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Non-abelian Cohomology and Extensions of Lie Algebras

N. Inassaridze, E. Khmaladze, and M. Ladra *

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Abstract. We introduce the second non-abelian cohomology of Lie algebras with coefficients in crossed modules using the generalised notion of Lie algebra of derivations, and establish a relationship of this cohomology with Lie algebra extensions by crossed modules. We extend the seven-term exact non-abelian cohomology sequence of Guin to a nine-term exact sequence.
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1. Introduction

From the 1960s, many authors have attempted to answer the question of what we should mean by non-abelian cohomology of various algebraic structures (see [6, 7, 18, 19]). A convincing answer for groups and Lie algebras was given by Guin [9, 10] in the 1980s. More recently, in the case of groups, H. Inassaridze [13, 14, 15] has demonstrated how Guin's definition can be naturally extended to higher dimensions. His non-abelian cohomology theory differs from that of Serre [19] and from the setting of various papers on non-abelian cohomology of groups [4, 5, 6].

In [17], using the non-abelian tensor product of Lie algebras of Ellis [8] and its non-abelian left derived functors, we introduced and investigated the non-abelian homology of Lie algebras, generalising the classical Chevalley-Eilenberg homology and Guin's low dimensional non-abelian homology of Lie algebras [10].

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In that paper we gave an application to cyclic homology. Namely, we established a relation of cyclic homology with Milnor cyclic homology of non-commutative associative algebras in terms of the long exact non-abelian homology sequence of Lie algebras.

This work is a continuation of [17] for setting up a theory of non-abelian (co)homology of Lie algebras similar to the non-abelian (co)homology theory of groups [9, 13, 14, 15, 16, 11]. Our aim is to construct the second non-abelian cohomology of Lie algebras, extending the zero and the first non-abelian cohomology of Guin [10], and to describe it by extensions of Lie algebras.

Further generalisation of non-abelian cohomology of Lie algebras is possible pursuing the line of [14, 15], that is, to make a definition in any dimension and for a wider class of coefficients.

The rest of the paper is organized as follows. Immediately below, we review some basic notations and conventions. In Section 2, for given Lie algebras P and R, we introduce the notions of P-(pre)crossed R-modules and we also generalise the construction of [10] of Lie algebra of derivations. In Section 3, following [10] and using ideas from [13], we introduce the second non-abelian cohomology $\mathfrak{H}^2(R,M)$ and the second non-abelian quasi-cohomology $\widetilde{\mathfrak{H}}^2(R,M)$ of a Lie algebra R with coefficients in a crossed R-module (M,μ), which generalise the classical second Chevalley-Eilenberg cohomology of Lie algebras. Then, for a coefficient short exact sequence of crossed R-modules having a linear splitting, we extend the seven-term exact non-abelian cohomology sequence of Guin [10] to a nine-term exact cohomology sequence. In Section 4, we define extensions of Lie algebras by crossed modules and describe the relationship between the pointed set of equivalence classes of extensions and our second non-abelian cohomology in terms of the short exact sequence of pointed sets.

Notations and Conventions. We denote by Λ a unital commutative ring. We use the term Lie algebra to mean a Lie algebra over Λ and $[\ ,\ ]$ to denote the Lie bracket. We use $|\ |$ to denote a coset of a quotient Lie algebra or a quotient set. For any Lie algebra P, $Z(P)$ denotes the center of P. Finally, Λ-mod and $\mathcal{L}ie$ denote the categories of modules and Lie algebras over Λ, respectively.

2. The Lie algebra of derivations

We begin this section by recalling the notion of (pre)crossed modules of Lie algebras (see [8]).

Let P and M be two Lie algebras. An action of P on M is a Λ-bilinear map $P \times M \to M$, $(p,m) \mapsto {}^{p}m$ satisfying the following conditions:

$$ {}^{[p,p']}m = {}^{p}({}^{p'}m) - {}^{p'}({}^{p}m), \qquad {}^{p}[m,m'] = [{}^{p}m,m'] + [m,{}^{p}m'] $$

for all $m,\ m' \in M$ and $p,\ p' \in P$. In other words, the action of P on M is a

Lie homomorphism $P \to \mathrm{Der}(M)$ to the Lie algebra of derivations of M. For example, if P is a subalgebra of some Lie algebra Q, and if M is an ideal in Q, then Lie multiplication in Q yields an action of P on M.

A precrossed P-module (M, μ) is a Lie homomorphism $\mu : M \to P$ together with an action of P on M satisfying the following condition:

$$\mu({}^p m) = [p, \mu(m)] \ .$$

If in addition the precrossed module (M, μ) satisfies the Peiffer identity:

$$^{\mu(m)}m' = [m, m'] \ ,$$

then it is said to be a crossed P-module. Note that for a crossed module (M, μ) the image of μ is necessarily an ideal in P and the kernel of μ is a P-invariant ideal in the center of M. Moreover, the action of P on $\mathrm{Ker}\,\mu$ induces an action of $P/\,\mathrm{Im}\,\mu$ on $\mathrm{Ker}\,\mu$, making $\mathrm{Ker}\,\mu$ a $P/\,\mathrm{Im}\,\mu$-module.

A morphism $f : (M, \mu) \to (N, \nu)$ of P-(pre)crossed modules is a Lie homomorphism $f : M \to N$ such that $f({}^p m) = {}^p f(m)$ and $\mu = \nu f$, $p \in P$, $m \in M$.

The following extended notion of (pre)crossed modules will be very useful in what follows.

Definition 2.1. Let P and R be Lie algebras acting on each other. A (pre)crossed R-module (M, μ) will be called a P-(pre)crossed R-module if the following conditions hold:

(i) $^{({}^r p)}r' = [r', {}^p r]$, $r, r' \in R$, $p \in P$;

(ii) P acts on M and μ is a P-equivariant Lie homomorphism, i.e.,

$$\mu({}^p m) = {}^p\mu(m) \ , \quad m \in M \ , \ \ p \in P \ ;$$

(iii) P and R act compatibly on M, that is,

$$^{({}^p r)}m = {}^p({}^r m) - {}^r({}^p m) = -{}^{({}^r p)}m \ , \quad r \in R \ , \ \ p \in P \ , \ \ m \in M \ .$$

Remark that any (pre)crossed P-module in a natural way can be thought as a P-(pre)crossed P-module, P acting on itself by Lie multiplication.

A morphism $f : (M, \mu) \to (N, \nu)$ of P-(pre)crossed R-modules is a morphism of (pre)crossed R-modules such that $f({}^p m) = {}^p f(m)$, $p \in P$, $m \in M$.

The rest of this section is devoted to construct a Lie algebra of derivations from P to (M, μ), $\mathrm{Der}(P, (M, \mu))$, for a given P-precrossed R-module (M, μ). This Lie algebra will be endowed with a P-precrossed R-module structure.

Definition 2.2. Let (M,μ) be a P-crossed R-module. Denote by $\mathrm{Der}(P,(M,\mu))$ the set of pairs (γ, r), where $\gamma : P \to M$ is a derivation, that means γ is a Λ-homomorphism satisfying the equality

$$\gamma([p,q]) = {}^p\gamma(q) - {}^q\gamma(p)\ , \quad p,q \in P$$

and r is an element of R such that

$$\mu\gamma(p) = -{}^p r, \ \ p \in P. \tag{1}$$

This set will be called the set of derivations from P to (M,μ).

Proposition 2.3. *The set $Der(P,(M,\mu))$ becomes a Lie algebra with the following operations*

$$\begin{aligned}&(\gamma, r) + (\gamma', s) = (\gamma+\gamma', r+s),\\&\lambda(\gamma, r) = (\lambda\gamma, \lambda r),\\&[(\gamma, r), (\gamma', s)] = (\gamma * \gamma', [r,s]),\end{aligned}$$

for all $(\gamma,r),(\gamma',s) \in Der(P,(M,\mu))$ and $\lambda \in \Lambda$, where $\gamma * \gamma'$ is given by $(\gamma*\gamma')(p) = \gamma({}^s p) - \gamma'({}^r p)$, $p \in P$.

Proof. We show that $(\gamma*\gamma', [r,s]) \in Der(P,(M,\mu))$. First we prove that $\gamma*\gamma'$ is a derivation. In fact,

$$\begin{aligned}&(\gamma*\gamma')([p,q]) = \gamma({}^s[p,q]) - \gamma'({}^r[p,q]) = \gamma([{}^s p,q]) + \gamma([p,{}^s q]) - \gamma'([{}^r p,q]) - \gamma'([p,{}^r q])\\&= {}^{({}^s p)}\gamma(q) - {}^q\gamma({}^s p) + {}^p\gamma({}^s q) - {}^{({}^s q)}\gamma(p) - {}^{({}^r p)}\gamma'(q) + {}^q\gamma'({}^r p) - {}^p\gamma'({}^r q) + {}^{({}^r q)}\gamma'(p)\ .\end{aligned}$$

On the other hand,

$${}^p(\gamma*\gamma')(q) - {}^q(\gamma*\gamma')(p) = {}^p\gamma({}^s q) - {}^p\gamma'({}^r q) - {}^q\gamma({}^s p) + {}^q\gamma'({}^r p)\ .$$

Moreover, using (iii) of Definition 2.1 and (1) we have

$${}^{({}^s p)}\gamma(q) - {}^{({}^s q)}\gamma(p) - {}^{({}^r p)}\gamma'(q) + {}^{({}^r q)}\gamma'(p) = 0\ .$$

Hence $(\gamma*\gamma')([p,q]) = {}^p(\gamma*\gamma')(q) - {}^q(\gamma*\gamma')(p)$.

Furthermore, by (i) of Definition 2.1 we have

$$\mu(\gamma*\gamma')(p) = \mu\gamma({}^s p) - \mu\gamma'({}^r p) = -{}^{({}^s p)}r + {}^{({}^r p)}s = -[r, {}^p s] + [s, {}^p r] = -{}^p[r,s]\ .$$

Thus $(\gamma*\gamma', [r,s]) \in \mathrm{Der}(P,(M,\mu))$.

The remaining details are easy to check and left to the reader. ∎

Remark. *Let* (N,ν) *and* (M,μ) *be precrossed and crossed* R*-modules, respectively. Then* (M,μ) *is an* N*-crossed* R*-module, where the actions of* N *on* M *and on* R *are induced via* ν*, and* $\mathrm{Der}(N,(M,\mu))$ *coincides with the Lie algebra* $\mathrm{Der}_R(N,M)$ *defined in [10]. In particular,* $\mathrm{Der}(R,(M,\mu))$ *coincides with the Lie algebra* $\mathrm{Der}_R(R,M)$ *from [10] when* (M,μ) *is viewed as an* R*-crossed* R*-module.*

Suppose at the same time (M,μ) is P-crossed R-module and P'-crossed R-module and $f: P' \to P$ is a Lie homomorphism such that

$$^{f(p')}m = {}^{p'}m\ , \quad {}^{f(p')}r = {}^{p'}r\ , \quad p' \in P'\ , r \in R\ , m \in M.$$

Then there is a Λ-homomorphism

$$\overline{f} : \mathrm{Der}(P,(M,\mu)) \longrightarrow \mathrm{Der}(P',(M,\mu))$$

given by $\overline{f}(\gamma,r) = (\gamma f, r)$, $(\gamma,r) \in \mathrm{Der}(P,(M,\mu))$. If in addition f satisfies the condition

$$f({}^{r}p') = {}^{r}f(p'), \quad p' \in P', \ r \in R,$$

then $\overline{f}$ is a Lie homomorphism.

Now assume that P and R are Lie algebras acting on each other compatibly, that is, the following conditions hold:

$$^{({}^{r}p)}r' = [r', {}^{p}r] \qquad \text{and} \qquad {}^{({}^{p}r)}p' = [p', {}^{r}p] \tag{2}$$

for all $p,p' \in P$ and $r,r' \in R$. Let (M,μ) be a P-crossed R-module, then there is an action of P on $\mathrm{Der}(P,(M,\mu))$ defined by

$$^{p}(\gamma,r) = (\gamma', {}^{p}r)\ , \ \ p \in P, \ (\gamma,r) \in \mathrm{Der}(P,(M,\mu))\ , \tag{3}$$

where $\gamma'(q) = {}^{q}\gamma(p)$, $q \in P$. There is also an action of R on $\mathrm{Der}(P,(M,\mu))$ given by

$$^{s}(\gamma,r) = (\gamma'', [s,r])\ , \ \ s \in R, \ (\gamma,r) \in \mathrm{Der}(P,(M,\mu))\ , \tag{4}$$

where $\gamma''(q) = {}^{s}\gamma(q) - \gamma({}^{s}q)$, $q \in P$. It is routine to show that the elements $(\gamma', {}^{p}r)$ and $(\gamma'', [s,r])$ belong to $\mathrm{Der}(P,(M,\mu))$ and that (3), (4) define Lie actions.

Proposition 2.4. *Let* (M,μ) *be a* P*-crossed* R*-module and the actions of* P *and* R *on each other satisfy the compatibility conditions (2). Then the Lie homomorphism* $\xi : \mathrm{Der}(P,(M,\mu)) \to R$ *given by* $(\gamma,r) \mapsto r$ *with the aforementioned actions of* P *and* R *on* $\mathrm{Der}(P,(M,\mu))$ *is a* P*-precrossed* R*-module.*

Proof. We only show the following equality

$$^{({}^{p}r)}(\gamma,s) = {}^{p}({}^{r}(\gamma,s)) - {}^{r}({}^{p}(\gamma,s))$$

for all $r \in R$, $p \in P$ and $(\gamma,s) \in \mathrm{Der}(P,(M,\mu))$.

In fact,

$$^{(^pr)}(\gamma, s) = (\gamma'', [^pr, s]) \ ,$$

where

$$\begin{aligned}\gamma''(q) &= {}^{(^pr)}\gamma(q) - \gamma({}^{(^pr)}q) = {}^{(^pr)}\gamma(q) - \gamma([q, {}^rp]) \\ &= {}^{(^pr)}\gamma(q) - {}^q\gamma({}^rp) + {}^{(^rp)}\gamma(q) = -{}^q\gamma({}^rp) \ .\end{aligned}$$

On the other hand,

$$^p(^r(\gamma, s)) - {}^r(^p(\gamma, s)) = (\gamma_1, {}^p[r, s]) - (\gamma_2, [r, {}^p s]) = (\gamma_1 - \gamma_2, [^pr, s]) \ ,$$

where $(\gamma_1 - \gamma_2)(q) = {}^q({}^r\gamma(p) - \gamma({}^rp)) - {}^r({}^q\gamma(p)) + {}^{(^rq)}\gamma(p) = -{}^q\gamma({}^rp)$.

The remaining details are omitted as they are routine. ■

3. Second non-abelian cohomology

Before introducing our original definition of the second non-abelian cohomology of Lie algebras we recall the definitions of Guin [10] of the zero and the first non-abelian cohomology of Lie algebras with coefficients in crossed modules.

Let R be a Lie algebra and (M, μ) a crossed R-module. The zero non-abelian cohomology is the ideal of M of all R-invariant elements,

$$\mathfrak{H}^0(R, M) = \{m \in M \mid {}^rm = 0 \ \text{ for all } \ r \in R\} \ .$$

The crossed module relation ${}^{\mu(m)}m' = [m, m']$ implies that $\mathfrak{H}^0(R, M)$ is contained in the center of M and therefore has only a Λ-module structure.

The first non-abelian cohomology is the Lie algebra defined by

$$\mathfrak{H}^1(R, M) = \mathrm{Der}_R(R, M)/\mathfrak{I} \ ,$$

where $\mathfrak{I}$ is the following ideal of the Lie algebra $\mathrm{Der}_R(R, M)$:

$$\mathfrak{I} = \{(\eta_m, -\mu(m) + c) \mid m \in M, c \in Z(R)\} \ ,$$

η_m is the principal derivation induced by m, namely $\eta_m(x) = {}^xm$, $x \in R$.

We need the following characterization of the classical second Chevalley-Eilenberg cohomology $H^2(R, M)$ of the Lie algebra R with coefficients in an R-module M. We assume the reader is familiar with cotriples, projective classes and non-abelian derived functors relative to cotriples and projective classes. See, for example, [12, Chapter 2] or [20, Section 8.6].

Let us consider the diagram of Lie algebras

$$P \underset{d_1}{\overset{d_0}{\rightrightarrows}} F \xrightarrow{\epsilon} R \ , \qquad (5)$$

where F is a free Lie algebra over some Λ-module, ϵ is a Lie homomorphism having a Λ-linear splitting and (P, d_0, d_1) is a simplicial kernel of ϵ in the category $\mathcal{L}ie$, i.e., $P = \{(x, y) \in F \times F \mid \epsilon(x) = \epsilon(y)\}$, $d_0(x, y) = x$ and $d_1(x, y) = y$. Suppose Δ denotes the Lie subalgebra $\{(x, x) \in F \times F \mid x \in F\}$ of P.

Let M be an R-module, and view M as F and P-modules via the Lie homomorphisms ϵ and ϵd_i $(i = 0,\ 1)$, respectively. Denote by $\mathrm{Der}(P, M)$ (resp. $\mathrm{Der}(F, M)$) the Λ-module of derivations from P to M (resp. from F to M). Let $\widetilde{\mathrm{Der}}(P, M)$ be the submodule of $\mathrm{Der}(P, M)$ of all derivations γ such that $\gamma(\Delta) = 0$. There is a Λ-homomorphism

$$\kappa : \mathrm{Der}(F, M) \longrightarrow \widetilde{\mathrm{Der}}(P, M),$$

given by $\beta \mapsto \beta d_0 - \beta d_1$.

Proposition 3.1. *There is a natural isomorphism*

$$H^2(R, M) \cong \mathrm{Coker}\,\kappa\ .$$

Proof. Let $\mathfrak{A}_M$ denote the category whose objects are all Lie algebras N together with an action of N on M and morphisms are Lie homomorphisms preserving the actions. There is a cotriple $\mathbb{F} = (\mathcal{F}, \tau, \delta)$ on the category $\mathfrak{A}_M$ (see also [17]), where $\mathcal{F} : \mathfrak{A}_M \to \mathfrak{A}_M$ is the endofunctor defined for every object $N \in \mathfrak{A}_M$ to be the free Lie algebra $\mathcal{F}(N)$ on the underlying Λ-module N with an action of $\mathcal{F}(N)$ on M induced by the action of N on M. The natural transformation $\tau : \mathcal{F} \to 1_{\mathfrak{A}_M}$ is the obvious and $\delta : \mathcal{F} \to \mathcal{F}^2$ is the natural transformation induced for every $N \in \mathfrak{A}_M$ by the inclusion $N \to \mathcal{F}(N)$ of Λ-modules. Let $\mathbb{P}$ denote the projective class on $\mathfrak{A}_M$ induced by the cotriple $\mathbb{F}$.

It is well known that the classical Chevalley-Eilenberg cohomology of the Lie algebra R with coefficients in the R-module M is isomorphic, up to dimension shift, to the non-abelian right derived functors $\mathcal{R}^k_{\mathbb{P}}\mathrm{Der}(-, M)(R)$, $k \geq 0$, of the contravariant functor $\mathrm{Der}(-, M) : \mathfrak{A}_M \to \Lambda\text{-mod}$ relative to the projective class $\mathbb{P}$ (cf. [17, Proposition 4]). The proof is given in [1] and it is similar to the case of group cohomology and Hochschild cohomology described as cotriple cohomology [2, 3]. Hence we only need to construct an isomorphism of Λ-modules

$$\mathcal{R}^1_{\mathbb{P}}\mathrm{Der}(-, M)(R) \cong \mathrm{Coker}\,\kappa\ .$$

Let us consider a $\mathbb{P}$-projective simplicial resolution of the object R in the category $\mathfrak{A}_M$

$$\cdots \; \vdots \; F_2 \underset{d_2^2}{\overset{d_0^2}{\rightrightarrows}} F_1 \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} F_0 \xrightarrow{\ \epsilon\ } R\ , \qquad F_1 \xrightarrow{\ d\ } P \underset{d_1}{\overset{d_0}{\rightrightarrows}} F_0 \tag{6}$$

where $F_0 = F$ and d is the unique Lie epimorphism such that $d_i^1 = d_i d$ $(i = 0,\ 1)$.

Applying the functor $\mathrm{Der}(-, M)$ to (6) yields a cochain complex of Λ-modules

$$\mathrm{Der}(F_0, M) \xrightarrow{\partial_0} \mathrm{Der}(F_1, M) \xrightarrow{\partial_1} \mathrm{Der}(F_2, M) \xrightarrow{\partial_2} \cdots ,$$

where $\partial_i = \sum\limits_{j=0}^{i+1} (-1)^j \,\mathrm{Der}(-, M)(d_j^{i+1})$.

Now define a Λ-homomorphism

$$\varphi : \widetilde{\mathrm{Der}}(P, M) \longrightarrow \mathrm{Ker}\,\partial_1$$

by $\varphi(\gamma) = \gamma d$, $\gamma \in \widetilde{\mathrm{Der}}(P, M)$. To show that the derivation $\gamma d : F_1 \to M$ belongs to $\mathrm{Ker}\,\partial_1$, we only need to examine the following lemma.

Lemma 3.2. *For $\gamma \in \widetilde{\mathrm{Der}}(P, M)$ there is an equality*

$$\gamma(x, y) = \gamma(x, z) + \gamma(z, y)$$

for all $x, y, z \in F$ such that $\epsilon(x) = \epsilon(y) = \epsilon(z)$.

Proof. Straightforward. ∎

Returning to the main proof, construct a Λ-homomorphism

$$\psi : \mathrm{Ker}\,\partial_1 \longrightarrow \widetilde{\mathrm{Der}}(P, M)$$

by $\psi(\beta) = \gamma$, $\beta \in \mathrm{Ker}\,\partial_1$, where the map $\gamma : P \to M$ is given by $\gamma(x, y) = \beta(z)$, $(x, y) \in P$ and $z \in F_1$ such that $d(z) = (x, y)$.

We have to show that γ is correctly defined. In fact, let $z' \in F_1$ such that $d(z') = (x, y)$. Then there exists an element $w \in F_2$ such that $d_0^2(w) = d_1^2(w) = 0$ and $d_2^2(w) = z - z'$. Hence,

$$\beta(z) - \beta(z') = \beta d_2^2(w) = \partial_1(\beta)(w) = 0 .$$

It is easy to show that γ is a derivation and $\psi\varphi$, $\varphi\psi$ are identity maps. Moreover, it is clear that the above-given isomorphism induces the isomorphism $H^2(R, M) \cong \mathrm{Coker}\,\kappa$. ∎

Now we are ready to construct our second non-abelian cohomology of Lie algebras.

Suppose that, in diagram (5), R acts on F, ϵ preserves the actions (here we mean that R acts on itself by Lie multiplication), implying the induced action of R on P. Note that all these conditions are satisfied when $F = \mathcal{F}(R)$ is the free

Lie algebra on the underlying Λ-module R with the canonical Lie homomorphism $\epsilon : \mathcal{F}(R) \to R$ and the action of R on $\mathcal{F}(R)$ induced by the action of R on itself. For the detailed construction of this action we refer to [17]. Hence, with no loss of generality, we can assume that $Z(R)$ acts trivially on F.

Let (M, μ) be a crossed R-module. Then (M, μ) can be viewed as a P-crossed R-module induced by ϵd_i $(i = 0,\ 1)$ and an F-crossed R-module induced by ϵ. Denote by $\widetilde{\mathrm{Der}}(P, (M, \mu))$ the subset of $\mathrm{Der}(P, (M, \mu)$ consisting of all elements of the form $(\gamma, 0)$ satisfying the condition $\gamma(\Delta) = 0$. Clearly $\widetilde{\mathrm{Der}}(P, (M, \mu))$ is a Λ-submodule of $\mathrm{Der}(P, (M, \mu))$, since in it the Lie multiplication of $\mathrm{Der}(P, (M, \mu))$ is killed.

Let us consider the Λ-submodule $B(P, (M, \mu))$ of $\widetilde{\mathrm{Der}}(P, (M, \mu))$ consisting of all elements $(\gamma, 0)$ for which there exists $(\beta, h) \in \mathrm{Der}(F, (M, \mu))$ such that $\beta d_0 - \beta d_1 = \gamma$. We also need the Λ-submodule $\widetilde{B}(P, (M, \mu))$ of $B(P, (M, \mu))$ consisting of all elements $(\gamma, 0) \in B(P, (M, \mu))$ for which the existing $(\beta, h) \in \mathrm{Der}(F, (M, \mu))$ satisfies the additional condition $\mathrm{Im}\, \beta \subseteq Z(M)$.

Proposition 3.3. *Let R be a Lie algebra and (M, μ) a crossed R-module. Then the Λ-modules*

$$\widetilde{\mathrm{Der}}(P, (M, \mu))/B(P, (M, \mu)) \quad \text{and} \quad \widetilde{\mathrm{Der}}(P, (M, \mu))/\widetilde{B}(P, (M, \mu))$$

are unique up to isomorphisms of the choice of the diagram (5) for the crossed R-module (M, μ).

Proof. Consider the diagram of Lie algebras

$$\begin{array}{ccccc} P & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & F & \xrightarrow{\ \epsilon\ } & R \\ {\scriptstyle \omega_1}\downarrow & & \downarrow{\scriptstyle v_1} & & \| \\ P' & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & F' & \xrightarrow{\ \epsilon\ } & R\ , \end{array}$$

where the bottom row is another diagram of the form (5), and $\epsilon v_1 = \epsilon$, $d_i \omega_1 = v_1 d_i$, $i = 0, 1$. The existence of such v_1 and ω_1, not preserving the actions of R in general, is clear. Suppose that there exists another $v_2 : F \to F'$ and $\omega_2 : P \to P'$ satisfying the conditions $\epsilon v_2 = \epsilon$ and $d_i \omega_2 = v_2 d_i$, $i = 0, 1$.

As it is indicated in Section 2 we have the induced Λ-homomorphisms which will be denoted by $\overline{\omega}_i : \mathrm{Der}(P', (M, \mu)) \to \mathrm{Der}(P, (M, \mu))$, $\overline{\omega}_i(\gamma, r) = (\gamma\omega_i, r)$, $i = 1, 2$.

It is easy to see that $(\gamma\omega_i, 0) \in \widetilde{\mathrm{Der}}(P, (M, \mu))$ if $(\gamma, 0) \in \widetilde{\mathrm{Der}}(P', (M, \mu))$. Let $(\gamma, 0) \in B(P', (M, \mu))$, i.e., there exists $(\beta, h) \in \mathrm{Der}(F', (M, \mu))$ such that $\beta d_0 - \beta d_1 = \gamma$, then $(\beta v_i, h) \in \mathrm{Der}(F, (M, \mu))$ and

$$\gamma\omega_i = (\beta d_0 - \beta d_1)\omega_i = \beta v_i d_0 - \beta v_i d_1\ ,\ \ i = 1, 2\ .$$

Thus $(\gamma\omega_i, 0) \in B(P, (M, \mu))$. Moreover, it is clear that if $(\gamma, 0) \in \widetilde{B}(P', (M, \mu))$ then $(\gamma\omega_i, 0) \in \widetilde{B}(P, (M, \mu))$, $i = 1, 2$. Hence we have the natural homomorphisms of Λ-modules

$$\chi_i : \widetilde{\mathrm{Der}}(P', (M, \mu))/B(P', (M, \mu)) \longrightarrow \widetilde{\mathrm{Der}}(P, (M, \mu))/B(P, (M, \mu)) \;,$$
$$\widetilde{\chi}_i : \widetilde{\mathrm{Der}}(P', (M, \mu))/\widetilde{B}(P', (M, \mu)) \longrightarrow \widetilde{\mathrm{Der}}(P, (M, \mu))/\widetilde{B}(P, (M, \mu)) \;,$$

induced by $\overline{\omega}_i$, $i = 1, 2$.

Now we show that $\chi_1 = \chi_2$ and $\widetilde{\chi}_1 = \widetilde{\chi}_2$. Take the Lie homomorphism $s : F \to P'$ given by $s(x) = (v_1(x), v_2(x))$. For $(\gamma, 0) \in \widetilde{\mathrm{Der}}(P', (M, \mu))$ we have $(\gamma s, 0) \in \mathrm{Der}(F, (M, \mu))$ and the equalities

$$\begin{aligned}
&(\gamma s d_0 - \gamma s d_1)(x, y) = \gamma s(x) - \gamma s(y) = \gamma(v_1(x), v_2(x)) - \gamma(v_1(y), v_2(y)) \\
&= \gamma(v_1(x) - v_1(y), v_2(x) - v_2(y)) + \gamma(v_1(y) - v_2(x), v_1(y) - v_2(x)) \\
&= \gamma(v_1(x) - v_2(x), v_1(y) - v_2(y)) = (\gamma\omega_1 - \gamma\omega_2)(x, y)
\end{aligned}$$

for every $(x, y) \in P$. Since $\mu\gamma s = 0$ then $\mathrm{Im}\,\gamma s \subseteq Z(M)$ and therefore $(\gamma\omega_1, 0) - (\gamma\omega_2, 0) \in \widetilde{B}(P, (M, \mu))$ implying $\chi_1 = \chi_2$ and $\widetilde{\chi}_1 = \widetilde{\chi}_2$.

The rest of the proof is standard. ∎

Proposition 3.4. *Let R be a Lie algebra and (M, μ) a crossed R-module.*

(i) There is a canonical epimorphism of Λ-modules

$$\vartheta : H^2(R, \mathrm{Ker}\,\mu) \longrightarrow \widetilde{\mathrm{Der}}(P, (M, \mu))/B(P, (M, \mu)) \;,$$

given by $\vartheta(|\gamma|) = |(\psi(\gamma), 0)|$, $|\gamma| \in H^2(R, \mathrm{Ker}\,\mu)$, where ψ is defined in Proposition 3.1.

(ii) If $h \in Z(R)$ for any element $(\beta, h) \in \mathrm{Der}(F, (M, \mu))$, then ϑ is an isomorphism.

Proof. Directly follows from Proposition 3.1. ∎

Note that the condition of Proposition 3.4 (ii) is fulfilled when either R is an abelian Lie algebra or M is an R-module thought as the crossed R-module $(M, 0)$. This assertion motivates our definition of the second non-abelian cohomology of Lie algebras with coefficients in crossed modules.

Definition 3.5. Let R be a Lie algebra and (M, μ) a crossed R-module. Then the Λ-module $\widetilde{\mathrm{Der}}(P, (M, \mu))/B(P, (M, \mu))$ will be called the second non-abelian cohomology of R with coefficients in (M, μ) and will be denoted by $\mathfrak{H}^2(R, M)$.

Now we introduce the second non-abelian quasi-cohomology of Lie algebras with coefficients in crossed modules. The reason for its introduction lies in the following argument: to find some modification of the second non-abelian cohomology which will be explicitly described in terms of Lie algebra extensions by crossed modules.

Definition 3.6. Let R be a Lie algebra and (M,μ) a crossed R-module. Then the Λ-module $\widetilde{\mathrm{Der}}(P,(M,\mu))/\widetilde{B}(P,(M,\mu))$ will be called the second non-abelian quasi-cohomology of R with coefficients in (M,μ) and will be denoted by $\widetilde{\mathfrak{H}}^2(R,M)$.

Note that given an R-module M we have $\mathfrak{H}^2(R,M) = \widetilde{\mathfrak{H}}^2(R,M) = H^2(R,M)$. Moreover, given a free Lie algebra R over some Λ-module, it is clear that $\mathfrak{H}^2(R,M) = \widetilde{\mathfrak{H}}^2(R,M) = 0$ for any crossed R-module (M,μ).

It is easy to see that $\mathfrak{H}^2(R,M)$ is a functor with respect to both arguments. Namely, for a morphism of crossed R-modules $\theta : (M,\mu) \to (N,\nu)$, there is a Λ-homomorphism

$$\theta^2 : \mathfrak{H}^2(R,M) \longrightarrow \mathfrak{H}^2(R,N)\ , \quad \theta^2(|(\alpha,0)|) = |(\theta\alpha,0)|\ .$$

Whilst $\widetilde{\mathfrak{H}}^2(R,M)$ is a functor only with respect to the first argument. Nevertheless, if θ is a surjective morphism of crossed R-modules, there is a Λ-homomorphism

$$\widetilde{\theta}^2 : \widetilde{\mathfrak{H}}^2(R,M) \longrightarrow \widetilde{\mathfrak{H}}^2(R,N)\ , \quad \widetilde{\theta}^2(|(\alpha,0)|) = |(\theta\alpha,0)|\ .$$

The rest of this section is devoted to obtain a nine-term exact non-abelian cohomology sequence which prolongs Guin's seven-term exact sequence. But, for the exactness, one additional necessary condition on coefficient short exact sequence is needed, not presented in [10, Theorem 2.8].

Theorem 3.7. *Let R be a Lie algebra and*

$$0 \longrightarrow (L,0) \xrightarrow{\xi} (M,\mu) \xrightarrow{\theta} (N,\nu) \longrightarrow 0$$

an exact sequence of crossed R-modules, having a Λ-linear splitting. Then there is an exact sequence of Λ-modules

$$\begin{aligned} 0 \longrightarrow \mathfrak{H}^0(R,L) \xrightarrow{\xi^0} \mathfrak{H}^0(R,M) \xrightarrow{\theta^0} \mathfrak{H}^0(R,N) \xrightarrow{\delta^0} \mathfrak{H}^1(R,L) \xrightarrow{\xi^1} \mathfrak{H}^1(R,M) \\ \xrightarrow{\theta^1} \mathfrak{H}^1(R,N) \xrightarrow{\delta^1} \mathfrak{H}^2(R,L) \xrightarrow{\xi^2} \mathfrak{H}^2(R,M) \xrightarrow{\theta^2} \mathfrak{H}^2(R,N)\ , \end{aligned}$$

where θ^1 is a Lie homomorphism and δ^1 is a derivation with the action of $\mathfrak{H}^1(R,N)$ on $\mathfrak{H}^2(R,L)$ induced by the action of R on P.

Proof. According to [10, Theorem 2.8] there is an exact sequence of Λ-modules

$$0 \longrightarrow \mathfrak{H}^0(R,L) \xrightarrow{\xi^0} \mathfrak{H}^0(R,M) \xrightarrow{\theta^0} \mathfrak{H}^0(R,N) \xrightarrow{\delta^0}$$
$$\mathfrak{H}^1(R,L) \xrightarrow{\xi^1} \mathfrak{H}^1(R,M) \xrightarrow{\theta^1} \mathfrak{H}^1(R,N)\ ,$$

where θ^1 is a Lie homomorphism.

Note that the Λ-linear splitting on coefficient sequence is needed to construct the connecting map δ^1.

We only have to define the derivation δ^1, the action of the Lie algebra $\mathfrak{H}^1(R,N)$ on the Λ-module $\mathfrak{H}^2(R,L)$ (in our setting) and to show the exactness of the following sequence

$$\mathfrak{H}^1(R,M) \xrightarrow{\theta^1} \mathfrak{H}^1(R,N) \xrightarrow{\delta^1} \mathfrak{H}^2(R,L) \xrightarrow{\xi^2} \mathfrak{H}^2(R,M) \xrightarrow{\theta^2} \mathfrak{H}^2(R,N)\ . \qquad (7)$$

Let $|(\alpha,r)| \in \mathfrak{H}^1(R,N)$ and consider the diagram

$$\begin{array}{ccccc} P & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & F & \xrightarrow{\epsilon} & R \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ L & \xrightarrow{\xi} & M & \xrightarrow{\theta} & N, \end{array} \qquad (8)$$

where $\beta : F \to M$ is a derivation such that $\theta\beta = \alpha\epsilon$. The existence of β follows from the following fact: let F be a free Lie algebra (over some Λ-module X) acting on a Lie algebra M, then any Λ-linear map from X to M can be naturally extended to a derivation from F to M.

Then there exists a (unique) derivation $\gamma : P \to L$ such that $\xi\gamma = \beta d_0 - \beta d_1$. It is clear that $\gamma(\Delta) = 0$. Define

$$\delta^1|(\alpha,r)| = |(\gamma,0)|\ .$$

We have to check that δ^1 is well defined. Let $\beta' : F \to M$ be another derivation such that $\theta\beta' = \alpha\epsilon$ and $\gamma' : P \to L$ be the induced derivation satisfying $\xi\gamma' = \beta' d_0 - \beta' d_1$. Then $\theta\beta' = \theta\beta$ and there is a derivation $\sigma : F \to L$ such that $\beta' = \beta + \xi\sigma$. Thus we have

$$\xi\gamma' = \beta' d_0 - \beta' d_1 = \beta d_0 + \xi\sigma d_0 - \beta d_1 - \xi\sigma d_1 = \xi\gamma + \xi\sigma d_0 - \xi\sigma d_1\ ,$$

implying $|(\gamma,0)| = |(\gamma',0)|$.

Now, if (α',r') is another representative of the class $|(\alpha,r)|$, then there exists $n \in N$ such that $\alpha' = \alpha + \eta_n$. Take $\beta' : F \to M$ such that $\beta' = \beta + \eta_m$, where $m \in M$ with $\theta(m) = n$ and $\theta\beta = \alpha\epsilon$. It is clear that $\theta\beta' = \alpha'\epsilon$. Moreover,

$$\xi\gamma' = \beta' d_0 - \beta' d_1 = \beta d_0 + \eta_m d_0 - \beta d_1 - \eta_m d_1 = \beta d_0 - \beta d_1 = \xi\gamma\ .$$

Whence $\gamma' = \gamma$ and the connecting map δ^1 is well defined.

Now we define the action of $\mathfrak{H}^1(R,N)$ on $\mathfrak{H}^2(R,L)$ by the formula

$$^{|(\alpha,r)|}|(\gamma,0)| = |(\widetilde{\gamma},0)| \;, \quad |(\alpha,r)| \in \mathfrak{H}^1(R,N) \;, \;\; |(\gamma,0)| \in \mathfrak{H}^2(R,L) \;,$$

where $\widetilde{\gamma}(x,y) = \gamma({}^r x, {}^r y)$, $(x,y) \in P$. The following equality in the Lie algebra $\mathrm{Der}(P,(M,\mu))$

$$(\xi\widetilde{\gamma},0) = [(\xi\gamma,0),(\beta d_0, r)] \;,$$

where $\beta : F \to M$ is a derivation as in diagram (8), implies that $\widetilde{\gamma} : P \to L$ is a derivation. Furthermore, it is obvious that $\widetilde{\gamma}(\Delta) = 0$. We have to check that this action is well defined. Suppose $|(\alpha',r')| = |(\alpha,r)| \in \mathfrak{H}^1(R,N)$, hence $\alpha' = \alpha - \eta_n$ and $r' = r + \nu(n) - c$ for some $n \in N$ and $c \in Z(R)$. We have

$$\gamma({}^{r'}x, {}^{r'}y) = \gamma({}^r x, {}^r y) + \gamma({}^{\nu(n)}x, {}^{\nu(n)}y) - \gamma({}^c x, {}^c y) \;.$$

As it is mentioned above we can assume, with no loss of generality, that $Z(R)$ acts trivially on F, hence $\gamma({}^c x, {}^c y) = \gamma(0,0) = 0$. Now, from the following lemma, we can deduce that this action is well defined.

Lemma 3.8. *The map $\overline{\beta} : F \to L$ given by $\overline{\beta}(x) = \gamma({}^{\nu(n)}x, [u\nu(n),x])$ is a derivation, where $u : R \to F$ is the required Λ-linear splitting, and the following equality holds:*

$$\gamma({}^{\nu(n)}x, {}^{\nu(n)}y) = (\overline{\beta}d_0 - \overline{\beta}d_1)(x,y) \;, \quad (x,y) \in P \;.$$

Proof. To show that $\overline{\beta}$ is a derivation we make the following calculations:

$$\begin{aligned}
&{}^x\overline{\beta}(y) - {}^y\overline{\beta}(x) = {}^x\gamma({}^{\nu(n)}y, [u\nu(n),y]) - {}^y\gamma({}^{\nu(n)}x, [u\nu(n),x]) \\
&= {}^{(x,x)}\gamma({}^{\nu(n)}y, [u\nu(n),y]) - {}^{(y,y)}\gamma({}^{\nu(n)}x, [u\nu(n),x]) \\
&= \gamma[(x,x),({}^{\nu(n)}y, [u\nu(n),y])] - \gamma[(y,y),({}^{\nu(n)}x, [u\nu(n),x])] \\
&= \gamma([x,{}^{\nu(n)}y],[x,[u\nu(n),y]]) - \gamma([y,{}^{\nu(n)}x],[y,[u\nu(n),x]]) \\
&= \gamma({}^{\nu(n)}[x,y],[u\nu(n),[x,y]]) = \overline{\beta}[x,y] \;.
\end{aligned}$$

Let $m \in M$ such that $\theta(m) = n$. Then

$$\begin{aligned}
&\gamma([u\nu(n),x],[u\nu(n),y]) = \gamma[(u\nu(n),u\nu(n)),(x,y)] \\
&= {}^{\nu(n)}\gamma(x,y) - {}^{(x,y)}\gamma(u\nu(n),u\nu(n)) = {}^{\mu(m)}\gamma(x,y) = [m,\gamma(x,y)] = 0 \;,
\end{aligned}$$

since L is contained in the center of M.

Thus by Lemma 3.2 we have

$$\begin{aligned}
&(\overline{\beta}d_0 - \overline{\beta}d_1)(x,y) = \overline{\beta}(x) - \overline{\beta}(y) = \gamma({}^{\nu(n)}x,[u\nu(n),x]) - \gamma({}^{\nu(n)}y,[u\nu(n),y]) \\
&= \gamma({}^{\nu(n)}x,[u\nu(n),x]) + \gamma([u\nu(n),x],[u\nu(n),y]) + \gamma([u\nu(n),y],{}^{\nu(n)}y) \\
&= \gamma({}^{\nu(n)}x,{}^{\nu(n)}y) \;.
\end{aligned}$$

■

It remains to prove the exactness of the sequence (7).

Let $|(\lambda, r)| \in \mathfrak{H}^1(R, M)$. Then $\delta^1\theta^1|(\lambda, r)| = \delta^1|(\theta\lambda, r)| = |(\gamma, 0)|$, where $\xi\gamma = \lambda\epsilon d_0 - \lambda\epsilon d_1 = 0$. Therefore $\operatorname{Im}\theta^1 \subseteq \operatorname{Ker}\delta^1$.

Let $|(\alpha, r)| \in \mathfrak{H}^1(R, N)$ such that $\delta^1|(\alpha, r)| = |(\gamma, 0)| = 0$, where $\xi\gamma = \beta d_0 - \beta d_1$ (see diagram (8)). Then there exists a derivation $\eta : F \to L$ satisfying $\gamma = \eta d_0 - \eta d_1$. Hence we get $(\beta - \xi\eta)d_0 = (\beta - \xi\eta)d_1$ implying the existence of $(\overline{\alpha}, r) \in \operatorname{Der}_R(R, M)$ with $\beta - \xi\eta = \overline{\alpha}\epsilon$. It is obvious that $\theta^1|(\overline{\alpha}, r)| = |(\alpha, r)|$. Hence $\operatorname{Ker}\delta^1 \subseteq \operatorname{Im}\theta^1$.

Let $|(\alpha, r)| \in \mathfrak{H}^1(R, N)$, then $\xi^2\delta^1|(\alpha, r)| = \xi^2|(\gamma, 0)| = |(\xi\gamma, 0)| = 0$, since there exists $(\beta, r) \in \operatorname{Der}(F, (M, \mu))$ such that $\xi\gamma = \beta d_0 - \beta d_1$. Therefore $\operatorname{Im}\delta^1 \subseteq \operatorname{Ker}\xi^2$.

Let $|(\gamma, 0)| \in \mathfrak{H}^2(R, L)$ such that $|(\xi\gamma, 0)| = 0 \in \mathfrak{H}^2(R, M)$. Then there exists $(\beta, s) \in \operatorname{Der}(F, (M, \mu))$ such that $\xi\gamma = \beta d_0 - \beta d_1$, whence $\theta\beta d_0 = \theta\beta d_1$. It follows that there is a unique derivation $\alpha : R \to N$ such that $\alpha\epsilon = \theta\beta$. It is easy to check that the pair (α, s) belongs to $\operatorname{Der}_R(R, N)$ and $\delta^1|(\alpha, s)| = |(\gamma, 0)|$. Therefore $\operatorname{Ker}\xi^2 \subseteq \operatorname{Im}\delta^1$.

It is obvious that $\operatorname{Im}\xi^2 \subseteq \operatorname{Ker}\theta^2$.

Let $|(\lambda, 0)| \in \mathfrak{H}^2(R, M)$ and $\theta^2(|(\lambda, 0)|) = |(\theta\lambda, 0)| = 0$. Then there exists $(\rho, h) \in \operatorname{Der}(F, (N, \nu))$ such that $\theta\lambda = \rho d_0 - \rho d_1$. Consider a derivation $\beta : F \to M$ such that $\theta\beta = \rho$. One can easily check that $(\beta, h) \in \operatorname{Der}(F, (M, \mu))$. The equality $\theta\lambda = \theta\beta d_0 - \theta\beta d_1$ implies $\operatorname{Im}(\lambda + \beta d_1 - \beta d_0) \subseteq \xi(L)$. Then the derivation $\gamma : P \to L$ given by $\xi\gamma = \lambda + \beta d_1 - \beta d_0$ satisfies the condition $\gamma(\Delta) = 0$ and clearly $\xi^2(|(\gamma, 0)|) = |(\lambda, 0)|$. Hence $\operatorname{Ker}\theta^2 \subseteq \operatorname{Im}\xi^2$. ∎

4. Extensions by crossed modules

In order to describe our second non-abelian quasi-cohomology in terms of extensions of Lie algebras by crossed modules and moreover, to establish the relationship between our second non-abelian cohomology and these extensions, we proceed in the same way as for groups [15].

We have to mention that the difference between the results obtained for Lie algebras and those given for groups in [15] is subtler than it seems. Namely, the notion of the second non-abelian quasi-cohomology of groups is not needed since the second non-abelian cohomology of groups with coefficients in crossed modules completely classifies extensions of groups by crossed modules.

We introduce just below the notion of extension of a Lie algebra by a crossed module.

Definition 4.1. Let R be a Lie algebra and (M, μ) a crossed R-module. An extension of R by (M, μ) is a pair $E = (0 \to M \xrightarrow{\sigma} X \xrightarrow{\psi} R \to 0,\ \varphi)$, where $0 \to M \xrightarrow{\sigma} X \xrightarrow{\psi} R \to 0$ is a short exact sequence of Lie algebras and φ is a

Λ-linear splitting of ψ such that the following equalities hold:

$$^{r}m = \sigma^{-1}[\varphi(r), \sigma(m)] \ , \quad r \in R, \ m \in M \tag{9}$$

and

$$\operatorname{Ker} \psi \cap \overline{\varphi(R)} \subseteq \operatorname{Ker} \mu \ , \tag{10}$$

where $\overline{\varphi(R)}$ denotes the Lie subalgebra of X generated by $\varphi(R)$.

For instance, the pair $(0 \to M \xrightarrow{\sigma_0} M \rtimes R \xrightarrow{\psi_0} R \to 0, \ \varphi_0)$, where $M \rtimes R$ denotes the semidirect product of the Lie algebras M and R, $\sigma_0(m) = (m, 0)$, $\psi_0(m, r) = r$, $\varphi_0(r) = (0, r)$ is an extension of R by (M, μ), called trivial.

Definition 4.2. Let R be a Lie algebra, (M, μ) a crossed R-module and $E = (0 \to M \xrightarrow{\sigma} X \xrightarrow{\psi} R \to 0, \ \varphi)$ and $E' = (0 \to M \xrightarrow{\sigma'} X' \xrightarrow{\psi'} R \to 0, \ \varphi')$ two extensions of R by (M, μ). The extensions E and E' will be called equivalent if there exists a Lie homomorphism $\theta : X \to X'$ and an element $h \in R$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\sigma} & X & \xrightarrow{\psi} & R & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\vartheta} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{\sigma'} & X' & \xrightarrow{\psi'} & R & \longrightarrow & 0 \end{array}$$

is commutative and the equality

$$\mu(\vartheta\varphi(r) - \varphi'(r)) = [h, r]$$

holds for any $r \in R$.

Note that, if it exists, ϑ is necessarily an isomorphism of Lie algebras and we can easily check that this relation of extensions is reflexive, symmetric and transitive. Let $E^1(R, M)$ denote the set of equivalence classes of extensions of the Lie algebra R by the crossed R-module (M, μ).

Theorem 4.3. *Let R be a Lie algebra and (M, μ) a crossed R-module. Then there is a bijection*

$$\widetilde{\mathfrak{H}}^2(R, M) \cong E^1(R, M) \ .$$

Proof. Let us define a map $\eta : \widetilde{\mathfrak{H}}^2(R, M) \to E^1(R, M)$ as follows. For any $|(\alpha, 0)| \in \widetilde{\mathfrak{H}}^2(R, M)$ consider the diagram of Lie algebras

$$\begin{array}{ccccc} P & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & \mathcal{F}(R) & \xrightarrow{\epsilon} & R \\ \downarrow{\scriptstyle\alpha} & & & & \\ M & & & & \end{array}$$

and take the semidirect product $M \rtimes \mathcal{F}(R)$, where $\mathcal{F}(R)$ acts on M via R. Then the subset $I_\alpha = \{(m,x) \mid \epsilon(x)=0, m=\alpha(x,0)\} \subseteq M \rtimes \mathcal{F}(R)$ is an ideal in $M \rtimes \mathcal{F}(R)$. In fact, I_α is clearly a submodule and for every $(m,x) \in I_\alpha$, $(n,y) \in M \rtimes \mathcal{F}(R)$ we have the equalities

$$\begin{aligned}
&{}^{x}n = {}^{\epsilon(x)}n = 0\ ,\\
&[m,n] = [\alpha(x,0),n] = {}^{\mu\alpha(x,0)}n = 0\ ,\\
&\alpha([x,y],0) = \alpha[(x,0),(y,y)] = {}^{(x,0)}\alpha(y,y) - {}^{(y,y)}\alpha(x,0) = -{}^{y}m\ .
\end{aligned}$$

Therefore

$$[(m,x),(n,y)] = ({}^{x}n - {}^{y}m + [m,n],[x,y]) = (-{}^{y}m,[x,y]) \in I_\alpha\ .$$

We get an exact sequence of Lie algebras

$$0 \longrightarrow M \xrightarrow{\ \sigma\ } (M \rtimes \mathcal{F}(R))/I_\alpha \xrightarrow{\ \psi\ } R \longrightarrow 0\ ,$$

where $\sigma(m) = |(m,0)|$, $\psi(|(m,x)|) = \epsilon(x)$ and the following commutative diagram

$$\begin{array}{ccccc}
P & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & \mathcal{F}(R) & \xrightarrow{\ \epsilon\ } & R\\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\delta}\downarrow & & \parallel\\
M & \xrightarrow{\ \sigma\ } & (M \rtimes \mathcal{F}(R))/I_\alpha & \xrightarrow{\ \psi\ } & R
\end{array}$$

with $\delta(x) = |(0,x)|$, that means $\sigma\alpha = \delta d_1 - \delta d_0$ and $\psi\delta = \epsilon$.

There is a Λ-linear splitting $\varphi : R \to (M \rtimes \mathcal{F}(R))/I_\alpha$ of ψ given by $\varphi(r) = |(0,\overline{r})|$, where $\overline{r} \in \mathcal{F}(R)$ is the image of r by the natural inclusion $R \hookrightarrow \mathcal{F}(R)$. It is easy to see that $\overline{\varphi(R)} = \operatorname{Im}\delta$. Therefore $\operatorname{Ker}\psi \cap \overline{\varphi(R)} = \delta(\operatorname{Ker}\epsilon)$. Then the equality $\sigma\alpha(0,x) = \delta(x)$ for $x \in \operatorname{Ker}\epsilon$, implies the condition (10). Hence the pair $E = (0 \to M \xrightarrow{\sigma} (M \rtimes \mathcal{F}(R))/I_\alpha \xrightarrow{\psi} R \to 0,\ \varphi)$ is an extension of R by (M,μ) and define $\eta(|(\alpha,0)|) = |E|$.

We have to show that η is well defined. Let $(\alpha',0)$ be another representative of $|(\alpha,0)|$, then there exists $(\beta,h) \in \operatorname{Der}(\mathcal{F}(R),(M,\mu))$ such that $\operatorname{Im}\beta \subseteq Z(M)$ and $\alpha' - \alpha = \beta d_0 - \beta d_1$. Let $E' = (0 \to M \xrightarrow{\sigma'} (M \rtimes \mathcal{F}(R))/I_{\alpha'} \xrightarrow{\psi'} R \to 0,\ \varphi')$ be the extension of R by (M,μ) corresponding to $(\alpha',0)$.

Consider a map $\vartheta : M \rtimes \mathcal{F}(R) \to M \rtimes \mathcal{F}(R)$ given by $\vartheta(m,x) = (m + \beta(x),x)$. Clearly ϑ is a homomorphism of Λ-modules and since $\operatorname{Im}\beta \subseteq Z(M)$ we have

$$\begin{aligned}
&[\vartheta(m,x),\vartheta(n,y)] = [(m+\beta(x),x),(n+\beta(y),y)]\\
&\ = ({}^{x}n +{}^{x}\beta(y) -{}^{y}m -{}^{y}\beta(x) + [m,n] + [m,\beta(y)] + [\beta(x),n] + [\beta(x),\beta(y)],[x,y])\\
&\ = ({}^{x}n -{}^{y}m + [m,n] + \beta[x,y],[x,y]) = \vartheta[(m,x),(n,y)]
\end{aligned}$$

for all $(m,x),(n,y) \in M \rtimes \mathcal{F}(R)$. Thus, ϑ is a Lie homomorphism. Moreover, if $(m,x) \in I_\alpha$, that is, $\epsilon(x) = 0$ and $m = \alpha(x,0)$, then

$$\vartheta(m,x) = (\alpha(x,0) + \beta(x), x) = (\alpha'(x,0), x) \in I_{\alpha'} \ .$$

Thus, ϑ induces a Lie homomorphism $\vartheta' : (M \rtimes \mathcal{F}(R))/I_\alpha \to (M \rtimes \mathcal{F}(R))/I_{\alpha'}$.

We have the following commutative diagram of Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\sigma} & (M \rtimes \mathcal{F}(R))/I_\alpha & \xrightarrow{\psi} & R & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \vartheta'} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{\sigma'} & (M \rtimes \mathcal{F}(R))/I_{\alpha'} & \xrightarrow{\psi'} & R & \longrightarrow & 0 \end{array}$$

with the above-defined splittings $\varphi : R \to (M \rtimes \mathcal{F}(R))/I_\alpha$ and $\varphi' : R \to (M \rtimes \mathcal{F}(R))/I_{\alpha'}$ and the equality

$$\mu(\vartheta'\varphi(r) - \varphi'(r)) = \mu(\vartheta'(|(0,\overline{r})|) - |(0,\overline{r})|) = \mu(|(\beta(\overline{r}),0)|) = -{}^{r}h = [h,r]$$

implying that $|E| = |E'|$.

Conversely, define a map $\eta' : E^1(R,M) \to \widetilde{\mathfrak{H}}^2(R,M)$ as follows. Let $|E| \in E^1(R,M)$, $E = (0 \to M \xrightarrow{\sigma} X \xrightarrow{\psi} R \to 0,\ \varphi)$. Then there is a commutative diagram

$$\begin{array}{ccccc} P & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & \mathcal{F}(R) & \xrightarrow{\epsilon} & R \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} & & \| \\ M & \xrightarrow{\sigma} & X & \xrightarrow{\psi} & R \ , \end{array}$$

where the Lie homomorphism δ is induced by the Λ-linear map φ and α is given by $\sigma\alpha = \delta d_1 - \delta d_0$. Using the equality (9) it is easy to check that α is a derivation. Clearly $\alpha(x,x) = 0$ for $x \in \mathcal{F}(R)$ and since $\delta(y) - \delta(x) \in \operatorname{Ker}\psi \cap \overline{\varphi(R)}$, by the condition (10) we have $\mu\alpha(x,y) = \mu(\delta(y) - \delta(x)) = 0$ for $(x,y) \in P$. Hence $(\alpha, 0) \in \widetilde{\operatorname{Der}}(P,(M,\mu))$ and we define $\eta'(|E|) = |(\alpha,0)|$.

We have to show that η' is also well defined. Suppose E is equivalent to $E' = (0 \to M \xrightarrow{\sigma'} X' \xrightarrow{\psi'} R \to 0,\ \varphi')$, then there exist a Lie homomorphism $\vartheta : X \to X'$ and an element $h \in R$ satisfying the conditions of Definition 4.2. Let $(\alpha', 0)$ be the element of $\widetilde{\operatorname{Der}}(P,(M,\mu))$ corresponding to E'.

Consider the derivation $\beta : \mathcal{F}(R) \to M$ given by $\sigma'\beta = \vartheta\delta - \delta'$. Then for $r, s \in R$ we have

$$\mu\beta(\overline{r}) = \mu(\vartheta\delta(\overline{r}) - \delta'(\overline{r})) = \mu(\vartheta\varphi(r) - \varphi'(r)) = [h,r]$$

and

$$\mu\beta([\overline{r},\overline{s}]) = \mu({}^{r}\beta(\overline{s})) - \mu({}^{s}\beta(\overline{r})) = [r, \mu\beta(\overline{s})] - [s, \mu\beta(\overline{r})] = [r,[h,s]] - [s,[h,r]]$$
$$= [h,[r,s]]$$

implying that $\mu\beta(x) = -{}^{x}h$ for all $x \in \mathcal{F}(R)$, i.e., $(\beta, h) \in \mathrm{Der}(\mathcal{F}(R), (M, \mu))$. The verification that $\beta d_0 - \beta d_1 = \alpha' - \alpha$ and $\mathrm{Im}\,\beta \subseteq Z(M)$ is now routine and will be omitted. Therefore $|(\alpha, 0)| = |(\alpha', 0)|$.

It is easily checked that $\eta\eta'$ and $\eta'\eta$ are identity maps. ■

Note that the bijection η maps the zero element of $\widetilde{\mathfrak{H}}^2(R, M)$ to the equivalence class of the trivial extension of R by (M, μ).

Now, suppose $S(R, M)$ denotes the subset of $E^1(R, M)$ consisting of all equivalence classes of extensions $E = (0 \to M \xrightarrow{\sigma} X \xrightarrow{\psi} R \to 0, \varphi)$ for that there exists a Λ-homomorphism $s : R \to M$ such that $(\sigma\beta + \delta)(x) = (\sigma\beta + \delta)(y)$ for $x, y \in \mathcal{F}(R)$ with $\epsilon(x) = \epsilon(y)$, β is the derivation induced by s and δ is the Lie homomorphism induced by φ.

The set $S(R, M)$ always contains at least one element, namely, the equivalence class of the trivial extension.

We arrive to the following

Proposition 4.4. *Let R be a Lie algebra and (M, μ) a crossed R-module. Then there is a short exact sequence of pointed sets*

$$0 \longrightarrow S(R, M) \longrightarrow E^1(R, M) \longrightarrow \mathfrak{H}^2(R, M) \longrightarrow 0\ .$$

Proof. As an easy consequence of Definitions 3.5 and 3.6, there is a short exact sequence of Λ-modules

$$0 \longrightarrow B(P, (M, \mu))/\widetilde{B}(P, (M, \mu)) \longrightarrow \widetilde{\mathfrak{H}}^2(R, M) \longrightarrow \mathfrak{H}^2(R, M) \longrightarrow 0\ .$$

It is easy to see that the restriction of the bijection η in the proof of Theorem 4.3 is the bijection from $B(P, (M, \mu))/\widetilde{B}(P, (M, \mu))$ to $S(R, M)$. Then the assertion follows. ■

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Nikoloz Inassaridze
National Center for Science and Technology, Georgia
A. Razmadze Math. Institute
M.Alexidze St. 1, Tbilisi 0193
Georgia
Dpto. Matemática Aplicada I
Universidad de Vigo
EUIT Forestal, 36005 Pontevedra, Spain
niko.inas@gmail.com

Emzar Khmaladze
National Center for Science and Technology (Georgia)
A. Razmadze Math. Institute
M.Alexidze St. 1, Tbilisi 0193
Georgia
khmal@rmi.acnet.ge

Manuel Ladra
Departamento de Álgebra
Universidad de Santiago
E-15782, Spain
ladra@usc.es

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