_{i2} & a_{i3} & a_{i4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{iN} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i = 1, 2, \ldots, m-1,$$

and

$$\Theta = \sum_{i=1}^{m-1} \Theta_i \overline{\Theta_i} + I_{N+1}.$$

Assume also that B_i is the first column of Θ_i, $i = 1, 2, \ldots, m-1$, and $B_m = (1, 0, \ldots, 0)^T$. Let $X_j = (x_{j0}, x_{j1}, \ldots, x_{jN})^T$ be the solution of the following linear system of algebraic equations:

$$\Theta \cdot X = B_j$$

(Θ is positive definite and has a *displacement structure of rank* m; therefore $O(mN^2)$ operations are required for its solution instead of traditional $O(N^3)$) and

$$v_{mj}(z) = \sum_{k=0}^{N} x_{jk} z^{-k}, \quad j = 1, 2, \ldots, m,$$

$$v_{ij}(z) = \left[\widetilde{\zeta}_i(z) v_{mj}(z)\right]_N^+ - \delta_{ij}, \quad i = 1, 2, \ldots, m-1,$$

where $[\,\cdot\,]_N^+$ stands for the projection operator,

$$\left[\sum_{k=-\infty}^{\infty} c_k z^k\right]_N^+ = \sum_{k=0}^{N} c_k z^k.$$

Then construct the $(m \times m)$ Laurent polynomial matrix

$$V(z) = (v_{ij}(z))_{i,j=1}^m,$$

and

$$U(z) = V(z)(V(1))^{-1}$$

will be the desired matrix polynomial (23). For a given $U(z)$, the corresponding $F(z)$ can also be explicitly computed as follows: if $u_{mj}(0) \neq 0$, then

$$\zeta_i(z) = \left[\frac{\widetilde{u}_{ij}(z)}{u_{mj}(z)}\right]_N^-, \quad i = 1, 2, \ldots, m-1,$$

where $[\,\cdot\,]_N^-$ stands for the projection operator,

$$\left[\sum_{k=-\infty}^{\infty} c_k z^k\right]_N^- = \sum_{k=-N}^{1} c_k z^k,$$

and under $1/u_{mj}$ a formal series expansion in a neighborhood of 0 is assumed. In consequence, we have a complete parametrization of compact wavelet matrices, i.e., one-to-one maps that can be effectively realized

$$\mathcal{W}_1(m, N, N) \longleftrightarrow \mathcal{U}_1(m, N, N) \longleftrightarrow \underbrace{\mathbb{C}^N \times \mathbb{C}^N \times \cdots \times \mathbb{C}^N}_{m-1}.$$

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