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SOME ASPECTS OF HOMOTOPIC ALGEBRA AND NON-ABELIAN (CO)HOMOLOGY THEORIES

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Abstract. This monograph is devoted to the study of homological and homotopic properties of various algebraic structures. The problems considered and line of investigation taken fall under the general headings of non-Abelian homological algebra and simplicial methods in category theory, with applications to K-theory and cyclic homology.

CONTENTS

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INTRODUCTION

This monograph is devoted to the study of homological and homotopic properties of various algebraic structures. The problems considered and line of investigation taken fall under the general headings of non-Abelian homological algebra and simplicial methods in category theory, with applications to K-theory and cyclic homology. Work in these areas has had fundamental applications in diverse fields of mathematics and has made a significant impact on the development of many areas of mathematics.

Homotopical algebra, in particular, simplicial algebra, including its non-Abelian and categorical aspects, is inspired by geometrical and topological constructions and plays a crucial role in the rapidly expanding areas of K-theory—cyclic homology and homotopy theory. While in classical (Abelian) homological algebra additive functors from Abelian (or additive) categories to Abelian categories are investigated, one powerful tool of simplicial algebra is the notion of derived functors of nonadditive functors, called non-Abelian derived functors, which has been applied to the simplicial-group approach to algebraic K-theory developed by Swan [120] and Keune [82] and advanced in non-Abelian homological algebra by Inassaridze [60] and others. On the other hand, most of the well-known (co)homology functors are described in terms of non-Abelian derived functors as (co)triple (co)homology (see the works of Barr and Beck [4, 5] and Duskin [40]).

The proposed results of this work are concentrated around homotopical algebra and (co)homology theories with special attention to the approach based on non-Abelian derived functors to (co)homology theories, their q-modular analogs, and n-fold Čech derived functors. It is a continuation of the author's previous investigations described in [68–70].

The main aims of this paper are to state and develop a general theory of n-fold Čech derived functors, $\mathcal{L}_*^{n\text{-fold}}T$, of group-valued functors $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$, generalizing to that of Čech derived functors introduced some 30 years ago by Pirashvili [103] (see also [60]) as an algebraic analog of the Čech (co)homology of open covers of topological spaces, and to illustrate the methods of this theory to generalize further the Brown–Ellis higher Hopf formulas for integral group homology [14].

Homology groups are the derived functors of the abelianization functor, which, of course, kills the commutator subgroup of a group. Our generalization handles the derived functors of the functors that kill higher commutators. More precisely, the "nilization of degree k" functor, $Z_k(G)$, $k \geq 2$, is given by $Z_k(G) = G/\Gamma_k(G)$, where $\{\Gamma_k(G),\ k \geq 1\}$ is the lower central series of a given group G. These Z_k are endofunctors on the category of groups and generalize the abelianization functor, so their non-Abelian left derived functors, $L_n^{\mathcal{P}} Z_k$, $n \geq 0$, with $\mathcal{P}$ a projective class of free groups, generalize the group homology functors H_n, $n \geq 1$ (cf., e.g., [5]). Namely, one of the main result of the paper (Theorem 3.9) says that for a given group G, its free exact n-presentation $\mathfrak{F}$ (see Definition 1.15) and $k \geq 2$, there is an isomorphism

$$L_n^{\mathcal{P}} Z_k(G) \cong \mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \frac{\bigcap\limits_{i \in \langle n \rangle} R_i \cap \Gamma_k(F)}{D_k(F; R_1, \dots, R_n)}, \quad n \geq 1,$$

where $(F; R_1, \dots, R_n)$ is the normal $(n+1)$-ad of groups induced by $\mathfrak{F}$.

Moreover, applying our results, we obtain the following Hopf type formula in algebraic K-theory.

Let $\mathfrak{R}$ be a ring with unit and $(F_*, d_0^0, GL(\mathfrak{R}))$ be a free pseudo-simplicial resolution of the general linear group $GL(\mathfrak{R})$. Then there is an exact sequence of Abelian groups

$$0 \longrightarrow \varprojlim_j{}^{(1)} \left(\frac{\Big(\bigcap\limits_{i\in\langle n+1\rangle} \operatorname{Ker} d_{i-1}^n\Big) \cap \Gamma_j(F_n)}{D_j(F_n; \operatorname{Ker} d_0^n, \ldots, \operatorname{Ker} d_n^n)} \right) \longrightarrow$$

$$\longrightarrow K_{n+1}(\mathfrak{R}) \longrightarrow \varprojlim_j \left(\frac{\Big(\bigcap\limits_{i\in\langle n\rangle} \operatorname{Ker} d_{i-1}^{n-1}\Big) \cap \Gamma_j(F_{n-1})}{D_j(F_{n-1}; \operatorname{Ker} d_0^{n-1}, \ldots, \operatorname{Ker} d_{n-1}^{n-1})} \right) \longrightarrow 0$$

for $n \geq 1$.

Particular attention in this paper is paid to the investigation of elegant algebraic models of connected CW-spaces whose homotopy groups are trivial in dimensions greater than $n+1$ introduced by Ellis and Steiner in [49] and called crossed n-cubes of groups. These models are more combinatorial algebraic systems, but equivalent to that previously invented by Loday in [87] and called cat n-groups. The origin of these notions comes from the late 1940s by the notion of crossed modules first given by Whitehead in [127] as a means of representing homotopy 2-types. A number of papers of the last years are dedicated to the investigation of homological properties of these objects. We mentioned here the papers [6, 21, 23, 46, 56, 85].

We study the diagonal of the n-simplicial multinerve $E_*^{(n)}$ of crossed n-cubes of groups in connection with the n-fold Čech complexes and with the abelianization of crossed n-cubes of groups $\mathfrak{A}b^{(n)}$. In particular, here we distinguish the following result for crossed n-cubes, which plays an essential role in obtaining generalized Hopf type formulas for the homology of crossed n-cubes.

Let $n \geq 0$, $m \geq 1$, and $\mathcal{M}$ be a crossed $(n+m)$-cube. Then there is an isomorphism of simplicial crossed n-cubes

$$\mathfrak{Ab}^{(n)} E^{(m)}(\mathcal{M})_* \cong E^{(m)} \mathfrak{Ab}^{(n+m)}(\mathcal{M})_*.$$

The universality of our new purely algebraic methods of n-fold Čech derived functors provides motivation to investigate other cotriple homology theories from a Hopf formula point of view. In fact, we study the tripleability of the category of crossed n-cubes (cf. [23]) and the leading cotriple homology of these homotopy $(n+1)$-types. Namely, there is an isomorphism

$$H_{m+1}(\mathcal{M}) \cong \frac{\bigcap\limits_{i\in\langle m\rangle} R^i_{\langle n\rangle} \cap \prod\limits_{B\cup C=\langle n\rangle} [\mathfrak{X}(\varnothing)_B, \mathfrak{X}(\varnothing)_C]}{\prod\limits_{A\subseteq\langle m\rangle} \Big(\prod\limits_{B\cup C=\langle n\rangle} \Big[\bigcap\limits_{i\in A} R^i_B, \bigcap\limits_{i\notin A} R^i_C \Big] \Big)}, \quad m \geq 1,$$

for any crossed n-cube $\mathcal{M}$ and its projective exact m-presentation $\mathfrak{X}$, where $R^i = \operatorname{Ker}(\mathfrak{X}(\varnothing) \longrightarrow \mathfrak{X}(\{i\}))$ for $i \in \langle m\rangle$. This result generalizes to that of Brown–Ellis [14] and the Hopf formula for the second CCG-homology of crossed modules [21].

The non-Abelian homology of groups with coefficients in groups was constructed and investigated in [67, 68] using the non-Abelian tensor product of groups of Brown and Loday [17, 18] and its non-Abelian left derived functors, generalizing the classical Eilenberg–MacLane homology of groups and extending Guin's low-dimensional non-Abelian homology of groups with coefficients in crossed modules [53], which has an interesting application to the algebraic K-theory of noncommutative local rings [53, 68]. In [61–63] Inassaridze developed a non-Abelian cohomology theory previously defined by Guin in low dimensions [53] generalizing the classical Eilenberg–MacLane cohomology of groups. In this paper, we continue the investigation of non-Abelian group homology theory establishing some interesting functorial properties and making some explicit computations of low-dimensional non-Abelian homologies.

Another goal of this paper is to set up a similar non-Abelian (co)homology theory for Lie algebras, stating and proving several desirable properties of this (co)homology theory.

In [45] Ellis introduced and studied the non-Abelian tensor product of Lie algebras, which is a Lie structural and purely algebraic analog of the non-Abelian tensor product of groups of Brown and Loday [17, 18]. Applying this tensor product of Lie algebras, Guin defined the low-dimensional non-Abelian homology of Lie algebras with coefficients in crossed modules [55].

We construct a non-Abelian homology $H_*(M,N)$ of a Lie algebra M with coefficients in a Lie algebra N as the non-Abelian left derived functors of the tensor product of Lie algebras, generalizing the classical Chevalley–Eilenberg homology of Lie algebras and extending Guin's non-Abelian homology of Lie algebras [55]. We give an application of our long exact homology sequence to cyclic homology of associative algebras, correcting the result of [55]. In fact, for a unital associative (noncommutative) algebra A we obtain a long exact non-Abelian homology sequence

$$\cdots \longrightarrow H_2(A,V(A),[A,A]) \longrightarrow H_2(A,V(A)) \longrightarrow H_2(A,[A,A]) \longrightarrow$$
$$\longrightarrow H_1(A,V(A),[A,A]) \longrightarrow H_1(A,V(A)) \longrightarrow H_1(A,[A,A]) \longrightarrow$$
$$\longrightarrow HC_1(A) \longrightarrow HC_1^M(A) \longrightarrow [A,A]/[A,[A,A]] \longrightarrow 0\ .$$

Following [55] and using ideas from [61], we introduce the second non-Abelian cohomology $H^2(R,M)$ of a Lie algebra R with coefficients in a crossed R-module (M,μ), generalizing the classical second cohomology of Lie algebras. Then, for a coefficient short exact sequence of crossed R-modules

$$0 \longrightarrow (L,0) \longrightarrow (M,\mu) \longrightarrow (N,\nu) \longrightarrow 0\ ,$$

having a module section over the ground ring, we give a nine-term exact non-Abelian cohomology sequence

$$0 \longrightarrow H^0(R,L) \longrightarrow H^0(R,M) \longrightarrow H^0(R,N) \longrightarrow H^1(R,L) \longrightarrow H^1(R,M) \longrightarrow$$
$$\longrightarrow H^1(R,N) \longrightarrow H^2(R,L) \longrightarrow H^2(R,M) \longrightarrow H^2(R,N),$$

extending the seven-term exact cohomology sequence of Guin [55], which exists under the aforementioned additional necessary condition on the coefficient sequence of crossed modules.

During the last twenty years many important works appeared investigating the mod q versions of algebraic and topological topics.

For example, in [101] Neisendorfer, following F. P. Peterson [102], constructed and studied a homotopy theory with $\mathbb{Z}/q$ coefficients (primary homotopy theory) having important applications to K-theory and homotopy theory.

In [9] Browder defined and investigated a mod q algebraic K-theory called the algebraic K-theory with $\mathbb{Z}/q$ coefficients.

In [119] Suslin and Voevodsky calculated the mod 2 algebraic K-theory of the integers as a result of Voevodsky's solution of the Milnor conjecture [124].

In [32] Conduché and Rodriguez–Fernández introduced and studied non-Abelian tensor and exterior products modulo q of crossed modules, generalizing definitions of Brown [13] and Ellis and Rodriguez [48] (see also [47, 112]) and having properties similar to the Brown–Loday's non-Abelian tensor product of crossed modules [17, 18].

In [81] Karoubi and Lambre introduced the mod q Hochschild homology as the homology of the mapping cone of the morphism given by the q multiplication on the standard Hochschild complex.

Then they constructed the Dennis trace map from mod q algebraic K-theory to mod q Hochschild homology and found an unexpected relationship with number theory.

The final goals of this work are to investigate the non-Abelian tensor (exterior) product modulo q of Conduché and Rodriguez–Fernández [32], to introduce and study mod q group homology and cohomology theories, and to unify them into a mod q Tate–Farrell–Vogel group cohomology theory.

We show that the "absolute" tensor product modulo q, $G \otimes {}^q H$, of two groups G and H with compatible actions is the quotient of non-Abelian tensor product $G \otimes H$ by $q(H_1(G,H) \cap H_1(H,G))$, where H_1 is the first non-Abelian group homology. We generalize Guin's isomorphism [12][12] for the tensor product modulo q by giving the short exact sequence of groups

$$0 \longrightarrow G \otimes {}^q A \longrightarrow I(G,q) \otimes_G A \longrightarrow q\mathbb{Z} \otimes_G A \longrightarrow 0, \qquad q \geq 0,$$

where G is a group, A is G-module, and $I(G,q)$ is the kernel of the morphism $\widetilde{\epsilon} \colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}_q$.

We give an application of tensor product modulo q to algebraic K-theory with $\mathbb{Z}_q$ coefficients [9] of noncommutative local rings. In particular, for a noncommutative local ring A such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$ and $q > 1$, there is an exact sequence of groups

$$((A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A))/q \oplus \operatorname{Tor}(K_2(A), \mathbb{Z}/q) \longrightarrow \operatorname{Ker} \xi'_{D_0(A)} \longrightarrow \operatorname{Ker} \xi'_{[A^*,A^*]} \longrightarrow$$
$$\longrightarrow K_2(A; \mathbb{Z}/q) \longrightarrow \operatorname{Sym}(A; \mathbb{Z}/q) \longrightarrow ([A^*,A^*]/[A^*,[A^*,A^*]])/q \longrightarrow 0.$$

This result is a q-modular analog of Guin's six-term exact sequence relating the non-Abelian homology of groups with Milnor's K_2 and the symbol group Sym (see [53]).

We introduce mod q homology, $H_*(G,A;\mathbb{Z}/q)$, and cohomology, $H^*(G,A;\mathbb{Z}/q)$, of a group G with coefficients in a G-module A, naturally inspired in the study of non-Abelian left derived functors of the "absolute" tensor product modulo q of groups, and which are the homologies of the mapping cones of the q multiplication on the standard homological and cohomological complexes, respectively, as in the case of the mod q Hochschild homology [81]. We have the following short exact sequences (universal coefficient formulas) for mod q group (co)homology relating them to classical group (co)homology:

$$0 \longrightarrow H_n(G,A) \otimes \mathbb{Z}/q \longrightarrow H_n(G,A;\mathbb{Z}/q) \longrightarrow \operatorname{Tor}(H_{n-1}(G,A), \mathbb{Z}/q) \longrightarrow 0,$$
$$0 \longrightarrow H^{n-1}(G,A) \otimes \mathbb{Z}/q \longrightarrow H^n(G,A;\mathbb{Z}/q) \longrightarrow \operatorname{Tor}(H^n(G,A), \mathbb{Z}/q) \longrightarrow 0,$$

for $n \geq 0$.

We introduce the notions of mod q version torsors and extensions to describe the first and second mod q cohomologies of groups, respectively.

We express the mod q cohomology of groups in terms of cotriple (right) derived functors of the kernels of higher dimensions of the mapping cone of the q multiplication on the standard cohomological complex. In fact, for a given group G and a G-module A we have the isomorphisms

$$R^0_{\mathcal{F}} Z^k(G,A;\mathbb{Z}/q) \cong Z^k(G,A;\mathbb{Z}/q),$$
$$R^n_{\mathcal{F}} Z^k(G,A;\mathbb{Z}/q) \cong H^{n+k}(G,A;\mathbb{Z}/q),$$

for $k > 1$ and $n > 0$.

We give an account of the Vogel cohomology theory [125]. In [52] Goichot gave a detailed exposition of Vogel's homology theory and its relations to Tate and Farrell theories. We shall give here the cohomological approach (see also [128, § 5]). Then the mod q Tate–Farrell–Vogel cohomology of groups is introduced unifying mod q homology and cohomology of groups. Finally we show how the periodicity properties of finite periodic groups extend to mod q Tate cohomology.

The results of this paper were published in [24, 31, 39, 65–67, 71–75].

Notation and Conventions. We denote the categories of sets, groups, and Abelian groups by **Set**, $\mathfrak{G}r$, and $Ab\mathfrak{G}r$, respectively.

For any set A, its cardinality is denoted by $|A|$.

For a nonnegative integer n, we denote by $\langle n\rangle$ the set of first n natural numbers $\{1,\cdots,n\}$.

Given a group G and n normal subgroups, $R_1,\ldots,R_n$, the tuple, $(G;R_1,\ldots,R_n)$, will be called a *normal* $(n+1)$*-ad of groups*, while $Z(G)$ and $[G,G]$ denote the center and the commutator subgroup of G, respectively. Moreover, for any elements $a,b\in G$, $[a,b]$ is the commutator $aba^{-1}b^{-1}$.

Given a (pseudo)-simplicial object X_* and a (pseudo)-simplicial morphism f_* in a category $\mathcal{U}$ and a functor $T:\mathcal{U}\longrightarrow\mathfrak{G}r$, denote by $T(X_*)$ and $T(f_*)$ the (pseudo)-simplicial group and (pseudo)-simplicial group morphism obtained by applying the functor T dimensionwise to X_* and to f_*, respectively.

Chapter 1

DERIVED FUNCTORS

In this chapter, we give a brief introduction to non-Abelian and Čech derived functors. For a fuller account of these derived functors, we refer the reader to [60]. We pay particular attention to Čech derived functors and develop its n-fold analog.

In Sec. 1, we recall the well-known notions and results on the non-Abelian derived functors of group-valued functors with respect to projective classes and cotriples.

In Sec. 2, we give a brief introduction to Čech derived functors (see also [60]); then their n-fold analogs are examined. The Čech derived functors of group-valued functors were introduced in [103] (see also [60]) as an algebraic analog of the Čech (co)homology construction of open covers of topological spaces with coefficients in sheaves of Abelian groups (see [115]). It is well known that the Čech cohomology of topological spaces with coefficients in sheaves is closely related to the sheaf cohomology of topological spaces; in particular, this relation is expressed by spectral sequences [115]. Some applications of Čech derived functors to group (co)homology theory and K-theory are given in [103–105].

The notion of Čech derived functors is generalized to that of the n-fold Čech derived functors of a group-valued functor, and their relationship to the non-Abelian derived functors is given in terms of spectral sequences. Later in Chap. 3, based on this notion, we get a new purely algebraic method for the investigation of higher integral group homology from a Hopf formula point of view and the further generalizations of these formulas. This method is universal and is valid for other algebraic structures.

1. Non-Abelian Derived Functors

In this section, we recall some well-known notions and results about derived functors of nonadditive functors from [60, 103, 123].

1.1. Pseudo-simplicial objects.

First some terminology and notation on pseudo-simplicial objects in a category are examined.

Definition 1.1. Let $\mathcal{U}$ be a category. By a *pseudo-simplicial object* X_* in $\mathcal{U}$ is meant a nonnegatively graded object with *face* morphisms $d_i^n:X_n\longrightarrow X_{n-1}$ and *pseudo-degeneracy* morphisms $s_i^n:X_n\longrightarrow X_{n+1}$, $0\le i\le n$, satisfying the following conditions:

$$d_i^{n-1}d_j^n=d_{j-1}^{n-1}d_i^n\quad\text{for}\quad i<j,$$
$$d_i^{n+1}s_j^n=s_{j-1}^{n-1}d_i^n\quad\text{for}\quad i<j,$$

$$d_j^{n+1}s_j^n = 1 = d_{j+1}^{n+1}s_j^n, \quad d_i^{n+1}s_j^n = s_j^{n-1}d_{i-1}^n \quad \text{for} \quad i > j+1.$$

The following identity on pseododegeneracies:

$$s_i^{n+1}s_j^n = s_{j+1}^{n+1}s_i^n \quad \text{for} \quad i \le j$$

is possibly not fulfilled. Otherwise we obtain the notion of a simplicial object in the category $\mathcal{U}$.

A *morphism* $f_* : X_* \longrightarrow X'_*$ of pseudo-simplicial objects is a morphism of degree zero of graded objects that commutes with the face and the degeneracy morphisms, i.e., a nonnegatively graded family of morphisms $\{f_n : X_n \longrightarrow X'_n,\ n \ge 0\}$ with $f_{n-1}d_i^n = d_i^n f_n$, $n > 0$, and $f_{n+1}s_i^n = s_i^n f_n$, $n \ge 0$ for all $0 \le i \le n$.

Definition 1.2. Let f_* and g_* be two morphisms from X_* to X'_*. We say that f_* is *pseudo-homotopic* to g_*, denoted by $f_* \simeq g_*$, if there exist morphisms $h_i^n : X_n \longrightarrow X'_{n+1}$, $0 \le i \le n$, such that the following conditions hold:

$$d_0^{n+1}h_0^n = f_n, \quad d_{n+1}^{n+1}h_n^n = g_n,$$
$$d_i^{n+1}h_j^n = h_{j-1}^{n-1}d_i^n \quad \text{for} \quad i < j,$$
$$d_{j+1}^{n+1}h_{j+1}^n = d_{j+1}^{n+1}h_j^n, \quad d_i^{n+1}h_j^n = h_j^{n-1}d_{i-1}^n \quad \text{for} \quad i > j+1.$$

If, in addition, the conditions

$$s_i^{n+1}h_j^n = h_{j+1}^{n+1}s_i^n \quad \text{for} \quad i \le j,$$
$$s_i^{n+1}h_j^n = h_j^{n+1}s_{i-1}^n \quad \text{for} \quad i > j$$

are satisfied, we say that f_* is homotopic to g_*.

An augmented pseudo-simplicial object (X_*, d_0^0, X) in the category $\mathcal{U}$ is a pseudo-simplicial object X_* with a morphism $d_0^0 : X_0 \longrightarrow X$ such that $d_0^1 d_0^0 = d_1^1 d_0^0$. It is (left) *contractible* if there exist morphisms $h_n : X_n \longrightarrow X_{n+1}$, $n \ge 0$, and $h : X \longrightarrow X_0$ such that $d_0^0 h = 1$, $d_0^{n+1}h_n = 1$, $n \ge 0$, $d_1^1 h_0 = h d_0^0$, and $d_i^{n+1}h_n = h_{n-1}d_{i-1}^n$ for $n \ge 1$, $1 \le i \le n+1$.

Now consider the pseudo-simplicial objects in the category $\mathfrak{Gr}$ (cf. [60] for the general theory). For examples of pseudo-simplicial groups, see [90].

Let G_* be a pseudo-simplicial group, $N_n(G_*) = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i^n$, $n \ge 0$, and ∂_n the restriction of d_n^n to $N_n(G_*)$, $n > 0$. Then $\operatorname{Im}\partial_n$ is a normal subgroup of G_{n-1}, and $\operatorname{Im}\partial_{n+1} \subseteq \operatorname{Ker}\partial_n$ for $n > 0$. This determines the Moore chain complex [96] $N(G_*) = \{N_n(G_*), \partial_*\}$, which clearly is independent of the pseudo-degeneracies and depends only on the face morphisms.

Definition 1.3. The nth homology group of the chain complex $N(G_*)$ is called the *nth homotopy group* $\pi_n(G_*)$ of the pseudo-simplicial group G_*, $n \ge 0$.

According to [96, Proposition 17.4] the nth homotopy group of a simplicial group coincides with the group $\pi_n(G_*)$. We also note that if an augmented pseudo-simplicial group (G_*, d_0^0, G) is contractible then $\pi_n(G_*) = 0$, $n \ge 1$, and d_0^0 induces an isomorphism $\pi_0(G_*) \cong G$.

Given a pseudo-simplicial group G_*, it is easy to verify that there are other chain complexes $N'(G_*) = \{N'_n(G_*), \partial'_*\}$ where $N'_n(G_*) = \bigcap_{i \in A} \operatorname{Ker} d_i^n$, $A = \{0, \dots, n\} \setminus k$, $0 \le k \le n$, and ∂_n is the restriction of d_k^n to $N'_n(G_*)$, $n > 0$. Then the nth homology group of the chain complex $N'(G_*)$ coincides with the group $\pi_n(G_*)$, $n \ge 0$ (see [60]).

Let $f_* : G_* \longrightarrow G'_*$ be a morphism of pseudo-simplicial groups. Then it is easy to see that it naturally induces group homomorphisms $\pi_n(f_*) : \pi_n(G_*) \longrightarrow \pi_n(G'_*)$, $n \ge 0$.

Theorem 1.4. *The homotopy groups $\pi_n(G_*)$ are Abelian for $n \geq 1$. If the morphisms of pseudo-simplicial groups $f_*, g_* : G_* \longrightarrow G'_*$ are pseudo-homotopic, then $\pi_n(f_*) = \pi_n(g_*)$, $n \geq 0$.*

Definition 1.5. A morphism $f_* : G_* \longrightarrow G'_*$ of pseudo-simplicial groups is called a *fibration* if the homomorphism $f_n : G_n \longrightarrow G'_n$ is surjective for all $n \geq 0$.

Theorem 1.6. *If $f_* : G_* \longrightarrow G'_*$ is a fibration, then the sequence of homotopy groups*

$$\cdots \longrightarrow \pi_{n+1}(G'_*) \longrightarrow \pi_n(\operatorname{Ker} f_*) \longrightarrow \pi_n(G_*) \longrightarrow \pi_n(G'_*) \longrightarrow \cdots$$

is exact, where $\operatorname{Ker} f_*$ *is the following pseudo-simplicial group* $\{\operatorname{Ker} f_n,\ n \geq 0\}$.

1.2. Comparison of non-Abelian and cotriple derived functors. Let $\mathcal{U}$ be an arbitrary category, and $\mathcal{P}$ a *projective class* of objects in $\mathcal{U}$ in the sense of Eilenberg–Moore. This is a class of objects in $\mathcal{U}$ such that for every object $X \in \mathcal{U}$ there exists a $\mathcal{P}$-epimorphism $\tau : P \longrightarrow X$, where P belongs to the class $\mathcal{P}$; a morphism $f : X \longrightarrow X'$ in $\mathcal{U}$ is said to be $\mathcal{P}$-*epimorphic* if the map $\operatorname{Hom}_{\mathcal{U}}(P, f) : \operatorname{Hom}_{\mathcal{U}}(P, X) \longrightarrow \operatorname{Hom}_{\mathcal{U}}(P, X')$ is surjective for every $P \in \mathcal{P}$.

Let X and Y be objects in the category $\mathcal{U}$, and

$$Y \ \overset{d_0}{\underset{d_n}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}}\ X$$

be sequence of $n+1$ morphisms, $n \geq 0$, in $\mathcal{U}$. A *simplicial kernel* of $(d_0, \ldots, d_n)$ is a sequence

$$K \ \overset{k_0}{\underset{k_{n+1}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}}\ Y$$

of $n+2$ morphisms in $\mathcal{U}$ satisfying $d_i k_j = d_{j-1} k_i$ for $0 \leq i, j \leq n+1$ and universal with respect to this property. That is, if

$$Z \ \overset{d'_0}{\underset{d'_{n+1}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}}\ Y$$

is any other sequence satisfying $d_i d'_j = d_{j-1} d'_i$ for $0 \leq i, j \leq n+1$, then there exists a unique $\partial : Z \longrightarrow K$ such that $k_i \partial = d'_i$, $0 \leq i \leq n+1$.

For any object $X \in \mathcal{U}$ we consider a $\mathcal{P}$-projective resolution (X_*, d_0^0, X) in the sense of Tierney–Vogel (see [123]). A nonnegatively graded object X_*, with face morphisms $d_i^n : X_n \longrightarrow X_{n-1}$, $0 \leq i \leq n$, and a morphism $d_0^0 : X_0 \longrightarrow X$, satisfying the condition $d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_i^n$, $0 \leq i < j \leq n$, is called $\mathcal{P}$-*projective* if each X_n belongs to the class $\mathcal{P}$ and is called $\mathcal{P}$-*exact* if each natural morphism from X_{n+1} to the simplicial kernel of $(d_0^n, \ldots, d_n^n)$ is $\mathcal{P}$-epimorphic. (X_*, d_0^0, X) is called the $\mathcal{P}$-*projective resolution* of X if it is $\mathcal{P}$-projective and $\mathcal{P}$-exact. We note that any $\mathcal{P}$-projective resolution admits pseudo-degeneracy morphisms $s_i^n : X_n \longrightarrow X_{n+1}$, $0 \leq i \leq n$ (see [123]), which was the reason to consider the theory of pseudo-simplicial groups in [59].

If in the category $\mathcal{U}$ there exist finite limits, then every object admits such a resolution.

Theorem 1.7. *Let (X_*, d_0^0, X) be $\mathcal{P}$-projective and (X'_*, d_0^0, X') be $\mathcal{P}$-exact. Then any morphism $f : X \longrightarrow X'$ in $\mathcal{U}$ can be extended to a morphism $f_* : X_* \longrightarrow X'_*$ over f, i.e., the diagram*

$$\begin{tikzcd} X_* \arrow[r, "d_0^0"] \arrow[d, "f_*"'] & X \arrow[d, "f"] \\ X'_* \arrow[r, "d_0^0"'] & X' \end{tikzcd}$$

is commutative. Furthermore, any two such extensions are pseudo-homotopic.

According to this theorem, we have the following definition of non-Abelian left derived functors.

Definition 1.8. Let $\mathcal{U}$ be a category with finite limits. For an arbitrary covariant functor $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$, define the nth *left derived functor* $L_n^{\mathcal{P}}T : \mathcal{U} \longrightarrow \mathfrak{Gr}$, $n \geq 0$, relative to the projective class $\mathcal{P}$, by choosing for each $X \in \mathcal{U}$, a $\mathcal{P}$-projective resolution (X_*, d_0^0, X) and setting

$$L_n^{\mathcal{P}}T(X) = \pi_n(T(X_*)) \quad \text{and} \quad L_n^{\mathcal{P}}T(f) = \pi_n(T(f_*))$$

for any object $X \in \mathcal{U}$ and any morphism $f \in \mathcal{U}$.

A *cotriple* $\mathcal{F} = (F, \tau, \delta)$ in a category $\mathcal{U}$ is an endofunctor $F : \mathcal{U} \longrightarrow \mathcal{U}$ together with natural transformations $\tau : F \longrightarrow 1_{\mathcal{U}}$ and $\delta : F \longrightarrow F^2$, satisfying the commutativity conditions

$$\begin{tikzcd} F \arrow[r, "\delta"] \arrow[d, "1_F"'] & F^2 \arrow[d, shift right, "\tau F"'] \arrow[d, shift left, "F\tau"] \\ F \arrow[r, equal] & F \end{tikzcd}, \qquad \begin{tikzcd} F \arrow[r, "\delta"] \arrow[d, "\delta"'] & F^2 \arrow[d, "F\delta"] \\ F^2 \arrow[r, "\delta F"'] & F^3 \end{tikzcd}.$$

Then the cotriple $\mathcal{F}$ induces the projective class $\mathcal{P}$: $X \in \mathcal{P}$ if and only if there exists a morphism $\vartheta : X \longrightarrow F(X)$ such that $\tau_X \vartheta = 1_X$. In fact, $F(X) \in \mathcal{P}$ and the morphism $\tau_X : F(X) \longrightarrow X$ is a $\mathcal{P}$-epimorphism for any object $X \in \mathcal{U}$.

Given an object $X \in \mathcal{U}$, consider the augmented simplicial object $(F_*(X), d_0^0, X)$ in the category $\mathcal{U}$, where

$$F_*(X) \equiv \cdots \begin{matrix}\longrightarrow \\ \vdots \\ \longrightarrow\end{matrix} F_n(X) \begin{matrix}\xrightarrow{d_0^n} \\ \vdots \\ \xrightarrow[d_n^n]{}\end{matrix} \cdots \begin{matrix}\xrightarrow{d_0^2} \\ \longrightarrow \\ \xrightarrow[d_2^2]{}\end{matrix} F_1(X) \begin{matrix}\xrightarrow{d_0^1} \\ \xrightarrow[d_1^1]{}\end{matrix} F_0(X) ,$$

$F_n(X) = F^{n+1}(X) = F(F^n(X))$, $d_i^n = F^i \tau F^{n-i}$, $s_i^n = F^i \delta F^{n-i}$, $0 \leq i \leq n$, which is called the *cotriple resolution* of X (see [120]).

Definition 1.9. Let $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ be a covariant functor. We define the nth *left derived functor* $L_n^{\mathcal{F}}T : \mathcal{U} \longrightarrow \mathfrak{Gr}$, $n \geq 0$, relative to the cotriple $\mathcal{F}$, by setting

$$L_n^{\mathcal{F}}T(X) = \pi_n(T(F_*(X))) \quad \text{and} \quad L_n^{\mathcal{F}}T(f) = \pi_n(T(F^{n+1}(f)))$$

for any object $X \in \mathcal{U}$ and any morphism $f \in \mathcal{U}$.

Now we compare these derived functors giving the following result of [103].

Proposition 1.10. *Let $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ be a covariant functor. There is an isomorphism*

$$L_n^{\mathcal{P}}T \cong L_n^{\mathcal{F}}T, \quad n \geq 0.$$

In spite of Proposition 1.10 the preferable usage of one of these derived functors is given in Chap. 4, Theorems 4.17 and 4.19 (derived functors relative to cotriple), and Theorem 4.23 (derived functors relative to projective class).

2. N-Fold Čech Derived Functors

In this section, $\mathcal{U}$ always denotes a category with finite limits and $\mathcal{P}$ a projective class in the category $\mathcal{U}$, unless otherwise stated.

2.1. Čech derived functors. Given an object X and a morphism $\alpha : P \longrightarrow X$ in the category $\mathcal{U}$, denote by $\underbrace{P \times_X \cdots \times_X P}_{(n+1)\text{-times}}$ the limit of the finite diagram $\{\alpha : P \longrightarrow X\}_{i \in \overline{0,n}}$ in the category $\mathcal{U}$ with natural morphisms $\alpha_i : \underbrace{P \times_X \cdots \times_X P}_{(n+1)\text{-times}} \longrightarrow P$, $0 \le i \le n$, such that $\alpha\alpha_i = \alpha\alpha_j$ for all $0 \le i, j \le n$.

An augmented simplicial object $(\check{C}(\alpha)_*, \alpha, X)$ in the category $\mathcal{U}$, where $\check{C}(\alpha)_n = \underbrace{P \times_X \cdots \times_X P}_{(n+1)\text{-times}}$ for $n \ge 0$, the face morphism $d_i^n : \check{C}(\alpha)_n \longrightarrow \check{C}(\alpha)_{n-1}$, $0 \le i \le n$, is induced by the morphisms $(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_n)$, and the degeneracy morphism $s_i^n : \check{C}(\alpha)_n \longrightarrow \check{C}(\alpha)_{n+1}$, $0 \le i \le n$, is induced by the morphisms $(\alpha_0, \dots, \alpha_i, \alpha_i, \alpha_{i+1}, \dots, \alpha_n)$, will be called the *Čech augmented complex* for α (see also [60, 103]). If P belongs to the projective class $\mathcal{P}$ and α is a $\mathcal{P}$-epimorphism, $(\check{C}(\alpha)_*, \alpha, X)$ will be called a *Čech resolution* of X. Note that the objects $\check{C}(\alpha)_n$, $n \ge 1$, do not usually belong to the projective class $\mathcal{P}$.

Definition 1.11. Let $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ be a covariant functor. Define the *ith Čech derived functor* $\mathcal{L}_i T : \mathcal{U} \longrightarrow \mathfrak{Gr}$, $i \ge 0$, of the functor T by choosing, for each object $X \in \mathcal{U}$, a $\mathcal{P}$-epimorphism $\alpha : P \longrightarrow X$ with $P \in \mathcal{P}$ and setting

$$\mathcal{L}_i T(X) = \pi_i(T\check{C}(\alpha)_*).$$

The next lemma shows that this definition does not depend on the choice of a $\mathcal{P}$-epimorphism $\alpha : P \longrightarrow X$ with $P \in \mathcal{P}$, and the functors $\mathcal{L}_i T$, $i \ge 0$, are well defined.

Lemma 1.12. *Let $\alpha : P \longrightarrow X$, $\beta : Q \longrightarrow Y$, and $\lambda : X \longrightarrow Y$ be any morphisms in the category $\mathcal{U}$, and f_*, $g_* : \check{C}(\alpha)_* \longrightarrow \check{C}(\beta)_*$ be morphisms of simplicial objects over λ. Then f_* and g_* are pseudo-simplicially homotopic, $f_* \simeq g_*$.*

Proof. We only construct the pseudo-simplicial homotopy. In fact, the morphism $h_i : \check{C}(\alpha)_n \longrightarrow \check{C}(\beta)_{n+1}$, $0 \le i \le n$, is naturally induced by the morphisms $(g\alpha_0, \dots, g\alpha_i, f\alpha_i, \dots, f\alpha_n)$, where $g = g_0$ and $f = f_0$. □

The prime usage of the Čech derived functors are in the classical group (co)homology theory [103–105]. Here is an illustration. It is a classical fact that the nth non-Abelian derived functor of the functor $H_1 : \mathfrak{Gr} \longrightarrow Ab\mathfrak{Gr}$ is isomorphic to $(n+1)$th group homology. So the functor H_1 determines all higher homologies. With the use of Čech derived functors, the following beautiful result of [105] leads one to think that the functor $H_2 : \mathfrak{Gr} \longrightarrow Ab\mathfrak{Gr}$ determines also all higher homologies.

Proposition 1.13. *For $n \ge 1$, there are isomorphisms*

$$H_{2n} \cong \mathcal{L}_n H_n \quad \textit{and} \quad H_{2n+1} \cong \mathcal{L}_n H_{n+1},$$

where $\mathcal{P}$ is the projective class of free groups in the category $\mathfrak{Gr}$.

In Chap. 3 the Čech derived functors of the group abelianization functor enlighten our motivation to develop the n-fold analog of some of this theory.

2.2. Construction of n-fold Čech derived functors. In this section, we generalize the notion of the Čech derived functors to that of the n-fold Čech derived functors of group-valued functors.

The subsets of $\langle n\rangle$ are ordered by inclusion. This ordered set determines in the usual way a category $\mathcal{C}_n$. For every pair (A,B) of subsets with $A\subseteq B\subseteq\langle n\rangle$, there is the unique morphism $\rho_B^A : A\longrightarrow B$ in $\mathcal{C}_n$. It is easy to see that any morphism in the category $\mathcal{C}_n$, not an identity, is generated by ρ_B^A for all $A\subseteq\langle n\rangle$, $A\neq\langle n\rangle$, $B=A\cup\{j\}$, $j\notin A$.

An *n-cube in the category $\mathcal{U}$* is a functor $\mathfrak{F}:\mathcal{C}_n\longrightarrow\mathcal{U}$, $A\longmapsto\mathfrak{F}_A$, $\rho_B^A\longmapsto\alpha_B^A$. A *morphism between n-cubes* $\mathfrak{F},\mathfrak{Q}:\mathcal{C}_n\longrightarrow\mathcal{U}$ is a natural transformation $\kappa:\mathfrak{F}\longrightarrow\mathfrak{Q}$.

Let $A\subseteq\langle n\rangle$ and consider two full subcategories of the category $\mathcal{C}_n$: $\mathcal{C}_n^A$ is the category of all subsets of $\langle n\rangle$ containing the subset A and $\mathcal{C}_n^{\overline{A}}$ is the category of all subsets of $\langle n\rangle$ having the trivial intersection with the subset A. For a given n-cube in the category $\mathcal{U}$, and A as above, denote by $\mathfrak{F}^A$ and $\mathfrak{F}^{\overline{A}}$ the functors induced by the restriction of the functor $\mathfrak{F}$ to the subcategories $\mathcal{C}_n^A$ and $\mathcal{C}_n^{\overline{A}}$ respectively. For a given morphism of n-cubes $\kappa:\mathfrak{F}\longrightarrow\mathfrak{Q}$ in the category $\mathcal{U}$ denote by $\kappa^A:\mathfrak{F}^A\longrightarrow\mathfrak{Q}^A$ the natural transformation induced by restriction of the natural transformation κ.

Example 1.14.

(a) Let $(G;R_1,\dots,R_n)$ be a normal $(n+1)$-ad of groups. These data naturally determine an n-cube of groups $\mathfrak{F}$ as follows: for any $A\subseteq\langle n\rangle$, let $\mathfrak{F}_A=G/\prod\limits_{i\in A}R_i$; for the inclusion $A\subseteq B$, let $\alpha_B^A:\mathfrak{F}_A\longrightarrow\mathfrak{F}_B$ be the natural homomorphism induced by 1_F. This n-cube of groups will be called the *n-cube of groups induced by the normal $(n+1)$-ad of groups*, $(G;R_1,\dots,R_n)$.

(b) Let (X_*,d_0^0,X) be an augmented pseudo-simplicial object in the category $\mathcal{U}$. A natural n-cube $X^{(n)}:\mathcal{C}_n\longrightarrow\mathcal{U}$, $n\geq 1$, in $\mathcal{U}$ is defined as follows:

$$\begin{aligned} X_A^{(n)} &= X_{n-1-|A|} && \text{for all } A\subseteq\langle n\rangle,\\ \alpha_{A\cup\{j\}}^A &= d_{k-1}^{n-1-|A|} && \text{for all } A\neq\langle n\rangle,\ j\notin A, \end{aligned}$$

where $X_{-1}=X$, $\delta(k)=j$, and $\delta:\langle n-|A|\rangle\longrightarrow\langle n\rangle\setminus A$ is the unique monotone bijection.

Given an n-cube $\mathfrak{F}$ in the category $\mathcal{U}$. It is easy to see that there exists a natural morphism $\mathfrak{F}_A\xrightarrow{\alpha_A}\lim_{B\supset A}\mathfrak{F}_B$ for any $A\subseteq\langle n\rangle$, $A\neq\langle n\rangle$.

Definition 1.15. Let X be an object in the category $\mathcal{U}$. An n-cube $\mathfrak{F}$ will be called an *n-presentation of X in the category $\mathcal{U}$* if $\mathfrak{F}_{\langle n\rangle}=X$. An n-presentation $\mathfrak{F}$ of X will be called *$\mathcal{P}$-projective* if the object $\mathfrak{F}_A$ belongs to the projective class $\mathcal{P}$ for all $A\neq\langle n\rangle$ and will be called *$\mathcal{P}$-exact* if the morphism α_A is $\mathcal{P}$-epimorphic for all $A\neq\langle n\rangle$.

Note that for any object of $\mathcal{U}$ we can construct step by step its $\mathcal{P}$-projective $\mathcal{P}$-exact n-presentation (see also fibrant n-presentations of a group in the sense of Brown–Ellis [14]). Moreover, we have the following proposition.

Proposition 1.16. *Let (X_*,d_0^0,X) be an augmented pseudo-simplicial object in the category $\mathcal{U}$.*

(i) *(X_*,d_0^0,X) is a $\mathcal{P}$-projective resolution of X if and only if the n-cube $X^{(n)}$ in $\mathcal{U}$ is a $\mathcal{P}$-exact n-presentation of X for all $n\geq 1$.*

(ii) *If $\mathcal{U}=\mathfrak{Gr}$ and the group morphism d_0^0 induces a natural isomorphism $\pi_0(X_*)\xrightarrow{\widetilde{d}_0^0}X$, then the n-cube of groups $X^{(n)}$, $n\geq 1$, is induced by the normal $(n+1)$-ad of groups $(X_{n-1},\operatorname{Ker}d_0^{n-1},\dots,\operatorname{Ker}d_{n-1}^{n-1})$, i.e.,*

$$X_A^{(n)}\cong X_{n-1}/\prod_{i\in A}\operatorname{Ker}d_{i-1}^{n-1},\quad A\subseteq\langle n\rangle.$$

Proof. (i) is proved by a straightforward calculation. (ii) is implied by the following well-known fact on pseudo-simplicial groups:

$$d_j^n(\operatorname{Ker} d_i^n) = \operatorname{Ker} d_i^{n-1} \quad \text{for} \quad n > 0, \quad 0 \le i < j \le n.$$

The proposition is proved. $\square$

Now let $\mathfrak{F}$ be an n-presentation of the object X in the category $\mathcal{U}$. Applying $\check{C}$ in the n-independent directions, we see that this construction leads naturally to an augmented n-simplicial object in $\mathcal{U}$. Taking the diagonal of this augmented n-simplicial object gives the augmented simplicial object $(\check{C}^{(n)}(\mathfrak{F})_*, \alpha, X)$, called an *augmented n-fold Čech complex for* $\mathfrak{F}$, where $\alpha = \alpha^{\varnothing}_{\langle n\rangle} : \mathfrak{F}_\varnothing \longrightarrow X$. If $\mathfrak{F}$ is a $\mathcal{P}$-projective $\mathcal{P}$-exact n-presentation of X, then $(\check{C}^{(n)}(\mathfrak{F})_*, \alpha, X)$ is called an *n-fold Čech resolution of* X.

Let $X,\ Y \in \mathcal{U}$, $\mathfrak{F}$ and $\mathfrak{Q}$ be n-presentations of X and Y respectively, and $\lambda : X \longrightarrow Y$ be a morphism in $\mathcal{U}$. A morphism $\kappa : \mathfrak{F} \longrightarrow \mathfrak{Q}$ of n-cubes will be called an *extension* of the morphism λ if $\kappa_{\langle n\rangle} = \lambda$.

Theorem 1.17. *Let $\mathfrak{F}$ and $\mathfrak{Q}$ be $\mathcal{P}$-projective and $\mathcal{P}$-exact n-presentations of given objects X and Y in $\mathcal{U}$, respectively. Then any morphism $\lambda : X \longrightarrow Y$ in $\mathcal{U}$ can be extended to a morphism $\kappa : \mathfrak{F} \longrightarrow \mathfrak{Q}$ of n-cubes that naturally induces a morphism $\widetilde{\kappa}_*$ of simplicial objects*

$$\begin{tikzcd}
\check{C}^{(n)}(\mathfrak{F})_* \arrow[r, "\alpha"] \arrow[d, "\widetilde{\kappa}_*"'] & X \arrow[d, "\lambda"] \\
\check{C}^{(n)}(\mathfrak{Q})_* \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

over λ. Furthermore, any two such extensions $\kappa, \pi : \mathfrak{F} \longrightarrow \mathfrak{Q}$ of λ induce pseudo-simplicially homotopic morphisms $\widetilde{\kappa}_*$, $\widetilde{\pi}_*$ of simplicial objects, $\widetilde{\kappa}_* \simeq \widetilde{\pi}_*$.

Proof. We begin by showing the existence of a morphism of n-cubes $\kappa : \mathfrak{F} \longrightarrow \mathfrak{Q}$ in $\mathcal{U}$ extending the morphism $\lambda : X \longrightarrow Y$.

Since $\mathfrak{F}$ is $\mathcal{P}$-projective and $\mathfrak{Q}$ is $\mathcal{P}$-exact, there exists a morphism $\kappa_{\langle n\rangle\setminus\{i\}} : \mathfrak{F}_{\langle n\rangle\setminus\{i\}} \longrightarrow \mathfrak{Q}_{\langle n\rangle\setminus\{i\}}$ for all $i \in \langle n\rangle$, such that $\alpha^{\langle n\rangle\setminus\{i\}}_{\langle n\rangle}\kappa_{\langle n\rangle\setminus\{i\}} = \lambda\alpha^{\langle n\rangle\setminus\{i\}}_{\langle n\rangle}$. Assume that for some $A \subsetneq \langle n\rangle$ and for all $B \supset A$, $B \subseteq \langle n\rangle$, there exists a morphism $\kappa_B : \mathfrak{F}_B \longrightarrow \mathfrak{Q}_B$ such that $\alpha^B_C\kappa_B = \kappa_C\alpha^B_C$, $C \supseteq B$. Then as an immediate consequence we have the induced morphism $\overline{\kappa} : \lim_{B\supset A}\mathfrak{F}_B \longrightarrow \lim_{B\supset A}\mathfrak{Q}_B$. Using again the facts that $\mathfrak{F}$ is $\mathcal{P}$-projective and $\mathfrak{Q}$ is $\mathcal{P}$-exact, we see that there exists a morphism $\kappa_A : \mathfrak{F}_A \longrightarrow \mathfrak{Q}_A$ such that $\alpha_A\kappa_A = \overline{\kappa}\alpha_A$. Clearly, the constructed morphism of n-cubes $\kappa : \mathfrak{F} \longrightarrow \mathfrak{Q}$ naturally induces a unique morphism of augmented n-simplicial objects, and applying the diagonal gives a morphism of simplicial objects $\widetilde{\kappa}_* : \check{C}^{(n)}(\mathfrak{F})_* \longrightarrow \check{C}^{(n)}(\mathfrak{Q})_*$ over the morphism λ.

We need to prove the remaining part of the assertion first in a particular case.

Particular Case. *Let $\kappa,\ \pi : \mathfrak{F} \longrightarrow \mathfrak{Q}$ be two extensions of the morphism $\lambda : X \longrightarrow Y$ and $l \in \langle n\rangle$. Let $\kappa^{\{l\}} = \pi^{\{l\}} : \mathfrak{F}^{\{l\}} \longrightarrow \mathfrak{Q}^{\{l\}}$; then the respective induced morphisms of simplicial objects $\widetilde{\kappa}_*, \widetilde{\pi}_* : \check{C}^{(n)}(\mathfrak{F})_* \longrightarrow \check{C}^{(n)}(\mathfrak{Q})_*$ over λ are pseudo-simplicially homotopic.*

The construction of $\check{C}^{(n)}$ directly implies that for any n-cube of groups $\mathfrak{F}$, $\check{C}^{(n)}(\mathfrak{F})_*$ is the diagonal of a bisimplicial object X_{**} induced by applying the ordinary Čech complex construction $\check{C}$ to the morphism of simplicial objects $\check{C}^{(n-1)}(\mathfrak{F}^{\overline{\{l\}}})_* \longrightarrow \check{C}^{(n-1)}(\mathfrak{F}^{\{l\}})_*$.

By assumption, the extensions κ and π of the morphism λ induce a commutative diagram of simplicial objects

$$\begin{array}{ccc}
\check{C}^{(n-1)}(\mathfrak{F}^{\overline{\{l\}}})_* & \overset{\widetilde{\kappa}'_*}{\underset{\widetilde{\pi}'_*}{\rightrightarrows}} & \check{C}^{(n-1)}(\mathfrak{Q}^{\overline{\{l\}}})_* \\
\downarrow & & \downarrow \\
\check{C}^{(n-1)}(\mathfrak{F}^{\{l\}})_* & \overset{\widetilde{\kappa}''_*}{\underset{\widetilde{\pi}''_*}{\rightrightarrows}} & \check{C}^{(n-1)}(\mathfrak{Q}^{\{l\}})_*,
\end{array}$$

where $\widetilde{\kappa}''_* = \widetilde{\pi}''_*$, which implies there are morphisms of simplicial objects of simplicial objects in $\mathcal{U}$

$$\begin{array}{ccc}
X_{**} & \overset{\overline{\kappa}_{**}}{\underset{\overline{\pi}_{**}}{\rightrightarrows}} & Y_{**} \\
\downarrow & & \downarrow \\
\check{C}^{(n-1)}(\mathfrak{F}^{\{l\}})_* & \xrightarrow[\widetilde{\kappa}''_*=\widetilde{\pi}''_*]{} & \check{C}^{(n-1)}(\mathfrak{Q}^{\{l\}})_*
\end{array}$$

over the morphism of simplicial objects $\widetilde{\kappa}''_* = \widetilde{\pi}''_*$ in $\mathcal{U}$.

The following lemma will be needed.

Lemma 1.18. *Let X_{**}, Y_{**} be bisimplicial objects in the category $\mathcal{U}$ and $\alpha_{**}, \beta_{**} : X_{**} \longrightarrow Y_{**}$ be morphisms of bisimplicial objects. Let there exist a vertical (horizontal) pseudo-simplicial homotopy h^v (h^h) between the induced morphisms of simplicial objects $\alpha_{m*}, \beta_{m*} : X_{m*} \longrightarrow Y_{m*}$ (α_{*m}, $\beta_{*m} : X_{*m} \longrightarrow Y_{*m}$) for all $m \geq 0$, such that the following conditions hold:*

$$d^h_j h^v_i = h^v_i d^h_j \qquad (d^v_j h^h_i = h^h_i d^v_j).$$

Then the induced morphisms of simplicial objects $\widetilde{\alpha}_*, \widetilde{\beta}_* : \Delta X_* \longrightarrow \Delta Y_*$ are pseudo-simplicially homotopic, $\widetilde{\alpha}_* \simeq \widetilde{\beta}_*$, where ΔX_* and ΔY_* are the diagonal simplicial objects of X_{**} and Y_{**}, respectively.

Proof. We can construct the required homotopy in the following way: $h'_i = h^v_i s^h_i : X_{nn} \longrightarrow Y_{n+1,n+1}$, $0 \leq i \leq n$.

Now we must verify the standard identities for pseudo-simplicial homotopy (see Definition 1.2). In fact,

$$\begin{aligned}
d^v_0 d^h_0 h^v_0 s^h_0 &= d^v_0 h^v_0 d^h_0 s^h_0 = d^v_0 h^v_0 = \alpha_{nn}, \\
d^v_{n+1} d^h_{n+1} h^v_n s^h_n &= d^v_{n+1} h^v_n d^h_{n+1} s^h_n = d^v_{n+1} h^v_n = \beta_{nn}, \\
d^v_i d^h_i h^v_j s^h_j &= d^v_i h^v_j d^h_i s^h_j = \begin{cases} h^v_{j-1} d^v_i s^h_{j-1} d^h_i = h^v_{j-1} s^h_{j-1} d^v_i d^h_i, & i < j, \\ h^v_j d^v_{i-1} s^h_j d^h_{i-1} = h^v_j s^h_j d^v_{i-1} d^h_{i-1}, & i > j+1, \end{cases} \\
d^v_{j+1} d^h_{j+1} h^v_{j+1} s^h_{j+1} &= d^v_{j+1} h^v_{j+1} d^h_{j+1} s^h_{j+1} = d^v_{j+1} h^v_{j+1} = d^v_{j+1} h^v_j d^h_{j+1} s^h_j = d^v_{j+1} d^h_{j+1} h^v_j s^h_j.
\end{aligned}$$

The lemma is proved. □

Returning to the main proof, using Lemma 1.12, it is easy to see that there exists a vertical homotopy h^v between the induced morphisms of simplicial objects $\overline{\kappa}_{m*}, \overline{\pi}_{m*} : F_{m*} \longrightarrow Q_{m*}$ for all $m \geq 0$, such that $d^h_j h^v_i = h^v_i d^h_j$. Applying Lemma 1.18, we see that there is a pseudo-simplicial homotopy between the morphisms of simplicial objects $\widetilde{\kappa}_*, \widetilde{\pi}_* : \check{C}^{(n)}(\mathfrak{F})_* \longrightarrow \check{C}^{(n)}(\mathfrak{Q})_*$ in the category $\mathcal{U}$.

Now we return to the *general case*, showing for any two extensions κ, $\pi : \mathfrak{F} \longrightarrow \mathfrak{Q}$ of amorphism $\lambda : X \longrightarrow Y$ the existence of extensions $\kappa_1, \ldots, \kappa_{n-1} : \mathfrak{F} \longrightarrow \mathfrak{Q}$ of λ such that $\widetilde{\kappa}_* \simeq \widetilde{\kappa}_{1_*}$, $\widetilde{\kappa}_{1_*} \simeq \widetilde{\kappa}_{2_*}, \ldots, \widetilde{\kappa}_{n-2_*} \simeq \widetilde{\kappa}_{n-1_*}$, $\widetilde{\kappa}_{n-1_*} \simeq \widetilde{\pi}_*$ which, of course, implies that $\widetilde{\kappa}_* \simeq \widetilde{\pi}_*$. In fact, we can construct an extension $\kappa_1 : \mathfrak{F} \longrightarrow \mathfrak{Q}$ in the following way: let $\kappa_1^{\{1\}} = \kappa^{\{1\}}$ and $\kappa_1^{\langle n\rangle\setminus\{1\}} = \pi^{\langle n\rangle\setminus\{1\}}$. We complete the construction of κ_1 using the technique above and the facts that $\mathfrak{F}$ is a $\mathcal{P}$-projective and $\mathfrak{Q}$ is an $\mathcal{P}$-exact n-presentation of the objects X and Y in $\mathcal{U}$, respectively.

We construct an extension κ_i for all $2 \le i \le n-1$ as follows: let $\kappa_i^{\{i\}} = \kappa_{i-1}^{\{i\}}$ and $\kappa_i^{\langle n\rangle\setminus\langle i\rangle} = \pi^{\langle n\rangle\setminus\langle i\rangle}$. We complete again the construction of κ_i using the above technique and the facts that $\mathfrak{F}$ is $\mathcal{P}$-projective and $\mathfrak{Q}$ is $\mathcal{P}$-exact.

The construction of κ_i, $1 \le i \le n-1$, and our already proved particular case imply that $\widetilde{\kappa}_* \simeq \widetilde{\kappa}_{1_*}$, $\widetilde{\kappa}_{1_*} \simeq \widetilde{\kappa}_{2_*}, \dots, \widetilde{\kappa}_{n-2_*} \simeq \widetilde{\kappa}_{n-1_*}$, $\widetilde{\kappa}_{n-1_*} \simeq \widetilde{\pi}_*$. □

Using this comparison theorem, we state the following definition.

Definition 1.19. Let $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ be a covariant functor. Define the ith n-fold Čech derived functor $\mathcal{L}_i^{n\text{-fold}}T : \mathcal{U} \longrightarrow \mathfrak{Gr}$, $i \ge 0$, of the functor T by choosing for each X in $\mathcal{U}$, a $\mathcal{P}$-projective $\mathcal{P}$-exact n-presentation $\mathfrak{F}$ and setting
$$\mathcal{L}_i^{n\text{-fold}}T(X) = \pi_i(T\check{C}^{(n)}(\mathfrak{F})_*),$$
where $(\check{C}^{(n)}(\mathfrak{F})_*, \alpha, X)$ is the n-fold Čech resolution of X.

Later, in Chap. 3, we provide explicit calculations of the n-fold Čech derived functors of "nilization of degree k" functor, Z_k, $k \ge 2$, and of the crossed n-cube abelianization functor.

2.3. Some properties of n-fold Čech derived functors. We recall the notion of cosheaf in the sense of [103] (see also [60]). A functor $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ is called *cosheaf* over $(\mathcal{U}, \mathcal{P})$ if for any $\mathcal{P}$-epimorphism $\alpha : Y \longrightarrow X$ the sequence of groups

$$T(Y \times_X Y) \rightrightarrows T(Y) \xrightarrow{T(\alpha)} T(X) \longrightarrow 1 \ ,$$

is exact. An important example of cosheaf is the functor $Z_P : \mathcal{U} \longrightarrow \mathfrak{Gr}$, $P \in \mathcal{P}$, defined as follows: for an object $X \in \mathcal{U}$ let $Z_P(X)$ be the free group generated by the set $\mathrm{Hom}_{\mathcal{U}}(P, X)$. Let us denote the category of cosheaves over $(\mathcal{U}, \mathcal{P})$ by $\mathbb{CS}(\mathcal{U}, \mathcal{P})$ and $\mathfrak{Q}$ be the projective class in the category $\mathbb{CS}(\mathcal{U}, \mathcal{P})$ generated by the cosheaves Z_P, which means that any object of $\mathfrak{Q}$ is a retract of coproducts of cosheaves of the form Z_P (see [103] and [60, Proposition 2.29]). Then, for any object $X \in \mathcal{U}$, we can define the section functor $\Gamma_X : \mathbb{CS}(\mathcal{U}, \mathcal{P}) \longrightarrow \mathfrak{Gr}$ by $\Gamma_X(T) = T(X)$ for all $T \in \mathbb{CS}(\mathcal{U}, \mathcal{P})$. The non-Abelian derived functors of this Γ_X functor will be considered in Chap. 4, Sec. 3.

Proposition 1.20 (see [103])**.** *The following conditions are equivalent*:

(i) *T is a cosheaf over $(\mathcal{U}, \mathcal{P})$*;

(ii) *for any $\mathcal{P}$-epimorphism $P \longrightarrow X$ there is an exact sequence of groups*

$$T(P \times_X P) \rightrightarrows T(P) \longrightarrow T(X) \longrightarrow 1,$$

where $P \in \mathcal{P}$;

(iii) *the natural transformation $\tau_T : L_0^{\mathcal{P}}T \longrightarrow T$ is an equivalence of functors.*

We recall the following notion from [103]. Given an object $X \in \mathcal{U}$, an augmented pseudo-simplicial object (X_*, d_0^0, X) is called a contractible $\mathcal{P}$-resolution of X if for all $P \in \mathcal{P}$ the augmented pseudo-simplicial set $(\mathrm{Hom}_{\mathcal{U}}(P, X_*), \mathrm{Hom}_{\mathcal{U}}(P, d_0^0), \mathrm{Hom}_{\mathcal{U}}(P, X))$ is contractible.

The next lemma is useful. The proof is routine.

Lemma 1.21. *Let $\mathfrak{F}$ be a $\mathcal{P}$-exact n-presentations of a given object X in the category $\mathcal{U}$. Then an augmented n-fold Čech complex $(\check{C}^{(n)}(\mathfrak{F})_*, \alpha, X)$ is a contractible $\mathcal{P}$-resolution of X.*

The following propositions establish a reasonable link between n-fold Čech derived functors and non-Abelian derived functors for a given cosheaf functor in terms of spectral sequences.

Proposition 1.22. *Let $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ be a cosheaf over $(\mathcal{U}, \mathcal{P})$, $X \in \mathcal{U}$ and $n \geq 1$. Then there exists a spectral sequence*

$$E^2_{p,q} = \mathcal{L}^{n\text{-}fold}_p(\mathcal{L}^{\mathcal{P}}_q T)(X) \Longrightarrow \mathcal{L}^{\mathcal{P}}_{p+q} T(X).$$

Moreover, for $E^2_{0,q} = 0$, $q > 0$, there are an isomorphism

$$\mathcal{L}^{n\text{-}fold}_1 T(X) \cong \mathcal{L}^{\mathcal{P}}_1 T(X)$$

and an epimorphism

$$\mathcal{L}^{\mathcal{P}}_2 T(X) \longrightarrow \mathcal{L}^{n\text{-}fold}_2 T(X).$$

Proof. The existence of such spectral sequence directly follows from Lemma 1.21 and [60, Theorem 2.35]. The rest of the assertion is obvious. □

Using Proposition 1.22, we easily see that for a cosheaf $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ there is an isomorphism

$$\mathcal{L}^{n\text{-fold}}_1 T \cong \mathcal{L}^{(n-1)\text{-fold}}_1 T, \quad n \geq 2.$$

Moreover, we have a natural connection between the higher n-fold and $(n-1)$-fold Čech derived functors. In fact, we have the following proposition.

Proposition 1.23. *Let $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ be a cosheaf over $(\mathcal{U}, \mathcal{P})$, $X \in \mathcal{U}$ and $n \geq 2$. Then there is a spectral sequence*

$$E^2_{pq} \Longrightarrow \mathcal{L}^{n\text{-}fold}_{p+q} T(X),$$

where $E^2_{0q} = 0$, $q > 0$ and $E^2_{p0} = \mathcal{L}^{(n-1)\text{-}fold}_p T(X)$, $p \geq 0$.

Proof. By the construction, for any $\mathcal{P}$-projective $\mathcal{P}$-exact n-presentation $\mathfrak{F}$ of G, $\check{C}^{(n)}(\mathfrak{F})_*$ is the diagonal of a bisimplicial object X_{**} induced by applying the ordinary Čech complex construction $\check{C}$ to the morphism of $(n-1)$-fold Čech complexes $\theta_* : \check{C}^{(n-1)}(\mathfrak{F}^{\overline{\{n\}}})_* \longrightarrow \check{C}^{(n-1)}(\mathfrak{F}^{\{n\}})_*$, where θ_i, $i \geq 0$, is a $\mathcal{P}$-epimorphism.

Now applying the cosheaf T dimensionwise, we denote the resulting bisimplicial object by $T(X_{**})$. By [107], there is a spectral sequence

$$E^2_{pq} \ \Longrightarrow \ \mathcal{L}^{n\text{-fold}}_{p+q} T(X).$$

Using Proposition 1.20, we have the isomorphism $E^2_{p0} \cong \mathcal{L}^{(n-1)\text{-fold}}_p T(X)$, $p \geq 0$. Moreover, since X_{00} belongs to the projective class $\mathcal{P}$, we have $E^2_{0q} = 0$, $q > 0$. □

Let $\mathcal{U}$ be a variety of groups with operators. This means that the objects are groups together with some additional operations satisfying some identities. Examples are the category of groups, nilpotent groups, or solvable groups of given degree, as well as rings, Lie algebras, and their subcategories of nilpotent or solvable objects.

Let $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ be a functor such that $T(0) = 0$. We recall (see [105]) that the simplicial degree of the functor T is less than or equal to d if for any $n \geq 0$ and any simplicial object X_* whose *length* is $\leq n$, denoted by $l(X_*) \leq n$ and which means that $NX_i = 1$ for $i > n$, we have $l(T(X_*)) \leq dn$. In this case we write $\mathrm{sdeg}(T) \leq d$.

Corollary 1.24. *Let $\mathcal{U}$ be a pointed variety of groups with operators, $T : \mathcal{U} \longrightarrow \mathfrak{Gr}$ a cosheaf over $(\mathcal{U}, \mathcal{P})$ whose simplicial degree is equal to* 1, $\mathrm{sdeg}(T) = 1$, *and $X \in \mathcal{U}$. Then $\mathcal{L}^{n\text{-}fold}_i T(X) = 0$, $i > n$ and for the spectral sequence of Proposition* 1.23, *there are an exact sequence of groups*

$$0 \longrightarrow E^2_{n-2,1} \longrightarrow \mathcal{L}^{n\text{-}fold}_{n-1} T(X) \longrightarrow \mathcal{L}^{(n-1)\text{-}fold}_{n-1} T(X) \longrightarrow$$

$$\longrightarrow E^2_{n-3,1} \longrightarrow \cdots \longrightarrow E^2_{11} \longrightarrow \mathcal{L}^{n\text{-}fold}_2 T(X) \longrightarrow \mathcal{L}^{(n-1)\text{-}fold}_2 T(X) \longrightarrow 0$$

and an isomorphism

$$E^2_{n-1,1} \cong \mathcal{L}_n^{n\text{-}fold} T(X).$$

Proof. Since $\mathrm{sdeg}(T) = 1$, $E^2_{pq} = 0$ either for $p > n-1$ or $q > 1$, which completes the proof. $\square$

We refer, as a good example, to Corollary 1.24 when $\mathcal{U} = \mathfrak{Gr}$ and T is the 'nilization of degree k' functor, Z_k, $k \geq 2$, which we examine in Chap. 3.

Chapter 2

HOMOTOPY $(n+1)$-TYPES AND HOMOLOGY

In the 1940s Whitehead [127] introduced the algebraic notion of a crossed module as a means of representing connected CW-spaces whose homotopy groups are trivial in dimensions ≥ 2 for solving some homotopical problems. Subsequently, MacLane and Whitehead used it to represent the third group cohomology [94]. Later, in [87], generalizing the notion of crossed modules, Loday gave the foundation of a theory of algebraic models of connected CW-spaces whose homotopy groups are trivial in dimensions greater than $n+1$, called cat n-groups. These algebraic structures have nice properties and satisfy a form of generalized Van Kampen theorem [18, 19]. Other equivalent algebraic models of homotopy $(n+1)$-types are more combinatorial algebraic systems, crossed n-cubes, invented by Ellis and Steiner in [49].

A number of papers of the last years are dedicated to the investigation of homological properties of these objects. Ellis [46] and Baues [6] introduced and investigated the (co)homology of crossed modules as the (co)homology of its classifying space, neglecting its algebraic structure. In [85] Ladra and Grandjeán gave the first approach to an internal homology theory of crossed modules taking into account its algebraic structure. Later, in [21] Carrasco, Cegarra, and Grandjeán made the observation that the category of crossed modules is an algebraic category, that is, there is a tripleable "underlying" functor from the category of crossed modules to the category of sets, implying a purely algebraic construction and study of cotriple (co)homology theory. In [56], Grandjeán, Ladra, and Pirashvili gave a connection of these two homology theories of crossed modules by the dimension shifting isomorphism, while Casas, Ellis, Ladra, and Pirashvili in [23] have recently generalized this result to higher patterns for cat n-groups.

This chapter is devoted to the investigation of crossed n-cubes, equivalently cat n-groups, in various aspects.

In Sec. 1, investigating the diagonal of the n-simplicial multinerve, $E^{(n)}(-)_*$, of crossed n-cubes of groups, we relate naturally this construction to the n-fold Čech complexes. Moreover, for an inclusion crossed n-cube of groups, $\mathcal{M}$, given by a normal $(n+1)$-ad of groups, we construct a new induced crossed n-cube $\mathcal{B}_k(\mathcal{M})$, $k \geq 2$ (Proposition 2.7) and show the existence of an isomorphism of simplicial groups $Z_k E^{(n)}(\mathcal{M})_* \cong E^{(n)}(\mathcal{B}_k(\mathcal{M}))_*$, where $E^{(n)}(\mathcal{M})_*$ denotes the diagonal of the n-simplicial nerve of the crossed n-cube of groups $\mathcal{M}$ (see Proposition 2.9). We provide a more general result, namely the commutation of the crossed n-cube abelianization functor $\mathfrak{Ab}^{(n)}$ with the diagonal of the m-simplicial multinerve $E^{(m)}$ (see Proposition 2.10), which plays an essential role in obtaining generalized Hopf type formulas for the homology of crossed n-cubes.

We study some properties of the mapping cone complex of a morphism of (non-Abelian) group complexes introduced in [87]. In particular, for a given morphism of pseudo-simplicial groups $\alpha : G_* \longrightarrow H_*$ the natural morphism $\kappa : NM_*(\alpha) \longrightarrow C_*(\widetilde{\alpha})$ induces isomorphisms of their homology groups, where $C_*(\widetilde{\alpha})$ is the mapping cone complex of the induced morphism of the Moore complexes and $NM_*(\alpha)$ is the Moore complex of a new pseudo-simplicial group constructed using α (see Proposition 2.11).

(Similar results have recently been found by Conduché, [29].) Using this result, we derive purely algebraically the result of [87, Proposition 3.4], giving for a crossed n-cube of groups $\mathcal{M}$ an isomorphism between the homotopy groups of $E^{(n)}(\mathcal{M})_*$ and the corresponding homology groups of a chain complex of groups $C_*(\mathcal{M})$ (see Proposition 2.13). In particular, we give an explicit computation of the nth homotopy group of the simplicial group $E^{(n)}(\mathcal{M})_*$.

In Sec. 2 we show that the category $\mathbf{Crs}^n$ is an algebraic category (see also [23]), that is, there is a tripleable forgetful functor from $\mathbf{Crs}^n$ to $\mathbf{Set}$ (Proposition 2.15). The leading cotriple homology of these homotopy $(n+1)$-types is constructed, which will be investigated in Chap. 4 from a Hopf formulas point of view.

Section 3 is devoted to the investigation of homological properties of precrossed modules pursuing the line of Conduché and Ellis [30]. Homology groups modulo q of a precrossed P-module in any dimensions are defined in terms of non-Abelian derived functors, where q is a nonnegative integer. The Hopf formula is proved for the second homology group modulo q of a precrossed P-module (Theorem 2.24). Some other properties of homologies of precrossed P-modules are investigated.

1. Crossed n-Cubes and $\operatorname{cat} n$-Groups

We begin by recalling the following algebraic concept of Whitehead [127].

A *precrossed P-module* (M,μ) over the group P is a group homomorphism $\mu : M \longrightarrow P$ together with an action of P on M, satisfying the following condition:

$$\mu({}^p m) = p\mu(m)p^{-1} \quad \text{for all} \quad dm \in M, \quad p \in P.$$

If, in addition, the precrossed module (M,μ) satisfies the Peiffer identity

$${}^{\mu(m)}m' = mm'm^{-1} \quad \text{for all} \quad m,\ m' \in M,$$

then it is said to be a crossed P-module. Given a crossed module (M,μ), the image of μ is necessarily an ideal in P and the kernel of μ is a P-invariant ideal in the center of M. Moreover, the action of P on $\operatorname{Ker}\mu$ induces an action of $P/\operatorname{Im}\mu$ on $\operatorname{Ker}\mu$, making $\operatorname{Ker}\mu$ a $P/\operatorname{Im}\mu$-module.

A morphism $(\varphi,\psi) : (M,\mu) \longrightarrow (N,\nu)$ of (pre)crossed modules is a commutative square

$$\begin{array}{ccc} M & \xrightarrow{\mu} & P \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ N & \xrightarrow[\nu]{} & Q, \end{array}$$

with $\varphi({}^p m) = {}^{\psi(p)}\varphi(m)$ for all $m \in M$, $p \in P$. Let us denote the category of crossed (precrossed) modules by $\mathcal{CM}$ ($\mathcal{PCM}$) and its subcategory of crossed (precrossed) P-modules with fixed group P by $\mathcal{CM}(P)$ ($\mathcal{PCM}(P)$).

Now we examine two equivalent algebraic models of homotopy $(n+1)$-types, $\operatorname{cat} n$-groups and crossed n-cubes [49, 87], generalizing the notion of crossed modules, and recall some well-known results and notions for our future purpose.

1.1. $\operatorname{cat} n$-group. A $\operatorname{cat} n$-group is a group G together with $2n$ endomorphisms $s_i,\ t_i : G \longrightarrow G$, $1 \le i \le n$, such that

$$t_i s_i = s_i, \quad s_i t_i = t_i, \quad [\operatorname{Ker} s_i, \operatorname{Ker} t_i] = 1 \quad \text{for all} \quad i,$$
$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \quad \text{for} \quad i \neq j.$$

A morphism of $\operatorname{cat} n$ groups $f : (G, s_i, t_i) \longrightarrow (G', s'_i, t'_i)$ is a group homomorphism $f : G \longrightarrow G'$ satisfying $s'_i f = f s_i$ and $t'_i f = f t_i$ for $1 \le i \le n$. We obtain the category of $\operatorname{cat} n$-groups denoted by $\mathbf{Cat}^n$.

Later, in [49], the higher-dimensional analogs of crossed modules were introduced, called crossed n-cubes. These generalize normal $(n+1)$-ads of groups in the same way that crossed modules generalize normal subgroups.

1.2. Crossed n-cube. A *crossed n-cube of groups* is a family $\mathcal{M} = \{\mathcal{M}_A : A \subseteq \langle n\rangle\}$ of groups together with homomorphisms $\mu_i : \mathcal{M}_A \longrightarrow \mathcal{M}_{A\setminus\{i\}}$ for $i \in \langle n\rangle$, $A \subseteq \langle n\rangle$ and functions $h : \mathcal{M}_A \times \mathcal{M}_B \longrightarrow \mathcal{M}_{A\cup B}$ for $A,\ B \subseteq \langle n\rangle$, such that if ${}^a b$ denotes $h(a,b)\cdot b$ for $a \in \mathcal{M}_A$ and $b \in \mathcal{M}_B$ with $A \subseteq B$, then for all $a, a' \in \mathcal{M}_A$, $b, b' \in \mathcal{M}_B$, $c \in \mathcal{M}_C$, and $i, j \in \langle n\rangle$, the following conditions hold:

$$
\begin{gathered}
\mu_i(a) = a \quad \text{if} \quad i \notin A,\\
\mu_i\mu_j(a) = \mu_j\mu_i(a),\\
\mu_i h(a,b) = h(\mu_i(a), \mu_i(b)),\\
h(a,b) = h(\mu_i(a), b) = h(a, \mu_i(b)) \quad \text{if} \quad i \in A \cap B,\\
h(a,a') = [a,a'],\\
h(a,b) = h(b,a)^{-1},\\
h(a,b) = 1 \quad \text{if} \quad a = 1 \quad \text{or} \quad b = 1,\\
h(aa',b) = {}^a h(a',b)h(a,b),\\
h(a,bb') = h(a,b)^b h(a,b'),\\
{}^a h(h(a^{-1},b),c)^c h(h(c^{-1},a),b)^b h(h(b^{-1},c),a) = 1,\\
{}^a h(b,c) = h({}^a b, {}^a c) \quad \text{if} \quad A \subseteq B \cap C.
\end{gathered}
$$

Warning: A crossed n-cube of groups gives an n-cube on forgetting structure, but note that there is a reversal of the role of the index A. The top corner of a crossed n-cube is $\mathcal{M}_{\langle n\rangle}$, and that in an n-cube is $\mathfrak{F}_{\varnothing}$. This is due to the fact that an n-cube of groups naturally yields a crossed n-cube as a sort of generalized kernel, as we have seen earlier.

A morphism of crossed n-cubes, $\alpha : \mathcal{M} \longrightarrow \mathcal{N}$, is a family of group homomorphisms $\{\alpha_A : \mathcal{M}_A \longrightarrow \mathcal{N}_A,\ A \subseteq \langle n\rangle\}$ commuting with the μ_i and the h-functions. The resulted category of crossed n-cubes of groups will be denoted by $\mathbf{Crs}^n$.

Now we give the notion of a crossed n-subcube, which is consistent with the categorical notion of subobject in the category $\mathbf{Crs}^n$. We say that a crossed n-cube $\mathcal{M}'$ is a *crossed n-subcube* of $\mathcal{M}$ if $\mathcal{M}'_A$ is a subgroup of $\mathcal{M}_A$, and the homomorphism $\mu'_i : \mathcal{M}'_A \longrightarrow \mathcal{M}'_{A\setminus\{i\}}$ and the function $h' : \mathcal{M}'_A \times \mathcal{M}'_B \longrightarrow \mathcal{M}'_{A\cup B}$ are the restrictions of $\mu_i : \mathcal{M}_A \longrightarrow \mathcal{M}_{A\setminus\{i\}}$ and $h : \mathcal{M}_A \times \mathcal{M}_B \longrightarrow \mathcal{M}_{A\cup B}$ respectively for every $i \in \langle n\rangle$, $A, B \subseteq \langle n\rangle$.

Moreover, a crossed n-subcube $\mathcal{M}'$ of $\mathcal{M}$ is said to be a *normal crossed n-subcube* if $h(a,b') \in \mathcal{M}'_{A\cup B}$ and $h(a',b) \in \mathcal{M}'_{A\cup B}$ for all $a \in \mathcal{M}_A$, $b' \in \mathcal{M}'_B$, $a' \in \mathcal{M}'_A$, $b \in \mathcal{M}_B$.

Let $\alpha : \mathcal{M} \longrightarrow \mathcal{N}$ be a morphism of crossed n-cubes and $\operatorname{Ker}\alpha$ denote the family $\{\operatorname{Ker}\alpha_A : A \subseteq \langle n\rangle\}$ of groups, which essentially is a normal crossed n-subcube of $\mathcal{M}$.

Example 2.1.

(i) A crossed 1-cube is the same as a crossed module, $\mathbf{Crs}^1 = \mathcal{CM}$.

(ii) A crossed 2-cube is the same as a crossed square (for the definition, see [18]). The detailed reformulation is easy.

(iii) Let G be a group and $N_1, \ldots, N_n$ be normal subgroups of G. Let $\mathcal{M}_A = \bigcap\limits_{i\in A} N_i$ for $A \subseteq \langle n\rangle$ (here $\mathcal{M}_{\varnothing}$ is understood to mean G); if $i \in \langle n\rangle$, define $\mu_i :\ \mathcal{M}_A \xrightarrow{i\in A} \mathcal{M}_{A\setminus\{i\}}$ to be the inclusion and given $A,\ B \subseteq \langle n\rangle$, let $h : \mathcal{M}_A \times \mathcal{M}_B \longrightarrow \mathcal{M}_{A\cup B}$ be given by the commutator: $h(a,b) = [a,b]$

for $a \in \mathcal{M}_A$, $b \in \mathcal{M}_B$ (here, of course, $\mathcal{M}_{A\cup B} = \mathcal{M}_A \cap \mathcal{M}_B$). Then $\{\mathcal{M}_A : A \subseteq \langle n\rangle,\ \mu_i,\ h\}$ is a crossed n-cube, called the *inclusion crossed n-cube* given by the *normal $(n+1)$-ad of groups* $(G; N_1, \ldots, N_n)$.

(iv) Let $\mathcal{N}$ be a crossed n-cube and $\mathcal{R}^1, \ldots, \mathcal{R}^m$ be normal crossed n-subcubes of $\mathcal{N}$. Let $A \subseteq \langle m+n\rangle$, $A_1 = A \cap \{n+1, \cdots, n+m\}$, $A_2 = A \cap \langle n\rangle$ and consider $\mathcal{M}_A = \bigcap\limits_{j\in A_1} R^{j-n}_{A_2}$ (here $\bigcap\limits_{j\in\varnothing} R^{j-n}_{A_2}$ is understood to mean $\mathcal{N}_{A_2}$); define $\mu_i : \mathcal{M}_A \xrightarrow{i\in A} \mathcal{M}_{A\setminus\{i\}}$ to be the inclusion $\bigcap\limits_{j\in A_1} R^{j-n}_{A_2} \hookrightarrow \bigcap\limits_{j\in A_1\setminus\{i\}} R^{j-n}_{A_2}$ if $i \in A_1$ and to be induced by $\mu_i : R^{j-n}_{A_2} \longrightarrow R^{j-n}_{A_2\setminus\{i\}}$ if $i \in A_2$; let $h : \mathcal{M}_A \times \mathcal{M}_B \longrightarrow \mathcal{M}_{A\cup B}$ be defined naturally by commutators and h-functions of the crossed n-cubes $\mathcal{N}, \mathcal{R}^1, \ldots, \mathcal{R}^m$. Then $\{\mathcal{M}_A : A \subseteq \langle n\rangle,\ \mu_i,\ h\}$ is a crossed $(m+n)$-cube, called the *crossed $(m+n)$-cube* of groups induced by the *normal $(m+1)$-ad of crossed n-cubes* $(\mathcal{N}; \mathcal{R}^1, \ldots, \mathcal{R}^m)$.

Remark 2.2. Note that for $n = 0$ the construction of (iv) agrees with that of (iii) if we assume that a crossed 0-cube is just a group.

According to [87] the category of cat 1-groups is equivalent to that of crossed modules, and the category of cat 2-groups to that of crossed squares. One of the main result of [49] says that the categories $\mathbf{Crs}^n$ and $\mathbf{Cat}^n$ are equivalent. Namely, we have the following

Theorem 2.3. *There are inverse equivalences of categories*

$$\mathbf{Crs}^n \underset{\Psi^n}{\overset{\Phi^n}{\rightleftarrows}} \mathbf{Cat}^n$$

given by

$$\Phi^n(\mathcal{M}) = \bigvee_{A\subseteq\langle n\rangle} \mathcal{M}_A \Big/ \Big\{ h(a,b) = [a,b] \text{ for all } a \in \mathcal{M}_A,\ b \in \mathcal{M}_B,\ A, B \subseteq \langle n\rangle \Big\}, \quad \mathcal{M} \in \mathbf{Crs}^n$$

and

$$\Psi^n(G)_A = \bigcap_{i\in A} \operatorname{Ker} s_i \cap \bigcap_{i\notin A} \operatorname{Im} s_i, \quad G \in \mathbf{Cat}^n, \quad A \subseteq \langle n\rangle.$$

Throughout this work, we mainly prefer to use crossed n-cubes instead of cat n-groups, except for those cases where using cat n-groups will make things easier to understand.

1.3. Nerve of crossed n-cubes. Given a crossed module, $\mathcal{M} = (\ M \xrightarrow{\mu} P\ )$, the corresponding cat^1-group is $(M \rtimes P, s, t)$, where $s(m,p) = p$ and $t(m,p) = \mu(m)p$. This cat^1-group has an internal category structure within the category $\mathfrak{Gr}$, and the nerve of its category structure forms the simplicial group $E(\mathcal{M})_*$, where $E(\mathcal{M})_n = M \rtimes (\cdots (M \rtimes P) \cdots)$ with n semidirect factors of M, and the face and degeneracy homomorphisms are defined by

$$\begin{aligned}
d_0(m_1, \ldots, m_n, p) &= (m_2, \ldots, m_n, p),\\
d_i(m_1, \ldots, m_n, p) &= (m_1, \ldots, m_i m_{i+1}, \ldots, m_n, p), \quad 0 < i < n,\\
d_n(m_1, \ldots, m_n, p) &= (m_1, \ldots, m_{n-1}, \mu(m_n)p),\\
s_i(m_1, \ldots, m_n, p) &= (m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_n, p), \quad 0 \le i \le n.
\end{aligned}$$

The simplicial group $E(\mathcal{M})_*$ is called the *nerve* of the crossed module $\mathcal{M}$, and its Moore complex is trivial in dimensions ≥ 2. In fact, its Moore complex is just the original crossed module up to isomorphism with M in dimension 1 and P in dimension 0.

For a given crossed n-cube $\mathcal{M}$, there is an associated cat^n-group and hence on applying the crossed module nerve structure E in the n-independent directions, this construction leads naturally to an n-simplicial group, called the *multinerve* of the crossed n-cube $\mathcal{M}$ and denoted by $\mathfrak{Ner}(\mathcal{M})$. Taking the diagonal of this n-simplicial group gives a simplicial group denoted by $E^{(n)}(\mathcal{M})_*$ (see [106]).

Given a crossed n-cube of groups $\mathcal{M} = \{\mathcal{M}_A : A \subseteq \langle n\rangle, \mu_i, h\}$, and any $i \in \langle n\rangle$, by [106, Proposition 5], there is a morphism $\mu_i : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ of crossed $(n-1)$-cubes of groups, where $\mathcal{M}_1 = \{\mathcal{M}_A : A \subseteq \langle n\rangle,\ i \in A\}$ and $\mathcal{M}_0 = \{\mathcal{M}_A : A \subseteq \langle n\rangle,\ i \notin A\}$, such that for each $B \subseteq \langle n-1\rangle$, $\operatorname{Ker}\mu_{i,B}$ is central in $\mathcal{M}_{1,B}$ and $\operatorname{Im}\mu_{i,B}$ is normal in $\mathcal{M}_{1,B}$. Moreover, there is an exact sequence of crossed $(n-1)$-cubes in the obvious sense

$$0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{M}_1 \xrightarrow{\mu_i} \mathcal{M}_0 \longrightarrow \mathcal{N}_0 \longrightarrow 1, \tag{2.1}$$

where $\mathcal{N}_1 = \{\operatorname{Ker}\mu_{i,B} :\ B \subseteq \langle n-1\rangle\}$ and $\mathcal{N}_0 = \{\operatorname{Coker}\mu_{i,B} :\ B \subseteq \langle n-1\rangle\}$ with the natural structure of crossed $(n-1)$-cubes.

The following result will be helpful in the sequel.

Proposition 2.4. *There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \pi_{n-2}(E^{(n-1)}(\mathcal{N}_1)_*) \longrightarrow \pi_{n-1}(E^{(n)}(\mathcal{M})_*) \longrightarrow \pi_{n-1}(E^{(n-1)}(\mathcal{N}_0)_*) \longrightarrow \\ \longrightarrow \pi_{n-3}(E^{(n-1)}(\mathcal{N}_1)_*) \longrightarrow \cdots \longrightarrow \pi_2(E^{(n-1)}(\mathcal{N}_0)_*) \longrightarrow \\ \longrightarrow \pi_0(E^{(n-1)}(\mathcal{N}_1)_*) \longrightarrow \pi_1(E^{(n)}(\mathcal{M})_*) \longrightarrow \pi_1(E^{(n-1)}(\mathcal{N}_0)_*) \longrightarrow 0 \end{aligned}$$

and isomorphisms

$$\pi_0(E^{(n)}(\mathcal{M})_*) \cong \pi_0(E^{(n-1)}(\mathcal{N}_0)_*), \qquad \pi_n(E^{(n)}(\mathcal{M})_*) \cong \pi_{n-1}(E^{(n-1)}(\mathcal{N}_1)_*).$$

Proof. By the construction, $E^{(n)}(\mathcal{M})_*$ is the diagonal of the bisimplicial group M_{**} induced by applying the crossed module nerve construction E to the morphism of simplicial groups $E^{(n-1)}(\mu_i) : E^{(n-1)}(\mathcal{M}_1)_* \longrightarrow E^{(n-1)}(\mathcal{M}_0)_*$. Moreover, applying Lemma B [106] to the exact sequence (2.1) of $(n-1)$-cubes, we have the following exact sequence of simplicial groups

$$0 \longrightarrow E^{(n-1)}(\mathcal{N}_1)_* \longrightarrow E^{(n-1)}(\mathcal{M}_1)_* \xrightarrow{E^{(n-1)}(\mu_i)} E^{(n-1)}(\mathcal{M}_0)_* \longrightarrow E^{(n-1)}(\mathcal{N}_0)_* \longrightarrow 1. \tag{2.2}$$

Hence by [107] there is a spectral sequence

$$E^2_{pq} \implies \pi_{p+q}(E^{(n)}(\mathcal{M})_*),$$

where $E^2_{p0} = \pi_p(E^{(n-1)}(\mathcal{N}_0)_*)$ and $E^2_{p1} = \pi_p(E^{(n-1)}(\mathcal{N}_1)_*)$, $p \geq 0$. Proposition 2.13 implies that $E^2_{pq} = 0$ either for $p > n-1$ or $q > 1$, which completes the proof. $\square$

Now we present a fresh view of the n-fold Čech complexes, relating them to the diagonal of the n-simplicial multinerve of crossed n-cubes of groups, which leads to some ideas that will be useful throughout the next chapter.

Given an n-cube of groups $\mathfrak{F}$, the normal $(n+1)$-ad of groups $(F; R_1, \dots, R_n)$, where $F = \mathfrak{F}_\varnothing$ and $R_i = \operatorname{Ker}\alpha^{\varnothing}_{\{i\}}$, $i \in \langle n\rangle$ will be called the *normal $(n+1)$-ad of groups induced by* $\mathfrak{F}$.

Lemma 2.5. *Let $\mathfrak{F}$ be an n-presentation of a group G in the category $\mathfrak{Gr}$. There is an isomorphism of simplicial groups*

$$E^{(n)}(\mathcal{M})_* \cong \check{C}^{(n)}(\mathfrak{F})_*,$$

where $\mathcal{M}$ is the inclusion crossed n-cube of groups given by the normal $(n+1)$-ad of groups $(F; R_1, \dots, R_n)$ induced by $\mathfrak{F}$.

Proof. For $n = 1$, we only construct the following isomorphism:

$$\lambda_* : \; E(R \hookrightarrow F)_* \xrightarrow{\cong} \check{C}(\alpha)_*,$$

where $F \xrightarrow{\alpha} G$ is a group homomorphism and $R = \operatorname{Ker}\alpha$. In fact, define $\lambda_0 = 1_F$ and $\lambda_n(r_1, \ldots, r_n, f) = (r_1 \cdots r_n f, r_2 \cdots r_n f, \ldots, r_n f, f)$, $n \geq 1$, for all $(r_1, \ldots, r_n, f) \in E(R \hookrightarrow F)_n$.

As the constructions are natural, we get, on repeated application, an isomorphism of n-simplicial groups. Applying the diagonal clearly gives the result. □

Now define the functor

$$E^{(m)} : \mathbf{Crs}^n \longrightarrow \mathfrak{SimplCrs}^{n-m} \quad \text{(simplicial crossed } (n-m)\text{-cubes)}, \tag{2.3}$$

$1 \leq m \leq n$, as follows: given a crossed n-cube $\mathcal{M}$, consider an associated cat n-group G, which is equivalent to a crossed $(n-m)$-cube endowed with m compatible category structures. Then, applying the nerve structure E to the m-independent directions, we see that this construction leads naturally to an m-simplicial crossed $(n-m)$-cube. Then the simplicial crossed $(n-m)$-cube $E^{(m)}(\mathcal{M})_*$ is the diagonal of this m-simplicial crossed $(n-m)$-cube.

Note that this construction depends upon the sequence of the m-independent directions.

An m-cube of crossed n-cubes $\mathfrak{X}$ determines a normal $(m+1)$-ad of crossed n-cubes $(\mathfrak{X}(\varnothing); R^1, \cdots, R^m)$, where $R^i = \operatorname{Ker}\mathfrak{X}(\rho^{\varnothing}_{\{i\}})$, $i \in \langle m \rangle$. This $(m+1)$-ad will be called the *normal $(m+1)$-ad of crossed n-cubes induced by $\mathfrak{X}$.*

The following assertion follows directly from Lemma 2.5.

Corollary 2.6. *Let $\mathfrak{X}$ be an m-presentation of a crossed n-cube $\mathcal{M}$ in the category $\mathbf{Crs}^n$. There is an isomorphism of simplicial crossed n-cubes*

$$\check{C}^{(m)}(\mathfrak{X})_* \cong E^{(m)}(\mathcal{N})_*,$$

where $\mathcal{N}$ is the crossed $(m+n)$-cube of groups given by the normal $(m+1)$-ad of crossed n-cubes $(\mathfrak{X}(\varnothing); R^1, \ldots, R^m)$ induced by $\mathfrak{X}$.

1.4. Abelianization and related functors. It is well known that for an algebraic category $\mathbf{C}$ the obvious inclusion functor of the category of Abelian group objects $\mathfrak{A}b\mathbf{C} \hookrightarrow \mathbf{C}$ has left adjoint $\mathfrak{A}b : \mathbf{C} \longrightarrow \mathfrak{A}b\mathbf{C}$, called the abelianization functor, which plays a fundamental role in the description of homology of objects in the category $\mathbf{C}$. Namely, the kth homology of an object $X \in \mathbf{C}$ is defined to be $\mathfrak{L}_k\mathfrak{A}b(X)$, where $\mathfrak{L}_k\mathfrak{A}b$ denotes the kth derived functor of $\mathfrak{A}b$ in the sense of Quillen [109].

An Abelian group object in $\mathbf{Crs}^n$, an *Abelian crossed n-cube*, is a crossed n-cube whose h maps are trivial. The abelianization functor

$$\mathfrak{Ab}^{(n)} : \mathbf{Crs}^n \longrightarrow \mathfrak{Ab}\mathbf{Crs^n}, \tag{2.4}$$

is given as follows:

(a) for $A \subseteq \langle n \rangle$

$$\mathfrak{Ab}^{(n)}(\mathcal{M})_A = \frac{\mathcal{M}_A}{\prod\limits_{B \cup C = A} D_{B,C}},$$

where $D_{B,C}$ is the subgroup of $\mathcal{M}_A$ generated by the elements $h(b,c)$, $h : \mathcal{M}_B \times \mathcal{M}_C \longrightarrow \mathcal{M}_{B \cup C = A}$ for all $b \in \mathcal{M}_B$, $c \in \mathcal{M}_C$;

(b) if $i \in \langle n \rangle$, the homomorphism $\widetilde{\mu}_i : \mathfrak{Ab}^{(n)}(\mathcal{M})_A \longrightarrow \mathfrak{Ab}^{(n)}(\mathcal{M})_{A \setminus \{i\}}$ is induced by the homomorphism μ_i;

(c) for $A, B \subseteq \langle n \rangle$, the function $\widetilde{h} : \mathfrak{Ab}^{(n)}(\mathcal{M})_A \times \mathfrak{Ab}^{(n)}(\mathcal{M})_B \longrightarrow \mathfrak{Ab}^{(n)}(\mathcal{M})_{A \cup B}$ is induced by h and therefore is trivial, for any $\mathcal{M} = \{\mathcal{M}_A : A \subseteq \langle n \rangle,\ \mu_i,\ h\} \in \mathbf{Crs}^n$.

The functor $\mathfrak{Ab}^{(n)}$ is left adjoint to the inclusion functor $i : \mathfrak{Ab}\mathbf{Crs}^n \hookrightarrow \mathbf{Crs}^n$, as is easily checked.

The equivalent Abelian group object to an Abelian crossed n-cube in the category $\mathbf{Cat}^n$ is just a cat n-group whose underlying group is Abelian, which is called an Abelian cat n-group (see also [23]). Moreover, the abelianization functor

$$\mathfrak{Ab}^{(n)} : \mathbf{Cat}^n \longrightarrow \mathfrak{Ab}\mathbf{Cat}^n \tag{2.5}$$

sends a cat n-group $G = (G, s_i, t_i)$ to the Abelian cat n-group $(G/[G,G], \overline{s_i}, \overline{t_i})$, where $\overline{s_i}$ and $\overline{t_i}$ are induced by s_i and t_i.

Proposition 2.7. *Let $\mathcal{M}$ be an inclusion crossed n-cube given by a normal $(n+1)$-ad of groups $(F; R_1, \dots, R_n)$ and $k \geq 2$. Then there is a crossed n-cube $\mathcal{B}_k(\mathcal{M})$ given as follows:*

(a) *for $A \subseteq \langle n \rangle$*

$$\mathcal{B}_k(\mathcal{M})_A = \bigcap_{i\in A} R_i / D_k(F; A),$$

where

$$D_k(F;A) = \prod_{A_1\cup A_2\cup\cdots\cup A_k = A} \Big[\bigcap_{i\in A_1} R_i, \Big[\bigcap_{i\in A_2} R_i, \dots, \Big[\bigcap_{i\in A_{k-1}} R_i, \bigcap_{i\in A_k} R_i\Big]\dots\Big]\Big], \quad A \subseteq \langle n \rangle;$$

(b) *if $j \in \langle n \rangle$, the homomorphism $\widetilde{\mu}_j : \mathcal{B}_k(\mathcal{M})_A \longrightarrow \mathcal{B}_k(\mathcal{M})_{A\setminus\{j\}}$ is induced by the inclusion homomorphism μ_j;*

(c) *representing an element in $\mathcal{B}_k(\mathcal{M})_A$ by $\overline{x}$, where $x \in \bigcap_{i\in A} R_i$ (the bar denotes a coset), for $A, B \subseteq \langle n \rangle$, the map*

$$\widetilde{h} : \mathcal{B}_k(\mathcal{M})_A \times \mathcal{B}_k(\mathcal{M})_B \longrightarrow \mathcal{B}_k(\mathcal{M})_{A\cup B}$$

is given by

$$\widetilde{h}(\overline{x}, \overline{y}) = \overline{h(x,y)} = \overline{[x,y]}$$

for all $\overline{x} \in \mathcal{B}_k(\mathcal{M})_A$, $\overline{y} \in \mathcal{B}_k(\mathcal{M})_B$.

Proof. Since

$$\Big[\bigcap_{i\in A_1} R_i, \Big[\bigcap_{i\in A_2} R_i, \dots, \Big[\bigcap_{i\in A_{k-1}} R_i, \bigcap_{i\in A_k} R_i\Big]\dots\Big]\Big]$$
$$\subseteq \Big[\bigcap_{i\in A_1\setminus\{j\}} R_i, \Big[\bigcap_{i\in A_2\setminus\{j\}} R_i, \dots, \Big[\bigcap_{i\in A_{k-1}\setminus\{j\}} R_i, \bigcap_{i\in A_k\setminus\{j\}} R_i\Big]\dots\Big]\Big]$$

for $A_1 \cup \cdots \cup A_k = A \subseteq \langle n \rangle$, the inclusion

$$\mu_j : \bigcap_{i\in A} R_i \hookrightarrow \bigcap_{i\in A\setminus\{j\}} R_i$$

induces the homomorphism $\widetilde{\mu}_j : \mathcal{B}_k(\mathcal{M})_A \longrightarrow \mathcal{B}_k(\mathcal{M})_{A\setminus\{j\}}$ for all $j \in \langle n \rangle$.

Now, what is left is to show that the function

$$\widetilde{h} : \mathcal{B}_k(\mathcal{M})_A \times \mathcal{B}_k(\mathcal{M})_B \longrightarrow \mathcal{B}_k(\mathcal{M})_{A\cup B}$$

for $A, B \subseteq \langle n \rangle$ is well defined. In fact, let $x' \in \bigcap_{i\in A} R_i$, $y' \in \bigcap_{i\in B} R_i$ be such that

$$xx'^{-1} \in \prod_{A_1\cup\cdots\cup A_k = A} \Big[\bigcap_{i\in A_1} R_i, \Big[\bigcap_{i\in A_2} R_i, \dots, \Big[\bigcap_{i\in A_{k-1}} R_i, \bigcap_{i\in A_k} R_i\Big]\dots\Big]\Big]$$

and
$$yy'^{-1} \in \prod_{A_1\cup\cdots\cup A_k=B}\Big[\bigcap_{i\in A_1} R_i, \Big[\bigcap_{i\in A_2} R_i, \ldots, \Big[\bigcap_{i\in A_{k-1}} R_i, \bigcap_{i\in A_k} R_i\Big]\ldots\Big]\Big].$$
The inclusion
$$\Big[\bigcap_{i\in A} R_i, \bigcap_{i\in B} R_i\Big] \subseteq \bigcap_{i\in A\cup B} R_i$$
for all $A, B \subseteq \langle n\rangle$ implies that
$$\begin{aligned}
[x,y][x',y']^{-1} &= xyx^{-1}y^{-1}y'x'y'^{-1}x'^{-1} \\
&= xy'\big[y'^{-1}y, x^{-1}\big]y'^{-1}x^{-1}x[y', x^{-1}x']x^{-1} \\
&\in \prod_{A_1\cup\cdots\cup A_k=A\cup B}\Big[\bigcap_{i\in A_1} R_i, \Big[\bigcap_{i\in A_2} R_i, \ldots, \Big[\bigcap_{i\in A_{k-1}} R_i, \bigcap_{i\in A_k} R_i\Big]\ldots\Big]\Big],
\end{aligned}$$
so $\overline{h(x,y)} = \overline{h(x',y')}$ and $\widetilde{h}$ is well defined. The verification that $\mathcal{B}_k(\mathcal{M})$ is a crossed n-cube is now routine and is omitted. $\square$

Remark 2.8. The functor $\mathcal{B}_2$ coincides on the subcategory of inclusion crossed n-cubes with the abelianization functor $\mathfrak{Ab}^{(n)}$.

For any inclusion crossed n-cube $\mathcal{M}$ given by a normal $(n+1)$-ad of groups $(F; R_1, \ldots, R_n)$ and $k \geq 2$, there is a natural morphism of crossed n-cubes $\mathcal{M} \longrightarrow \mathcal{B}_k(\mathcal{M})$ inducing the natural fibration of simplicial groups $E^{(n)}(\mathcal{M})_* \xrightarrow{\Delta_*^{n,k}} E^{(n)}(\mathcal{B}_k(\mathcal{M}))_*$ defined by
$$\Delta_m^{n,k}(x_1, \ldots, x_l) = (\overline{x_1}, \ldots, \overline{x_l})$$
for all $(x_1, \ldots, x_l) \in E^{(n)}(\mathcal{M})_m = \big(\bigcap_{i\in A_1} R_i\big) \rtimes \cdots \rtimes \big(\bigcap_{i\in A_l} R_i\big)$ and $m \geq 0$, where $A_1, \ldots, A_l \subseteq \langle n\rangle$ and $l = (m+1)^n$. It is easy to see that $\operatorname{Ker}\Delta_m^{n,k} = D_k(F; A_1) \rtimes \cdots \rtimes D_k(F; A_l)$.

Proposition 2.9. *Let $\mathcal{M}$ be an inclusion crossed n-cube given by a normal $(n+1)$-ad of groups $(F; R_1, \ldots, R_n)$ and $k \geq 2$. Then there is an isomorphism of simplicial groups*
$$Z_k E^{(n)}(\mathcal{M})_* \cong E^{(n)}(\mathcal{B}_k(\mathcal{M}))_*.$$

Proof. For any inclusion crossed module $R \hookrightarrow F$, It is easy to verify the following equalities in the group $R \rtimes \cdots \rtimes R \rtimes F$:
$$\begin{aligned}
\big[(1, \ldots, 1, x), (1, \ldots, 1, x')\big] &= \big(1, \ldots, 1, [x, x']\big), \\
\big[(1, \ldots, 1, \underset{s}{r}, 1, \ldots, 1), (1, \ldots, 1, x)\big] &= \big(1, \ldots, 1, \underset{s}{[r, x]}, 1, \ldots, 1\big), \\
\big[(1, \ldots, 1, \underset{s}{r}, 1, \ldots, 1), (1, \ldots, 1, \underset{t}{r'}, 1, \ldots, 1)\big] &= \big(1, \ldots, 1, \underset{\min\{s,\ t\}}{[r, r']}, 1, \ldots, 1\big)
\end{aligned}$$
for all $x, x' \in F$, $r, r' \in R$.

There are further generalizations of these equalities, namely for any inclusion crossed n-cube $\mathcal{M}$ given by the normal $n+1$-ad of groups $(F, R_1, \ldots, R_n)$ we have the following facts, the proof of which is routine and will be omitted.

(A) Let s and t be any fixed elements of the set $\langle (m+1)^n\rangle$. Then there exists a unique $\lambda = \lambda(s,t) \in \langle (m+1)^n\rangle$ such that $A_\lambda = A_s \cup A_t$ and in the group $E^{(n)}(\mathcal{M})_m$ the equality
$$\big[(1, \ldots, 1, \underset{s}{x}, 1, \ldots, 1), (1, \ldots, 1, \underset{t}{y}, 1, \ldots, 1)\big] = \big(1, \ldots, 1, \underset{\lambda}{[x, y]}, 1, \ldots, 1\big)$$

holds for all $x \in \bigcap_{i \in A_s} R_i$, $y \in \bigcap_{i \in A_t} R_i$.

(B) Let $s \in \langle (m+1)^n \rangle$ and $A, B \subseteq A_s$ with $A \cup B = A_s$. Then there exists $p, q \in \langle (m+1)^n \rangle$ such that $A_p = A$, $A_q = B$, and $\lambda(p,q) = s$.

We must only show the equality

$$\Gamma_k\big(E^{(n)}(\mathcal{M})_m\big) = \operatorname{Ker} \Delta_m^{n,k}, \tag{2.6}$$

which will be done by induction on k, using facts (A) and (B) above.

Let $k=1$; then it is clear that $\Gamma_1(E^{(n)}(\mathcal{M})_m) = \operatorname{Ker} \Delta_m^{n,1}$.

Proceeding by induction, we assume that (2.6) is true for $k-1$ and we will prove it for k.

First, we will show the inclusion $\operatorname{Ker} \Delta_m^{n,k} \subseteq \Gamma_k(E^{(n)}(\mathcal{M})_m)$. It suffices to show that

$$1 \rtimes \cdots \rtimes 1 \rtimes D_k(F, A_s) \rtimes 1 \rtimes \cdots \rtimes 1 \subseteq \Gamma_k(E^{(n)}(\mathcal{M})_m) \quad \text{for all} \quad s \in \langle (m+1)^n \rangle.$$

In fact, any generator w of $D_k(F, A_s)$ has the form $w = [x, y]$, where $x \in \bigcap_{i \in A} R_i$, $y \in D_{k-1}(F, B)$ and $A \cup B = A_s$.

Now (B) implies that there exist $p, q \in \langle (m+1)^n \rangle$ such that $A_p = A$, $A_q = B$ and $\lambda(p,q) = s$. Thus we have

$$\Big[(1, \ldots, 1, \underset{p}{x}, 1, \ldots, 1), (1, \ldots, 1, \underset{q}{y}, 1, \ldots, 1)\Big] = (1, \ldots, 1, \underset{s}{w}, 1, \ldots, 1),$$

which means that

$$1 \rtimes \cdots \rtimes 1 \rtimes D_k(F, A_s) \rtimes 1 \rtimes \cdots \rtimes 1 \subseteq \Big[E^{(n)}(\mathcal{M})_m, \operatorname{Ker} \Delta_m^{n,k-1}\Big].$$

Therefore, by the inductive hypothesis we obtain

$$1 \rtimes \cdots \rtimes 1 \rtimes D_k(F, A_s) \rtimes 1 \rtimes \cdots \rtimes 1 \subseteq \Big[E^{(n)}(\mathcal{M})_m, \Gamma_{k-1}(E^{(n)}(\mathcal{M})_m)\Big] = \Gamma_k(E^{(n)}(\mathcal{M})_m).$$

Finally, we will show the inverse inclusion $\Gamma_k(E^{(n)}(\mathcal{M})_m) \subseteq \operatorname{Ker} \Delta_m^{n,k}$. In fact, any generator w of $\Gamma_k(E^{(n)}(\mathcal{M})_m)$ can be written in the form $w = [w_1, w_2]$, where $w_1 \in E^{(n)}(\mathcal{M})_m$ and $w_2 \in \Gamma_{k-1}(E^{(n)}(\mathcal{M})_m)$. Using again the inductive hypothesis, we have $w_2 \in \operatorname{Ker} \Delta_m^{n,k-1}$. Thus,

$$w_1 = \prod_{s=1}^{(m+1)^n} (1, \ldots, 1, \underset{s}{x_s}, 1, \ldots, 1), \quad x_s \in \bigcap_{i \in A_s} R_i,$$
$$w_2 = \prod_{t=1}^{(m+1)^n} (1, \ldots, 1, \underset{t}{y_t}, 1, \ldots, 1), \quad y_t \in D_{k-1}(F, A_t).$$

We know that $[x_s, y_t] \in D_k(F, A_s \cup A_t)$, so (A) implies that we have

$$\begin{aligned}\Big[(1, \ldots, 1, \underset{s}{x_s}, 1, \ldots, 1), (1, \ldots, 1, \underset{t}{y_t}, 1, \ldots, 1)\Big] &= \Big(1, \ldots, 1, \underset{\lambda(s,t)}{[x_s, y_t]}, 1, \ldots, 1\Big) \\ &\in 1 \rtimes \cdots \rtimes 1 \rtimes D_k(F, A_{\lambda(s,t)}) \rtimes 1 \rtimes \cdots \rtimes 1 \subseteq \operatorname{Ker} \Delta_m^{n,k}.\end{aligned}$$

and the Witt–Hall identities on commutators imply that $w \in \operatorname{Ker} \Delta_m^{n,k}$. $\square$

From Proposition 2.9 we can deduce that the abelianization of a crossed module commutes with its nerve. We provide a more general result for functors (2.4) and (2.3), which plays an essential role in obtaining generalized Hopf-type formulas for the homology of crossed n-cubes.

Proposition 2.10. *Let $n \geq 0$, $m \geq 1$ and $\mathcal{M}$ be a crossed $(n+m)$-cube. Then there is an isomorphism of simplicial crossed n-cubes*

$$\mathfrak{Ab}^{(n)} E^{(m)}(\mathcal{M})_* \cong E^{(m)} \mathfrak{Ab}^{(n+m)}(\mathcal{M})_*,$$

where $E^{(m)}$ functors in both sides of the isomorphism are applied to the same directions.

Proof. To simplify things, according to Theorem 2.3, instead of the crossed $(n+m)$-cube $\mathcal{M}$ we use its equivalent object, the cat $(n+m)$-group, $G = (G, s_i, t_i) = \Phi^{n+m}(\mathcal{M})$. The proof will be done by induction on m.

Let $m = 1$ and $n = 0$; then the assertion reduces to Proposition 2.9. This case plays the key role in the whole proof.

In fact, for $m = 1$, $n \geq 1$ and for the cat $(n+1)$-group G, let us fix some $k \in \langle n+1 \rangle$ and apply the functor $E^{(1)}$ to this "direction." By the definition, the simplicial cat n-group, $E^{(1)}(\mathcal{M})_*$, is just the simplicial group $E(\Psi^1(G, s_k, t_k))_*$ endowed with n compatible category structures induced by the respective structural endomorphisms s_j, t_j $(0 \leq j \leq n+1,\ j \neq k)$ of the cat $(n+1)$-group G. The fact that the abelianization of a cat n-group is just the abelianization of the underlying group endowed with the induced structural endomorphism and our key fact above completes the assertion in this case.

Proceeding by induction, we assume that the assertion is true for $m-1$, and we will prove it for m.

By the construction, $E^{(m)}(\mathcal{M})_*$ is the diagonal of the bisimplicial crossed n-cube induced by applying the crossed module nerve construction $E^{(1)}$ to the simplicial crossed $(n+1)$-cube $E^{(m-1)}(\mathcal{M})_*$. Hence we have

$$E^{(m)}(\mathcal{M})_k = E^{(1)}\big(E^{(m-1)}(\mathcal{M})_k\big)_k,$$

for all $k \geq 0$. Using the inductive hypothesis, we have the isomorphisms

$$\begin{aligned}\mathfrak{Ab}^{(n)} E^{(m)}(\mathcal{M})_k &= \mathfrak{Ab}^{(n)} E^{(1)}\big(E^{(m-1)}(\mathcal{M})_k\big)_k \cong E^{(1)} \mathfrak{Ab}^{(n+1)}\big(E^{(m-1)}(\mathcal{M})_k\big)_k \\ &\cong E^{(1)}\big(E^{(m-1)}\big(\mathfrak{Ab}^{(n+m)}(\mathcal{M})\big)_k\big)_k = E^{(m)}\big(\mathfrak{Ab}^{(n+m)}(\mathcal{M})\big)_k.\end{aligned}$$

□

1.5. Non-Abelian mapping cone complex. This section is devoted to the investigation of some properties of the mapping cone complex of a morphism of (non-Abelian) group complexes as introduced in [87].

A complex of (non-Abelian) groups (A_*, d_*) of length n is a sequence of group homomorphisms

$$A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} A_0$$

such that $\operatorname{Im} d_{i+1}$ is normal in $\operatorname{Ker} d_i$. Now we recall the following definition from [87].

Let $f : (A_*, d_*) \longrightarrow (B_*, d'_*)$ be a morphism of chain complexes of groups. Let f satisfy the following conditions $(*)$:

each $f_i : A_i \longrightarrow B_i$ is a crossed module

and

the maps (d_i, d'_i) form a morphism of crossed modules.

Then the mapping cone of f is a complex of (non-Abelian) groups $(C_*(f), \partial_*)$ defined by $C_i(f) = A_{i-1} \rtimes B_i$, where the action of B_i on A_{i-1} is induced by the action of B_{i-1} on A_{i-1} via the homomorphism d'_i; and

$$\partial_i(a, b) = \big(d_{i-1}(a)^{-1}, f_{i-1}(a) d'_i(b)\big)$$

for all $a \in A_{i-1}$, $b \in B_i$. By [87, Proposition 3.2], there is a long exact sequence of groups

$$\cdots \longrightarrow H_i(A_*) \longrightarrow H_i(B_*) \longrightarrow H_i(C_*(f)) \longrightarrow H_{i-1}(A_*) \longrightarrow \cdots. \tag{2.7}$$

Now let us consider a morphism of pseudo-simplicial groups $\alpha : (G_*, d_i^*, s_i^*) \longrightarrow (H_*, d_i'^*, s_i'^*)$ satisfying the following conditions $(**)$:

each $\alpha_n : G_n \longrightarrow H_n$ *is a crossed module*

and

the maps $(d_i^*, d_i'^*)$ *and* $(s_i^*, s_i'^*)$ *form morphisms of crossed modules.*

Define a new pseudo-simplicial group $M_*(\alpha)$ as follows:

$$M_n(\alpha) = \underbrace{G_n \rtimes G_n \rtimes \cdots \rtimes G_n}_{n\text{-times}} \rtimes H_n,$$

$$\begin{aligned}
d_0^n(g_1, \dots, g_n, h) &= \big(d_0^n(g_2), \dots, d_0^n(g_n), d_0'^n(h)\big),\\
d_i^n(g_1, \dots, g_n, h) &= \big(d_i^n(g_1), \dots, d_i^n(g_i)d_i^n(g_{i+1}), \dots, d_i^n(g_n), d_i'^n(h)\big), \quad 0 < i < n,\\
d_n^n(g_1, \dots, g_n, h) &= \big(d_n^n(g_1), \dots, d_n^n(g_{n-1}), \alpha_{n-1}d_n^n(g_n)d_n'^n(h)\big),\\
s_i^n(g_1, \dots, g_n, h) &= \big(s_i^n(g_1), \dots, s_i^n(g_i), 1, s_i^n(g_{i+1}), \dots, s_i^n(g_n), s_i'^n(h)\big), \quad 0 \le i \le n.
\end{aligned}$$

It is easy to see that the induced morphism $\widetilde{\alpha} : NG_* \longrightarrow NH_*$, where NG_* and NH_* are the Moore complexes of G_* and H_* respectively, satisfies the conditions $(*)$. Therefore, we can consider the mapping cone complex $C_*(\widetilde{\alpha})$ of $\widetilde{\alpha}$.

Proposition 2.11. *The natural morphism of complexes* $\kappa : NM_*(\alpha) \longrightarrow C_*(\widetilde{\alpha})$, *given by* $\kappa_n(g_1, g_2, \dots, g_n, h) = (d_n^n(g_n), h)$, $n \ge 0$, *induces an isomorphism of groups*

$$\pi_n(M_*(\alpha)) \cong H_n(C_*(\widetilde{\alpha})), \quad n \ge 0.$$

Proof. The verification that κ_n, $n \ge 0$ is a homomorphism and commuting with differentials is easy. Let $(g, h) \in NG_{n-1} \rtimes NH_n = C_n(\widetilde{\alpha})$; then it is easy to verify that $(s_0^{n-1}(g)^{\epsilon(n-1)}, \dots, s_{n-2}^{n-1}(g)^{-1}, s_{n-1}^{n-1}(g), h) \in NM_n(\alpha)$, where $\epsilon(i) = (-1)^i$. It is clear that

$$\kappa_n\Big(s_0^{n-1}(g)^{\epsilon(n-1)}, \dots, s_{n-2}^{n-1}(g)^{-1}, s_{n-1}^{n-1}(g), h\Big) = (g, h).$$

Hence κ_n is surjective for all $n \ge 0$.

Consider the kernel complex $(\mathfrak{G}_*, \overline{\partial}_*)$ of κ. Note that $\operatorname{Im}\overline{\partial}_n$ is not normal in $\operatorname{Ker}\overline{\partial}_{n-1}$ in general, $\mathfrak{G}_0 = 1$ and

$$\mathfrak{G}_n = \left\{ (g_1, g_2, \dots, g_n) \in \underbrace{G_n \rtimes G_n \rtimes \cdots \rtimes G_n}_{n\text{-times}} \;\middle|\; \begin{array}{ll} d_0^n(g_j) = 1, & 2 \le j \le n;\\ d_i^n(g_j) = d_i^n(g_i)d_i^n(g_{i+1}) = 1, & 1 \le i \le n-1,\\ & 1 \le j \le n \quad \text{and} \quad i \ne j-1, j;\\ d_n^n(g_n) = 1. & \end{array} \right\}$$

Furthermore, it is easy to verify that for an element $(g_1, \dots, g_{n-1}) \in \operatorname{Ker}\overline{\partial_{n-1}}$, the element $(g_1', \dots, g_{n-1}', g_n')$, defined by the formulas

$$g_i' = \begin{cases} s_{n-1}^{n-1}(g_i)s_{n-2}^{n-1}(g_i^{-1}) \cdots s_i^{n-1}(g_i^{\epsilon(n-i-1)})s_{i-1}^{n-1}(g_i^{\epsilon(n-i)}g_{i+1}^{\epsilon(n-i)} \cdots g_{n-1}^{\epsilon(n-i)}), & i \text{ is even},\\ s_{i-1}^{n-1}(g_{n-1}^{\epsilon(n-i)} \cdots g_{i+1}^{\epsilon(n-i)}g_i^{\epsilon(n-i)})s_i^{n-1}(g_i^{\epsilon(n-i-1)}) \cdots s_{n-2}^{n-1}(g_i^{-1})s_{n-1}^{n-1}(g_i), & i \text{ is odd}, \end{cases}$$

for all $1 \le i \le n-1$ and $g_n' = 1$, belongs to $\mathfrak{G}_n$ and

$$\overline{\partial_n}(g_1', \dots, g_{n-1}', g_n') = (g_1, \dots, g_{n-1}).$$

Now the proposition follows from the long exact homology sequence induced by the short exact sequence of complexes $1 \longrightarrow \mathfrak{G}_* \longrightarrow NM_*(\alpha) \xrightarrow{\kappa} C_*(\widetilde{\alpha}) \longrightarrow 1$. $\square$

Given a pseudo-simplicial group G_*, we will say that the *length of* G_* *is* $\leq n$, denoted by $l(G_*) \leq n$, if $NG_i = 1$ for $i > n$.

Remark 2.12. Let $\alpha : (G_*, d_i^*, s_i^*) \longrightarrow (H_*, d_i'^*, s_i'^*)$ be a morphism of pseudo-simplicial groups satisfying the conditions $(**)$ and $n \geq 2$. Assume $l(G_*) \leq n-1$ and $l(H_*) \leq n-1$. Consider an element $(g_1, g_2, \ldots, g_k, h) \in NM_k(\alpha)$, $k > n$; then

$$\begin{aligned}
&d_0^k(g_j) = 1, \quad 2 \leq j \leq k, \\
&d_i^k(g_j) = d_i^k(g_i) d_i^k(g_{i+1}) = 1, \quad 1 \leq i \leq k-1, \quad 1 \leq j \leq k \quad \text{and} \quad i \neq j-1, j, \\
&d_i'^k(h) = 1, \quad 0 \leq i \leq k-1.
\end{aligned}$$

Using the well-known result given as Lemma 3.5 below, we can easily show that $g_i = 1$, $1 \leq i \leq k$ and $h = 1$, so $NM_k(\alpha) = 1$ for $k > n$. Thus $l(M_*(\alpha)) \leq n$.

Now using the mapping cone construction, for a given crossed n-cube $\mathcal{M}$, we construct inductively a complex of groups $C_*(\mathcal{M})$ of length n, always having in mind that $\mathcal{M}$ is thought of as a crossed module of crossed $(n-1)$-cubes, $\mathcal{M}_1 \longrightarrow \mathcal{M}_0$. In fact, for $n = 1$, and $\mathcal{M} = (\ M \xrightarrow{\mu} P\ )$, $C_*(\mathcal{M})$ is the complex $M \longrightarrow P$ of length 1. Let $n = 2$ and $\mathcal{M}$ be a crossed square, considered as a crossed module of crossed modules or a morphism of complexes of length 1 satisfying the conditions $(*)$. The construction above gives a complex $C_*(\mathcal{M})$ of length 2. (It has a 2-crossed module structure, [27], as noted by Conduché; see also [100].) Proceeding by induction, assume that for any crossed $(n-1)$-cube $\mathcal{M}$, we have constructed a complex $C_*(\mathcal{M})$ of length $n-1$. Now let $\mathcal{M}$ be a crossed n-cube and consider it as a crossed module of crossed $(n-1)$-cubes $\mathcal{M}_1 \longrightarrow \mathcal{M}_0$. This implies that there is a morphism of complexes of groups $C_*(\mathcal{M}_1) \xrightarrow{\delta} C_*(\mathcal{M}_0)$ of length $n-1$ satisfying the conditions $(*)$. So using again the above-mentioned construction, we obtain a chain complex of groups $C_*(\mathcal{M}) = C_*(\delta)$ of length n.

Proposition 2.13 (see [87])**.** *Let $\mathcal{M}$ be a crossed n-cube of groups. Then $l(E^n(\mathcal{M})_*) \leq n$ and there is a natural morphism of complexes $NE^{(n)}(\mathcal{M})_* \longrightarrow C_*(\mathcal{M})$ that induces isomorphisms of groups*

$$\pi_i(E^{(n)}(\mathcal{M})_*) \cong H_i(C_*(\mathcal{M})), \quad i \geq 0.$$

Moreover,

$$\pi_n(E^{(n)}(\mathcal{M})_*) \cong \bigcap_{i=1}^{n} \operatorname{Ker}\left(\mathcal{M}_{\langle n \rangle} \xrightarrow{\mu_i} \mathcal{M}_{\langle n \rangle \setminus \{i\}}\right).$$

Proof. This is obvious for $n = 1$. Let $n = 2$ and $\mathcal{M}$ be a crossed square. If we consider $\mathcal{M}$ as a crossed module of crossed modules $\mathcal{M}_1 \longrightarrow \mathcal{M}_0$, inducing the natural morphism of simplicial groups $E^{(1)}(\mathcal{M}_1)_* \xrightarrow{\alpha} E^{(1)}(\mathcal{M}_0)_*$, which satisfies the conditions $(**)$, then by definition $E^{(2)}(\mathcal{M})_* = M_*(\alpha)$, and by Proposition 2.11 and the corresponding remark, $l(E^{(2)}(\mathcal{M})_*) \leq 2$, and there exists a natural morphism of complexes $NE^{(2)}(\mathcal{M})_* \longrightarrow C_*(\widetilde{\alpha})$ inducing an isomorphism

$$\pi_i(E^{(2)}(\mathcal{M})_*) \cong H_i(C_*(\widetilde{\alpha})), \quad i \geq 0.$$

Clearly, $C_*(\widetilde{\alpha}) \cong C_*(\mathcal{M})$.

Proceeding by induction, we assume that the assertion is valid for $n-1$ and we will show it for n.

Let us consider any crossed n-cube $\mathcal{M}$ as a crossed module of crossed $(n-1)$-cubes $\mathcal{M}_1 \longrightarrow \mathcal{M}_0$. This implies a morphism of simplicial groups $E^{(n-1)}(\mathcal{M}_1)_* \xrightarrow{\alpha} E^{(n-1)}(\mathcal{M}_0)_*$ satisfying the conditions $(**)$ and a morphism of complexes $C_*(\mathcal{M}_1) \xrightarrow{\delta} C_*(\mathcal{M}_0)$ satisfying the conditions $(*)$. By definition, $E^{(n)}(\mathcal{M})_* = M_*(\alpha)$; hence Proposition 2.11 and Remark 2.12 imply that $l(E^{(n)}(\mathcal{M})_*) \leq n$ and there exists a natural morphism of complexes $NE^{(n)}(\mathcal{M})_* \xrightarrow{\kappa} C_*(\widetilde{\alpha})$ inducing isomorphisms

$$\pi_i(E^{(n)}(\mathcal{M})_*) \cong H_i(C_*(\widetilde{\alpha})), \quad i \geq 0.$$

Using the inductive hypothesis, we see that there exist natural morphisms of complexes

$$NE^{(n-1)}(\mathcal{M}_1)_* \xrightarrow{\kappa'} C_*(\mathcal{M}_1) \quad \text{and} \quad NE^{(n-1)}(\mathcal{M}_0)_* \xrightarrow{\kappa''} C_*(\mathcal{M}_0) ,$$

that induce isomorphisms

$$\pi_i(E^{(n-1)}(\mathcal{M}_1)_*) \cong H_i(C_*(\mathcal{M}_1)),$$
$$\pi_i(E^{(n-1)}(\mathcal{M}_0)_*) \cong H_i(C_*(\mathcal{M}_0)),$$

for $i \geq 0$. It is easy to verify that $\kappa''\widetilde{\alpha} = \delta\kappa'$ and that (κ'_i, κ''_i) is a morphism of crossed modules for all $i \geq 0$. Then the natural morphism of complexes

$$C_*(\widetilde{\alpha}) \xrightarrow{\kappa' \rtimes \kappa''} C_*(\delta) = C_*(\mathcal{M}),$$

by (2.7) and the five lemma, induces $H_i(C_*(\widetilde{\alpha})) \cong H_i(C_*(\mathcal{M}))$, $i \geq 0$. Therefore the morphism of complexes

$$NE^{(n)}(\mathcal{M})_* \xrightarrow{(\kappa' \rtimes \kappa'')\circ\kappa} C_*(\mathcal{M})$$

induces isomorphisms

$$\pi_i(E^{(n)}(\mathcal{M})_*) \cong H_i(C_*(\mathcal{M})), \quad i \geq 0.$$

These isomorphisms and the construction of $C_*(\mathcal{M})$ imply that

$$\pi_n(E^{(n)}(\mathcal{M})_*) \cong \bigcap_{i=1}^{n} \mathrm{Ker}\left(\mathcal{M}_{\langle n\rangle} \xrightarrow{\mu_i} \mathcal{M}_{\langle n\rangle\setminus\{i\}}\right).$$

$\square$

2. Homology of Crossed n-Cubes

In this section, we give the construction of the (cotriple) homology of homotopy $(n+1)$-types, which will be investigated in the next chapter from a Hopf formulas point of view.

First, we show that the category $\mathbf{Crs}^n$ is an algebraic category (see also [23]), that is, there is a tripleable forgetful functor from $\mathbf{Crs}^n$ to $\mathbf{Set}$. In fact, we need only to construct a 'free' cotriple in the category $\mathbf{Crs}^n$.

We begin by constructing the adjoint pair of functors $\mathbf{Crs}^n \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Gr}$.

Assume that the functor $U : \mathbf{Crs^n} \longrightarrow \mathbf{Gr}$ assigns to any crossed n-cube $\mathcal{M} = \{\mathcal{M}_A : A \subseteq \langle n\rangle\}$ the direct product of groups $\mathcal{M}_A$, $A \subseteq \langle n\rangle$, i.e.,

$$U(\mathcal{M}) = \prod_{A\subseteq\langle n\rangle} \mathcal{M}_A.$$

Now define the functor $F : \mathbf{Gr} \longrightarrow \mathbf{Crs}^n$ as follows: for any group G, let $F(G)$ denote the inclusion crossed n-cube induced by the normal $(n+1)$-ad of groups $\left(\bigvee_{A\subseteq\langle n\rangle} G_A; \mathrm{Ker}\, p_1, \ldots, \mathrm{Ker}\, p_n\right)$ (see Example 2.1), where $\bigvee_{A\subseteq\langle n\rangle} G_A$ is the sum of groups $G_A = G$, $A \subseteq \langle n\rangle$ and

$$p_i : \bigvee_{A\subseteq\langle n\rangle} G_A \longrightarrow \bigvee_{B\subseteq\langle n-1\rangle} G_B, \quad i \in \langle n\rangle,$$

are natural projections given by

$$p_i = \begin{cases} 1_G : G_A \longrightarrow G_B & \text{if} \quad A \subseteq \langle n \rangle \setminus \{i\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta_i : \langle n \rangle \setminus \{i\} \longrightarrow \langle n-1 \rangle$ is the unique monotone bijection.

Proposition 2.14. *The functor F is left adjoint to the functor U.*

To prove this proposition we use the following easily verified facts requiring only care over the notation. Given a crossed n-cube $\mathcal{M} = \{\mathcal{M}_A : \ A \subseteq \langle n \rangle\}$, for any $B \subseteq \langle n \rangle$ denote by $\mathcal{M}^B$ and $\mathcal{M}^{\overline{B}}$ the families $\{\mathcal{M}_A : \ A \subseteq \langle n \rangle, \ B \subseteq A\}$ and $\{\mathcal{M}_A : \ A \subseteq \langle n \rangle, \ B \cap A = \varnothing\}$, respectively. Then $\mathcal{M}^B$ and $\mathcal{M}^{\overline{B}}$ have the structure of crossed $(n-|B|)$-cubes (see [106, Proposition 5]).

Proof of Proposition 2.14. We claim that for any group G, the homomorphism

$$u = \{u_A\} : G \longrightarrow \prod_{A \subseteq \langle n \rangle} F(G)_A = UF(G),$$

where $u_A : G \longrightarrow F(G)_A = \bigcap\limits_{i \in A} \operatorname{Ker} p_i$ is given by the identity from G to G_A, is a universal arrow from G to the functor U.

Let $\mathcal{M}$ be a crossed n-cube and let $\alpha_A : G \longrightarrow \mathcal{M}_A$, $A \subseteq \langle n \rangle$ be homomorphisms defining a homomorphism $\alpha : G \longrightarrow \prod\limits_{A \subseteq \langle n \rangle} \mathcal{M}_A = U(\mathcal{M})$. Then there is a commutative diagram with splitting short exact sequences of groups:

$$\begin{tikzcd} \operatorname{Ker} p_i \arrow[r, tail] \arrow[d, "\widetilde{\gamma}_i"'] & \bigvee\limits_{A \subseteq \langle n \rangle} G_A \arrow[r, two heads, "p_i"] \arrow[d, "\gamma"'] & \bigvee\limits_{B \subseteq \langle n-1 \rangle} G_B \arrow[d, "\gamma_i"'] \\ \Phi^{n-1}(\mathcal{M}^{\{i\}}) \arrow[r, tail] & \Phi^n(\mathcal{M}) \arrow[r, two heads] & \Phi^{n-1}(\mathcal{M}^{\overline{\{i\}}}) \end{tikzcd},$$

where Φ^* is the equivalence given in Theorem 2.3, γ_i is induced by $G_B \xrightarrow{\alpha_A} \mathcal{M}_A$ with $A \subseteq \langle n \rangle \setminus \{i\}$ such that $\delta_i(A) = B$, γ is induced by $G_A \xrightarrow{\alpha_A} \mathcal{M}_A$, $A \subseteq \langle n \rangle$, and $\widetilde{\gamma}_i$ is the restriction of γ. It is easy to see that the homomorphisms $\widetilde{\gamma}_i$ induce the homomorphisms $\widetilde{\gamma}_A : \bigcap\limits_{i \in A} \operatorname{Ker} p_i \longrightarrow \Phi^{n-|A|}(\mathcal{M}^A)$.

Now define the homomorphism $\widetilde{\widetilde{\gamma}}_A : \bigcap\limits_{i \in A} \operatorname{Ker} p_i \longrightarrow \mathcal{M}_A$, $A \subseteq \langle n \rangle$ as the composition of $\widetilde{\gamma}_A$ and $\beta_A : \Phi^{n-|A|}(\mathcal{M}^A) \longrightarrow \mathcal{M}_A$ given by $\mathcal{M}_B \xrightarrow{\mu_{B \setminus A}} \mathcal{M}_A$ for $B \supseteq A$, where $\mu_{B \setminus A}$ is the composition of the homomorphisms μ_{i_j}, $j = 1, \dots, |B \setminus A|$, with any $i_j \in (B \setminus A) \setminus \bigcup\limits_{k=1}^{j-1} \{i_k\}$. Finally, it is easy to verify that $\widetilde{\widetilde{\gamma}} = \{\widetilde{\widetilde{\gamma}}_A\} : F(G) \longrightarrow \mathcal{M}$ is the unique morphism of crossed n-cubes with $U(\widetilde{\widetilde{\gamma}})u = \alpha$. $\square$

We denote by $U_1 : \mathbf{Gr} \longrightarrow \mathbf{Set}$ the usual forgetful functor and by $F_1 : \mathbf{Set} \longrightarrow \mathbf{Gr}$ its left adjoint, the free group functor. Composing these two adjunctions,

$$\mathbf{Crs}^n \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Gr} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathbf{Set},$$

we deduce the following proposition.

Proposition 2.15. *The underlying set functor $\mathcal{U} = U_1 \circ U : \mathbf{Crs}^n \longrightarrow \mathbf{Set}$ has a left adjoint $\mathcal{F} = F \circ F_1 : \mathbf{Set} \longrightarrow \mathbf{Crs}^n$.*

It is routine to verify that the category $\mathbf{Crs}^n$, $n \geq 2$, similarly to that of crossed modules (i.e., $n = 1$) [21], has kernel pairs and coequalizers preserved and reflected by the functor $\mathcal{U}$. Then by Proposition 2.15 and Linton's criterion on tripleability [86] the underlying set functor $\mathcal{U} : \mathbf{Crs}^n \longrightarrow \mathbf{Set}$ is tripleable.

Now we construct the cotriple homology of crossed n-cubes (cat n-groups). We refer the reader to the work of Barr and Beck [5] for the background on cotriple (co)homology.

The above constructed pair of adjoint functors $\mathbf{Set} \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathbf{Crs^n}$ induces the cotriple $\mathbb{F} \equiv (\mathbb{F}, \delta, \tau)$ on the category $\mathbf{Crs}^n$ by the obvious way: $\mathbb{F} = \mathcal{F}\mathcal{U} : \mathbf{Crs}^n \longrightarrow \mathbf{Crs}^n$, $\tau : \mathbb{F} \longrightarrow 1^n_{\mathbf{Crs}}$ is the counit and $\delta = \mathcal{F}u\mathcal{U} : \mathbb{F} \longrightarrow \mathbb{F}^2$, where $u : 1_{\mathbf{Set}} \longrightarrow \mathcal{U}\mathcal{F}$ is the unit of the adjunction.

Using the general theory of cotriple homology due to [5], we have the following definition.

Definition 2.16. Let $\mathcal{M}$ be a crossed n-cube. Define *the kth homology* of $\mathcal{M}$ by setting

$$\mathcal{H}_k(\mathcal{M}) = \mathcal{L}^{\mathbb{F}}_{k-1}\mathfrak{Ab}^{(n)}(\mathcal{M}), \quad k \geq 1.$$

Let $\mathbb{P}$ be the projective class induced by the "free" cotriple $\mathbb{F}$ (see Chap. 1, Sec. 1.2). According to Proposition 1.10 the derived functors relative to the cotriple are isomorphic to the derived functors relative to the projective class induced by the cotriple [60]. Thus there is an isomorphism

$$\mathcal{L}^{\mathbb{F}}_k\mathfrak{Ab}^{(n)} \cong \mathcal{L}^{\mathbb{P}}_k\mathfrak{Ab}^{(n)}.$$

Recall also that an object P of a category $\mathbf{C}$ is projective if, given a regular epimorphism $f : X \twoheadrightarrow Y$, each morphism $g : P \longrightarrow Y$ can be lifted to a morphism $h : P \longrightarrow X$ such that $fh = g$. We say that $\mathbf{C}$ has enough projective objects if any object X admits a projective presentation, i.e., there exists a regular epimorphism $P \longrightarrow X$ with P a projective object. If $\mathbf{C}$ is a tripleable category with the adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{C}$, then $F(X)$, $X \in \mathbf{Set}$, is a projective object and the natural morphism $FU(C) \longrightarrow C$, $C \in \mathbf{C}$, is a regular epimorphism in $\mathbf{C}$, implying that $\mathbf{C}$ has enough projectives. It is also known that the projective class of all projective objects in the algebraic category $\mathbf{C}$ coincides with the projective class $\overline{\mathbb{P}}$ induced by the adjunction, and regular epimorphisms are just $\overline{\mathbb{P}}$-epimorphisms.

It is easy to verify that if $\mathcal{M}_*$ is a $\mathbb{F}$-cotriple resolution of a crossed n-cube $\mathcal{M}$, then $\mathcal{M}_*^{\overline{\langle n\rangle\setminus A}}$ is a projective resolution of $\mathcal{M}^{\overline{\langle n\rangle\setminus A}}$ for $A \subseteq \langle n\rangle$, $A \neq \langle n\rangle$. Hence

$$\mathcal{H}_k(\mathcal{M})_A = \mathcal{H}_k(\mathcal{M}^{\overline{\langle n\rangle\setminus A}})_{\langle |A|\rangle}, \quad k \geq 1.$$

Therefore, the interest of our investigation is the group $\mathcal{H}_k(\mathcal{M})_{\langle n\rangle}$, which we denote by $H_k(\mathcal{M})$. If we define the functor $\sigma : \mathbf{Crs}^n \longrightarrow \mathbf{Gr}$ by $\sigma(\mathcal{M}) = \mathcal{M}_{\langle n\rangle}$ for $\mathcal{M} \in \mathbf{Crs}^n$, then

$$H_k(\mathcal{M}) = \mathcal{L}^{\mathbb{F}}_{k-1}(\sigma\mathfrak{Ab}^{(n)})(\mathcal{M}), \quad k \geq 1.$$

3. Homology of Precrossed Modules

Precrossed modules form a model of homotopy type in dimensions 1 and 2 for connected CW-complexes. Precisely, Kan's $\mathbb{G}$ functor establishes an equivalence relation between the category of connected CW-complexes and the category of free simplicial groups [79] and the first two terms of the Moore chain complex associated to the simplicial group gives a precrossed module.

The homology of precrossed modules was introduced by Conduché and Ellis in [30]. The aim of this section is to pursue their line of investigation of homological properties of precrossed modules.

Let (M, μ) be a precrossed P-module. The following type elements in M

$$\langle m, m'\rangle = mm'm^{-1\mu(m)}m'^{-1}, \quad m, m' \in M,$$

are called Peiffer commutators, and now we give some identities for them from [7]

$$\langle m, m'm'' \rangle = \langle m, m' \rangle^{\mu(m)} m' \langle m, m'' \rangle^{\mu(m)} m'^{-1}, \tag{2.8}$$

$$\langle mm', m'' \rangle = m \langle m', m'' \rangle m^{-1} \langle m, {}^{\mu(m')} m'' \rangle, \tag{2.9}$$

$$^{p}\langle m, m' \rangle = \langle {}^{p}m, {}^{p}m' \rangle, \tag{2.10}$$

$$\langle k, m \rangle = kmk^{-1}m^{-1}, \tag{2.11}$$

$$\langle k, m \rangle \langle m, k \rangle = k^{\mu(m)} k^{-1} \tag{2.12}$$

for all $m, m', m'' \in M$, $p \in P$ and $k \in \operatorname{Ker} \mu$.

The Peiffer commutator subgroup $\langle M, M \rangle$, which is a subgroup of the group M generated by the Peiffer commutators, plays the same role for precrossed modules as the commutator subgroup plays for groups. Analogously, as a lower central series in a group, a lower Peiffer central series in a precrossed P-module is defined by Baues and Conduché [7]:

$$M^{(1)} = M \supset M^{(2)} \supset \cdots .$$

This series has properties similar to classical central series, giving one hope to generalize some methods of Curtis [34, 35] and Quillen [108] for nonsimply connected spaces.

The crossed P-module $\mu' : M/\langle M, M \rangle \longrightarrow P$ associated to the precrossed P-module $\mu : M \longrightarrow P$, where $M/\langle M, M \rangle$ is a factor group of M by the Peiffer commutator subgroup, and the homomorphism μ' and the action of P on $M/\langle M, M \rangle$ are induced by μ and the action of P on M, respectively, are further called *Peiffer abelianization.* As an analog of the classical first group homology, Conduché and Ellis [30] defined the first homology of a precrossed P-module (M, μ) by Peiffer abelianization,

$$H_1(M)_P = M/\langle M, M \rangle.$$

We point out that despite its name, the Peiffer abelianization can be non-Abelian.

Let X be a set and $\delta : X \longrightarrow P$ a map to the group P. Then the *free precrossed P-module* $\partial : F \longrightarrow P$ with base (X, δ) is defined as follows: F is the free group generated by the set $X \times P$, ∂ is defined on generators by $\partial(x, p) = p\delta(x)p^{-1}$, and the action of P on F is given by ${}^{p}(x, p') = (x, pp')$.

Conduché and Ellis in [30] also defined the second homology group of a precrossed P-module (M, μ) by the Hopf formula

$$H_2(M)_P = R \cap \langle F, F \rangle / \langle \langle F, R \rangle \rangle,$$

where $1 \longrightarrow R \longrightarrow F \longrightarrow M \longrightarrow 1$ is a short exact sequence of precrossed P-modules, and (F, ∂) is a free precrossed P-module with some base (X, δ), which is called the free presentation of the precrossed P-module (M, μ). They studied some properties of so-defined low-dimensional homology groups of precrossed P-modules and hoped that higher homologies could be defined analogously using Hopf formulas for higher homology groups (see [14]). Using this method to define all homology groups of a precrossed P-module (M, μ), $H_n(M)_P$, one encounters some difficulties, for $n \geq 3$, in proving that the definition does not depend on the free presentation of the precrossed P-module (M, μ).

We have another concept to define all homology groups of a precrossed P-module, particulary the use of non-Abelian derived functors.

We consider all treatments with homology of precrossed P-modules in the q modular aspect, where q is a nonnegative integer, and for $q = 0$ this gives the homology groups of precrossed modules introduced in [30]. Thus, for nonnegative integer q, we define homology groups modulo q of precrossed P-module (M, μ) in any dimension $n \geq 1$, denoted by $H_n(M, q)_P$, and study their properties generalizing the classical homology of groups with coefficients in $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. *Note that q modular aspects of some other theories will be treated in Chap.* 6.

3.1. Construction. Let us denote by $\mathrm{Set}(P)$ the category of sets over the group P, whose objects are all sets with a map to P and whose morphisms are all maps of sets such that the corresponding triangles are commutative.

Consider the functor $\mathcal{F} : \mathrm{Set}(P) \longrightarrow \mathcal{PCM}(P)$ defined as follows: for an object $X \xrightarrow{\alpha} P$ of the category $\mathrm{Set}(P)$, let $\mathcal{F}(X \xrightarrow{\alpha} P)$ be a free precrossed P-module with base (X, α); for a morphism $X \xrightarrow{\kappa} X'$, let $\mathcal{F}(\kappa)$ be the canonical homomorphism induced by κ.

It is known that the forgetful functor from the category $\mathcal{PCM}(P)$ to the category $\mathrm{Set}(P)$ is a right adjoint of the functor $\mathcal{F}$. This adjunction induces the cotriple $(\mathcal{F}, \tau, \delta)$ in the category $\mathcal{PCM}(P)$. Let $\mathcal{P}$ be the projective class in the category $\mathcal{PCM}(P)$ induced by the cotriple $(\mathcal{F}, \tau, \delta)$ (see [60, 123]).

First, we describe the projective class $\mathcal{P}$ and the corresponding $\mathcal{P}$-epimorphisms.

Proposition 2.17. *A morphism $M \xrightarrow{\varphi} N$ of the category $\mathcal{PCM}(P)$ is a $\mathcal{P}$-epimorphism if and only if φ is surjective (as map of sets).*

Proposition 2.18. *In the category $\mathcal{PCM}(P)$ the following conditions are equivalent:*

(i) *A precrossed P-module (Q, ν) belongs to the projective class $\mathcal{P}$;*

(ii) *(Q, ν) is a free precrossed P-module with base (X, α) for some object $X \xrightarrow{\alpha} P$ of the category $\mathrm{Set}(P)$.*

The proof of these propositions is easy and we omit it.

A precrossed P-module (N, ν) is a precrossed P-submodule of a precrossed P-module (M, μ) if N is a subgroup of M, the action of P on N is induced by the action of P on M, and ν is the restriction of μ on N. If, in addition, N is a normal subgroup of the group M, then we write $N <_P M$.

Let (M, μ) be a precrossed P-module, N, N' be two subgroups of M, and q be a nonnegative integer. We denote by $\langle N, N' \rangle_{(q)}$ the subgroup of M generated by the elements $\langle n, n' \rangle$ and k^q for all $n \in N$, $n' \in N'$, $k \in N \cap N' \cap \operatorname{Ker} \mu$. Let $\langle\langle N, N' \rangle\rangle_{(q)} = \langle N, N' \rangle_{(q)} \langle N', N \rangle_{(q)}$. We have the following lemma.

Lemma 2.19.

(i) *If N and N' are precrossed P-submodules of M, then $\langle N, N' \rangle_{(q)}$ and $\langle\langle N, N' \rangle\rangle_{(q)}$ are precrossed P-submodules of M.*

(ii) *If $N <_P M$, then $\langle M, N \rangle_{(q)} <_P M$, $\langle N, M \rangle_{(q)} <_P M$, $\langle\langle M, N \rangle\rangle_{(q)} <_P M$.*

Proof. (i) Follows from relation (2.10) and the equality ${}^p(k^q) = ({}^p k)^q$, $p \in P$, $k \in N \cap N' \cap \operatorname{Ker} \mu$.

(ii) follows from relations (2.8) and (2.9) and the equality $mk^q m^{-1} = (mkm^{-1})^q$, $m \in M$, $k \in N \cap \operatorname{Ker} \mu$. □

Using Lemma 2.19, we can define a covariant functor $T_{(q)}$ from the category $\mathcal{PCM}(P)$ to the category $\mathfrak{Gr}$ of groups by the following way: for any precrossed P-module (M, μ), let $T_{(q)}(M) = M / \langle\langle M, M \rangle\rangle_{(q)} = M / \langle M, M \rangle_{(q)}$; for a morphism $(M, \mu) \xrightarrow{\varphi} (M', \mu')$, let $T_{(q)}(\varphi)$ be a group homomorphism induced by φ. Note that for $q = 0$ the functor $T_{(q)}$ is the Peiffer abelianization functor.

In the category $\mathcal{PCM}(P)$, there exist finite limits (easy to show). Let us consider the non-Abelian left derived functors $\mathcal{L}_n^{\mathcal{P}} T_{(q)}$, $n \geq 0$, of the functor $T_{(q)} : \mathcal{PCM}(P) \longrightarrow \mathfrak{Gr}$ relative to the projective class $\mathcal{P}$ induced by the cotriple $(\mathcal{F}, \tau, \delta)$ in the category $\mathcal{PCM}(P)$ (see [60]).

Definition 2.20. Let P be a group, (M, μ) be a precrossed P-module, and q be a nonnegative integer. Define the nth homology group modulo q of the precrossed P-module (M, μ) by

$$H_n(M, q)_P = \mathcal{L}_{n-1}^{\mathcal{P}} T_{(q)}(M), \quad n \geq 1.$$

Proposition 2.21. *Let $\mu : M \longrightarrow P$ be a precrossed P-module such that $\mu(m) = 1$ for all $m \in M$. Then we have*

$$H_n(M, q)_P = H_n(M, \mathbb{Z}_q), \quad n \geq 1.$$

Proof. Consider a $\mathcal{P}$-projective pseudo-simplicial resolution (see Chap. 1, Sec. 1.2) of (M, μ) in the category $\mathcal{PCM}(P)$

$$\cdots \ \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}} \ F_2 \xrightarrow{\kappa_2} Y_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} F_1 \xrightarrow{\kappa_1} Y_0 \rightrightarrows F_0 \longrightarrow M \ , \tag{2.13}$$

where $F_n \in \mathcal{P}$ and Y_n is a simplicial kernel in the category $\mathcal{PCM}(P)$. By Propositions 2.17 and 2.18 all F_n are free groups and all κ_n are surjective group homomorphisms, implying that (2.13) is a projective resolution of the group M in the category $\mathfrak{Gr}$. Since μ is a trivial group homomorphism, $T_{(q)}(F_n) = F_n^{ab}/qF_n^{ab}$. Using [5] we obtain the assertion. $\square$

3.2. Main properties. We investigate the functor $T_{(q)}$ and prove a Hopf type formula for the second homology modulo q of precrossed P-modules, generalizing the classical one (see [3, 47]).

Lemma 2.22. *Let P be a group and q be a nonnegative integer. Then the functor $T_{(q)} : \mathcal{PCM}(P) \longrightarrow \mathfrak{Gr}$ is a cosheaf over $(\mathcal{PCM}(P), \mathcal{P})$, where $\mathcal{P}$ is the projective class induced by the cotriple $(\mathcal{F}, \tau, \delta)$.*

Proof. It is easy to verify that for a short exact sequence of precrossed P-modules

$$1 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 1$$

there is an exact sequence of groups

$$T_{(q)}(L) \longrightarrow T_{(q)}(M) \longrightarrow T_{(q)}(N) \longrightarrow 1. \tag{2.14}$$

Consider a $\mathcal{P}$-epimorphism $Q \xrightarrow{\alpha} M$ in the category $\mathcal{PCM}(P)$. We must show that the diagram of groups

$$T_{(q)}(Q \times_M Q) \underset{d_1}{\overset{d_0}{\rightrightarrows}} T_{(q)}(Q) \xrightarrow{T_{(q)}(\alpha)} T_{(q)}(M) \longrightarrow 1$$

is exact. In effect, we have the following commutative diagram of groups:

$$\begin{tikzcd}
T_{(q)}(R) \arrow[r] \arrow[d, "\lambda"'] & T_{(q)}(Q) \arrow[r, "T_{(q)}(\alpha)"] \arrow[d, equal] & T_{(q)}(M) \arrow[r] \arrow[d, equal] & 1 \\
\operatorname{Ker} d_0 \arrow[r, "d_1"'] & T_{(q)}(Q) \arrow[r, "T_{(q)}(\alpha)"'] & T_{(q)}(M) \arrow[r] & 1
\end{tikzcd},$$

where R is the kernel of $\alpha : Q \longrightarrow M$, λ is a homomorphism induced by the inclusion $R \hookrightarrow Q \times_M Q$, $r \longmapsto (r, 1)$, and the top row is exact by (2.14). Hence the bottom row of this diagram is also exact. $\square$

Proposition 2.23. *Let P be a group, (M, μ) be a precrossed P-module, and q be a nonnegative integer. Then there is a natural isomorphism*

$$H_1(M, q)_P \cong M/\langle M, M \rangle_{(q)}.$$

Proof. The proposition follows from Lemma 2.22 and Proposition 1.20. $\square$

Theorem 2.24 (Hopf's formula)**.** *Let P be a group, (M, μ) be a precrossed P-module, and q be a nonnegative integer. Then there is an isomorphism*

$$H_2(M, q)_P \approx R \cap \langle F, F \rangle_{(q)} / \langle\langle F, R \rangle\rangle_{(q)},$$

where $1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 1$ *is any free presentation of the precrossed* P*-module* (M,μ) *i.e., using Propositions* 2.17 *and* 2.18*, we have that* F *is an object of the projective class* $\mathcal{P}$ *and* φ *is a* $\mathcal{P}$*-epimorphism.*

Proof. Consider the augmented Čech resolution $(\check{C}(\varphi),\varphi,(M,\mu))$ of $(M,\mu) \in \mathcal{P}CM(P)$ for $\varphi : F \longrightarrow M$.

By Lemma 2.22, $T_{(q)}$ is a cosheaf over $(\mathcal{P}CM(P),\mathcal{P})$, and using [103] or [60, Theorem 2.39(ii)], we see that there is an isomorphism

$$\mathcal{L}_1^{\mathcal{P}} T_{(q)}(M) \cong \pi_1 C_*,$$

where C_* is the following simplicial group

$$C_* \equiv \ \cdots \ \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} T_{(q)}(F \times_M F \times_M F) \ \substack{\xrightarrow{T_q(d_0)}\\[-2pt]\longrightarrow\\[-2pt]\xrightarrow[T_q(d_2)]{}} T_{(q)}(F \times_M F) \ \substack{\xrightarrow{T_q(d_0)}\\[-2pt]\xrightarrow[T_q(d_1)]{}} T_{(q)}(F) \ .$$

The Moore complex NC_* of the simplicial group C_* has length 1, i.e., $(NC_*)_n = 0$, $n \geq 2$. This follows from the fact that the Moore complex of the Ĉech resolution has length 1. Hence

$$\pi_1 C_* = \operatorname{Ker} T_q(d_0) \cap \operatorname{Ker} T_q(d_1).$$

Furthermore, we have the following isomorphism of precrossed P-modules $F \times_M F \xrightarrow{\cong} R \rtimes F$, defined by $(r,f) \longmapsto (rf,f)$, where the precrossed P-module structure on the group $R \rtimes F$ is given by the following way: a homomorphism $R \rtimes F \longrightarrow P$ is defined by $(r,f) \longmapsto \mu\varphi(f)$ and an action of P on $R \rtimes F$ by ${}^p(r,f) = ({}^p r, {}^p f)$ for all $p \in P$, $r \in R$, $f \in F$. We obtain $R \rtimes F \ \substack{\xrightarrow{d_0}\\[-2pt]\xrightarrow[d_1]{}} \ F$, $d_0(r,f) = f$, $d_1(r,f) = rf$.

It only remains to prove that the homomorphism

$$\alpha : (R/\langle\langle F,R\rangle\rangle_{(q)}) \times F/\langle F,F\rangle_{(q)} \longrightarrow T_{(q)}(R \rtimes F),$$

defined by $\alpha([r],[f]) = [(r,f)]$, is an isomorphism. □

Remark 2.25. For $\mu = 0$, Theorem 2.24 generalizes the classical Hopf formula from [3] and for $q = 0$ Proposition 2.23 and Theorem 2.24 show that we can obtain the first and the second homology of precrossed P-modules of Conduché and Ellis [30] as non-Abelian derived functors of the Peiffer abelianization functor.

3.3. Some other properties. In this section, we investigate low-dimensional, first and second, homologies modulo q of precrossed P-modules, always having in mind Proposition 2.23 and Theorem 2.24, and give some results generalizing, in the q modular aspect, the results of Conduché and Ellis (see [30]).

Proposition 2.26. *Let* P *be a group,* q *be a nonnegative integer, and*

$$1 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 1$$

be a short exact sequence of precrossed P*-modules. Then there is an exact sequence of groups*

$$H_2(M,q)_P \longrightarrow H_2(N,q)_P \longrightarrow L/\langle\langle M,L\rangle\rangle_{(q)} \longrightarrow H_1(M,q)_P \longrightarrow H_1(N,q)_P \longrightarrow 1. \qquad (2.15)$$

Proof. Assume $1 \longrightarrow R \longrightarrow F \longrightarrow M \longrightarrow 1$ is a free presentation of the precrossed P-module M, and hence $1 \longrightarrow R' \longrightarrow F \longrightarrow N \longrightarrow 1$ is a free presentation of the precrossed P-module N. Therefore, $R \subset R'$, implying $R \cap \langle F,F\rangle_{(q)} \subset R' \cap \langle F,F\rangle_{(q)}$, $\langle\langle F,R\rangle\rangle_{(q)} \subset \langle\langle F,R'\rangle\rangle_{(q)}$, and there is the canonical group homomorphism $H_2(M,q)_P \longrightarrow H_2(N,q)_P$.

The following commutative diagram of groups with exact rows

$$\begin{tikzcd}
1 \arrow[r] & R' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & L \arrow[r] & {} \arrow[r] & N \arrow[r] & 1
\end{tikzcd}$$

induces a homomorphism $H_2(N,q)_P \longrightarrow L/\langle\langle M,L\rangle\rangle_{(q)}$.

Other homomorphisms are defined naturally and it is easy to verify that the sequence (2.15) is exact. $\square$

Remark 2.27. We can extend the sequence (2.15) to any dimensions using the long exact sequence of the non-Abelian derived functors and recover for $\mu = 0$ the eight-term exact homology sequence of groups with coefficients in $\mathbb{Z}_q$ (see [48]).

The following result generalizes the classical group result and uses the standard proof, originally due to [116].

For any precrossed P-module (M,μ) and any nonnegative integer q, there is the following family of precrossed P-submodules:

$$M^{(1)}_{(q)} = M, M^{(2)}_{(q)} = \langle\langle M,M\rangle\rangle_{(q)}, \dots, M^{(n+1)}_{(q)} = \big\langle\big\langle M, M^{(n)}_{(q)}\big\rangle\big\rangle_{(q)}.$$

Theorem 2.28. *Let P be a group, q a nonnegative integer, and $\varphi : M \longrightarrow N$ be a morphism of precrossed P-modules such that the following properties hold*:

(i) *the natural homomorphism $H_1(M,q)_P \longrightarrow H_1(N,q)_P$, induced by φ, is an isomorphism*;
(ii) *the natural homomorphism $H_2(M,q)_P \longrightarrow H_2(N,q)_P$, induced by φ, is a surjection.*

Then φ induces a natural isomorphism of precrossed P-modules

$$M/M^{(n)}_{(q)} \xrightarrow{\cong} N/N^{(n)}_{(q)} \quad \textit{for} \quad n \geq 2.$$

Proof. By induction. For $n = 2$, the theorem is obvious. Assume that it is valid for n. By Proposition 2.26 and the commutative diagram of groups with exact rows

$$\begin{tikzcd}
1 \arrow[r] & M^{(n)}_{(q)} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\varphi"'] & M/M^{(n)}_{(q)} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N^{(n)}_{(q)} \arrow[r] & N \arrow[r] & N/N^{(n)}_{(q)} \arrow[r] & 1
\end{tikzcd},$$

we have the following commutative diagram of groups with exact rows

$$\begin{tikzcd}
H_2(M,q)_P \arrow[r] \arrow[d] & H_2(M/M^{(n)}_{(q)},q)_P \arrow[r] \arrow[d] & M^{(n)}_{(q)}/M^{(n+1)}_{(q)} \arrow[r] \arrow[d] & H_1(M,q)_P \arrow[r] \arrow[d] & H_1(M/M^{(n)}_{(q)},q)_P \arrow[r] \arrow[d] & 1 \\
H_2(N,q)_P \arrow[r] & H_2(N/N^{(n)}_{(q)},q)_P \arrow[r] & N^{(n)}_{(q)}/N^{(n+1)}_{(q)} \arrow[r] & H_1(N,q)_P \arrow[r] & H_1(N/N^{(n)}_{(q)},q)_P \arrow[r] & 1
\end{tikzcd}.$$

Using the five-lemma, we see that $M^{(n)}_{(q)}/M^{(n+1)}_{(q)}$ is isomorphic to $N^{(n)}_{(q)}/N^{(n+1)}_{(q)}$. Then the commutative diagram of groups

$$\begin{tikzcd}
1 \arrow[r] & M^{(n)}_{(q)}/M^{(n+1)}_{(q)} \arrow[r] \arrow[d] & M/M^{(n+1)}_{(q)} \arrow[r] \arrow[d] & M/M^{(n)}_{(q)} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N^{(n)}_{(q)}/N^{(n+1)}_{(q)} \arrow[r] & N/N^{(n+1)}_{(q)} \arrow[r] & N/N^{(n)}_{(q)} \arrow[r] & 1
\end{tikzcd}$$

gives the result for $n+1$. ☐

For any precrossed P-module $\mu : M \longrightarrow P$, let $M \wedge^q_P M$ be the group generated by the symbols $m \wedge m'$ and $\{k\}$, $m, m' \in M$, $k \in \operatorname{Ker}\mu$ subject to the following relations:

$$m \wedge m'm'' = (m \wedge m')(m \wedge m'')\big(\langle m, m''\rangle^{-1} \wedge {}^{\mu m}m'\big), \tag{2.16}$$

$$mm' \wedge m'' = (m \wedge m'm''m'^{-1})\big({}^{\mu m}m' \wedge {}^{\mu m}m''\big), \tag{2.17}$$

$$\langle m, m'\rangle \wedge \langle n, n'\rangle = (m \wedge m')(n \wedge n')(m \wedge m')^{-1}(n \wedge n')^{-1}, \tag{2.18}$$

$$(\langle m, m'\rangle \wedge m'')(m'' \wedge \langle m, m'\rangle) = (m \wedge m')\big({}^{\mu m''}m \wedge {}^{\mu m''}m'\big)^{-1}, \tag{2.19}$$

$$k \wedge k = 1, \tag{2.20}$$

$$\{k\}(m \wedge m')\{k\}^{-1} = (k^q m \wedge m')(k^q \wedge {}^{\mu m}m')^{-1}, \tag{2.21}$$

$$\{kk'\} = \{k\} \prod_{i=1}^{q-1} \big(k^{-1} \wedge k^{1-q+i}(k')^i k^{q-1-i}\big)\{k'\}, \tag{2.22}$$

$$\{k\}\{k'\}\{k\}^{-1}\{k'\}^{-1} = k^q \wedge k'^q, \tag{2.23}$$

$$\{\langle m, m'\rangle\} = (m \wedge m')^q \tag{2.24}$$

for all $m, m', m'', n, n' \in M$, and $k, k' \in \operatorname{Ker}\mu$.

Note that (2.16)–(2.20) are the defining relations for the group $M \wedge_P M$ defined in [30]. Furthermore, when $P = 1$ or $\mu = 0$, the group $M \wedge^q_P M$ coincides with the non-Abelian exterior product modulo q, $M \wedge^q M$, introduced by Conduché and Rodriguez–Fernández [32] (see Chap. 6 and also [13, 47, 48]).

There is an action of the group P on the group $M \wedge^q_P M$ given by ${}^p(m \wedge m') = {}^pm \wedge {}^pm'$ and ${}^p\{k\} = \{{}^pk\}$ for all $m, m' \in M$, $k \in \operatorname{Ker}\mu$. Moreover, there exists a P-equivariant group homomorphism $\partial^q_2 : M \wedge^q_P M \longrightarrow M$ defined by $\partial^q_2(m \wedge m') = \langle m, m'\rangle$ and $\partial^q_2(\{k\}) = k^q$. It is clear that

$$\partial^q_2(M \wedge^q_M M) = M^{(2)}_{(q)}.$$

Note that the complex of groups $M \wedge^q_P M \xrightarrow{\partial^q_2} M \xrightarrow{\mu} P$ is a 2-crossed module in the sense of Conduché [27].

Proposition 2.29. *Let (M, μ) be a precrossed P-module, $q > 0$, and*

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 1$$

be a short exact sequence of precrossed P-modules, where (F, ν) is a free precrossed P-module. If the homomorphism $\partial^q_2 : F \wedge^q_P F \longrightarrow F$ is injective, then the group $M \wedge^q_P M$ is isomorphic to the group $F^{(2)}_{(q)}/\langle\langle F, R\rangle\rangle_{(q)}$.

Proof. Let L_F (respectively, L_M) be the free group generated by the set $(F\times F)\cup \operatorname{Ker}\nu$ (respectively, $(M\times M)\cup \operatorname{Ker}\mu$). There is a commutative diagram of groups

$$\begin{array}{ccc} L_F & \longrightarrow & L_M \\ {\scriptstyle \pi_F}\big\downarrow & & \big\downarrow{\scriptstyle \pi_M} \\ F\wedge_P^q F & \longrightarrow & M\wedge_P^q M \end{array},$$

where the horizontal homomorphisms are surjective and π_F and π_M are canonical homomorphisms defined by $\pi_F(f,f')=f\wedge f'$, $\pi_F(g)=\{g\}$ and $\pi_F(m,m')=m\wedge m'$, $\pi_F(k)=\{k\}$ for all $f,f'\in F$, $g\in \operatorname{Ker}\nu$, $m,m'\in M$ and $k\in\operatorname{Ker}\mu$. It is easy to obtain that $\operatorname{Ker}(F\wedge_P^q F\longrightarrow M\wedge_P^q M)$ is the homomorphic image of $\operatorname{Ker}(L_F\longrightarrow L_M)$ by π_F. It is also easy to verify that $\operatorname{Ker}(L_F\longrightarrow L_M)$ is the normal subgroup of L_F generated by the elements $(f_1,f_2)({f'}_1,{f'}_2)^{-1}$ and $f_3{f'}_3^{-1}$ such that $\varphi f_i=\varphi {f'}_i$, $f_i,f_i'\in F$ $(i=1,2)$ and $\varphi f_3=\varphi {f'}_3$, $f_3,{f'}_3\in Ker\nu$. Thus, its image in $F\wedge_P^q F$ is the normal subgroup generated by the elements $(f_1\wedge f_2)({f'}_1\wedge {f'}_2)^{-1}$ and $\{f_3\}\{{f'}_3\}^{-1}$, which by the formulas (2.16), (2.17), and (2.23) coincides with the normal subgroup of $F\wedge_P^q F$ generated by the elements $f\wedge r$, $r\wedge f$ and $\{r\}$, $f\in F$, $r\in R$. Then the image of this subgroup by the isomorphism $F\wedge_P^q F\cong \partial_2^q(F\wedge_P^q F)=F_{(q)}^{(2)}$ is $\langle\langle F,R\rangle\rangle_{(q)}$ and thus $F_{(q)}^{(2)}/\langle\langle F,R\rangle\rangle_{(q)}\cong M\wedge_P^q M$. □

Lemma 2.30. *Let $h:A\longrightarrow B$ and $g:B\longrightarrow C$ be group homomorphisms. If h is surjective, then the following sequence of groups is exact:*

$$1\longrightarrow \operatorname{Ker}(h)\longrightarrow \operatorname{Ker}(gh)\longrightarrow \operatorname{Ker}(g)\longrightarrow 1.$$

Theorem 2.31. *Let $\mu: M\longrightarrow P$ be a precrossed P-module, $q>0$, and*

$$1\longrightarrow R\longrightarrow F\xrightarrow{\varphi} M\longrightarrow 1$$

be a short exact sequence of precrossed P-modules, where F is a free precrossed P-module. If the homomorphism $\partial_2^q: F\wedge_P^q F\longrightarrow F_{(q)}^{(2)}$ is an isomorphism, then there is an isomorphism of groups

$$H_2(M,q)\cong \operatorname{Ker}\big(M\wedge_P^q M\longrightarrow M\big).$$

Proof. By Lemma 2.30, we have the following exact sequence of groups:

$$1\longrightarrow \operatorname{Ker}(\varphi\wedge_P^q\varphi)\longrightarrow \operatorname{Ker}(\partial_2^q(\varphi\wedge_P^q\varphi))\longrightarrow \operatorname{Ker}\partial_2^q\longrightarrow 1.$$

From the commutative diagram of groups

$$\begin{array}{ccc} F\wedge_P^q F & \xrightarrow{\varphi\wedge_P^q\varphi} & M\wedge_P^q M \\ {\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \partial_2^q} \\ F_{(q)}^{(2)} & \xrightarrow[\varphi_{(q)}^{(2)}]{} & M_{(q)}^{(2)} \end{array}$$

we obtain

$$\operatorname{Ker}(\partial_2^q(\varphi\wedge_P^q\varphi))\cong \operatorname{Ker}\varphi_{(q)}^{(2)}=R\cap F_{(q)}^{(2)}.$$

Then

$$\operatorname{Ker}(\partial_2^q: M\wedge_P^q M\longrightarrow M)\cong R\cap F_{(q)}^{(2)}/\langle\langle F,R\rangle\rangle_{(q)}=H_2(M,q)_P.$$

□

Finally, we give an example showing that there exists a group P and a free precrossed P-module F such that the homomorphism $\partial_2^q: F\wedge_P^q F\longrightarrow F$ is injective.

Lemma 2.32. *Let (M, μ) be a precrossed P-module and q a nonnegative integer. Then there is an exact sequence of groups*

$$M \wedge_P M \xrightarrow{\varphi} M \wedge_P^q M \longrightarrow \ker \mu / \langle M, M \rangle \longrightarrow 1.$$

Proof. The homomorphism φ is given by $\varphi(m \wedge m') = m \wedge m'$. The required exactness can be easily verified. $\square$

Proposition-Example 2.33. *Let P be a free group, $\mu : F \longrightarrow P$ a free precrossed P-module and $q > 0$. Then the homomorphism $\partial_2^q : F \wedge_P^q F \longrightarrow F_{(q)}^{(2)}$ is an isomorphism.*

Proof. Using Lemma 2.32, we have the following commutative diagram of groups with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & F \wedge_P F \arrow[r, "\varphi"] \arrow[d, "\partial_2"'] & F \wedge_P^q F \arrow[r] \arrow[d, "\partial_2^q"'] & \ker \mu / F^{(2)} \arrow[r] \arrow[d, "\alpha"'] & 1 \\
1 \arrow[r] & F^{(2)} \arrow[r] & F_{(q)}^{(2)} \arrow[r] & F_{(q)}^{(2)} / F^{(2)} \arrow[r] & 1
\end{tikzcd},$$

where $F^{(2)} = \langle F, F \rangle$, ∂_2 is an isomorphism [7], proved by applying a theorem of Whitehead [126] on 2-dimensional CW-complexes and the theorem of Kan [79] (see above), and hence φ is injective. We can directly check that α is an isomorphism, and so is ∂_2^q. $\square$

Chapter 3

HOPF-TYPE FORMULAS

One of the basic results of the theory of group homology is the well-known Hopf formula for the second integral group homology, relating homology to an elementary formula involving a presentation of the group being studied. In particular, it asserts that for a given group G there is an isomorphism

$$H_2(G) \cong \frac{R \cap [F, F]}{[F, R]},$$

where $R \rightarrowtail F \twoheadrightarrow G$ is a free presentation of the group G.

Several alternative generalizations of this classical Hopf formula to higher dimensions were made in various papers (see, e.g., [33, 111, 117]), but perhaps the most successful one, giving formulas in all dimensions, was by Brown and Ellis [14]. They used topological methods, in particular, the Hurewicz theorem for n-cubes of spaces (see [19]), which itself is an application of the generalized van Kampen theorem for diagrams of spaces [18]. The final result is as follows.

Theorem (see [14])**.** *Let $R_1, \dots, R_n$ be normal subgroups of a group F such that*

$$H_2(F) = 0, \quad H_r\Big(F / \prod_{i \in A} R_i\Big) = 0 \quad \textit{for} \quad r = |A| + 1, \quad r = |A| + 2,$$

where A is a nonempty proper subset of $\langle n \rangle = \{1, \dots, n\}$ (for example, if the groups $F / \prod_{i \in A} R_i$ are free for $A \neq \langle n \rangle$) and $F / \prod_{1 \le i \le n} R_i \cong G$. Then there is an isomorphism

$$H_{n+1}(G) \cong \frac{\bigcap_{i=1}^{n} R_i \cap [F, F]}{\prod_{A \subseteq \langle n \rangle} \big[\bigcap_{i \in A} R_i, \bigcap_{i \notin A} R_i \big]}.$$

Later Ellis [44] gave a purely algebraic proof of the formula using hyper-relative derived functors. His results are clearly related to those presented here, and a comparison between them may yield some general links between the two theories of derived functors being used. In the original paper [14] and in the paper with an algebraic approach [44], a technical assumption was omitted. An erratum is available from Ellis' homepage.

In this chapter, using the general theory of n-fold Čech derived functors, we establish a new purely algebraic method for investigating higher integral group homology from a Hopf formula point of view and the further generalizations of these formulas. This method is universal and is valid for other algebraic structures.

Section 1 is devoted to the study of normal $(n+1)$-ads of groups arising from simplicial groups and shows how we pass naturally from simplicial groups to Hopf type formulas (Theorem 3.6).

Our generalization, in Sec. 2, handles the non-Abelian derived functors of the "nilization of degree k" functor, $Z_k(G) : \mathfrak{Gr} \longrightarrow \mathfrak{Gr}$, $k \geq 2$, where Z_2 coincides with the group abelianization functor $\mathfrak{A}b$. We give Hopf type formulas for these derived functors (Theorems 3.8 and 3.9). Finally, we apply these results to algebraic K-theory and obtain Hopf type formula for algebraic K-theory (Theorem 3.14).

In Sec. 3, the m-fold Čech derived functors of group-valued functors from the category of crossed n-cubes is treated. In particular, we calculate the mth m-fold Čech derived functor of the certain abelianization functor $\sigma\mathfrak{Ab}$ from the category of crossed n-cubes to the category of groups (Theorem 3.15), implying the expression of the cotriple homology of crossed n-cubes (cat n-groups) as generalized Hopf type formulas (Theorem 3.16).

1. From Simplicial Groups to Hopf-Type Formulas

We start by developing some techniques for handling $(n+1)$-ads of groups, relating them to iterated commutators.

Definition 3.1. Let j be given, $1 \leq j \leq n$. A normal $(n+1)$-ad of groups $(F; R_1, \dots, R_n)$ is called *simple relative to R_j* if there exists a subgroup F' of the group F such that

$$F' \cap R_j = 1, \quad \bigcap_{i \in A} R_i = \Big(\bigcap_{i \in A} R_i \cap F'\Big)\Big(\bigcap_{i \in A} R_i \cap R_j\Big)$$

for all $A \subseteq \langle n \rangle \setminus \{j\}$.

For a given $(n+1)$-ad of groups $(F; R_1, \dots, R_n)$, $A \subseteq \langle n \rangle$, and $k \geq 1$ recall that we have denoted above by $D_k(F; A)$ (see Proposition 2.7) the following normal subgroup of the group F:

$$\prod_{A_1 \cup A_2 \cup \cdots \cup A_k = A} \Big[\bigcap_{i \in A_1} R_i, \Big[\bigcap_{i \in A_2} R_i, \dots, \Big[\bigcap_{i \in A_{k-1}} R_i, \bigcap_{i \in A_k} R_i\Big] \dots\Big]\Big].$$

Sometimes, we write $D_k(F; R_1, \dots, R_n)$ instead of $D_k(F; \langle n \rangle)$.

Lemma 3.2. *Let $(F; R_1, \dots, R_n)$ be a normal $(n+1)$-ad of groups which is simple relative to R_j, $1 \leq j \leq n$ and let $k \geq 1$. Then*

$$D_k(F; A) = (D_k(F; A) \cap F') D_k(F; A \cup \{j\})$$

for all $A \subseteq \langle n \rangle \setminus \{j\}$.

Proof. We use induction on k. Let $k = 1$; then

$$D_1(F; A) = \bigcap_{i \in A} R_i = \Big(\bigcap_{i \in A} R_i \cap F'\Big)\Big(\bigcap_{i \in A} R_i \cap R_j\Big) = \Big(\bigcap_{i \in A} R_i \cap F') D_1(F; A \cup \{j\}\Big)$$

for $A \subseteq \langle n \rangle \setminus \{j\}$.

Proceeding by induction, we assume that the assertion is true for $k-1$ and we will prove it for k.

The inclusion $(D_k(F;A) \cap F')D_k(F;A \cup \{j\}) \subseteq D_k(F;A)$ is obvious. It is easy to see that a generator of $D_k(F;A)$ has the form $[x,w]$, where $x \in \bigcap_{i \in B} R_i$, $w \in D_{k-1}(F;C)$, $B, C \subseteq A \subseteq \langle n \rangle \setminus \{j\}$, and $B \cup C = A$. There exist elements $y \in \bigcap_{i \in B} R_i \cap F'$ and $z \in \bigcap_{i \in B} R_i \cap R_j$ such that $x = yz$. We have

$$[x,w] = [yz,w] = y[z,w]y^{-1}[y,w].$$

Clearly,

$$[z,w] \in D_k(F; B \cup C \cup \{j\}) = D_k(F; A \cup \{j\})$$

and hence $y[z,w]y^{-1} \in D_k(F;A \cup \{j\})$. By the inductive hypothesis, there exist $w' \in D_{k-1}(F;C \cup \{j\})$ and $x' \in D_{k-1}(F;C) \cap F'$ such that $w = x'w'$. We have

$$[y,w] = [y,x'w'] = [y,x']x'[y,w']x'^{-1}.$$

Clearly,

$$[y,w'] \in D_k(F; B \cup C \cup \{j\}) = D_k(F; A \cup \{j\})$$

and hence

$$x'[y,w']x'^{-1} \in D_k(F; A \cup \{j\}).$$

Therefore, there is an element $w'' \in D_k(F;A \cup \{j\})$ such that $[x,w] = [y,x']w''$ where $[y,x'] \in D_k(F;A) \cap F'$. □

For a given group G, the (lower) central series ($\Gamma_k = \Gamma_k(G)$)

$$G = \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k \supseteq \cdots$$

of G is defined inductively by

$$\Gamma_k = \prod_{i+j=k} [\Gamma_i, \Gamma_j].$$

The well-known Witt–Hall identities on commutators (see, e.g., [7]) imply that $\Gamma_k = [G, \Gamma_{k-1}]$.

Let us define the nilization of degree k functor $Z_k : \mathfrak{Gr} \longrightarrow \mathfrak{Gr}$, $k \geq 2$ by $Z_k(G) = G/\Gamma_k(G)$ for any $G \in \mathfrak{Gr}$ and where $Z_k(\alpha)$ is the natural homomorphism induced by a group homomorphism α. Of course, Z_2 is the ordinary abelianization functor of groups.

Proposition 3.3. *Let $(F; R_1, \ldots, R_n)$ be a normal $(n+1)$-ad of groups and $k \geq 2$. Assume that $(F; R_1, \ldots, R_j)$ is a simple normal $(j+1)$-ad of groups relative to R_j for all $1 \leq j \leq n$. Then*

$$\bigcap_{i \in \langle j \rangle} R_i \cap \Gamma_k(F) = D_k(F; \langle j \rangle), \quad 1 \leq j \leq n.$$

Proof. Since the inclusion

$$D_k(F; \langle j \rangle) \subseteq \bigcap_{i \in \langle j \rangle} R_i \cap \Gamma_k(F)$$

is clear, we must only show the inclusion

$$\bigcap_{i \in \langle j \rangle} R_i \cap \Gamma_k(F) \subseteq D_k(F; \langle j \rangle),$$

which will be done by induction on j.

Let $j = 1$, then there exists a subgroup F_1 of the group F such that $R_1 \cap F_1 = 1$ and $F = F_1 R_1$. Let $w \in R_1 \cap \Gamma_k(F) \subseteq \Gamma_k(F) = D_k(F; \varnothing)$. Using Lemma 3.2, we have elements $x' \in D_k(F;\varnothing) \cap F_1$ and $w' \in D_k(F; \langle 1 \rangle)$ such that $w = x'w'$. But $x' = ww'^{-1} \in R_1$ and hence $x' = 1$. Thus,

$$R_1 \cap \Gamma_k(F) \subseteq D_k(F; \langle 1 \rangle).$$

Proceeding by induction, we assume that the result is true for $j-1$ and we will prove it for j.

There exists a subgroup F_j of the group F such that $R_j \cap F_j = 1$ and

$$\bigcap_{i\in A} R_i = \Big(\bigcap_{i\in A} R_i \cap F_j\Big)\Big(\bigcap_{i\in A} R_i \cap R_j\Big)$$

for all $A \subseteq \{1, \dots, j-1\}$. Let

$$w \in \bigcap_{i\in\langle j\rangle} R_i \cap \Gamma_k(F) \subseteq \bigcap_{i\in\langle j-1\rangle} R_i \cap \Gamma_k(F).$$

Using the inductive hypothesis, we have the equality

$$\bigcap_{i\in\langle j-1\rangle} R_i \cap \Gamma_k(F) = D_k\big(F; \langle j-1\rangle\big).$$

By Lemma 3.2, there are elements $x' \in D_k(F; \langle j-1\rangle) \cap F_j$ and $w' \in D_k(F; \langle j\rangle)$ such that $w = x'w'$. Certainly, $x' = ww'^{-1} \in R_j$ and hence $x' = 1$. Therefore,

$$\bigcap_{i\in\langle j\rangle} R_i \cap \Gamma_k(F) \subseteq D_k(F; \langle j\rangle).$$

The proposition is proved. □

These conditions of "simplicity" may seem rather restrictive, but the following observation shows that examples of simple normal $(n+1)$-ads of groups appear naturally, and that moreover these examples satisfy the conditions of Proposition 3.3.

Proposition 3.4. *Let F_* be a pseudo-simplicial group. Then $(F_n; \operatorname{Ker} d_0^n, \dots, \operatorname{Ker} d_{j-1}^n)$ is a simple normal $(j+1)$-ad of groups relative to $\operatorname{Ker} d_{j-1}^n$ for all $1 \le j \le n$.*

Proof. Since $d_{j-1}^n s_{j-1}^{n-1} = 1$,

$$s_{j-1}^{n-1}(F_{n-1}) \cap \operatorname{Ker} d_{j-1}^n = 1, \quad s_{j-1}^{n-1}(F_{n-1}) \operatorname{Ker} d_{j-1}^n = F_n$$

for all $n \ge 1$. Hence for $j = 1$, $(F_n; \operatorname{Ker} d_0^n)$ is a simple normal 2-ad of groups relative to $\operatorname{Ker} d_0^n$ and the F' of the definition of simplicity is $s_0^{n-1}(F_{n-1})$.

Now assume that $j > 1$. We will show the following equality:

$$\bigcap_{i\in A} \operatorname{Ker} d_i^n = \Big(\bigcap_{i\in A} \operatorname{Ker} d_i^n \cap s_{j-1}^{n-1}(F_{n-1})\Big)\Big(\bigcap_{i\in A} \operatorname{Ker} d_i^n \cap \operatorname{Ker} d_{j-1}^n\Big)$$

for all $A \subseteq \{0, \dots, j-2\}$ and $A \neq \varnothing$, so again the F' of the definition of simplicity is $s_{j-1}^{n-1}(F_{n-1})$.

Let

$$x = s_{j-1}^{n-1}(x_{n-1}) r_{j-1} \in \bigcap_{i\in A} \operatorname{Ker} d_i^n,$$

where $x_{n-1} \in F_{n-1}$, $r_{j-1} \in \operatorname{Ker} d_{j-1}^n$. Thus

$$d_i^n(x) = d_i^n s_{j-1}^{n-1}(x_{n-1}) d_i^n(r_{j-1}) = 1$$

for all $i \in A$. Since $i < j-1$, we have

$$d_i^n(r_{j-1}) = s_{j-2}^{n-2} d_i^{n-1}(x_{n-1})^{-1}.$$

Hence

$$1 = d_i^{n-1} d_{j-1}^n(r_{j-1}) = d_{j-2}^{n-1} d_i^n(r_{j-1}) = d_{j-2}^{n-1} s_{j-2}^{n-2} d_i^{n-1}(x_{n-1})^{-1} = d_i^{n-1}(x_{n-1})^{-1}.$$

Therefore, $d_i^n(r_{j-1}) = 1$ and $d_i^n s_{j-1}^{n-1}(x_{n-1}) = 1$ for all $i \in A$. □

The next lemma is well known but very useful. The proof is routine.

Lemma 3.5. *Let G_* be a pseudo-simplicial group and $A \subseteq \langle n\rangle$, $A \neq \langle n\rangle$. Then*

$$d_n^n\Big(\bigcap_{i\in A} \operatorname{Ker} d_{i-1}^n\Big) = \bigcap_{i\in A} \operatorname{Ker} d_{i-1}^{n-1}, \quad n \geq 2.$$

Now we give the main result of this section.

Theorem 3.6. *Let (F_*, d_0^0, G) be an aspherical augmented pseudo-simplicial group. Then there is a natural isomorphism*

$$\pi_n Z_k(F_*) \cong \frac{\bigcap\limits_{i=0}^{n-1} \operatorname{Ker} d_i^{n-1} \cap \Gamma_k(F_{n-1})}{D_k(F_{n-1}; \operatorname{Ker} d_0^{n-1}, \ldots, \operatorname{Ker} d_{n-1}^{n-1})}, \quad k \geq 2, \quad n \geq 1.$$

Proof. Let us consider the short exact sequence of augmented pseudo-simplicial groups

$$\begin{tikzcd}
 & 1 \arrow[d] & & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & \Gamma_k(F_n) \arrow[r, shift left, "\widetilde{d}_0^n"] \arrow[r, shift right, "\widetilde{d}_n^n"'] \arrow[d] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \Gamma_k(F_1) \arrow[r, shift left, "\widetilde{d}_0^1"] \arrow[r, shift right, "\widetilde{d}_1^1"'] \arrow[d] & \Gamma_k(F_0) \arrow[r, "\widetilde{d}_0^0"] \arrow[d] & \Gamma_k(G) \arrow[d] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & F_n \arrow[r, shift left, "d_0^n"] \arrow[r, shift right, "d_n^n"'] \arrow[d] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F_1 \arrow[r, shift left, "d_0^1"] \arrow[r, shift right, "d_1^1"'] \arrow[d] & F_0 \arrow[r, "d_0^0"] \arrow[d] & G \arrow[d] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & Z_k(F_n) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Z_k(F_1) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Z_k(F_0) \arrow[r] \arrow[d] & Z_k(G) \arrow[d] \\
 & 1 & & 1 & 1 & 1
\end{tikzcd}.$$

By the induced long exact homotopy sequence, we have isomorphisms of groups

$$\pi_n Z_k(F_*) \cong \frac{\bigcap\limits_{i=0}^{n-1} \operatorname{Ker} \widetilde{d}_i^{n-1}}{\widetilde{d}_n^n\big(\bigcap\limits_{i=0}^{n-1} \operatorname{Ker} \widetilde{d}_i^n\big)}, \quad n \geq 1.$$

Since $\widetilde{d}_i^n$ is the restriction of d_i^n to $\Gamma_k(F_n)$, $\operatorname{Ker} \widetilde{d}_i^n = \operatorname{Ker} d_i^n \cap \Gamma_k(F_n)$. Hence

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} \widetilde{d}_i^{n-1} = \Big(\bigcap_{i=0}^{n-1} \operatorname{Ker} d_i^{n-1}) \cap \Gamma_k(F_{n-1}\Big)$$

and

$$\bigcap_{i=0}^{n-1} \operatorname{Ker} \widetilde{d}_i^n = \Big(\bigcap_{i=0}^{n-1} \operatorname{Ker} d_i^n\Big) \cap \Gamma_k(F_n).$$

Using Propositions 3.3 and 3.4, we have

$$\Big(\bigcap_{i\in\langle n\rangle} \operatorname{Ker} d_{i-1}^n\Big) \cap \Gamma_k(F_n) = D_k(F_n; \operatorname{Ker} d_0^n, \ldots, \operatorname{Ker} d_{n-1}^n), \quad n \geq 1.$$

Since (F_*, d_0^0, G) is an aspherical augmented pseudo-simplicial group,

$$d_n^n\Big(\bigcap_{i\in\langle n\rangle} \operatorname{Ker} d_{i-1}^n\Big) = \bigcap_{i\in\langle n\rangle} \operatorname{Ker} d_{i-1}^{n-1}, \quad n \geq 1.$$

Using this fact and Lemma 3.5, it is now easy to see that we have an equality

$$\begin{aligned}\widetilde{d}_n^n\Big(\bigcap_{i=0}^{n-1} \operatorname{Ker} \widetilde{d}_i^n\Big) &= d_n^n\big(D_k(F_n; \operatorname{Ker} d_0^n, \ldots, \operatorname{Ker} d_{n-1}^n)\big)\\ &= D_k\big(F_{n-1}; \operatorname{Ker} d_0^{n-1}, \ldots, \operatorname{Ker} d_{n-1}^{n-1}\big).\end{aligned}$$

□

In the case where (F_*, d_0^0, G) is a free pseudo-simplicial resolution of G, the homotopy groups of $Z_k(F_*)$ will be the left non-Abelian derived functors $L_n^{\mathcal{P}} Z_k(G)$ of Z_k, evaluated at G, where $\mathcal{P}$ is the projective class of free groups (see Chap. 1, Sec. 1.2). We thus have the following formal result.

Corollary 3.7. *Let G be a group and (F_*, d_0^0, G) an aspherical augmented pseudo-simplicial group and $k \geq 2$. If F_n is a free group for all $n \geq 0$, i.e., (F_*, d_0^0, G) is a free pseudo-simplicial resolution of the group G, then there is a natural isomorphism*

$$L_n^{\mathcal{P}} Z_k(G) \cong \frac{\bigcap_{i=0}^{n-1} \operatorname{Ker} d_i^{n-1} \cap \Gamma_k(F_{n-1})}{D_k(F_{n-1}; \operatorname{Ker} d_0^{n-1}, \ldots, \operatorname{Ker} d_{n-1}^{n-1})}, \quad n \geq 1.$$

2. Generalized Hopf-Type Formulas

In this section, we focus our attention on the investigation of the n-fold Čech derived functors of the functor $Z_k : \mathfrak{Gr} \longrightarrow \mathfrak{Gr}$, $k \geq 2$. Our method gives the possibility of finding a new purely algebraic proof of the generalized Hopf formula of Brown and Ellis; moreover, we express $L_n^{\mathcal{P}} Z_k(G)$, $n \geq 1$, $k \geq 2$ by a Hopf type formula, where $\mathcal{P}$ is the projective class of free groups.

Note that a $\mathcal{P}$-projective $\mathcal{P}$-exact n-presentation $\mathfrak{F}$ of a group G is called a free exact n-presentation of the group G.

The Quillen algebraic K-functors K_{n+1}, $n \geq 1$, are described in terms of a short exact sequence including the higher Hopf-type formulas for free exact n-presentations induced by a free simplicial resolution of the general linear group.

2.1. Hopf-type formulas for derived functors of the functors Z_k.

Using the fact that Z_k is a right exact functor, we easily show that $\mathcal{L}_0^{n\text{-fold}} Z_k \cong Z_k$. Moreover, Propositions 2.9 and 2.13 and Lemma 2.5 or alternatively Corollary 1.24 imply that $\mathcal{L}_i^{n\text{-fold}} Z_k = 0$ for $i > n$. Then the following theorem gives the nth n-fold Čech derived functor of the functor $Z_k : \mathfrak{Gr} \longrightarrow \mathfrak{Gr}$, $k \geq 2$.

Theorem 3.8. *Let G be a group and $k \geq 2$. Then there is an isomorphism*

$$\mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \frac{\bigcap_{i\in\langle n\rangle} R_i \cap \Gamma_k(F)}{D_k(F; R_1, \ldots, R_n)}, \quad n \geq 1,$$

where $(F; R_1, \ldots, R_n)$ is the normal $(n+1)$-ad of groups induced by some free exact n-presentation $\mathfrak{F}$ of the group G.

Proof. By their definition and Lemma 2.5 we obtain an isomorphism

$$\mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \pi_n(Z_k E^{(n)}(\mathcal{M}))_*,$$

where $\mathcal{M}$ is the inclusion crossed n-cube of groups given by the normal $(n+1)$-ad of groups $(F; R_1, \dots, R_n)$. Hence using Proposition 2.9 we have an isomorphism

$$\mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \pi_n(E^{(n)} \mathcal{B}_k(\mathcal{M})_*).$$

Then, by Proposition 2.13,

$$\mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \bigcap_{l \in \langle n \rangle} \operatorname{Ker}(\mathcal{B}_k(\mathcal{M})_{\langle n \rangle} \xrightarrow{\widetilde{\mu}_{l,\langle n \rangle}} \mathcal{B}_k(\mathcal{M})_{\langle n \rangle \setminus \{l\}}). \tag{3.1}$$

Now we set up the inductive hypothesis. Let $n = 1$, then

$$\mathcal{L}_1^{1\text{-fold}} Z_k(G) \cong \operatorname{Ker}\Big(\frac{R_1}{D_k(F; R_1)} \longrightarrow \frac{F}{\Gamma_k(F)}\Big) = \frac{R_1 \cap \Gamma_k(F)}{D_k(F; R_1)}.$$

Proceeding by induction, we assume that the result is true for $n-1$ and we will prove it for n.

Let us consider $l \in \langle n \rangle$ and the restriction $\mathfrak{F}^{\overline{\{l\}}}$ of $\mathfrak{F}$ to the subcategory of $\mathcal{C}_n$ consisting of those $A \subset \langle n \rangle$ not containing l (recall the discussion in Chap. 1, Sec. 2.2). It is easy to verify that $\mathfrak{F}^{\overline{\{l\}}}$ is a free exact $(n-1)$-presentation of the free group $\mathfrak{F}_{\langle n \rangle \setminus \{l\}}$. Here we use the fact that if G is a free group, then $\mathcal{L}_i^{n\text{-fold}} T(G) = 0$, $i > 0$ and $\mathcal{L}_0^{n\text{-fold}} T(G) \cong T(G)$ for any functor $T : \mathfrak{Gr} \longrightarrow \mathfrak{Gr}$. Thus, because of our inductive hypothesis,

$$\mathcal{L}_{n-1}^{(n-1)\text{-fold}} Z_k(\mathfrak{F}_{\langle n \rangle \setminus \{l\}}) \cong \frac{\bigcap\limits_{i \in \langle n \rangle \setminus \{l\}} R_i \cap \Gamma_k(F)}{D_k(F; R_1, \dots, R_{l-1}, R_{l+1}, \dots, R_n)} = 0. \tag{3.2}$$

Now from (3.1) and (3.2) we can easily deduce that there is the isomorphism

$$\mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \frac{\bigcap\limits_{i \in \langle n \rangle} R_i \cap \Gamma_k(F)}{D_k(F; R_1, \dots, R_n)}.$$

□

Now we are ready to express, by generalized Hopf type formulas not only the non-Abelian derived functors of the functor Z_2, i.e., group homology functors, but also the derived functors of all the functors Z_k, $k \geq 2$.

Theorem 3.9. *Let G be a group, $\mathfrak{F}$ be a free exact n-presentation of G and $k \geq 2$. Then*

$$L_n^{\mathcal{P}} Z_k(G) \cong \mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \frac{\bigcap\limits_{i \in \langle n \rangle} R_i \cap \Gamma_k(F)}{D_k(F; R_1, \dots, R_n)}, \quad n \geq 1,$$

where $(F; R_1, \dots, R_n)$ is the normal $(n+1)$-ad of groups induced by $\mathfrak{F}$.

Proof. It directly follows from Corollary 3.7, Proposition 1.16(i), and Theorem 3.8. □

Remark 3.10. One technical condition was omitted in the statement of the generalized Hopf formula as originally formulated in [14]. The result here corrects and generalizes that Brown–Ellis higher Hopf formulas.

Proposition 3.11. *There is an isomorphism*

$$\mathcal{L}_i^{n\text{-fold}} Z_k \cong \mathcal{L}_i^{(n-1)\text{-fold}} Z_k, \quad 0 \leq i \leq n-1.$$

Proof. Assume that G is a group and $\mathfrak{F}$ a free exact n-presentation of G. By Definition 1.19, Proposition 2.9, and Lemma 2.5,

$$\mathcal{L}_i^{n\text{-fold}} Z_k(G) \cong \pi_i(E^{(n)}\mathcal{B}_k(\mathcal{M})_*), \quad i \geq 0,$$

where $\mathcal{M}$ is the inclusion crossed n-cube of groups given by the normal $(n+1)$-ad of groups $(F; R_1, \ldots, R_n)$.

Now applying Proposition 2.4 to the crossed n-cube of groups $\mathcal{B}_k(\mathcal{M})$ we have an exact sequence

$$0 \longrightarrow \pi_{n-2}(E^{(n-1)}(\mathcal{N}_1)_*) \longrightarrow \mathcal{L}_{n-1}^{n\text{-fold}} Z_k(G) \longrightarrow \pi_{n-1}(E^{(n-1)}(\mathcal{N}_0)_*) \longrightarrow$$
$$\longrightarrow \pi_{n-3}(E^{(n-1)}(\mathcal{N}_1)_*) \longrightarrow \cdots \longrightarrow \pi_2(E^{(n-1)}(\mathcal{N}_0)_*) \longrightarrow$$
$$\longrightarrow \pi_0(E^{(n-1)}(\mathcal{N}_1)_*) \longrightarrow \mathcal{L}_1^{n\text{-fold}} Z_k(G) \longrightarrow \pi_1(E^{(n-1)}(\mathcal{N}_0)_*) \longrightarrow 0$$

and isomorphisms

$$\mathcal{L}_0^{n\text{-fold}} Z_k(G) \cong \pi_0(E^{(n-1)}(\mathcal{N}_0)_*), \quad \mathcal{L}_n^{n\text{-fold}} Z_k(G) \cong \pi_{n-1}(E^{(n-1)}(\mathcal{N}_1)_*).$$

Since Z_k, $k \geq 2$, is a right exact functor, we easily show that

$$\pi_i(E^{(n-1)}(\mathcal{N}_0)_*) \cong \mathcal{L}_i^{(n-1)\text{-fold}} Z_k(G), \quad i \geq 0.$$

It only remains to show that $\pi_i(E^{(n-1)}(\mathcal{N}_1)_*) = 0$ for $0 \leq i \leq n-2$. In fact, by the construction, the crossed $(n-1)$-cube of groups $\mathcal{N}_1$ is the kernel of the morphism of crossed $(n-1)$-cubes $\mu_l : \mathcal{B}_k(\mathcal{M})_1 \longrightarrow \mathcal{B}_k(\mathcal{M})_0$. It is easy to verify that $\mathfrak{F}^{\overline{\langle n \rangle \setminus A}}$ is a free exact m-presentation of the free group $\mathfrak{F}_A$ where $m = |A|$. Using Theorem 3.8 we have

$$\mathcal{L}_m^{m\text{-fold}} Z_k(\mathfrak{F}_A) \cong \frac{\bigcap\limits_{i \in A} R_i \cap \Gamma_k(F)}{D_k(F; A)} = 0 \quad \text{for} \quad A \neq \langle n \rangle,$$

which implies that $\mathcal{N}_{1,B} = \operatorname{Ker} \mu_{l,B} = 0$ for all $B \subseteq \langle n-1 \rangle$ and $B \neq \langle n-1 \rangle$ and

$$\mathcal{N}_{1,\langle n-1 \rangle} = \operatorname{Ker} \mu_{l,\langle n-1 \rangle} \cong \mathcal{L}_n^{n\text{-fold}} Z_k(G).$$

Hence $\pi_i(E^{(n-1)}(\mathcal{N}_1)_*) = 0$ for $0 \leq i \leq n-2$. $\square$

Now from Theorem 3.9 and Proposition 3.11 we deduce the following result.

Theorem 3.12. *There is an isomorphism*

$$\mathcal{L}_i^{n\text{-fold}} Z_k \cong L_i^{\mathcal{P}} Z_k, \quad 0 \leq i \leq n.$$

We obtain an interesting formula for $n = 2$. For this we need the following lemma.

Lemma 3.13 (see [28, 100])**.** *Let*

$$\mathcal{M} = \left\{ \begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\nu]{} & P \end{array} \right\}$$

be a crossed square. Then

$$H_0(C_*(\mathcal{M})) = P/\operatorname{Im}\mu \operatorname{Im}\nu,$$
$$H_1(C_*(\mathcal{M})) \cong M \times_P N/\operatorname{Im}\kappa,$$
$$H_2(C_*(\mathcal{M})) = \operatorname{Ker}\lambda \cap \operatorname{Ker}\lambda',$$

where $C_*(\mathcal{M})$ is the mapping cone complex of groups

$$L \xrightarrow{\alpha} M \rtimes N \xrightarrow{\beta} P$$

with $\alpha(l) = (\lambda(l)^{-1}, \lambda'(l))$, $\beta(m,n) = \mu(m)\nu(n)$ for all $l \in L$, $(m,n) \in M \rtimes N$, and κ is the natural homomorphism from L to $M \times_P N$.

Proof. We only prove that $H_1(C_*(\mathcal{M})) \cong M \times_P N / \operatorname{Im} \kappa$. It is easy to verify that $f : \operatorname{Ker} \beta \longrightarrow M \times_P N$, given by $f(m,n) = (m^{-1}, n)$ for all $(m,n) \in \operatorname{Ker} \beta$, is an isomorphism and $\operatorname{Im} f\alpha = \operatorname{Im} \kappa$. The other results are as easy as this part to verify. □

Using Theorem 3.12 and Lemma 3.13, for a given group G and $k \geq 2$, we see that there are isomorphisms of groups

$$L_1^{\mathcal{P}} Z_k(G) \cong \frac{R_1 \Gamma_k(F) \cap R_2 \Gamma_k(F)}{(R_1 \cap R_2)\Gamma_k(F)},$$

where $(F; R_1, R_2)$ is a normal 3-ad of groups induced by some free exact 2-presentation $\mathfrak{F}$ of G.

Note that for group-abelianization functor $\mathfrak{Ab} = Z_2$ we have the following apparently new interpretation of the second integral group homology:

$$H_2(G) \cong \frac{R_1[F,F] \cap R_2[F,F]}{(R_1 \cap R_2)[F,F]}.$$

2.2. Hopf-type formulas in algebraic K-theory. In this section, we will give an application of our generalized Hopf type formulas to algebraic K-theory.

First, recall the well-known definition of $\varprojlim^{(1)}$, the first derived functor of the functor $\varprojlim$ (inverse limit in the category of groups)(see, for example, [60]). Let $\{A_k,\ p^k_{k+1}\}_k$ be a countable inverse system of groups; then

$$\varprojlim^{(1)} \{A_k,\ p^k_{k+1}\} = \prod_k A_k / \sim,$$

where $\sim$ is an equivalence relation on the set $\prod_k A_k$ defined as follows: $\{a_k\} \sim \{a'_k\}$ if there exists $\{h_k\}$ such that $\{h_k\}\{a_k\}\{p^k_{k+1}(h^{-1}_{k+1})\} = \{a'_k\}$.

Theorem 3.14. *Let $\mathfrak{R}$ be a ring with unit and $(F_*, d_0^0, GL(\mathfrak{R}))$ be a free pseudo-simplicial resolution of the general linear group $GL(\mathfrak{R})$. Then there is an exact sequence of Abelian groups*

$$0 \longrightarrow \varprojlim_j{}^{(1)} \left(\frac{(\bigcap_{i \in \langle n+1 \rangle} \operatorname{Ker} d^n_{i-1}) \cap \Gamma_j(F_n)}{D_j(F_n; \operatorname{Ker} d^n_0, \ldots, \operatorname{Ker} d^n_n)} \right) \longrightarrow$$

$$\longrightarrow K_{n+1}(\mathfrak{R}) \longrightarrow \varprojlim_j \left(\frac{(\bigcap_{i \in \langle n \rangle} \operatorname{Ker} d^{n-1}_{i-1}) \cap \Gamma_j(F_{n-1})}{D_j(F_{n-1}; \operatorname{Ker} d^{n-1}_0, \ldots, \operatorname{Ker} d^{n-1}_{n-1})} \right) \longrightarrow 0$$

for $n \geq 1$.

Proof. Using [60, Theorem 2.15], we see that there is a short exact sequence of groups

$$0 \longrightarrow \varprojlim_k{}^{(1)} L^{\mathcal{P}}_{n+1} Z_k(GL(\mathfrak{R})) \longrightarrow L^{\mathcal{P}}_n Z_\infty(GL(\mathfrak{R})) \longrightarrow \varprojlim_k L^{\mathcal{P}}_n Z_k(GL(\mathfrak{R})) \longrightarrow 0 \quad (3.3)$$

for all $n \geq 0$, where the functor $Z_\infty : \mathfrak{Gr} \longrightarrow \mathfrak{Gr}$ is given by $Z_\infty(G) = \varprojlim_k Z_k(G)$, $G \in \mathfrak{Gr}$.

It is known from [82] (see also [60]) that the values of the non-Abelian left derived functors $L_*^{\mathcal{P}} Z_\infty$ of the functor $Z_\infty : \mathfrak{Gr} \longrightarrow \mathfrak{Gr}$ on $GL(\mathfrak{R})$ are isomorphic to Quillen's K-groups. Thus from (3.3) we deduce that there is a short exact sequence of Abelian groups

$$0 \longrightarrow \varprojlim_k{}^{(1)} L_{n+1}^{\mathcal{P}} Z_k(GL(\mathfrak{R})) \longrightarrow K_{n+1}(\mathfrak{R}) \longrightarrow \varprojlim_k L_n^{\mathcal{P}} Z_k(GL(\mathfrak{R})) \longrightarrow 0, \quad n \geq 0.$$

Now Corollary 3.7 directly implies the result. $\square$

Note that using Theorem 3.9 and Remark 3.10, we can express $K_{n+1}(\mathfrak{R})$ in data coming from exact $(n+1)$ and n-presentations of the group $GL(\mathfrak{R})$.

3. Hopf Type Formulas for the Homology of Homotopy $(n+1)$-Types

The aim of this section is to investigate the homology of the homotopy $(n+1)$-types, given in previous chapter, from a Hopf formulas point of view, using our purely algebraic method of m-fold Čech derived functors.

Now we consider the m-fold Čech derivatives of functors from the category of crossed n-cubes to the category of groups, while the general situation has been dealt in Chap. 1, Sec. 2. In particular, we give an explicit computation of the m-fold Čech derived functors of the functor $\sigma\mathfrak{Ab}^{(n)} : \mathbf{Crs}^n \longrightarrow \mathfrak{Ab}\mathbf{Gr}$, implying a purely algebraic approach to the homology groups of crossed n-cubes from a Hopf type formula point of view.

The following theorem gives the calculation of the mth m-fold Čech derived functors of the functor

$$\sigma\mathfrak{Ab}^{(n)} : \mathbf{Crs}^n \longrightarrow \mathfrak{Ab}\mathbf{Gr} \subseteq \mathbf{Gr}.$$

Theorem 3.15. *Let $\mathcal{M}$ be a crossed n-cube and $\mathfrak{X}$ its $\mathbb{P}$-projective $\mathbb{P}$-exact m-presentation, where $\mathbb{P}$ is the projective class induced by the "free" cotriple $\mathbb{F}$ in the category $\mathbf{Crs}^n$ (see Chap. 2, Sec. 2). Then there is an isomorphism*

$$\mathcal{L}_m^{m\text{-}fold}(\sigma\mathfrak{Ab}^{(n)})(\mathcal{M}) \cong \frac{\bigcap_{i\in\langle m\rangle} R^i_{\langle n\rangle} \cap \prod_{B\cup C=\langle n\rangle} [\mathfrak{X}(\varnothing)_B, \mathfrak{X}(\varnothing)_C]}{\prod_{A\subseteq\langle m\rangle} \Big(\prod_{B\cup C=\langle n\rangle} \big[\bigcap_{i\in A} R^i_B, \bigcap_{i\notin A} R^i_C\big]\Big)}, \quad m \geq 1,$$

where $R^i = \operatorname{Ker}(\mathfrak{X}(\varnothing) \longrightarrow \mathfrak{X}(\{i\}))$ for $i \in \langle m\rangle$.

Proof. Using Corollary 2.6, we have

$$\mathcal{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathcal{M}) \cong \pi_m(\sigma\mathfrak{Ab}^{(n)} E^{(m)}(\mathcal{N})_*),$$

where $\mathcal{N}$ is the crossed $(n+m)$-cube of groups induced by the normal $(m+1)$-ad of crossed n-cubes $(\mathfrak{X}(\varnothing); R^1, \dots, R^m)$. Hence Proposition 2.10 implies an isomorphism

$$\mathcal{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathcal{M}) \cong \pi_m(\sigma E^{(m)}\mathfrak{Ab}^{(n+m)}(\mathcal{N})_*).$$

Then, by Proposition 2.13 (see also [87, Proposition 3.4]),

$$\mathcal{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathcal{M}) \cong \bigcap_{l\in\langle m\rangle} \operatorname{Ker}(\mathfrak{Ab}^{(n+m)}(\mathcal{N})_{\langle n+m\rangle} \longrightarrow \widetilde{\mathfrak{Ab}}^{(n+m)}_l(\mathcal{N}_{\langle n+m\rangle\setminus\{l\}}). \tag{3.4}$$

Now we set up the inductive hypothesis. Let $m = 1$; then

$$\mathcal{L}_1^{1\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(\mathcal{M})$$

$$\cong \operatorname{Ker}\left(\frac{R^1_{\langle n\rangle}}{\prod_{A\subseteq\langle 1\rangle} \big(\prod_{B\cup C=\langle n\rangle} \big[\bigcap_{i\in A} R^i_B, \bigcap_{i\notin A} R^i_C\big]\big)} \longrightarrow \frac{\mathfrak{X}(\varnothing)_{\langle n\rangle}}{\prod_{B\cup C=\langle n\rangle} [\mathfrak{X}(\varnothing)_B, \mathfrak{X}(\varnothing)_C]} \right)$$

$$
= \frac{R^1_{\langle n\rangle} \cap \prod\limits_{B\cup C=\langle n\rangle} [\mathfrak{X}(\varnothing)_B, \mathfrak{X}(\varnothing)_C]}{\prod\limits_{A\subseteq\langle 1\rangle} \Big(\prod\limits_{B\cup C=\langle n\rangle} \big[\bigcap\limits_{i\in A} R^i_B, \bigcap\limits_{i\notin A} R^i_C \big]\Big)}.
$$

Proceeding by induction, we assume that the result is true for $m-1$ and we will prove it for m.

Let us consider $l \in \langle m\rangle$ and denote by $\mathfrak{X}^{\overline{\{l\}}}$ the restriction of the functor $\mathfrak{X} : \underline{C_m} \longrightarrow \mathbf{Crs}^n$ to the subcategory of $\underline{C_m}$ consisting of all subsets $A \subseteq \langle m\rangle$ with $l \notin A$. It is easy to check that $\mathfrak{X}^{\overline{\{l\}}}$ is a projective exact $(m-1)$-presentation of the crossed n-cube $\mathfrak{X}(\langle m\rangle \setminus \{l\})$, which itself is a projective crossed n-cube. Since the values of m-fold Čech derived functors of any functor for an object belonging to the projective class are trivial, our inductive hypothesis implies that

$$
\mathcal{L}^{(m-1)\text{-fold}}_{m-1}(\sigma\mathfrak{Ab}^{(n)})\big(\mathfrak{X}(\langle m\rangle \setminus \{l\})\big) \cong \frac{\bigcap\limits_{i\in\langle m\rangle\setminus\{l\}} R^i_{\langle n\rangle} \cap \prod\limits_{B\cup C=\langle n\rangle} [\mathfrak{X}(\varnothing)_B, \mathfrak{X}(\varnothing)_C]}{\prod\limits_{A\subseteq\langle m\rangle\setminus\{l\}} \Big(\prod\limits_{B\cup C=\langle n\rangle} \big[\bigcap\limits_{i\in A} R^i_B, \bigcap\limits_{i\notin A} R^i_C \big]\Big)} = 1. \tag{3.5}
$$

Now from (3.4) and (3.5) we can easily deduce the required isomorphism. $\square$

Now we give the result which expresses the homology of crossed n-cubes as Hopf type formulas generalizing the Hopf formula for the second CCG-homology of crossed modules [21].

Theorem 3.16. *Let $\mathcal{M}$ be a crossed n-cube and $\mathfrak{X}$ its $\mathbb{P}$-projective $\mathbb{P}$-exact m-presentation. Then there is an isomorphism*

$$
H_{m+1}(\mathcal{M}) \cong \frac{\bigcap\limits_{i\in\langle m\rangle} R^i_{\langle n\rangle} \cap \prod\limits_{B\cup C=\langle n\rangle} [\mathfrak{X}(\varnothing)_B, \mathfrak{X}(\varnothing)_C]}{\prod\limits_{A\subseteq\langle m\rangle} \Big(\prod\limits_{B\cup C=\langle n\rangle} \big[\bigcap\limits_{i\in A} R^i_B, \bigcap\limits_{i\notin A} R^i_C \big]\Big)}, \quad m \geq 1,
$$

where $R^i = \operatorname{Ker}(\mathfrak{X}(\varnothing) \longrightarrow \mathfrak{X}(\{i\}))$ for $i \in \langle m\rangle$.

Proof. Let $(F_*, d^0_0, \mathcal{M})$ be a $\mathbb{P}$-projective pseudo-simplicial resolution of $\mathcal{M}$ in the category $\mathbf{Crs}^n$ and consider the short exact sequence of augmented pseudo-simplicial groups

$$
\begin{tikzcd}
 & 1 \arrow[d] & & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & D(F_m) \arrow[d] \arrow[r, shift left=2, "\widetilde{d}^m_{0,\langle n\rangle}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\widetilde{d}^m_{m,\langle n\rangle}"'] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & D(F_1) \arrow[d] \arrow[r, shift left, "\widetilde{d}^1_{0,\langle n\rangle}"] \arrow[r, shift right, "\widetilde{d}^1_{1,\langle n\rangle}"'] & D(F_0) \arrow[d] \arrow[r, "\widetilde{d}^0_{0,\langle n\rangle}"] & D(\mathcal{M}) \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & F_{m,\langle n\rangle} \arrow[d] \arrow[r, shift left=2, "d^m_{0,\langle n\rangle}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "d^m_{m,\langle n\rangle}"'] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F_{1,\langle n\rangle} \arrow[d] \arrow[r, shift left, "d^1_{0,\langle n\rangle}"] \arrow[r, shift right, "d^1_{1,\langle n\rangle}"'] & F_{0,\langle n\rangle} \arrow[d] \arrow[r, "d^0_{0,\langle n\rangle}"] & \mathcal{M}_{\langle n\rangle} \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \sigma\mathfrak{Ab}^{(n)}(F_m) \arrow[d] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \sigma\mathfrak{Ab}^{(n)}(F_1) \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & \sigma\mathfrak{Ab}^{(n)}(F_0) \arrow[d] \arrow[r] & \sigma\mathfrak{Ab}^{(n)}(\mathcal{M}) \arrow[d] \\
 & 1 & & 1 & 1 & 1
\end{tikzcd}
,
$$

where $D(\mathcal{M})$ denotes the group $\prod\limits_{B\cup C=\langle n\rangle} [\mathcal{M}_B, \mathcal{M}_C]$ for any crossed n-cube $\mathcal{M}$.

By the induced long exact homotopy sequence, we have the isomorphisms of groups

$$\pi_m \sigma\mathfrak{Ab}^{(n)}(F_*) \cong \frac{\bigcap\limits_{i=0}^{m-1} \operatorname{Ker} \widetilde{d}_{i,\langle n\rangle}^{m-1}}{\widetilde{d}_{m,\langle n\rangle}^{m}\big(\bigcap\limits_{i=0}^{m-1} \operatorname{Ker} \widetilde{d}_{i,\langle n\rangle}^{m}\big)}, \quad m \geq 1. \tag{3.6}$$

Since $\widetilde{d}_{i,\langle n\rangle}^{m}$ is the restriction of $d_{i,\langle n\rangle}^{m}$ to $D(F_m)$, $\operatorname{Ker} \widetilde{d}_{i,\langle n\rangle}^{m} = \operatorname{Ker} d_{i,\langle n\rangle}^{m} \cap D(F_m)$. Hence

$$\bigcap_{i=0}^{m-1} \operatorname{Ker} \widetilde{d}_{i,\langle n\rangle}^{m-1} = \Big(\bigcap_{i=0}^{m-1} \operatorname{Ker} d_{i,\langle n\rangle}^{m-1}\Big) \cap D(F_{m-1})$$

and

$$\bigcap_{i=0}^{m-1} \operatorname{Ker} \widetilde{d}_{i,\langle n\rangle}^{m} = \Big(\bigcap_{i=0}^{m-1} \operatorname{Ker} d_{i,\langle n\rangle}^{m}\Big) \cap D(F_m).$$

Since the shift of pseudo-simplicial object F_* is the contractible augmented pseudo-simplicial object $(\mathrm{Dec}(F_*), d_0^1, F_0)$ (see [40]), by Proposition 1.16(i) the m-cube of crossed n-cubes $\mathrm{Dec}(F)^{(m)}$ is a projective exact m-presentation of F_0. Hence, by Theorem 3.15 we have

$$\mathcal{L}_m^{m\text{-fold}}(\sigma\mathfrak{Ab}^{(n)})(F_0) \cong \frac{\bigcap\limits_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1,\langle n\rangle}^{m} \cap \prod\limits_{B\cup C=\langle n\rangle} [F_{m,B}, F_{m,C}]}{\prod\limits_{A\subseteq\langle m\rangle}\big(\prod\limits_{B\cup C=\langle n\rangle}\big[\bigcap\limits_{i\in A}\operatorname{Ker} d_{i-1,B}^{m}, \bigcap\limits_{i\notin A}\operatorname{Ker} d_{i-1,C}^{m}\big]\big)} = 1, \quad m \geq 1,$$

implying the equality

$$\begin{aligned}\bigcap_{i\in\langle m\rangle} \operatorname{Ker} d_{i-1,\langle n\rangle}^{m} \cap \prod_{B\cup C=\langle n\rangle} [F_{m,B}, F_{m,C}]& \\ = \prod_{A\subseteq\langle m\rangle}\Big(\prod_{B\cup C=\langle n\rangle}\Big[\bigcap_{i\in A}\operatorname{Ker} d_{i-1,B}^{m}, \bigcap_{i\notin A}\operatorname{Ker} d_{i-1,C}^{m}\Big]\Big),& \quad m \geq 1.\end{aligned} \tag{3.7}$$

Since $(F_{*,\langle n\rangle}, d_{0,\langle n\rangle}^{0}, \mathcal{M}_{\langle n\rangle})$ is an aspherical augmented pseudo-simplicial group,

$$d_{m,\langle n\rangle}^{m}\Big(\bigcap_{i\in\langle m\rangle}\operatorname{Ker} d_{i-1,\langle n\rangle}^{m}\Big) = \bigcap_{i\in\langle m\rangle}\operatorname{Ker} d_{i-1,\langle n\rangle}^{m-1}, \quad m \geq 1.$$

Using this fact and Lemma 3.5, by (3.7) it is easy to see that we have an equality

$$\begin{aligned}\widetilde{d}_{m,\langle n\rangle}^{m}\Big(\bigcap_{i=0}^{m-1}\operatorname{Ker}\widetilde{d}_{i,\langle n\rangle}^{m}\Big) &= d_{m,\langle n\rangle}^{m}\Big(\prod_{A\subseteq\langle m\rangle}\Big(\prod_{B\cup C=\langle n\rangle}\Big[\bigcap_{i\in A}\operatorname{Ker} d_{i-1,B}^{m}, \bigcap_{i\notin A}\operatorname{Ker} d_{i-1,C}^{m}\Big]\Big)\Big) \\ &= \prod_{A\subseteq\langle m\rangle}\Big(\prod_{B\cup C=\langle n\rangle}\Big[\bigcap_{i\in A}\operatorname{Ker} d_{i-1,\,B}^{m-1}, \bigcap_{i\notin A}\operatorname{Ker} d_{i-1,\,C}^{m-1}\Big]\Big).\end{aligned}$$

Thus by (3.6) we have

$$H_{m+1}(\mathcal{M}) \cong \frac{\big(\bigcap\limits_{i=0}^{m-1}\operatorname{Ker} d_{i,\langle n\rangle}^{m-1}\big) \cap \prod\limits_{B\cup C=\langle n\rangle}[F_{m-1,B}, F_{m-1,C}]}{\prod\limits_{A\subseteq\langle m\rangle}\big(\prod\limits_{B\cup C=\langle n\rangle}\big[\bigcap\limits_{i\in A}\operatorname{Ker} d_{i-1,\,B}^{m-1}, \bigcap\limits_{i\notin A}\operatorname{Ker} d_{i-1,\,C}^{m-1}\big]\big)}.$$

Using again Proposition 1.16(i) and Theorem 3.15, we complete the proof. □

Chapter 4

NON-ABELIAN HOMOLOGY OF GROUPS

The non-Abelian homology of groups with coefficients in any group in any dimension was introduced in [68] as the non-Abelian left derived functors of the non-Abelian tensor product of groups. It generalizes the classical Eilenberg–MacLane homology of groups [41, 42] and extends Guin's low-dimensional H_0 and H_1 non-Abelian homology groups with coefficients in crossed modules [53], which has important applications in the algebraic K-theory of noncommutative local rings.

The non-Abelian tensor product of groups was introduced by Brown and Loday in [17, 18] following the works of Lue [92] and Dennis [37]. It arose in applications in the homotopy theory of a generalized Van Kampen theorem. It was defined for a pair A, B of groups which act on themselves by conjugation and on each other such that the certain compatibility conditions hold. During the last twenty years the non-Abelian tensor product has been the subject of a number of papers. We refer to here the web page of Brown for a full account of this subject.

In [61–63], H. Inassaridze developed a non-Abelian cohomology theory previously defined by Guin in low dimensions [53] that differs from the classical first non-Abelian cohomology pointed set of Serre [114] and from the setting of various papers on non-Abelian cohomology [7, 26, 37] extending the classical exact non-Abelian cohomology sequence from lower dimensions [114] to higher dimensions. This non-Abelian cohomology theory of groups will not be treated in this work.

This chapter is devoted to the investigation of the non-Abelian homology of groups.

In Sec. 1, we give a short review of the results on the non-Abelian tensor product of groups of Brown–Loday [17–19] and its generalization in the sense of [68], which will be useful in the sequel.

Section 2 is devoted to the construction of the non-Abelian homology of groups with coefficients in any groups, which generalize the classical Eilenberg–MacLane group homology theory.

In Sec. 3, some properties on the non-Abelian homology of groups are established. In particular, various exact sequences of the non-Abelian homology $H_*(G, A)$ of groups with respect to the both variables are given (Theorems 4.15, 4.17, and 4.19 (Mayer–Vietoris sequence)). Then non-Abelian homology groups are described as the non-Abelian left derived functors of the functor $H_1(-, A)$ (Theorem 4.20), as well as of the section functor Γ in the category of cosheaves (Theorem 4.22). Sufficient conditions are established for the non-Abelian homology groups to be finitely generated, finite, p-groups, torsion groups, or groups of exponent q (Theorem 4.23).

In Sec. 4, special attention is given to the investigation of the second and third non-Abelian homology of groups. In particular, the explicit formulas for them are obtained by using Čech resolutions (Theorems 4.24, 4.25, and 4.28).

1. The Non-Abelian Tensor Product of Groups

Let a pair G, H of groups act on themselves by conjugation (${}^{x}y = xyx^{-1}$) and on each other such that the following compatibility conditions hold:

$$^{({}^{g}h)}(g') = {}^{g}({}^{h}({}^{g^{-1}}g')), \quad {}^{({}^{h}g)}(h') = {}^{h}({}^{g}({}^{h^{-1}}h')) \tag{4.1}$$

for all $g, g' \in G$ and $h, h' \in H$.

Example 4.1. Let $\alpha : G \longrightarrow P$ and $\beta : H \longrightarrow P$ be crossed modules over a group P. Let G and H act on each other via P and on themselves by conjugation. Then these actions satisfy the conditions (4.1).

Now a slightly modified version of the non-Abelian tensor product of groups will be given, which was studied in [68–70], in order to construct its non-Abelian derived functors.

Let G and H be arbitrary groups that act on each other and on themselves by conjugation. The compatibility conditions (4.1) are not assumed to hold.

Definition 4.2. The *non-Abelian tensor product* $G \otimes H$ is the group generated by the symbols $g \otimes h$, $g \in G$, $h \in H$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h), \tag{4.2}$$

$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h'), \tag{4.3}$$

$$(g \otimes h)(g' \otimes h') = ({}^{[g,h]}g' \otimes {}^{[g,h]}h')(g \otimes h) \tag{4.4}$$

for all $g, g' \in G$ and $h, h' \in H$, where $[g, h] = ghg^{-1}h^{-1} \in G * H$.

Remark 4.3.

(a) If the groups G and H satisfy the compatibility conditions (4.1) then this definition coincides with that of given by Brown–Loday [17, 18].

(b) In [67–70] the relation

$$(g' \otimes h')(g \otimes h) = (g \otimes h)({}^{[h,g]}g' \otimes {}^{[h,g]}h')$$

was included in the definition of $G \otimes H$. It is easy to check that this relation is redundant (it directly follows from (4.4)).

Assume that $\Theta : G \longrightarrow A$, $\Phi : H \longrightarrow B$ are homomorphism of groups, A and B act on each other, and Θ and Φ preserve the actions in the sense that

$$\Phi({}^{g}h) = {}^{\Theta g}(\Phi h), \quad \Theta({}^{h}g) = {}^{\Phi h}(\Theta g)$$

for all $g \in G$, $h \in H$. Then there is a unique homomorphism $\Theta \otimes \Phi : G \otimes H \longrightarrow A \otimes B$ such that $(\Theta \otimes \Phi)(g \otimes h) = \Theta g \otimes \Phi h$ for all $g \in G$, $h \in H$. Further, if Θ, Φ are onto, so also is $\Theta \otimes \Phi$.

It is easy to verify that there are natural isomorphisms

$$G \otimes H \cong H \otimes G, \quad \text{given by} \quad g \otimes h \longmapsto (h \otimes g)^{-1}$$

and, if G and H act trivially on each other [18],

$$G \otimes H \cong G^{ab} \otimes_{\mathbb{Z}} H^{ab}, \quad \text{given by} \quad g \otimes h \longmapsto [g] \otimes [h].$$

The following properties of the non-Abelian tensor product of groups are well known but useful in the sequel, and the relevant proofs are omitted.

Proposition 4.4 (see [18])**.** *Let M and N be groups equipped with compatible actions on each other.*

(a) *The free product $M * N$ acts on $M \otimes N$ so that*

$${}^{p}(m \otimes n) = {}^{p}m \otimes {}^{p}n, \quad m \in M, \quad n \in N, \quad p \in M * N.$$

(b) *There are homomorphisms*

$$\lambda : M \otimes N \longrightarrow M, \quad \lambda' : M \otimes N \longrightarrow N$$

such that $\lambda(m \otimes n) = m \cdot {}^{n}m^{-1}$, $\lambda'(m \otimes n) = {}^{m}nn^{-1}$.

(c) *The homomorphism λ, λ', with the given actions, are crossed modules.*

(d) *If $l \in M \otimes N$, $m' \in M$, $n' \in N$, then*

$$(\lambda l) \otimes n' = l \cdot {}^{n'}l^{-1} m' \otimes (\lambda' l) = {}^{m'}ll^{-1}.$$

(e) *The actions of M on $\operatorname{Ker} \lambda'$, N on $\operatorname{Ker} \lambda$, are trivial.*

(f) *If $l, l' \in M \otimes N$, then*

$$[l, l'] = (\lambda l) \otimes (\lambda' l')$$

and, in particular, $[m \otimes n, m' \otimes n'] = (m \cdot {}^{n}m^{-1}) \otimes ({}^{m'}n'n'^{-1})$.

Proposition 4.5 (see [16]). *Let G be any group and let*

$$1 \longrightarrow A \xrightarrow{i} K \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension. Then there is a homomorphism $\xi : G \otimes G \longrightarrow K$ such that $\pi\xi$ is the commutator map κ. If G is perfect ($G = [G, G]$), then ξ is unique, i.e., $\kappa : G \otimes G \longrightarrow G$ is the universal central extension of the perfect group G.

The non-Abelian tensor product of groups is a right exact functor. In particular, the following theorem holds.

Theorem 4.6.

(a) *Assume that*

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$$

is a short exact sequence of groups, D is an arbitrary group that acts on A, B, and C, the groups A, B, and C also act on D, and f and g preserve the actions. Then we have the following exact sequence of groups:

$$D \otimes A \xrightarrow{f'} D \otimes B \xrightarrow{g'} D \otimes C \longrightarrow 1,$$

where $f' = 1 \otimes f$, $g' = 1 \otimes g$.

(b) *Assume that*

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1, \tag{4.5}$$

$$1 \longrightarrow D \xrightarrow{\phi} E \xrightarrow{\psi} F \longrightarrow 1 \tag{4.6}$$

are short exact sequences of groups, where A and D, B and E, and C and F act on each other, and f and ϕ and g and ψ preserve the actions. Then the sequence of groups

$$(A \otimes E) \times (B \otimes D) \xrightarrow{\alpha} B \otimes E \xrightarrow{g \otimes \psi} C \otimes F \longrightarrow 1$$

is exact, where α is a map of sets.

Later the following calculation of the non-Abelian tensor product of groups will be needed.

Proposition 4.7. *Let A be any group acting trivially on $\mathbb{Z}$ and $\mathbb{Z}$ acts on A such that the compatibility conditions* (4.1) *hold. Then there is a natural isomorphism*

$$\mathbb{Z} \otimes A \xrightarrow{\cong} A^{ab}, \quad n \otimes a \longmapsto [{}^{n-1}a] \cdot [{}^{n-2}a] \cdots [{}^{1}a] \cdot [a].$$

Proof. By [18, Proposition 2.3], $\mathbb{Z} \otimes A$ is an Abelian group and $n \otimes {}^{a}a' = n \otimes a'$. Therefore,

$$n \otimes aa' = (n \otimes a)(n \otimes a'). \tag{4.7}$$

Let define a homomorphism $f : \mathbb{Z} \otimes A \longrightarrow A^{ab}$ as follows:

$$n \otimes a \longmapsto [{}^{n-1}a] \cdot [{}^{n-2}a] \cdots [{}^{1}a] \cdot [a].$$

We must show that this map preserves the defining relations of the non-Abelian tensor product. In fact,

$$f((n+m) \otimes a) = [{}^{n+m-1}a] \cdots [{}^{1}a] \cdot [a],$$
$$f(m, {}^{n}a) = [{}^{m-1}({}^{n}a)] \cdots [{}^{1}({}^{n}a)] \cdot [{}^{n}a] = [{}^{n+m-1}a] \cdots [{}^{n+1}a] \cdot [{}^{n}a].$$

Therefore, $f((n+m)\otimes a) = f(m\otimes {}^n a) f(n\otimes a)$. Next,

$$f(n\otimes aa') = [{}^{n-1}(aa')\cdots[{}^{1}(aa')]\cdot[aa'],$$
$$f(n\otimes {}^a a') = [{}^{n-1}a']\cdots[{}^{1}a']\cdot[a'].$$

Therefore, $f(n\otimes aa') = f(n\otimes a)f(n\otimes {}^a a')$.

Now define $g : A^{ab} \longrightarrow \mathbb{Z}\otimes A$ by $[a] \longmapsto 1\otimes a$. It is easy to see that g is a well-defined homomorphism. We must show that gf is the identity map. In effect, by (4.7) and the formula

$$(n+1)\otimes a = (1\otimes {}^n a)(n\otimes a)$$

we have

$$gf(n\otimes a) = 1\otimes({}^{n-1}a\cdots{}^{1}a\cdot a) = (1\otimes {}^{n-1}a)\cdots(1\otimes {}^{1}a)(1\otimes a)$$
$$= (n\otimes a)((n-1)\otimes a)^{-1}\cdots((n-1)\otimes a)((n-2)\otimes a)^{-1}\cdots(2\otimes a)\cdot(1\otimes a)^{-1}(1\otimes a) = n\otimes a.$$

It is easy to see that fg is also the identity map. □

Example 4.8. Let A be a metabelian group and the action of $\mathbb{Z}$ on A be defined by inner automorphism, i.e., ${}^n a' = a^n a' a^{-n}$ for all $n\in\mathbb{Z}$, $a'\in A$ for some $a\in A$. If A acts on $\mathbb{Z}$ trivially, then in this case the groups $\mathbb{Z}$ and A act on each other compatibly, and Proposition 4.7 can be applied.

Proposition 4.9. *Let A be a metabelian group acting on $\mathbb{Z}$ trivially and $\mathbb{Z}$ act on A by inner automorphism (see above). Then there is a natural isomorphism*

$$\mathbb{Z}^2\otimes A \cong (A^{ab})^2.$$

Proof. Check the conditions of [16, Proposition 10] in our case. We have

(a): ${}^m({}^n a') = {}^n({}^m a')$.

(b)–(c): ${}^n(m\otimes a') = m\otimes {}^n a'$ is an action and since $= m\otimes a^n a' a^{-n} = m\otimes a'$, it is trivial.

Therefore by [16, Proposition 10] and Proposition 4.7 we have natural isomorphisms

$$\mathbb{Z}^2\otimes A \cong (\mathbb{Z}\otimes A)\times(\mathbb{Z}\otimes A)\cong (A^{ab})^2.$$

The proposition is proved. □

2. Construction of Non-Abelian Homology of Groups

Let A denote a group. Let $\mathcal{A}_A$ denote the category whose objects are all groups G together with an action of G on A by automorphisms of A and an action of A on G by automorphisms of G. Morphisms in the category $\mathcal{A}_A$ are all group homomorphisms $\alpha : G\longrightarrow H$ that preserve the actions, namely $\alpha({}^a g) = {}^a\alpha(g)$ and ${}^g a = {}^{\alpha(g)}a$, for all $a\in A$ and $g\in G$.

Let $F : \mathcal{A}_A \longrightarrow \mathcal{A}_A$ be the endofunctor defined as follows: for an object G of $\mathcal{A}_A$, let $F(G)$ denote the free group generated by G with actions: ${}^{|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s}}a = {}^{g_1^{\epsilon_1}}(\cdots({}^{g_s^{\epsilon_s}}a)\cdots)$, and ${}^a(|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s}) = |{}^a g_1|^{\epsilon_1}\cdots|{}^a g_s|^{\epsilon_s}$ where $a\in A$, $|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s}\in F(G)$ and $\epsilon_i = \pm 1$; for a morphism $\alpha : G\longrightarrow G'$ of $\mathcal{A}_A$, let $F(\alpha)$ be the canonical homomorphism from $F(G)$ to $F(G')$ induced by α.

Let $\tau : F\longrightarrow 1_{\mathcal{A}_A}$ be the obvious natural transformation and let $\delta : F\longrightarrow F^2$ be the natural transformation induced for every $G\in\mathcal{A}_A$ by the injection $G\longrightarrow F(G)$. We obtain a cotriple $\mathcal{F} = (F,\tau,\delta)$, which we call the free cotriple in the category $\mathcal{A}_A$. Let $\mathcal{P}$ be the projective class induced by the free cotriple $\mathcal{F}$ (see Chap. 1, Sec. 1.2).

First we describe in the category $\mathcal{A}_A$ the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$ and the corresponding $\mathcal{P}$-epimorphisms.

Proposition 4.10. *A morphism $\alpha : G \longrightarrow G'$ of the category $\mathcal{A}_A$ is a $\mathcal{P}$- epimorphism if and only if there exists a section map $\beta : G' \longrightarrow G$ that preserves the actions of A, i.e., $\beta({}^a g') = {}^a\beta(g')$ for all $a \in A$, $g' \in G'$.*

Proposition 4.11. *In the category $\mathcal{A}_A$ the following conditions are equivalent*:

(i) *F belongs to the projective class $\mathcal{P}$.*

(ii) *F is a free group generated by a set X such that ${}^a x \in X$ for all $a \in A$ and $x \in X$.*

(iii) *For any morphism $\alpha : G \longrightarrow G'$ that has a section map $\beta : G' \longrightarrow G$ preserving the actions of A and any morphism $\kappa' : F \longrightarrow G'$, there exists a morphism $\kappa : F \longrightarrow G$ such that $\alpha\kappa = \kappa'$.*

The proof of these propositions is easy and is omitted.

The non-Abelian tensor product of groups defines a covariant functor $-\otimes A$ from the category $\mathcal{A}_A$ to the category $\mathfrak{Gr}$. Consider the non-Abelian left derived functors $L_n^{\mathcal{P}}(-\otimes A)$, $n \geq 0$, of the functor $-\otimes A$ relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$.

It is easy to verify that there is a natural isomorphism $L_0^{\mathbf{P}}(-\otimes A) \cong -\otimes A$ [68] and by Proposition 1.20 the functor $-\otimes A$ is a cosheaf over $(\mathcal{A}_A, \mathcal{P})$. The following result is given in [68].

Theorem 4.12. *If A is an Abelian group that acts trivially on a group $G \in \mathcal{A}_A$, then we have natural isomorphisms*

$$L_n^{\mathcal{P}}(G \otimes A) \cong H_{n+1}(G, A), \quad n \geq 1,$$
$$\operatorname{Ker} \lambda' \cong H_1(G, A), \quad \operatorname{Coker} \lambda' \cong H_0(G, A),$$

where $\lambda' : G \otimes A \longrightarrow A$, $\lambda'(g \otimes a) = {}^g a \cdot a^{-1}$.

Proof. Let A be a G-module, let $\mathfrak{Gr}_G$ be the category of groups over G, and let $\operatorname{Diff}_G(W) = \mathbf{Z}[G] \otimes_W IW$ for W be a group over G. By Guin's Proposition 3.2 [53] $L_n^{\mathcal{P}}(-\otimes A)$ is isomorphic to the nth left derived functor of $A \otimes_{\mathbf{Z}[G]} \operatorname{Diff}_G(-) : \mathfrak{Gr}_G \longrightarrow \mathfrak{Gr}$ that gives the Eilenberg–Maclane homology group $H_{n+1}(G, A)$ if $n \geq 1$ (see [5]). □

This theorem enables us to introduce *non-Abelian homology of groups*. In fact, we have the following definition.

Definition 4.13. Let G and A be groups acting on each other. Then we define

$$H_n(G, A) = L_{n-1}^{\mathcal{P}}(G \otimes A), \quad n \geq 2,$$
$$H_1(G, A) = \operatorname{Ker} \lambda', \quad H_0(G, A) = \operatorname{Coker} \lambda',$$

where $\lambda' : G \otimes A \longrightarrow A/H'$, $\lambda'(g \otimes a) = [{}^g a a^{-1}]$, and H' is the normal subgroup generated by the elements ${}^{({}^a g)}a' {}^{a g a^{-1}}a'^{-1}$, for each $a, a' \in A$, $g \in G$.

It is easy to see that if G and A are any groups acting on each other trivially, then $H_n(G, A) \cong H_n(G, A^{ab})$ for $n \geq 1$, where A^{ab} is the abelianization of the group A.

Remark 4.14. It is clear that the groups $H_n(G, A)$ are Abelian for $n \geq 2$. We will show that $\operatorname{Im} \lambda'$ is a normal subgroup of A/H', and when for the actions of G and A the compatibility conditons (4.1) hold, then $H_1(G, A)$ is also Abelian.

3. Some Properties of the Non-Abelian Homology of Groups

We begin this section by setting exact sequences of the non-Abelian homology $H_*(G, A)$ of groups with respect to both variables.

Theorem 4.15. *Let G, A_1, A, A_2 be arbitrary groups, and G act on A_1, A, and A_2, which act on G. Let $1 \longrightarrow A_1 \xrightarrow{f} A \xrightarrow{g} A_2 \longrightarrow 1$ be an exact sequence of groups, where f and g homomorphisms preserve the actions. Then there exist exact sequences of the non-Abelian homology*

$$
\begin{aligned}
\cdots \longrightarrow H_3(G,A_2) \longrightarrow H_2(G,A,A_2) \longrightarrow H_2(G,A) \longrightarrow \\
\longrightarrow H_2(G,A_2) \xrightarrow{l_1} H_1(G,A,A_2) \xrightarrow{l_2} H_1(G,A) \longrightarrow \\
\longrightarrow H_1(G,A_2) \xrightarrow{l_3} H_0(G,A,A_2) \xrightarrow{l_4} H_0(G,A) \longrightarrow H_0(G,A_2) \longrightarrow 1,
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
H_n(G,A,A_2) &= \pi_{n-1}(\operatorname{Ker}(1_{F_*(G)} \otimes g)), \quad n \geq 2,\\
\operatorname{Ker}(1_{F_*(G)} \otimes g) &= \big\{ \operatorname{Ker}(1_{F^n(G)} \otimes g),\ n \geq 1 \big\};\\
H_1(G,A,A_2) &= \frac{\big[\operatorname{Ker}(1_{F^1(G)} \otimes g) \cap {\partial_0^0}^{-1}(\operatorname{Ker}(1_G \otimes g) \cap \operatorname{Ker}\lambda')\big]}{\partial_1^1(\operatorname{Ker}(1_{F^2(G)} \otimes g) \cap \operatorname{Ker}\partial_0^1)},\\
H_0(G,A,A_2) &= \operatorname{Ker}(\widetilde{g})/\lambda'(\operatorname{Ker}(1_G \otimes g)) \quad (\textit{the set of left cosets}),
\end{aligned}
$$

and

$$
\begin{aligned}
\cdots \longrightarrow H_3(G,A,A_2) \longrightarrow H_2(G,A_1,A) \longrightarrow H_2(G,A_1) \longrightarrow \\
\longrightarrow H_2(G,A,A_2) \longrightarrow H_1(G,A_1,A) \longrightarrow H_1(G,A_1) \longrightarrow H_1(G,A,A_2) \longrightarrow \\
\longrightarrow H_0(G,A_1,A) \longrightarrow H_0(G,A_1) \longrightarrow H_0(G,A,A_2) \longrightarrow 1\ ,
\end{aligned}
\tag{4.9}
$$

where the groups $H_n(G,A_1,A)$ are defined analogously.

Remark 4.16.

(a) The sequence (4.8) generalizes the well-known classical exact sequence of the homology of groups with Abelian coefficients. If A_1, A, A_2 are G-modules, then the groups $H_n(G,A_1,A)$ are trivial.

(b) If G and A act on each other compatibly (in this case G, A_1 and G, A_2 act on each other compatibly), then $H_0(G,A,A_2) = H_0(G,A_1)$.

(c) Let $1 \longrightarrow (A_1,1) \longrightarrow (A,\mu) \longrightarrow (A_2,\lambda) \longrightarrow 1$ be an exact sequence of crossed G-modules. Then Guin has obtained [9] the following exact sequence of the non-Abelian homology

$$
\begin{aligned}
H_1(G,A_1) \longrightarrow H_1(G,A) \longrightarrow H_1(G,A_2) \longrightarrow \\
\longrightarrow H_0(G,A_1) \longrightarrow H_0(G,A) \longrightarrow H_0(G,A_2) \longrightarrow 1\ .
\end{aligned}
\tag{4.10}
$$

The first five terms of the sequence (4.8) coincide with the sequence (4.10).

(d) we have a natural homomorphism $H_1(G,A_1) \longrightarrow H_1(G,A,A_2)$ such that the diagram

$$
\begin{tikzcd}
H_1(G,A_1) \arrow[r] \arrow[d] & H_1(G,A) \arrow[d, equal] \\
H_1(G,A,A_2) \arrow[r, "l_2"'] & H_1(G,A)
\end{tikzcd}
$$

is commutative. When the actions are compatible, this natural homomorphism is surjective.

For noncommutative local rings, the relationship of Milnor's algebraic K-functor K_2 with the symbol group Sym is given in terms of a long exact sequence of non-Abelian homology of groups [68] extending Guin's six-term exact sequence [53]. In fact, let R be a noncommutative local ring such that $R/\operatorname{Rad} R \neq \mathbb{F}_2$. Then we have the following exact sequence of groups

$$\begin{aligned} \cdots \longrightarrow H_3(R^*,[R^*,R^*]) \longrightarrow H_2(R^*,U(R),[R^*,R^*]) \longrightarrow H_2(R,U(R)) \longrightarrow \\ \longrightarrow H_2(R^*,[R^*,R^*]) \longrightarrow H_1(R^*,U(R),[R^*,R^*]) \longrightarrow H_1(R,U(R)) \longrightarrow \\ \longrightarrow H_1(R^*,[R^*,R^*]) \longrightarrow K_2(R) \longrightarrow \operatorname{Sym}(R) \longrightarrow [R^*,R^*]/[R^*,[R^*,R^*]] \longrightarrow 1\,. \end{aligned}$$

Theorem 4.17. *Let G_1, G, G_2, A be arbitrary groups. Assume that A acts on G_1, G, and G_2, and all groups act on A. Let $1 \longrightarrow G_1 \xrightarrow{\alpha} G \xrightarrow{\beta} G_2 \longrightarrow 1$ be an exact sequence of groups such that the homomorphisms α and β preserve the actions. Then $H_0(G,A) \cong H_0(G_2,A)$ and there is a long exact sequence of non-Abelian homology groups*

$$\begin{aligned} \cdots \longrightarrow H_4(G_2,A) \longrightarrow H_3(G,G_2,A) \longrightarrow H_3(G,A) \longrightarrow \\ \longrightarrow H_3(G_2,A) \longrightarrow H_2(G,G_2,A) \longrightarrow H_2(G,A) \longrightarrow H_2(G_2,A) \longrightarrow \\ \longrightarrow H_1(G,G_2,A) \longrightarrow H_1(G,A) \longrightarrow H_1(G_2,A) \longrightarrow 1, \end{aligned} \tag{4.11}$$

where $H_n(G,G_2,A) = \pi_{n-1}(\operatorname{Ker}(F_*(\beta)\otimes 1_A))$ *and* $\operatorname{Ker}(F_*(\beta)\otimes 1_A) = \{\operatorname{Ker}(F^n(\beta)\otimes 1_A)$ *for* $n \geq 1$.

Proof. The proof follows from the commutative diagram of groups

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \operatorname{Ker}(F^2(\beta)\otimes 1_A) \arrow[r, shift left, "\overline{\partial}_0^1"] \arrow[r, shift right, "\overline{\partial}_1^1"'] \arrow[d] & \operatorname{Ker}(F^1(\beta)\otimes 1_A) \arrow[d] & & \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(G)\otimes A \arrow[r, shift left, "\partial_0^1"] \arrow[r, shift right, "\partial_1^1"'] \arrow[d, "F^2(\beta)\otimes 1_A"] & F^1(G)\otimes A \arrow[r, "\partial_0^0"] \arrow[d, "F^1(\beta)\otimes 1_A"] & G\otimes A \arrow[r, "\lambda'"] \arrow[d, "\beta\otimes 1_A"] & A/H' \arrow[d, equal] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(G_2)\otimes A \arrow[r, shift left, "\partial_0^1"] \arrow[r, shift right, "\partial_1^1"'] & F^1(G_2)\otimes A \arrow[r, "\partial_0^0"] & G_2\otimes A \arrow[r, "\lambda'"] & A/H_2'
\end{tikzcd}$$

□

Remark 4.18. In the exact sequence (4.11) the groups $H_1(G,A)$ and $H_1(G_2,A)$ can be replaced by the groups $\pi_0(F_*(G)\otimes A)$ and $\pi_0(F_*(G_2)\otimes A)$ respectively.

Let

$$\mathcal{D} = \begin{tikzcd} & G_2 \arrow[d, "\alpha_2"] \\ G_1 \arrow[r, "\alpha_1"'] & G \end{tikzcd} \tag{4.12}$$

be a diagram in the category $\mathcal{A}_A$ with surjective α_1. Let $L_*(\mathcal{D},A)$ be the pullback of the induced diagram

$$\begin{tikzcd} & F_*(G_2)\otimes A \arrow[d, "F_*(\alpha_2)\otimes 1_A"] \\ F_*(G_1)\otimes A \arrow[r, "F_*(\alpha_1)\otimes 1_A"'] & F_*(G)\otimes A \end{tikzcd}.$$

Define $H_n(\mathcal{D},A)=\pi_{n-1}L_*(\mathcal{D},A)$, $n\geq 2$.

Theorem 4.19 (Mayer–Vietoris sequence)**.** *For any diagram* (4.12) *there is a long exact sequence*

$$\begin{aligned}\cdots \longrightarrow H_{n+1}(G,A) \longrightarrow H_n(\mathcal{D},A) \longrightarrow H_n(G_1,A)\oplus H_n(G_2,A) \longrightarrow H_n(G,A) \longrightarrow \\ \longrightarrow \cdots \longrightarrow H_2(\mathcal{D},A) \longrightarrow H_2(G_1,A)\oplus H_2(G_2,A) \longrightarrow H_2(G,A) \longrightarrow \\ \longrightarrow \pi_0 L_*(\mathcal{D},A) \longrightarrow \pi_0(F_*(G_1)\otimes A)\times \pi_0(F_*(G_2)\otimes A) \longrightarrow \pi_0(F_*(G)\otimes A) \longrightarrow 1.\end{aligned} \tag{4.13}$$

Proof. We have the following commutative diagram of simplicial groups with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{I}_* \arrow[r, "\sigma_*"] \arrow[d, equal] & L_*(\mathcal{D},A) \arrow[r, "p_*"] \arrow[d, "q_*"] & F_*(G_2)\otimes A \arrow[r] \arrow[d, "F_*(\alpha_2)\otimes 1_A"] & 1 \\ 1 \arrow[r] & \mathcal{I}_* \arrow[r] & F_*(G_1)\otimes A \arrow[r, "F_*(\alpha_1)\otimes 1_A"] & F_*(G)\otimes A \arrow[r] & 1 \end{tikzcd}, \tag{4.14}$$

where $\mathcal{I}_*=\operatorname{Ker}(F_*(\alpha_1)\otimes 1_A)$. Diagram (4.14) induces the following commutative diagram with exact rows

$$\begin{tikzcd} \cdots \arrow[r] & \pi_1(F_*(G_2)\otimes A) \arrow[r] \arrow[d] & \pi_0(\mathcal{I}_*) \arrow[r] \arrow[d, equal] & \pi_0 L_*(\mathcal{D},A) \arrow[r] \arrow[d] & \pi_0(F_*(G_2)\otimes A) \arrow[r] \arrow[d] & 1 \\ \cdots \arrow[r] & \pi_1(F_*(G)\otimes A) \arrow[r, "\delta_1"'] & \pi_0(\mathcal{I}_*) \arrow[r] & \pi_0(F_*(G_1)\otimes A) \arrow[r] & \pi_0(F_*(G)\otimes A) \arrow[r] & 1 \end{tikzcd}. \tag{4.15}$$

The connecting homomorphism $\pi_n(F_*(G)\otimes A)\longrightarrow \pi_{n-1}L_*(\mathcal{D},A)$, $n\geq 1$, is the composite map $\pi_{n-1}(\sigma_*)\delta_n$. The homomorphism $\pi_n(L_*(\mathcal{D},A))\longrightarrow \pi_n(F_*(G_1)\otimes A)\times\pi_n(F_*(G_2)\otimes A)$, $n\geq 0$, is induced by $\pi_n(q_*)$ and $\pi_n(p_*)$. The map $\pi_n(F_*(G_1)\otimes A)\times\pi_n(F_*(G_2)\otimes A)\longrightarrow \pi_n(F_*(G)\otimes A)$, $n\geq 0$, is given by $\pi_n(F_*(\alpha_1)\otimes 1_A)\pi_n(F_*(\alpha_2)\otimes 1_A)^{-1}$. To get the exactness of the sequence (4.13), it remains to apply diagram (4.15). $\square$

Note that if the group G_2 is trivial, then we recover the sequence (4.11) (see Remark 4.13).

Let G and A be any groups that act on each other. Let us consider $H_1(-,A)$ as a functor from the category $\mathcal{A}_A$ to the category $\mathfrak{Gr}$ of groups and its left derived functors $L_n^{\mathcal{P}}(H_1(-,A))$ relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$. Then we have

Theorem 4.20. *There is a natural isomorphism*

$$H_n(G,A)\cong L_{n-1}^{\mathcal{P}}(H_1(G,A)),\quad n\geq 1.$$

Proof. We have a short exact sequence of groups

$$1 \longrightarrow H_1(G,A) \longrightarrow G\otimes A \xrightarrow{\lambda'} \operatorname{Im}\lambda' \longrightarrow 1,$$

and for any surjective morphism $\alpha: G\longrightarrow G'$ of $\mathcal{A}_A$ the commutative diagram of groups

$$\begin{tikzcd} G\otimes A \arrow[r, "\lambda'"] \arrow[d] & A/H \arrow[d, equal] \\ G'\otimes A \arrow[r, "\lambda'"'] & A/H' \end{tikzcd},$$

where $H = H'$ and $\lambda'(G \otimes A) = \lambda'(G' \otimes A)$. Then the assertion follows from the long exact sequence of homotopy groups of the following short exact sequence of simplicial groups with $\lambda' : G \otimes A \longrightarrow A/H$:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & H_1(F^3(G), A) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & H_1(F^2(G), A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & H_1(F^1(G), A) \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & F^3(G) \otimes A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & F^2(G) \otimes A \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(G) \otimes A \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \operatorname{Im} \lambda' \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \operatorname{Im} \lambda' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \operatorname{Im} \lambda' \arrow[d] \\
 & 1 & 1 & 1
\end{tikzcd}
\tag{4.16}
$$

□

Note that Theorem 4.20 generalizes the well-known fact that the integral homology can be obtained as the left derived functors of the abelianization functor.

Proposition 4.21.

(i) *$H_1(-, A)$ is a cosheaf over $(\mathcal{A}_A, \mathcal{P})$* (for the definition see Chap. 1, Sec. 2.3).

(ii) *If the actions satisfy compatibility conditions* (4.1)*, then $H_1(-, A)$ is a right exact functor.*

Proof.

(i) From the commutative diagram of groups (4.16) it follows that $H_1(-, A) \cong L_0^{\mathcal{P}} H_1(-, A)$. Then the assertion follows by Proposition 1.20.

(ii) Follows from the commutative diagram of groups

$$
\begin{tikzcd}
1 \arrow[r] & H_1(G_2, A) \arrow[r] \arrow[d] & G_2 \otimes A \arrow[r] \arrow[d] & A \arrow[d, equal] \\
1 \arrow[r] & H_1(G, A) \arrow[r] \arrow[d] & G \otimes A \arrow[r] \arrow[d] & A \arrow[d, equal] \\
1 \arrow[r] & H_1(G_1, A) \arrow[r] \arrow[d] & G_1 \otimes A \arrow[r] \arrow[d] & A \\
 & 1 & 1 &
\end{tikzcd}
.
$$

□

Now a new description of the non-Abelian homology of groups will be given in terms of the non-Abelian left derived functors of the section functor $\Gamma_G : \mathbb{CS}(\mathcal{A}_A, \mathcal{P}) \longrightarrow \mathfrak{Gr}$ (see Chap.1, Sec. 2.3).

Theorem 4.22. *Let G and A be groups acting on each other. Then there are isomorphisms $H_n(G, A) \cong L_{n-1}^{Q} \Gamma_G(- \otimes A)$, $n \geq 2$ and $H_n(G, A) \cong L_{n-1}^{Q} \Gamma_G(H_1(-, A))$, $n \geq 1$.*

Proof. Since Theorem 4.6(a) and Proposition 4.21(i) say that the functors $(- \otimes A)$ and $H_1(-, A)$ are cosheaves over $(\mathcal{A}_A, \mathcal{P})$, respectively, the isomorphisms follow from [60, Theorem 2.34] and Theorem 4.20. □

Theorem 4.23. *Let G and A be any groups. Let A act on G trivially and G act on A.*

(i) *If G is a finite group and A is a finite group (or p-group or finitely generated group), then $H_n(G,A)$, $n \geq 2$, is a finite group (or p-group or finitely generated group);*

(ii) *If A is a torsion group (or a group of exponent q), then $H_n(G,A)$, $n \geq 2$, is a torsion group (or a group of exponent q).*

Proof. Let us consider a $\mathcal{P}$-projective pseudo-simplicial resolution $X_* \xrightarrow{\partial_0^0} G$ of G in the category $\mathcal{A}_A$. Apply the Quillen's construction [60, 107]

$$1 \longrightarrow \Omega X_* \longrightarrow EX_* \longrightarrow X_* \longrightarrow C\pi_0 X_* \longrightarrow 1$$

to the pseudo-simplicial group X_*. We obtain the commutative diagram of groups

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1}) \arrow[d, "\partial_{n+1}^{n+1}"] \arrow[r, shift left=2, "\partial_0^{n+1}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\partial_n^{n+1}"'] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Ker}(\partial_0^1 \partial_0^2) \arrow[d, "\partial_2^2"] \arrow[r, shift left, "\partial_0^2"] \arrow[r, shift right, "\partial_1^2"'] & \mathrm{Ker}(\partial_0^1) \equiv EX_* \arrow[d, "\partial_1^1"] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & X_n \arrow[d] \arrow[r, shift left=2, "\partial_0^n"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\partial_n^n"'] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[d] \arrow[r, shift left, "\partial_0^1"] \arrow[r, shift right, "\partial_1^1"'] & X_0 \equiv X_* \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & G \arrow[d] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & G \equiv C\pi_0 X_* \arrow[d] \\
 & 1 & & 1 & 1
\end{tikzcd}$$

where $C\pi_0 X_*$ is the constant simplicial group. Therefore we have the following commutative diagram of groups:

$$\begin{tikzcd}
 & 1 \arrow[d] & & 1 \arrow[d] & 1 \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathrm{Ker}(\partial_{n+1}^{n+1} \otimes 1_A) \arrow[d] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Ker}(\partial_2^2 \otimes 1_A) \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Ker}(\partial_1^1 \otimes 1_A) \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1}) \otimes A \arrow[d, "\partial_{n+1}^{n+1}\otimes 1_A"] \arrow[r, shift left=2, "\partial_0^{n+1}\otimes 1_A"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\partial_n^{n+1}\otimes 1_A"'] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Ker}(\partial_0^1 \partial_0^2) \otimes A \arrow[d, "\partial_2^2\otimes 1_A"] \arrow[r, shift left, "\partial_0^2\otimes 1_A"] \arrow[r, shift right, "\partial_1^2\otimes 1_A"'] & \mathrm{Ker}(\partial_0^1) \otimes A \arrow[d, "\partial_1^1\otimes 1_A"] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & X_n \otimes A \arrow[d] \arrow[r, shift left=2, "\partial_0^n\otimes 1_A"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\partial_n^n\otimes 1_A"'] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \otimes A \arrow[d] \arrow[r, shift left, "\partial_0^1\otimes 1_A"] \arrow[r, shift right, "\partial_1^1\otimes 1_A"'] & X_0 \otimes A \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & G \otimes A \arrow[d] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \otimes A \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & G \otimes A \arrow[d] \\
 & 1 & & 1 & 1
\end{tikzcd} \tag{4.17}$$

Thus the sequence of simplicial groups

$$1 \longrightarrow \mathrm{Ker}(\vartheta \otimes 1_A) \longrightarrow EX_* \otimes A \xrightarrow{\vartheta\otimes 1_A} X_* \otimes A \xrightarrow{j\otimes 1_A} C\pi_0 X_* \otimes A \longrightarrow 1,$$

is exact, where $\vartheta_n : (EX_*)_n \longrightarrow X_n$ is induced by the homomorphism $\partial_{n+1}^{n+1} : X_{n+1} \longrightarrow X_n$ and $j_n : X_n \longrightarrow (C\pi_0 X_*)$ is the composite homomorphism $\partial_0^0 \partial_0^1 \cdots \partial_0^n$. Therefore, we obtain two long exact sequences of groups:

$$\cdots \longrightarrow \pi_{n+1}\,\mathrm{Ker}(j \otimes 1_A) \longrightarrow \pi_n\,\mathrm{Ker}(\vartheta \otimes 1_A) \longrightarrow \pi_n(EX_* \otimes A) \longrightarrow$$
$$\longrightarrow \pi_n\,\mathrm{Ker}(j \otimes 1_A) \longrightarrow \pi_{n-1}\,\mathrm{Ker}(\vartheta \otimes 1_A) \longrightarrow \cdots \longrightarrow \pi_1\,\mathrm{Ker}(j \otimes 1_A) \longrightarrow$$
$$\longrightarrow \pi_0\,\mathrm{Ker}(\vartheta \otimes 1_A) \longrightarrow \pi_0(EX_* \otimes A) \longrightarrow \pi_0\,\mathrm{Ker}(j \otimes 1_A) \longrightarrow 1$$

and

$$\cdots \longrightarrow \pi_{n+1}(C\pi_0 X_* \otimes A) \longrightarrow \pi_n\,\mathrm{Ker}(j \otimes 1_A) \longrightarrow \pi_n(X_* \otimes A) \longrightarrow$$
$$\longrightarrow \pi_n(C\pi_0 X_* \otimes A) \longrightarrow \pi_{n-1}\,\mathrm{Ker}(j \otimes 1_A) \longrightarrow \cdots \longrightarrow \pi_1(C\pi_0 X_* \otimes A) \longrightarrow$$
$$\longrightarrow \pi_0\,\mathrm{Ker}(j \otimes 1_A) \longrightarrow \pi_0(X_* \otimes A) \longrightarrow \pi_0(C\pi_0 X_* \otimes A) \longrightarrow 1\ .$$

But the homotopy groups $\pi_n(EX_* \otimes A)$, $n \geq 0$, are trivial, since the augmented pseudo-simplicial group $(EX_*, \epsilon, 1)$ is right contractible with contractions $h = 0$ and $h_n = s_{n+1}^{n+1}$ for $n \geq 0$ [60, 107]. This implies that

$$H_{n+1}(G, A) = \pi_n(X_* \otimes A) \cong \pi_{n-1}(\mathrm{Ker}(\vartheta \otimes 1_A)), \quad n \geq 1. \tag{4.18}$$

Assume now with no loss of generality that A acts on X_* trivially. Any $\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$, $n \geq 0$, acts trivially on A and since it is a free group, by [18, Proposition 2.4] we have

$$\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1}) \otimes A \cong \mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})^{ab} \otimes A^{ab} \cong \sum_{\alpha} A^{ab}, \tag{4.19}$$

where α runs over the basis of $\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$.

First, we prove (i): If G is a finite group and A acts trivially on G then we can construct a new $\mathcal{P}$-projective simplicial resolution G_* of the object G in the category $\mathcal{A}_A$ such that every G_n will be a finitely generated free group [70]. This can be done as follows.

Recall the definition of the loop functor $\mathbb{G}$ from the category of reduced complexes to the category of simplicial groups [78].

Let K be a reduced complex (i.e., K is a simplicial set that has only one 0-simplex ϕ). Then we define a simplicial group $\mathbb{G}K$ as follows: the group of the n-simplices is a group that has

(i) one generator $\overline{\sigma}$ for every $n+1$-simplex $\sigma \in K_{n+1}$,

(ii) one relation $\overline{s_n^n \tau} = e_n$ for every n-simplex $\tau \in K_n$.

The face and degeneracy homomorphisms $\partial_i^n : \mathbb{G}_n K \longrightarrow \mathbb{G}_{n-1} K$ and $s_i^n : \mathbb{G}_n K \longrightarrow \mathbb{G}_{n+1} K$ are given by the formulas

$$\begin{aligned}
\partial_i^n \overline{\sigma} &= \overline{\partial_i^n \sigma}, \quad 0 \leq i < n,\\
\partial_n^n \overline{\sigma} &= \overline{\partial_n^n \sigma} \cdot \overline{\partial_{n+1}^n \sigma}^{-1},\\
s_i^n \overline{\sigma} &= \overline{s_i^n \sigma}, \quad 0 \leq i \leq n.
\end{aligned}$$

Clearly, the groups $\mathbb{G}_n K$ are free and from [78] it is known that $\pi_n(K) \cong \pi_{n-1}(\mathbb{G}K)$ for $n > 0$.

Let $\underline{G}$ be a category that has only one object O and $\mathrm{Hom}(O, O) = G$, and consider the nerve $M_*(\underline{G})$ of the category $\underline{G}$. It is easy to see that $M_*(\underline{G})$ is a reduced complex. It is known that $\pi_1(M_*(\underline{G})) \cong G$

and $\pi_n(M_*(\underline{G})) = 0$, $n \neq 1$. Let us consider the simplicial group $\mathbb{G}M_*(\underline{G})$. Then $\pi_0(\mathbb{G}M_*(\underline{G})) \cong \pi_1(M_*(\underline{G})) \cong G$ and $\pi_{n-1}(\mathbb{G}M_*(\underline{G})) \cong \pi_n(M_*(\underline{G})) = 0$, $n > 1$. Hence $\mathbb{G}M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is a $\mathcal{P}$-projective simplicial resolution in the category of groups, where ∂_0^0 is the natural epimorphism, and $\mathcal{P}$ is the projective class in the category of groups induced by the free cotriple.

Thus, if we define the action of A on $\mathbb{G}_n M_*(\underline{G})$ trivially and the action of $\mathbb{G}_n M_*(\underline{G})$ on A as follows:

$$^{m}a = {}^{\partial_0^0\partial_0^1\cdots\partial_0^n m}a, \quad m \in \mathbb{G}_n M_*(\underline{G}),$$

then by Propositions 4.10 and 4.11 we deduce that $\mathbb{G}_n M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is the $\mathcal{P}$-projective simplicial resolution in the category $\mathcal{A}_A$.

Therefore the group $\operatorname{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$ is the subgroup with finite index of the finitely generated free group and by [113] $\operatorname{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$ is also a finitely generated free group. Hence by (10) if A is finite group (or p-group or finitely generated group) so is $\operatorname{Ker}(\partial_0^1 \cdots \partial_0^{n+1}) \otimes A$, $n \geq 0$. By (4.17) and (4.18) it follows that $H_n(G, A)$, $n \geq 2$ is a finite group (or p-group or finitely generated group).

(ii) is proved analogously by (4.17)–(4.19) and the fact that the property of a group to be torsion (or of exponent q) is stable under coproducts, subgroups, and quotient groups. $\square$

4. Second and Third Non-Abelian Homologies of Groups

Now new descriptions of the second and the third non-Abelian homology of groups will be given using the Čech derived functors (see Chap. 1, Sec. 2.1).

Let G and A be arbitrary groups that act on each other. Let P be an object of the projective class $\mathcal{P}$ and $\alpha : P \longrightarrow G$ be a $\mathcal{P}$-epimorphism in the category $\mathcal{A}_A$. Consider the Čech resolution $(\check{C}(\alpha)_*, \alpha, G)$ of G. The actions of A and $\check{C}(\alpha)_n = \underbrace{P \times_G \cdots \times_G P}_{(n+1)\text{-times}}$, $n \geq 1$, on each other are induced in a natural way by the actions of A and P on each other.

Denote by $\check{C}(\alpha)_* \otimes A$ the simplicial group obtained by applying the functor $- \otimes A$ dimension-wise to the simplicial group $\check{C}(\alpha)_*$.

Theorem 4.24.

(i) *There is an isomorphism*

$$H_2(G, A) \cong \operatorname{Ker}(d_0^1 \otimes 1_A) \cap \operatorname{Ker}(d_1^1 \otimes 1_A) \big/ \big[\operatorname{Ker}(d_0^1 \otimes 1_A), \operatorname{Ker}(d_1^1 \otimes 1_A)\big];$$

(ii) *there is an epimorphism*

$$H_3(G, A) \longrightarrow \bigcap_{i \in [2]} \operatorname{Ker}(d_i^2 \otimes 1_A) \Big/ \prod_{I,J} [K_I, K_J],$$

where $\varnothing \neq I, J \subset [2] = \{0, 1, 2\}$ *with* $I \cup J = [2]$, $K_I = \bigcap_{i \in I} \operatorname{Ker}(d_i^2 \otimes 1_A)$ *and* $K_J = \bigcap_{j \in J} \operatorname{Ker}(d_j^2 \otimes 1_A)$.

Proof. By [103] (see also [60, Theorem 2.39(ii)]) we know that there is an isomorphism

$$H_2(G, A) = L_1^{\mathcal{P}}(G \otimes A) \cong \pi_1(\check{C}(\alpha)_* \otimes A) \tag{4.20}$$

and an epimorphism

$$H_3(G, A) = L_2^{\mathcal{P}}(G \otimes A) \longrightarrow \pi_2(\check{C}(\alpha)_* \otimes A). \tag{4.21}$$

Now we must show that $\check{C}(\alpha)_2 \otimes A$ and $\check{C}(\alpha)_3 \otimes A$ are generated by degenerate elements. In fact, for any $(x, y, z) \in \check{C}(\alpha)_2$ and $a \in A$ we have

$$
\begin{aligned}
(x,y,z)\otimes a &= (x,x,x)(1,x^{-1}y,x^{-1}y)(1,1,y^{-1}z)\otimes a = \\
&= \big({}^{(x,x,x)}(1,x^{-1}y,x^{-1}y)\cdot {}^{(x,x,x)}(1,1,y^{-1}z)\otimes {}^{\alpha(x)}a\big)((x,x,x)\otimes a) = \\
&= \big((1,yx^{-1},yx^{-1})(1,1,xy^{-1}zx^{-1})\otimes {}^{\alpha(x)}a\big)((x,x,x)\otimes a) = \\
&= \big({}^{(1,yx^{-1},yx^{-1})}(1,1,xy^{-1}zx^{-1})\otimes {}^{\alpha(x)}a\big)\big((1,yx^{-1},yx^{-1})\otimes {}^{\alpha(x)}a\big)((x,x,x)\otimes a) = \\
&= \big((1,1,zy^{-1})\otimes {}^{\alpha(x)}a\big)\big((1,yx^{-1},yx^{-1})\otimes {}^{\alpha(x)}a\big)((x,x,x)\otimes a) = \\
&= (s_0\otimes 1_A)\big((1,zy^{-1})\otimes {}^{\alpha(x)}a\big)\cdot(s_1\otimes 1_A)\big((1,yx^{-1})\otimes {}^{\alpha(x)}a\big)\cdot(s_0\otimes 1_A)((x,x)\otimes a).
\end{aligned}
$$

For $\check{C}(\alpha)_3\otimes A$ the proof is similar.

By [18, Lemma 5.7] and [99, Theorem 4.1] there are equalities

$$
\begin{aligned}
\operatorname{Im}\partial_2 &= \big[\operatorname{Ker}(d_0\otimes 1_A),\operatorname{Ker}(d_1\otimes 1_A)\big],\\
\operatorname{Im}\partial_3 &= \prod_{I,J}[K_I,K_J],
\end{aligned}
\tag{4.22}
$$

where $\varnothing\neq I,J\subset[2]=\{0,1,2\}$ with $I\cup J=[2]$,

$$K_I=\bigcap_{i\in I}\operatorname{Ker}(d_i^2\otimes 1_A),$$

$$K_J=\bigcap_{j\in J}\operatorname{Ker}(d_j^2\otimes 1_A),$$

and ∂_2 and ∂_3 are Moore complex homomorphisms of $\check{C}(\alpha)_*\otimes A$.

From (4.20)–(4.22) follows the assertion. □

Let us consider an exact sequence of groups

$$1\longrightarrow R\xrightarrow{\sigma}F\xrightarrow{\alpha}G\longrightarrow 1,$$

where $F\in\mathcal{P}$ and α is a $\mathcal{P}$-epimorphism. We have commutative diagrams of groups with exact rows and columns

$$
\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & R \arrow[r, equal] \arrow[d, "\sigma_1"'] & R \arrow[d, "\sigma"] & \\
1 \arrow[r] & R \arrow[r, "\sigma_0"] \arrow[d, equal] & F\times_G F \arrow[r, "\partial_0"] \arrow[d, "d_1"'] & F \arrow[r] \arrow[d, "\alpha"] & 1\ , \\
1 \arrow[r] & R \arrow[r, "\sigma"'] & F \arrow[r, "\alpha"'] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}
\tag{4.23}
$$

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathrm{Ker}(\sigma_1 \otimes 1_A) \arrow[r, "\beta"] \arrow[d] & \mathrm{Ker}(\sigma \otimes 1_A) \arrow[r] \arrow[d] & \mathrm{Coker}\,\beta \\
 & R \otimes A \arrow[r, equal] \arrow[d, "\sigma_1 \otimes 1_A"'] & R \otimes A \arrow[r] \arrow[d, "\sigma \otimes 1_A"] & {} \\
R \otimes A \arrow[r, "\sigma_0 \otimes 1_A"] \arrow[d, equal] & (F \times_G F) \otimes A \arrow[r, "\partial_0 \otimes 1_A"] \arrow[d, "d_1 \otimes 1_A"'] & F \otimes A \arrow[r] \arrow[d, "\alpha \otimes 1_A"] & 1 \\
R \otimes A \arrow[r, "\sigma \otimes 1_A"] & F \otimes A \arrow[r, "\alpha \otimes 1_A"] \arrow[d] & G \otimes A \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 &
\end{tikzcd}
\qquad . \tag{4.24}
$$

Define a homomorphism

$$\delta : \mathrm{Ker}(d_0 \otimes 1_A) \cap \mathrm{Ker}(d_1 \otimes 1_A) \longrightarrow \mathrm{Coker}\,\beta,$$

by $\delta(x) = [y]$ for all $x \in \mathrm{Ker}(d_0 \otimes 1_A) \cap \mathrm{Ker}(d_1 \otimes 1_A)$ and with $(\sigma_1 \otimes 1_A)(y) = x$. It is easy to verify that δ is correctly defined and is an isomorphism. From Theorem 4.24(i) we obtain the following theorem.

Theorem 4.25. *Let G and A be any groups acting on each other. Then*

$$H_2(G, A) \cong \mathrm{Coker}\,\beta / \delta\big(\big[\,\mathrm{Ker}(d_0 \otimes 1_A), \mathrm{Ker}(d_1 \otimes 1_A)\big]\big).$$

Corollary 4.26. *If A acts trivially on G, G acts on A and the actions are compatible, then there is an isomorphism*

$$H_2(G, A) \cong \mathrm{Coker}\,\beta.$$

Proof. It is obvious, since by [18, Proposition 2.3] in this case the group $(F \times_G F) \otimes A$ is Abelian. □

Note that if G acts on A and A acts on G trivially such that the actions are compatible, then $H_n(G, A) \cong H_n(G, A^{ab})$, $n \geq 2$, since $G \otimes A \cong G \otimes A^{ab}$.

Let $\mathbb{Z}_n$ act on a group A and A act on $\mathbb{Z}_n$ trivially such that the actions are compatible. Further we assume that $\mathbb{Z}$ acts on A via the canonical homomorphism $\alpha : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ and A acts trivially on $\mathbb{Z}$. Consider the isomorphism $\kappa : A^{ab} \longrightarrow n\mathbb{Z} \otimes A$ given by $a \longmapsto n \otimes a$ and the homomorphism $\sigma_1 \otimes 1_A : n\mathbb{Z} \otimes A \longrightarrow (\mathbb{Z} \times_\alpha \mathbb{Z}) \otimes A$ (see diagram (4.24)). Then we have

Proposition 4.27. *There is an isomorphism*

$$H_2(\mathbb{Z}_n, A) \cong {}_N(A^{ab}) / \mathrm{Ker}((\sigma_1 \otimes 1_A)\kappa),$$

where ${}_N(A^{ab}) = \mathrm{Ker}\,N$, $N : A^{ab} \longrightarrow A^{ab}$ *with* $N([a]) = \sum\limits_{x \in \mathbb{Z}_n} {}^x[a]$.

Proof. It is clear that α is a $\mathcal{P}$-epimophism (see Proposition 4.10). Using Proposition 4.7 we have the following commutative diagram:

$$
\begin{tikzcd}
n\mathbb{Z} \otimes A \arrow[r, "\kappa^{-1}"] \arrow[d, "\sigma \otimes 1_A"'] & A^{ab} \arrow[d, "N"] \\
\mathbb{Z} \otimes A \arrow[r, "\cong"'] & A^{ab}
\end{tikzcd}
\; .
$$

It remains to apply Corollary 4.26. □

Let us consider the category $A_{\mathfrak{Mod}}$ of A-modules. In this category consider the following projective class $\mathfrak{P}$:

An object of $\mathfrak{P}$ is a free Abelian group having a basis X such that ${}^a x \in X$ for any $x \in X$, $a \in A$.

It is clear that the projective class $\mathfrak{P}$ includes all free A-modules. For any additive functor $T : A_{\mathfrak{Mod}} \longrightarrow Ab\mathfrak{Gr}$ consider its left derived functors $L_n^{\mathfrak{P}} T$, $n \geq 0$.

For any short exact sequence of A-modules $0 \longrightarrow B_1 \xrightarrow{i} B \xrightarrow{j} B_2 \longrightarrow 0$, where j is a $\mathfrak{P}$-epimorphism, i.e., with an action preserving section, we have a long exact sequence of the left derived functors $L_n^{\mathfrak{P}} T$ of T.

Consider the following commutative diagram of groups with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & R \arrow[r, "\sigma"] \arrow[d, "\tau'"'] & F \arrow[r, "\alpha"] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Ker}(\alpha^{ab}) \arrow[r, "\sigma'"'] & F^{ab} \arrow[r, "\alpha^{ab}"'] & G^{ab} \arrow[r] & 1,
\end{tikzcd}$$

where $G \in \mathcal{A}_A$, F is an object of the projective class $\mathcal{P}$ in the category $\mathcal{A}_A$ (see above) and α is a $\mathcal{P}$-epimorphism.

Theorem 4.28. *Let G be an Abelian group acting on a group A trivially, and A act on G. Then we have an exact sequence of groups*

$$(\mathrm{Ker}(\sigma \otimes 1_A) \cap \mathrm{Ker}(\tau' \otimes 1_A))^{ab} \longrightarrow H_2(G, A) \longrightarrow \mathrm{Tor}_1^{\mathfrak{P}}(G, IA) \longrightarrow 0,$$

where IA is the augmentation ideal of A.

Proof. It is easy to show that in this case τ' is surjective. So we obtain the following commutative diagram of groups with exact rows and columns:

$$\begin{tikzcd}
 & & {[F,F] \otimes A} \arrow[r, equal] \arrow[d] & {[F,F] \otimes A} \arrow[d] & & \\
1 \arrow[r] & \mathrm{Ker}(\sigma \otimes 1_A) \arrow[r] \arrow[d, "\lambda"'] & R \otimes A \arrow[r, "\sigma \otimes 1_A"] \arrow[d, "\tau' \otimes 1_A"'] & F \otimes A \arrow[r, "\alpha \otimes 1_A"] \arrow[d] & G \otimes A \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{Ker}(\sigma' \otimes 1_A) \arrow[r] & \mathrm{Ker}(\alpha^{ab}) \otimes A \arrow[r, "\sigma' \otimes 1_A"'] \arrow[d] & F^{ab} \otimes A \arrow[r, "\alpha^{ab} \otimes 1_A"'] \arrow[d] & G \otimes A \arrow[r] & 1 \\
 & & 1 & 1 & &
\end{tikzcd}.$$

It will be shown that the natural epimorphism

$$\lambda : \mathrm{Ker}(\sigma \otimes 1_A) \longrightarrow \mathrm{Ker}(\sigma' \otimes 1_A)$$

reduced to $\mathrm{Ker}(\sigma_1 \otimes 1_A)$ (see diagram (4.24)) and to $\delta([\mathrm{Ker}(d_0^1 \otimes 1_A), \mathrm{Ker}(d_1^1 \otimes 1_A)])$ is trivial. In effect, the commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & R \arrow[r, "\sigma_1"] \arrow[d] & F \times_G F \arrow[r, "d_1^1"] \arrow[d] & F \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Ker}({d_1^1}^{ab}) \arrow[r] & (F \times_G F)^{ab} \arrow[r, "{d_1^1}^{ab}"'] & F^{ab} \arrow[r] & 1
\end{tikzcd}$$

induces the following commutative diagram:

$$\begin{tikzcd} R\otimes A \arrow[r, "\sigma_1\otimes 1_A"] \arrow[d] & (F\times_G F)\otimes A \arrow[d, "d_1^1\otimes 1_A"'] & F\otimes A \arrow[r] \arrow[d] & 1 \\ \mathrm{Ker}(d_1^{1\,ab})\otimes A \arrow[r] & (F\times_G F)^{ab}\otimes A \arrow[r, "d_1^{1\,ab}\otimes 1_A"'] & F^{ab}\otimes A \arrow[r] & 1 \end{tikzcd} \tag{4.25}$$

which maps to the diagram

$$\begin{tikzcd} R\otimes A \arrow[r, "\sigma\otimes 1_A"] \arrow[d, "\tau'\otimes 1_A"'] & F\otimes A \arrow[r, "\alpha\otimes 1_A"] \arrow[d, "\mathrm{Ker}(\alpha^{ab})\otimes A"'] & G\otimes A \arrow[r] \arrow[d, equal] & 1 \\ \mathrm{Ker}(\alpha^{ab})\otimes A \arrow[r, "\sigma'\otimes 1_A"'] & F^{ab}\otimes A \arrow[r, "\alpha^{ab}\otimes 1_A"'] & G\otimes A \arrow[r] & 1 \end{tikzcd} \tag{4.26}$$

using the triple $(1_R, d_0^1, \alpha)$ (see diagram (4.23)).

To this end we need the fact that the homomorphism $\mathrm{Ker}(d_1^{1\,ab})\otimes A \longrightarrow (F\times_G F)^{ab}\otimes A$ is injective. The short exact sequence of A-modules

$$1 \longrightarrow \mathrm{Ker}(d_1^{1\,ab}) \longrightarrow (F\times_G F)^{ab} \xrightarrow{d_1^{1\,ab}} F^{ab} \longrightarrow 1$$

has a section γ given by $\gamma[f] = [(f,f)]$ that preserves the actions of A. Therefore it induces a long exact sequence of the left derived functors of $-\otimes_{\mathbb{Z}[A]} IA$ with relative to the projective class $\mathfrak{P}$

$$\cdots \longrightarrow \mathrm{Tor}_1^{\mathfrak{P}}(F^{ab}, IA) \longrightarrow \mathrm{Ker}(d_1^{1\,ab})\otimes_{\mathbb{Z}[A]} IA \longrightarrow$$
$$\longrightarrow (F\times_G F)^{ab}\otimes_{\mathbb{Z}[A]} IA \longrightarrow F^{ab}\otimes_{\mathbb{Z}[A]} IA \longrightarrow 0.$$

Since F^{ab} belongs to the projective class $\mathfrak{P}$, we have $\mathrm{Tor}_1^{\mathfrak{P}}(F^{ab}, IA) = 0$. By Guin's isomorphism [53] the groups $\mathrm{Ker}(d_1^{1\,ab})\otimes_{\mathbb{Z}[A]} IA$, $(F\times_G F)^{ab}\otimes_{\mathbb{Z}[A]} IA$ and $F^{ab}\otimes_{\mathbb{Z}[A]} IA$ are naturally isomorphic to the groups $\mathrm{Ker}(d_1^{1\,ab})\otimes A$, $(F\times_G F)^{ab}\otimes A$ and $F^{ab}\otimes A$ respectively. This implies that the homomorphism $\mathrm{Ker}(d_1^{1\,ab})\otimes A \longrightarrow (F\times_G F)^{ab}\otimes A$ is injective.

From diagrams (4.25) and (4.26), the triviality of λ on $\mathrm{Ker}(\sigma_1\otimes 1_A)$ and $\delta([\mathrm{Ker}(d_0\otimes 1_A), \mathrm{Ker}(d_1\otimes 1_A)])$ is now clear.

By the same reason the short exact sequence of A modules

$$0 \longrightarrow \mathrm{Ker}(\alpha^{ab}) \longrightarrow F^{ab} \xrightarrow{\alpha^{ab}} G \longrightarrow 0,$$

which has a natural needed section, gives the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathfrak{P}}(G, IA) \longrightarrow \mathrm{Ker}(\alpha^{ab})\otimes A \xrightarrow{\sigma'\otimes 1_A} F^{ab}\otimes A \xrightarrow{\alpha^{ab}\otimes 1_A} G\otimes A \longrightarrow 0.$$

It follows by Theorem 4.25 that λ induces a natural epimorphism $H_2(G,A)$ on $\mathrm{Tor}_1^{\mathfrak{P}}(G, IA)$, and by the homomorphism δ the group $(\mathrm{Ker}(\sigma\otimes 1_A)\cap \mathrm{Ker}(\tau'\otimes 1_A))$ maps on its kernel. $\square$

Note also that $\mathrm{Tor}_1^{\mathfrak{P}}(G, IA) = \mathrm{Coker}\,\mathrm{Tor}_1(\alpha^{ab}, IA)$.

Chapter 5

NON-ABELIAN (CO)HOMOLOGY OF LIE ALGEBRAS

The purpose of this chapter is to set up a similar non-Abelian (co)homology theory for Lie algebras and mainly dedicated to state and prove several desirable properties of this (co)homology theory.

In [45] Ellis introduced and studied the non-Abelian tensor product of Lie algebras which is a Lie structural and purely algebraic analog of the non-Abelian tensor product of groups of Brown and Loday [17, 18], treated slightly in Chap. 4, Sec. 1.

Applying this tensor product of Lie algebras, Guin defined the low-dimensional non-Abelian homology of Lie algebras with coefficients in crossed modules [55].

In Sec. 1, we recall the notion of the non-Abelian tensor product of Lie algebras due to Ellis [45] and give some needed properties.

In Sec. 2, we construct a non-Abelian homology $H_*(M, N)$ of a Lie algebra M with coefficients in a Lie algebra N as the non-Abelian left derived functors of the tensor product of Lie algebras, generalizing the classical homology of Lie algebras and extending Guin's non-Abelian homology of Lie algebras [55] (Proposition 5.6 and Definition 5.7).

In Sec. 3, we investigate the non-Abelian homology of Lie algebras in various aspects, establishing its some functorial properties. In particular, long exact non-Abelian homology sequences are established (Theorems 5.9, 5.11 and Corollary 5.12). Moreover, the non-Abelian homology of Lie algebras is expressed in terms of first non-Abelian homology (Theorem 5.13) and its compatibility with direct limits of Lie algebras is established (Proposition 5.14). Some explicit formulas for the second and the third non-Abelian homology of Lie algebras are obtained using Čech derived functors (Theorem 5.15).

In Sec. 4, we give an application of the long exact non-Abelian homology sequence of Lie algebras to cyclic homology of associative algebras (Theorem 5.18), correcting the result of [55].

Sections 5 and 6 are dedicated to the non-Abelian cohomology of Lie algebras. Following ideas from [55, 61], using the generalized notion of the Lie algebra of derivations (Definition 5.20 and Proposition 5.21), we introduce the second non-Abelian cohomology $H^2(R, M)$ of a Lie algebra R with coefficients in a crossed R-module (M, μ) (Proposition 5.26, Definition 5.28), generalizing the classical second cohomology of Lie algebras (Propositions 5.24 and 5.27). Then, for a coefficient short exact sequence of crossed R-modules having a module section over the ground ring, we give a nine-term exact non-Abelian cohomology sequence extending the seven-term exact cohomology sequence of Guin [55], which exists under the aforementioned additional necessary condition on the coefficient sequence of crossed modules (Proposition 5.29).

Further generalizations of non-Abelian cohomology of Lie algebras is possible pursuing the line of [62, 63], in particular, in the direction of making a definition in any dimension and for a wider class of coefficients.

In this chapter, we denote by Λ a unital commutative ring unless otherwise stated. We shall use the term Lie algebra to mean a Lie algebra over Λ and $[\ ,\ ]$ and $|\ |$ to denote the Lie bracket and the coset of the quotient Lie algebra respectively. We denote the category of Lie algebras over Λ by $\mathcal{L}ie$.

We mean under the classical (co)homology of a Lie algebra M with coefficients in an M-module N the (co)homology groups in the sense of Chevalley–Eilenberg (see, e.g., [22]) that are the homology of the complexes obtained by applying the functors $\mathrm{Hom}_{U(M)}(-, N)$ and $- \otimes_{U(M)} N$ to the following standard complex of $U(M)$-modules

$$\cdots \xrightarrow{\partial} V_n(M) \xrightarrow{\partial} \cdots \xrightarrow{\partial} V_1(M) \xrightarrow{\partial} V_0(M) \xrightarrow{\epsilon} \Lambda,$$

where $U(M)$ is the universal enveloping algebra of M, $V_n(M) = U(M) \otimes_\Lambda E_n(M)$, $n \geq 0$, $E_n(M) = \underbrace{M \wedge_\Lambda \cdots \wedge_\Lambda M}_{n\text{-times}}$, $n \geq 1$, and $E_0(M) = \Lambda$, and the chain boundary is given by the formula

$$\partial\langle x_1, \dots, x_n\rangle = \sum_{i=1}^{n}(-1)^{i+1}x_i\langle x_1, \dots, \widehat{x_i}, \dots, x_n\rangle + \\ + \sum_{1\leq i<j\leq n}(-1)^{i+j}\big\langle [x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n\big\rangle,$$

and $\epsilon\langle \quad \rangle = 1$.

1. The Tensor Products of Lie Algebras

Let P and M be two Lie algebras. By an action of P on M we mean a Λ-bilinear map $P \times M \longrightarrow M$, $(p, m) \longmapsto {}^p m$ satisfying the following conditions:

$${}^{[p,p']}m = {}^p({}^{p'}m) - {}^{p'}({}^p m), \quad {}^p[m, m'] = [{}^p m, m'] + [m, {}^p m']$$

for all $m,\ m' \in M$ and $p,\ p' \in P$. For example, if P is a subalgebra of some Lie algebra Q, and if M is an ideal in Q, then Lie multiplication in Q yields an action of P on M.

Now we give the definition of the tensor product of Lie algebras due to Ellis [45] (see also [25, 55]).

Definition 5.1. Let M and N be two Lie algebras acting on each other. The tensor product $M \otimes N$ of the Lie algebras M and N is the Lie algebra generated by the symbols $m \otimes n$, $m \in M$, $n \in N$, and subject to the following relations:

(i) $\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n$,

(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n$,
$m \otimes (n + n') = m \otimes n + m \otimes n'$,

(iii) $[m, m'] \otimes n = m \otimes ({}^{m'}n) - m' \otimes ({}^m n)$,
$m \otimes [n, n'] = ({}^{n'}m) \otimes n - ({}^n m) \otimes n'$,

(iv) $[(m \otimes n), (m' \otimes n')] = -({}^n m) \otimes ({}^{m'}n')$

for all $\lambda \in \Lambda$, $m, m' \in M$, $n, n' \in N$.

Assume that $\phi : M \longrightarrow A$, $\psi : N \longrightarrow B$ are Lie homomorphisms, A, B act on each other, and ϕ, ψ preserve the actions in the following sense:

$$\phi({}^n m) = {}^{\psi(n)}\phi(m), \quad \psi({}^m n) = {}^{\phi(m)}\psi(n), \quad m \in M, \quad n \in N.$$

Then, by [45], there is a unique homomorphism $\phi \otimes \psi : M \otimes N \longrightarrow A \otimes B$ such that $(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$ for all $m \in M$, $n \in N$. Furthermore, if ϕ, ψ are onto, so also is $\phi \otimes \psi$.

The tensor product of Lie algebras is symmetric in the sense of the isomorphism $M \otimes N \longrightarrow N \otimes M$ given by $m \otimes n \longmapsto -n \otimes m$ [45].

A precrossed P-module (M, μ) is a Lie homomorphism $\mu : M \longrightarrow P$ together with an action of P on M satisfying the following condition:

$$\mu({}^p m) = [p, \mu(m)] \quad \text{for all} \quad m \in M, \quad p \in P.$$

If in addition the precrossed module (M, μ) satisfies the Peiffer identity:

$${}^{\mu(m)}m' = [m, m'] \quad \text{for all} \quad m, m' \in M,$$

then it is said to be a crossed P-module. Note that, as in the group case, for a crossed module (M, μ) the image of μ is an ideal in P, the kernel of μ is a P-invariant ideal in the center of M, and the action of P on $\operatorname{Ker}\mu$ induces an action of $P/\operatorname{Im}\mu$ on $\operatorname{Ker}\mu$, making $\operatorname{Ker}\mu$ a $P/\operatorname{Im}\mu$-module.

In [45] the results on the tensor product $M \otimes N$ are obtained assuming the actions of M and N on each other compatible, i.e.,

$$^{({}^n m)}n' = [n', {}^m n] \quad \text{and} \quad ^{({}^m n)}m' = [m', {}^n m] \tag{5.1}$$

for all $m, m' \in M$ and $n, n' \in N$. This is the case, for example, if (M, μ) and (N, ν) are crossed P-modules, M and N act on each other via the action of P. These compatibility conditions are not assumed to hold except the following

Proposition 5.2. *Let M and N be Lie algebras acting on each other such that the compatibility conditions* (5.1) *hold. Then there is a natural isomorphism of Lie algebras*

$$M \otimes N \cong (M \otimes_\Lambda N)/D(M, N),$$

where $D(M, N)$ is the Λ-submodule of $M \otimes_\Lambda N$ generated by the elements

$$\begin{aligned}
&[m, m'] \otimes n - m \otimes ({}^{m'}n) + m' \otimes ({}^m n),\\
&m \otimes [n, n'] - ({}^{n'}m) \otimes n + ({}^n m) \otimes n',\\
&({}^n m) \otimes ({}^m n),\\
&({}^n m) \otimes ({}^{m'}n') + ({}^{n'}m') \otimes ({}^m n),\\
&[{}^n m, {}^{n'}m'] \otimes ({}^{m''}n'') + [{}^{n'}m', {}^{n''}m''] \otimes ({}^m n) + [{}^{n''}m'', {}^n m] \otimes ({}^{m'}n')
\end{aligned}$$

for all $m,\ m',\ m'' \in M$ and $n,\ n',\ n'' \in N$.

Proof. Let us introduce in the Λ-module $(M \otimes_\Lambda N)/D(M, N)$ a Lie structure by the following formula:

$$[m \otimes n, m' \otimes n'] = -({}^n m) \otimes ({}^{m'}n').$$

To show that this multiplication can be extended from generators to any elements of $(M \otimes_\Lambda N)/D(M, N)$, we must verify its compatibility with the defining relations of $(M \otimes_\Lambda N)/D(M, N)$, which is routine and will be omitted. Now it is easy to see the required isomorphism of Lie algebras. $\square$

The interesting properties of the tensor product of Lie algebras, in particular its compatibility with the direct limits and the right exactness, will be given.

Proposition 5.3. *Let $\{M_\alpha,\ \phi_\alpha^\beta,\ \alpha \leq \beta\}$ be a direct system of Lie algebras. Let N be a Lie algebra, and let for every α the Lie algebras M_α, N act on each other and the homomorphisms ϕ_α^β preserve the actions. Then there is a natural isomorphism of Lie algebras*

$$\Big(\varinjlim_\alpha \{M_\alpha\}\Big) \otimes N \cong \varinjlim_\alpha \{M_\alpha \otimes N\}.$$

Proof. We only define the actions of $\varinjlim_\alpha \{M_\alpha\}$ and N on each other by the following way:

$$^{|m_\alpha|}n = {}^{m_\alpha}n \quad \text{and} \quad {}^n|m_\alpha| = |{}^n m_\alpha|$$

for all $m_\alpha \in M_\alpha$, $n \in N$, and the natural isomorphism of Lie algebras

$$f : \Big(\varinjlim_\alpha \{M_\alpha\}\Big) \otimes N \longrightarrow \varinjlim_\alpha \{M_\alpha \otimes N\} \quad \text{by} \quad f(|m_\alpha| \otimes n) = |m_\alpha \otimes n|.$$

The details of the proof are straightforward. $\square$

Proposition 5.4. *Assume that* $0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$ *is a short exact sequence of Lie algebras, and* N *is an arbitrary Lie algebra acting on* M', M *and* M''*; the Lie algebras* M', M, M'' *act on* N *and* ϕ, ψ *preserve these actions. Then there is an exact sequence of Lie algebras*

$$M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \longrightarrow 0.$$

Proof. Similarly to the proof of [45, Proposition 9], since it does not use compatibility conditions (5.1). ☐

2. Construction of Non-Abelian Homology

We begin this section by recalling the well-known construction of the free Lie algebra on some Λ-module.

Let M be a Λ-module. Let $\mathcal{A}_1(M) = M$, $\mathcal{A}_k(M) = \sum\limits_{0<i<k} \mathcal{A}_i(M) \otimes_\Lambda \mathcal{A}_{k-i}(M)$ and $\mathcal{A}(M) = \sum\limits_{0<k} \mathcal{A}_k(M)$. The inclusion maps $\mathcal{A}_i(M) \otimes_\Lambda \mathcal{A}_k(M) \longrightarrow \mathcal{A}_{i+k}(M)$ give rise to a nonassociative multiplication on $\mathcal{A}(M)$, turning it into an algebra over Λ.

Let $\mathcal{B}(M)$ be the two-sided ideal of $\mathcal{A}(M)$ generated by the elements

$$xx \quad \text{and} \quad x(yz) + y(zx) + z(xy),$$

for all $x, y, z \in \mathcal{A}(M)$.

We obtain the Lie algebra $\mathcal{F}(M) = \mathcal{A}(M)/\mathcal{B}(M)$ which is the free Lie algebra on the Λ-module M satisfying the following universal property: there is a natural Λ-homomorphism $i : M \longrightarrow \mathcal{F}(M)$ such that for any Lie algebra L and a Λ-homomorphism $\alpha : M \longrightarrow L$ there exists a unique Lie homomorphism $\kappa : \mathcal{F}(M) \longrightarrow L$ such that $\kappa i = \alpha$.

Let N be a Lie algebra and $\alpha : M \longrightarrow \mathrm{Der}(N)$ a Λ-homomorphism, where $\mathrm{Der}(N)$ is the Lie algebra of derivations of N. Then there exists a unique Lie homomorphism $\kappa : \mathcal{F}(M) \longrightarrow \mathrm{Der}(N)$ such that $\kappa i = \alpha$, which means there is an action of the Lie algebra $\mathcal{F}(M)$ on the Lie algebra N.

Now if in addition M is an N-module, then the module action of N on M yields an N-module structure on $\mathcal{A}_k(M)$: if $x \otimes y \in \mathcal{A}_i(M) \otimes_\Lambda \mathcal{A}_{k-i}(M)$ and $n \in N$ then, inductively, we define

$$n(x \otimes y) = nx \otimes y + x \otimes ny,$$

and this extends linearly to an action of n on an arbitrary element of $\mathcal{A}_k(M)$. The action of N on $\mathcal{A}_k(M)$ extends linearly to an action of N on $\mathcal{A}(M)$, making $\mathcal{A}(M)$ an N-module. Since $\mathcal{B}(M)$ is N-invariant, the action of N on $\mathcal{A}(M)$ induces a Lie action of N on $\mathcal{F}(M)$.

Let $\mathfrak{A}_N$ denote, for a fixed Lie algebra N, the category whose objects are all Lie algebras M together with an action of M on N by derivations of N and an action of N on M by derivations of M. Morphisms in the category $\mathfrak{A}_N$ are all Lie homomorphisms $\alpha : M \longrightarrow M'$ preserving the actions, namely $\alpha({}^n m) = {}^n\alpha(m)$ and ${}^m n = {}^{\alpha(m)}n$ for all $m \in M$, $n \in N$.

Let $\mathcal{F} : \mathfrak{A}_N \longrightarrow \mathfrak{A}_N$ be the endofunctor defined as follows: for an object M of $\mathfrak{A}_N$, let $\mathcal{F}(M)$ denote the free Lie algebra on the underlying Λ-module M with the above-mentioned actions of N on $\mathcal{F}(M)$ and $\mathcal{F}(M)$ on N; for a morphism $\alpha : M \longrightarrow M'$ of $\mathfrak{A}_N$, let $\mathcal{F}(\alpha)$ be the canonical Lie homomorphism from $\mathcal{F}(M)$ to $\mathcal{F}(M')$ induced by α.

Let $\tau : \mathcal{F} \longrightarrow 1_{\mathfrak{A}_N}$ be the obvious natural transformation and let $\delta : \mathcal{F} \longrightarrow \mathcal{F}^2$ be the natural transformation induced for every $M \in \mathfrak{A}_N$ by the natural inclusion of Λ-modules $M \longrightarrow \mathcal{F}(M)$. We obtain the cotriple $\mathbb{F} = (\mathcal{F}, \tau, \delta)$. Let $\mathbb{P}$ be the projective class in the category $\mathfrak{A}_N$ induced by the cotriple $\mathbb{F}$. It is easy to see that in the category $\mathfrak{A}_N$ there exist finite limits. Therefore every object M of the category $\mathfrak{A}_N$ has a $\mathbb{P}$-projective pseudo-simplicial resolution (F_*, d_0^0, M) (see Chap. 1, Sec. 1.2).

Let $\mathcal{T} : \mathfrak{A}_N \longrightarrow \mathcal{L}ie$ be a covariant functor. Applying $\mathcal{T}$ dimension-wise to F_* yields the simplicial Lie algebra $\mathcal{T}F_*$. Define the *kth derived functor* $\mathcal{L}_k^{\mathbb{P}}\mathcal{T} : \mathfrak{A}_N \longrightarrow \mathcal{L}ie$, $k \geq 0$, of the functor $\mathcal{T}$ relative to the projective class $\mathbb{P}$ as the kth homotopy of $\mathcal{T}F_*$. Note that $\mathcal{L}_k^{\mathbb{P}}\mathcal{T}(M)$, $k \geq 1$, is an Abelian Lie algebra and will be thought as a Λ-module. Hence, forgetting the Lie algebra structure, we see that Proposition 1.10 implies that there is an isomorphism of functors

$$\mathcal{L}_k^{\mathbb{P}}\mathcal{T} \cong \mathcal{L}_k^{\mathbb{F}}\mathcal{T}, \quad k \geq 0,$$

where $\mathcal{L}_k^{\mathbb{F}}\mathcal{T}$ is kth cotriple derived functor of the functor $\mathcal{T}$.

The next lemma is useful. The proof is easy and is omitted.

Lemma 5.5. *A morphism $\alpha : M \longrightarrow M'$ of the category $\mathfrak{A}_N$ is a $\mathbb{P}$-epimorphism if and only if there exists a Λ-linear splitting $\beta : M' \longrightarrow M$ that preserves the actions of N, i.e., $\beta({}^n m) = {}^n\beta(m)$ for all $m \in M$, $n \in N$.*

The non-Abelian tensor product of Lie algebras defines a covariant functor $-\otimes N$ from the category $\mathfrak{A}_N$ to the category $\mathcal{L}ie$. Consider the left derived functors $\mathcal{L}_k^{\mathbb{P}}(-\otimes N)$, $k \geq 0$, of the functor $-\otimes N$ relative to the projective class $\mathbb{P}$.

Proposition 5.6. *Let M be a Lie algebra and N a module over the Lie algebra M. Then there are natural isomorphisms*

$$\mathcal{L}_k^{\mathbb{P}}(-\otimes N)(M) \cong H_{k+1}(M,N), \quad k \geq 1,$$
$$\operatorname{Ker}\nu \cong H_1(M,N), \quad \operatorname{Coker}\nu \cong H_0(M,N),$$

where N is regarded as an Abelian Lie algebra acting trivially on M and $\nu : M \otimes N \longrightarrow N$ is a Lie homomorphism given by $\nu(m \otimes n) = {}^m n$, $m \in M$, $n \in N$.

Proof. Let $\mathcal{L}ie_M$ denote the category of Lie algebras over M, and $\operatorname{Diff}_M : \mathcal{L}ie_M \longrightarrow U(M) - mod$ (category of $U(M)$-modules) a functor given by

$$\operatorname{Diff}_M(W) = I(W) \otimes_{U(W)} U(M),$$

where $U(M)$ and $U(W)$ are the universal enveloping algebras of M and W respectively and $I(W)$ is the augmentation ideal. By [25, Proposition 13], $\mathcal{L}_*^{\mathbb{F}}(-\otimes N)(M)$ are isomorphic to the values of the non-Abelian left derived functors of the functor $\operatorname{Diff}_M(-) \otimes_{U(M)} N : \mathcal{L}ie_M \longrightarrow \Lambda - \mathrm{mod}$ (category of Λ-modules) for the object 1_M of the category $\mathcal{L}ie_M$, which give the classical homology $H_*(M,N)$ of Lie algebras with the usual dimension shift, similarly to the cases of group (co)homology and Hochschild (co)homology described as cotriple (co)homology [4, 5]. $\square$

Using this proposition we make the following

Definition 5.7. Let M and N be Lie algebras acting on each other. Define *the non-Abelian homology* of M with coefficients in N by setting

$$H_k(M,N) = \mathcal{L}_{k-1}^{\mathbb{P}}(-\otimes N)(M), \quad k \geq 2,$$
$$H_1(M,N) = \operatorname{Ker}\nu, \quad H_0(M,N) = \operatorname{Coker}\nu,$$

where $\nu : M \otimes N \longrightarrow N/H$, $\nu(m \otimes n) = |{}^m n|$, and H is the ideal of the Lie algebra N generated by the elements ${}^{({}^n m)}n' - [n', {}^m n]$ for all $m \in M$, $n, n' \in N$.

Remark 5.8.

(a) It is clear that $H_k(M,N)$, $k \geq 2$, are only Λ-modules, while $H_1(M,N)$ and $H_0(M,N)$ are Lie algebras. If the actions of M and N satisfy the compatibility conditions (5.1), then $H_1(M,N)$ is also an Abelian Lie algebra.

(b) Let N be a crossed M-module; then $H_0(M,N)$ and $H_1(M,N)$ coincides with zero and first non-Abelian homology Λ-modules of the Lie algebra M with coefficients in the crossed M-module N introduced by Guin [55].

We can define another non-Abelian homology theory of Lie algebras using the non-Abelian left derived functors of the non-Abelian tensor product relative to the cotriple over sets which coincides with our theory for Lie algebras being free Λ-modules.

3. Some Properties of Non-Abelian Homology

In this section we give some functorial properties of the non-Abelian homology of Lie algebras.

Now several long exact non-Abelian homology sequences with respect to both variables will be given.

Theorem 5.9. *Let $\alpha : N \longrightarrow N'$ be a surjective Lie homomorphism, M an arbitrary Lie algebra acting on N and N' which act on M and α preserve the actions. Then there is a long exact sequence of non-Abelian homology*

$$\begin{aligned} \cdots \longrightarrow H_3(M,N') \xrightarrow{\delta_3} H_2(M,N,N') \xrightarrow{j_2} H_2(M,N) \xrightarrow{i_2} H_2(M,N') \xrightarrow{\delta_2} \\ \xrightarrow{\delta_2} H_1(M,N,N') \xrightarrow{j_1} H_1(M,N) \xrightarrow{i_1} H_1(M,N') \xrightarrow{\delta_1} \\ \xrightarrow{\delta_1} H_0(M,N,N') \xrightarrow{j_0} H_0(M,N) \xrightarrow{i_0} H_0(M,N') \longrightarrow 0\,, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} &H_k(M,N,N') = \pi_{k-1}(\mathrm{Ker}(1_{\mathcal{F}^*(M)} \otimes \alpha)), \quad k \geq 2,\\ &H_1(M,N,N') = \frac{\left\{ \mathrm{Ker}(1_{\mathcal{F}^1(M)} \otimes \alpha) \cap (d_0^0 \otimes 1_N)^{-1}(\mathrm{Ker}(1_M \otimes \alpha) \cap \mathrm{Ker}\,\nu)\right\}}{(d_1^1 \otimes 1_N)(\mathrm{Ker}(1_{\mathcal{F}^2(M)} \otimes \alpha) \cap \mathrm{Ker}\,(d_0^1 \otimes 1_N))},\\ &H_0(M,N,N') = \mathrm{Ker}\,\widetilde{\alpha}/\nu(\mathrm{Ker}(1_M \otimes \alpha)), \end{aligned}$$

$(\mathcal{F}^*(M), d_0^0, M)$ is the $\mathbb{F}$ cotriple resolution of the object M of the category $\mathfrak{A}_N$, and $\widetilde{\alpha} : N/H \longrightarrow N'/H'$ is the homomorphism induced by α.

Proof. The following commutative diagram of Lie algebras with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & \mathrm{Ker}(1_{\mathcal{F}^2(M)} \otimes \alpha) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Ker}(1_{\mathcal{F}^1(M)} \otimes \alpha) \arrow[r] \arrow[d] & \mathrm{Ker}(1_M \otimes \alpha) \arrow[r] \arrow[d] & \mathrm{Ker}\,\widetilde{\alpha} \arrow[d] \\
\cdots \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & \mathcal{F}^2(M) \otimes N \arrow[r, shift left, "d_0^1 \otimes 1_N"] \arrow[r, shift right, "d_1^1 \otimes 1_N"'] \arrow[d, "1_{\mathcal{F}^2(M)} \otimes \alpha"] & \mathcal{F}^1(M) \otimes N \arrow[r, "d_0^0 \otimes 1_N"] \arrow[d, "1_{\mathcal{F}^1(M)} \otimes \alpha"] & M \otimes N \arrow[r, "\nu"] \arrow[d, "1_M \otimes \alpha"] & N/H \arrow[d, "\widetilde{\alpha}"] \\
\cdots \arrow[r, shift left] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & \mathcal{F}^2(M) \otimes N' \arrow[r, shift left, "d_0^1 \otimes 1_{N'}"] \arrow[r, shift right, "d_1^1 \otimes 1_{N'}"'] \arrow[d] & \mathcal{F}^1(M) \otimes N' \arrow[r, "d_0^0 \otimes 1_{N'}"] \arrow[d] & M \otimes N' \arrow[r, "\nu'"] \arrow[d] & N'/H' \arrow[d] \\
 & 0 & 0 & 0 & 0
\end{tikzcd}$$

immediately induces the exactness of the sequence

$$\cdots \longrightarrow H_3(M,N') \xrightarrow{\delta_3} H_2(M,N,N') \xrightarrow{j_2} H_2(M,N) \xrightarrow{i_2} H_2(M,N').$$

Using the "snake lemma" in the last two columns of this diagram we have the following exact sequence:

$$H_1(M,N) \xrightarrow{i_1} H_1(M,N') \xrightarrow{\delta_1} H_0(M,N,N') \xrightarrow{j_0} H_0(M,N) \xrightarrow{i_0} H_0(M,N') \longrightarrow 0\ .$$

We define the homomorphisms j_1 and δ_2 by

$$j_1(|x|) = (d_0^0 \otimes 1_N)(x)$$

for $x \in \{\mathrm{Ker}(1_{\mathcal{F}^1(M)} \otimes \alpha) \cap (d_0^0 \otimes 1_N)^{-1}(\mathrm{Ker}(1_M \otimes \alpha) \cap \mathrm{Ker}\,\nu)\}$ and

$$\delta_2(|y|) = \big|(d_1^1 \otimes 1_N)(y') - (d_0^1 \otimes 1_N)(y')\big|$$

for $y \in \mathrm{Ker}\,(d_1^1 \otimes 1_{N'}) \cap \mathrm{Ker}\,(d_0^1 \otimes 1_{N'})$, where $y' \in \mathcal{F}^2(M) \otimes N$ such that $(1_{\mathcal{F}^2(M)} \otimes \alpha)(y') = y$. It is easy to verify that j_1 and δ_2 are well defined and that the sequence (5.2) is exact in terms $H_2(M,N')$, $H_1(M,N,N')$ and $H_1(M,N)$ by virtue Proposition 5.4. □

Remark 5.10.

(a) If the actions of M and N satisfy the compatibility conditions (5.1), then $H_0(M,N,N') = H_0(M,N'')$, where $N'' = \mathrm{Ker}\,\alpha$.

(b) Let $0 \longrightarrow (N'',0) \longrightarrow (N,\mu) \longrightarrow (N',\nu) \longrightarrow 0$ be an exact sequence of crossed M-modules. Thanks to the result in [55], there is a six-term exact non-Abelian homology sequence

$$\begin{aligned} &H_1(M,N'') \longrightarrow H_1(M,N) \longrightarrow H_1(M,N') \longrightarrow \\ &\qquad \longrightarrow H_0(M,N') \longrightarrow H_0(M,N) \longrightarrow H_0(M,N') \longrightarrow 0. \end{aligned} \tag{5.3}$$

The first five terms of the sequence (5.3) coincide with the first five terms of the sequence (5.2) and there is a natural homomorphism of Λ-modules

$$H_1(M,N'') \longrightarrow H_1(M,N,N').$$

Let

$$\mathcal{D} = \begin{tikzcd} & M_2 \arrow[d, "\alpha_2"] \\ M_1 \arrow[r, "\alpha_1"'] & M \end{tikzcd} \tag{5.4}$$

be a diagram in the category $\mathfrak{A}_N$ with surjective α_1. Let $L_*(\mathcal{D},N)$ be the pullback of the induced diagram

$$\begin{tikzcd} & \mathcal{F}^*(M_2) \otimes A \arrow[d, "\mathcal{F}^*(\alpha_2)\otimes 1_N"] \\ \mathcal{F}^*(M_1) \otimes N \arrow[r, "\mathcal{F}^*(\alpha_1)\otimes 1_N"'] & \mathcal{F}^*(M) \otimes N \end{tikzcd}\ .$$

Define $H_k(\mathcal{D},N) = \pi_{k-1}L_*(\mathcal{D},N)$, $k \geq 2$.

Theorem 5.11 (Mayer–Vietoris sequence)**.** *For any diagram* (5.4) *there is a long exact sequence of* Λ*-modules*

$$\cdots \longrightarrow H_{k+1}(M,N) \longrightarrow H_k(\mathcal{D},N) \longrightarrow H_k(M_1,N) \oplus H_k(M_2,N) \longrightarrow H_k(M,N) \longrightarrow$$

$$\longrightarrow \cdots \longrightarrow H_2(\mathcal{D},N) \longrightarrow H_2(M_1,N)\oplus H_2(M_2,N) \longrightarrow H_2(M,N) \longrightarrow \pi_0 L_*(\mathcal{D},N) \longrightarrow$$
$$\longrightarrow \pi_0(\mathcal{F}^*(M_1)\otimes N)\oplus \pi_0(\mathcal{F}^*(M_2)\otimes N) \longrightarrow \pi_0(\mathcal{F}^*(M)\otimes N) \longrightarrow 0. \tag{5.5}$$

Proof. There is a commutative diagram of simplicial Lie algebras with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_* \arrow[r, "\sigma_*"] \arrow[d, equal] & L_*(\mathcal{D},N) \arrow[r, "p_*"] \arrow[d, "q_*"] & \mathcal{F}^*(M_2)\otimes N \arrow[r] \arrow[d, "\mathcal{F}^*(\alpha_2)\otimes 1_N"] & 0 \\
0 \arrow[r] & \mathcal{I}_* \arrow[r] & \mathcal{F}^*(M_1)\otimes N \arrow[r, "\mathcal{F}^*(\alpha_1)\otimes 1_N"] & \mathcal{F}^*(M)\otimes N \arrow[r] & 0
\end{tikzcd},$$

where $\mathcal{I}_* = \mathrm{Ker}(\mathcal{F}^*(\alpha_1)\otimes 1_N)$. Hence we have the following commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(\mathcal{F}^*(M_2)\otimes N) \arrow[r] \arrow[d] & \pi_0(\mathcal{I}_*) \arrow[r] \arrow[d, equal] & \pi_0 L_*(\mathcal{D},N) \arrow[r] \arrow[d] & \pi_0(\mathcal{F}^*(M_2)\otimes N) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \pi_1(\mathcal{F}^*(M)\otimes N) \arrow[r, "\delta_1"'] & \pi_0(\mathcal{I}_*) \arrow[r] & \pi_0(\mathcal{F}^*(M_1)\otimes N) \arrow[r] & \pi_0(\mathcal{F}^*(M)\otimes N) \arrow[r] & 0
\end{tikzcd}. \tag{5.6}$$

The connecting homomorphism $\pi_k(\mathcal{F}^*(M)\otimes N) \longrightarrow \pi_{k-1}L_*(\mathcal{D},N)$, $k\geq 1$, is the composite map $\pi_{k-1}(\sigma_*)\delta_k$. The homomorphism $\pi_k(L_*(\mathcal{D},N)) \longrightarrow \pi_k(\mathcal{F}^*(M_1)\otimes N)\oplus \pi_k(\mathcal{F}^*(M_2)\otimes N)$, $k\geq 0$, is induced by $\pi_k(q_*)$ and $\pi_k(p_*)$. The homomorphism $\pi_k(\mathcal{F}^*(M_1)\otimes N)\oplus\pi_k(\mathcal{F}^*(M_2)\otimes N) \longrightarrow \pi_k(\mathcal{F}^*(M)\otimes N)$, $k\geq 0$, is given by $\pi_k(\mathcal{F}^*(\alpha_1)\otimes 1_N) - \pi_k(\mathcal{F}^*(\alpha_2)\otimes 1_N)$. To get the exactness of the sequence (5.5), it remains to apply the diagram (5.6). □

Corollary 5.12. *There is a long exact sequence of the non-Abelian homology of Lie algebras with respect to the first variable.*

Proof. It follows by applying Theorem 5.11 for $M_2 = 0$. □

Let us consider $H_1(-,N)$ as a functor from the category $\mathfrak{A}_N$ to the category $\mathcal{L}ie$ of Lie algebras and its non-Abelian left derived functors $\mathcal{L}^{\mathbb{F}}_k(H_1(-,N))$ relative to the cotriple $\mathbb{F}$.

Theorem 5.13. *There is a natural isomorphism*

$$H_k(-,N) \cong \mathcal{L}^{\mathbb{F}}_{k-1}(H_1(-,N)), \quad k\geq 1.$$

Proof. It follows from the long exact homotopy sequence of the following short exact sequence of simplicial Lie algebras

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\cdots \ \vdots & H_1(\mathcal{F}^3(M),N) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & H_1(\mathcal{F}^2(M),N) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & H_1(\mathcal{F}^1(M),N) \arrow[d] \\
\cdots \ \vdots & \mathcal{F}^3(M)\otimes N \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\nu"] & \mathcal{F}^2(M)\otimes N \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\nu"] & \mathcal{F}^1(M)\otimes N \arrow[d, "\nu"] \\
\cdots \ \vdots & \mathrm{Im}\,\nu \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \mathrm{Im}\,\nu \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Im}\,\nu \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd},$$

where the bottom simplicial Lie algebra is a constant simplicial Lie algebra and $\nu : M\otimes N \longrightarrow N/H$ is a homomorphism given in Definition 5.7. □

Proposition 5.14. *Let $\{M_\alpha,\ \phi_\alpha^\beta,\ \alpha\le\beta\}$ and $\{N_\alpha,\ \psi_\alpha^\beta,\ \alpha\le\beta\}$ be direct systems of Lie algebras. Let M and N be Lie algebras and for every α the Lie algebras M_α, N and M, N_α act on each other and the homomorphisms ϕ_α^β, ψ_α^β preserve the actions. Then there are natural isomorphisms*

$$H_k\Big(M,\varinjlim_\alpha\{N_\alpha\}\Big)\cong\varinjlim_\alpha\{H_k(M,N_\alpha)\},\quad k\ge 0,$$
$$H_k\Big(\varinjlim_\alpha\{M_\alpha\},N\Big)\cong\varinjlim_\alpha\{H_k(M_\alpha,N\}),\quad k\ge 0.$$

Proof. The constructions of both isomorphisms are similar, and only the first one will be given. In fact, for $k=0$ the homomorphism

$$f: H_0\Big(M,\varinjlim_\alpha\{N_\alpha\}\Big)\longrightarrow\varinjlim_\alpha\big\{H_0(M,N_\alpha)\big\}$$

is defined by $f(|\{n_\alpha\}|)=\{|n_\alpha|\}$, $\{n_\alpha\}\in\varinjlim_\alpha\{N_\alpha\}$, and the homomorphism

$$g:\varinjlim_\alpha\big\{H_0(M,N_\alpha)\big\}\longrightarrow H_0\Big(M,\varinjlim_\alpha\{N_\alpha\}\Big)$$

is induced by the homomorphisms

$$g_\alpha: H_0(M,N_\alpha)\longrightarrow H_0\Big(M,\varinjlim_\alpha\{N_\alpha\}\Big),$$
$$|n_\alpha|\longmapsto|\{n_\alpha\}|.$$

It is easy to see that these homomorphisms are well defined and fg and gf are identity maps.

For $k=1$ the isomorphism

$$H_1\Big(M,\varinjlim_\alpha\{N_\alpha\}\Big)\xrightarrow{\ \cong\ }\varinjlim_\alpha\{H_1(M,N_\alpha)\}$$

is induced by the isomorphism $M\otimes\varinjlim_\alpha\{N_\alpha\}\xrightarrow{\ \cong\ }\varinjlim_\alpha\{M\otimes N_\alpha\}$ (see Proposition 5.3).

Finally, for $k\ge 2$ the required isomorphism can be obtained applying the well-known assertions

$$\mathcal{F}\Big(\varinjlim_\alpha\{N_\alpha\}\Big)\cong\varinjlim_\alpha\{\mathcal{F}(N_\alpha)\}$$

and

$$\pi_k\Big(\varinjlim_\alpha\{D_\alpha\}\Big)\cong\varinjlim_\alpha\{\pi_k(D_\alpha)\},$$

where D_α is a simplicial Lie algebra. $\square$

We end this section with explicit descriptions of the second and the third non-Abelian homology of Lie algebras using Čech derived functors.

Let M and N be Lie algebras acting on each other. Let F be an object of the projective class $\mathbb{P}$ and $F\xrightarrow{\ \varepsilon\ }M$ be a $\mathbb{P}$-epimorphism in the category $\mathfrak{A}_N$. Let us consider the *augmented Čech resolution* $(\check{C}(\varepsilon)_*,\varepsilon,M)$ of the object M in the category $\mathfrak{A}_N$ (see Chap. 1, Sec. 2.1), where

$$\check{C}(\varepsilon)_k=\underbrace{F\underset{M}{\times}\cdots\underset{M}{\times}F}_{(k+1)\text{-times}},\quad k\ge 0,$$
$$d_i^k(x_0,\ldots,x_k)=(x_0,\ldots,\widehat{x_i},\ldots,x_k),\quad k\ge 1,\quad 0\le i\le k,$$
$$s_i^k(x_0,\ldots,x_k)=(x_0,\ldots,x_i,x_i,\ldots,x_k),\quad k\ge 0,\quad 0\le i\le k.$$

Applying the functor $-\otimes N$ dimension-wise to the Čech resolution of M yields the augmented simplicial Lie algebra $(\check{C}(\varepsilon)_* \otimes N, \varepsilon \otimes 1_N, M \otimes N)$.

Theorem 5.15.

(i) *There is an isomorphism of* Λ*-modules*

$$H_2(M,N) \cong \big\{ \operatorname{Ker}(d_0^1 \otimes 1_N) \cap \operatorname{Ker}(d_1^1 \otimes 1_N) \big\} \big/ \big[\operatorname{Ker}(d_0^1 \otimes 1_N), \operatorname{Ker}(d_1^1 \otimes 1_N) \big];$$

(ii) *there is an epimorphism of* Λ*-modules*

$$H_3(M,N) \longrightarrow \bigcap_{i=0}^{2} \operatorname{Ker}(d_i^2 \otimes 1_N) \Big/ \sum_{I,J} [K_I, K_J],$$

where $\varnothing \neq I, J \subset \{0,1,2\}$ *such that* $I \cup J = \{0,1,2\}$ *and* $K_I = \bigcap_{i\in I} \operatorname{Ker}(d_i^2 \otimes 1_N)$, $K_J = \bigcap_{j\in J} \operatorname{Ker}(d_j^2 \otimes 1_N)$.

Proof. We have an isomorphism

$$H_2(M,N) = \mathcal{L}_1^{\mathbb{P}}(-\otimes N)(M) \cong \pi_1(\check{C}(\varepsilon)_* \otimes N) \tag{5.7}$$

and an epimorphism

$$H_3(M,N) = \mathcal{L}_2^{\mathbb{P}}(-\otimes N)(M) \longrightarrow \pi_2(\check{C}(\varepsilon)_* \otimes N) \tag{5.8}$$

(see e.g. [60, Theorem 2.39(ii)]).

The Lie algebra $\check{C}(\varepsilon)_2 \otimes N$ coincides with its ideal generated by the degenerate elements. In fact, for any $(x,y,z) \otimes n \in \check{C}(\varepsilon)_2 \otimes N$ there is an equality

$$\begin{aligned}(x,y,z)\otimes n &= (x,x,x)\otimes n + (0, y-x, y-x)\otimes n + (0,0,z-y)\otimes n \\ &= (s_0^1 \otimes 1_N)((x,x)\otimes n) + (s_1^1 \otimes 1_N)((0,x-y)\otimes n) + (s_0^1 \otimes 1_N)((0,z-y)\otimes n).\end{aligned}$$

It is easy to verify the similar fact for $\check{C}(\varepsilon)_3 \otimes N$. Then by [1, Theorem 1]

$$\begin{aligned}\operatorname{Im}\partial_2 &= \big[\operatorname{Ker}(d_0^1 \otimes 1_N), \operatorname{Ker}(d_1^1 \otimes 1_N) \big], \\ \operatorname{Im}\partial_3 &= \sum_{I,J} [K_I, K_J],\end{aligned}$$

where ∂_2 and ∂_3 are differentials of the Moore complex of $\check{C}(\varepsilon)_* \otimes N$. Hence the assertion follows from (5.7) and (5.8). □

4. Application to Cyclic Homology

In this section the relation of cyclic homology to Milnor cyclic homology of associative algebras is established in terms of the long exact non-Abelian homology sequence of Lie algebras.

It is well known from the result of Loday and Quillen [88, 91] that for a unital associative algebra A over a characteristic zero field Λ the cyclic homology $HC_{*-1}(A)$ of A is the primitive part of the Hopf algebra $H_*(\mathfrak{gl}(A), \Lambda)$ of the Lie algebra $\mathfrak{gl}(A)$ of matrices with coefficients in A. Analogously, the rational algebraic K-theory of A is the primitive part of the homology $H_*(GL(A), \mathbb{Q})$ of the general linear group $GL(A)$ of A, which makes one think that cyclic homology is an additive version of the algebraic K-theory.

The following stabilization result is also known [88, 91] for the homology of Lie algebra $\mathfrak{gl}_i(A)$ for any unital associative algebra A over a characteristic zero field Λ:

$$H_k(\mathfrak{gl}_i(A), \Lambda) \cong H_k(\mathfrak{gl}_{i+1}(A), \Lambda), \quad i \geq k,$$

and for the computation of the first obstruction to stability

$$\mathrm{Coker}(H_k(\mathfrak{gl}_{k-1}(A),\Lambda) \longrightarrow H_k(\mathfrak{gl}_k(A),\Lambda)) \cong HC^M_{k-1}(A),$$

where $HC^M_{*-1}(A)$ are the Milnor cyclic homlogy groups [88], which coincides with $\Omega^{*-1}_{A|\Lambda}/d\Omega^{*-2}_{A|\Lambda}$ for commutative A, where $\Omega^*_\Lambda A$ are the Kähler differentials forms of A. Similar results on the homology of the linear group $GL_i(A)$ of a ring A under certain conditions (see [54, 118])

$$H_k(GL_i(A),\mathbb{Z}) \cong H_k(GL_{i+1}(A),\mathbb{Z}), \quad i \geq k$$

and

$$\mathrm{Coker}(H_k(GL_{k-1}(A),\mathbb{Z}) \longrightarrow H_k(GL_k(A),\mathbb{Z})) \cong K^M_k(A),$$

where $K^M_*(A)$ denotes the Milnor K-theory of A, give one thought to consider Milnor cyclic homology as the additive version of the Milnor K-theory.

Using the non-Abelian group homology, we establish the relation of algebraic K-functor K_2 to Milnor K-functor K^M_2 for noncommutative local rings [53, 68]. Now we give an additive version of this result. In particular, the relation of the first cyclic homology HC_1 and the first Milnor cyclic homology HC^M_1 of unital associative algebras is expressed in terms of a long exact non-Abelian homology sequence of Lie algebras, which corrects and extends the six-term exact sequence of [55, Theorem 5.7].

First we introduce the definition of the first Milnor cyclic homology.

Definition 5.16. Let A be a unital associative Λ-algebra. The first Milnor cyclic homology $HC^M_1(A)$ of A is the quotient of $A \otimes_\Lambda A$ by the relations

$$\begin{aligned} a\otimes b + b\otimes a &= 0,\\ ab\otimes c - a\otimes bc + ca\otimes b &= 0,\\ a\otimes bc - a\otimes cb &= 0 \end{aligned}$$

for all $a,b,c \in A$.

Our definition of $HC^M_1(A)$ coincides with the definition in the sense of [88] when Λ is a field of characteristic $\neq 2$.

It is well known that the first cyclic homology $HC_1(A)$ of a unital associative Λ-algebra A is the kernel of the homomorphism of Λ-modules

$$\begin{aligned} A\otimes_\Lambda A/J(A) &\longrightarrow [A,A],\\ a\otimes b &\longmapsto ab-ba, \end{aligned}$$

where $[A,A]$ is the additive commutator of A and $J(A)$ is the submodule of $A\otimes_\Lambda A$ generated by the elements

$$\begin{aligned} &a\otimes b + b\otimes a,\\ &ab\otimes c - a\otimes bc + ca\otimes b \end{aligned}$$

for all $a,b,c\in A$.

It is clear that $HC^M_1(A)$ coincides with $HC_1(A)$ when A is commutative.

Given a unital associative (noncommutative) Λ-algebra A, consider A as the Lie algebra with the usual induced Lie structure $[a,b]=ab-ba$, $a,\ b\in A$. Denote by $V(A)$ the quotient Lie algebra of the non-Abelian tensor square $A\otimes A$ by the ideal generated by the elements

$$\begin{aligned} &a\otimes b + b\otimes a,\\ &ab\otimes c - a\otimes bc + ca\otimes b \end{aligned}$$

for all $a,b,c\in A$. We compile the results of [55] on the Lie algebra $V(A)$ into the following proposition.

Proposition 5.17. *Let A be a unital associative Λ-algebra.*

(i) *There is an action of the Lie algebra A on the Lie algebra $V(A)$ defined by the formula*

$$^{a'}(a \otimes b) = [a', a] \otimes b + a \otimes [a', b]$$

and a homomorphism $\mu : V(A) \longrightarrow A$ given by $a \otimes b \longmapsto [a, b]$ has the structure of a crossed A-module;

(ii) *There is a natural isomorphism of Λ-modules*

$$V(A) \cong A \otimes_\Lambda A/J(A);$$

(iii) *A acts trivially on $HC_1(A)$;*

(iv) *There is a short exact sequence of crossed A-modules of Lie algebras*

$$0 \longrightarrow HC_1(A) \longrightarrow V(A) \longrightarrow [A, A] \longrightarrow 0\ .$$

Proof. For the proof of (i) and (iii), see [55]. To prove (ii), we can show that $J(A) \supseteq D(A, A)$ and then examine similar arguments as in Proposition 5.2. The proof of (iv) is straightforward from (i)–(iii). □

We have the following assertion.

Theorem 5.18. *Let A be a unital associative (noncommutative) Λ-algebra. Then there is an exact sequence of Λ-modules*

$$\cdots \longrightarrow H_2(A, V(A), [A, A]) \longrightarrow H_2(A, V(A)) \longrightarrow H_2(A, [A, A]) \longrightarrow$$
$$\longrightarrow H_1(A, V(A), [A, A]) \longrightarrow H_1(A, V(A)) \longrightarrow H_1(A, [A, A]) \longrightarrow$$
$$\longrightarrow HC_1(A) \longrightarrow HC_1^M(A) \longrightarrow [A, A]/[A, [A, A]] \longrightarrow 0,$$

Proof. Proposition 5.17, Theorem 5.9, and Remark 5.10 yield the following long exact sequence of Λ-modules

$$\cdots \longrightarrow H_2(A, V(A), [A, A]) \longrightarrow H_2(A, V(A)) \longrightarrow H_2(A, [A, A]) \longrightarrow$$
$$\longrightarrow H_1(A, V(A), [A, A]) \longrightarrow H_1(A, V(A)) \longrightarrow H_1(A, [A, A]) \longrightarrow$$
$$\longrightarrow H_0(A, HC_1(A)) \longrightarrow H_0(A, V(A)) \longrightarrow H_0(A, [A, A]) \longrightarrow 0.$$

It is easy to see that

$$H_0(A, HC_1(A)) = HC_1(A),$$
$$H_0(A, [A, A]) = [A, A]/[A, [A, A]].$$

Moreover, $H_0(A, V(A)) = \operatorname{Coker} \nu$, where $\nu : A \otimes V(A) \longrightarrow V(A)$ is the Lie homomorphism given by $a \otimes (b \otimes c) = {}^a(b \otimes c)$. Calculations in the Lie algebra $V(A)$ [55, Lemma 5.4] say that

$$^a(b \otimes c) = a \otimes [b, c] = a \otimes bc - a \otimes cb, \quad a, b, c \in A.$$

Now we easily deduce that there is a natural isomorphism of Λ-modules

$$H_0(A, V(A)) \cong HC_1^M(A).$$

□

5. The Lie Algebra of Derivations

In the remaining two sections of this chapter we deal with the non-Abelian cohomology of Lie algebras. In particular, we construct the second non-Abelian cohomology of Lie algebras with coefficients in crossed modules extending Guin's low-dimensional non-Abelian cohomology of Lie algebras. For this we need to modify the Lie algebra of derivations introduced in [55].

We begin this section by introducing the extended notion of crossed modules of Lie algebras in the following sense.

Definition 5.19. Let P and R be Lie algebras acting on each other, and (M, μ) a (pre)crossed R-module. (M, μ) will be called a P-(pre)crossed R-module if the following conditions hold:

(i) ${}^{({}^r p)}r' = [r', {}^p r]$, $r, r' \in R$, $p \in P$;

(ii) P acts on M and μ is a P-equivariant Lie homomorphism, i.e.,

$$\mu({}^p m) = {}^p\mu(m), \quad m \in M, \quad p \in P;$$

(iii) ${}^{({}^p r)}m = {}^p({}^r m) - {}^r({}^p m) = -{}^{({}^r p)}m$, $r \in R$, $p \in P$, $m \in M$.

It is easy to see that any (pre)crossed P-module in a natural way can be thought as a P-(pre)crossed P-module, P acting on itself by Lie multiplication.

A morphism $f : (M, \mu) \longrightarrow (N, \nu)$ of P-(pre)crossed R-modules is a morphism of (pre)crossed R-modules such that $f({}^p m) = {}^p f(m)$, $p \in P$, $m \in M$.

Definition 5.20. Let (M, μ) be a P-crossed R-module. Denote by $\mathrm{Der}(P, (M, \mu))$ the set of pairs (γ, r), where $\gamma : P \longrightarrow M$ is a crossed homomorphism (or derivation), which means that γ is a Λ-homomorphism satisfying the equality

$$\gamma([p, p']) = {}^p\gamma(p') - {}^{p'}\gamma(p), \quad p, p' \in P,$$

and r is an element of R such that

$$\mu\gamma(p) = -{}^p r, \quad p \in P. \tag{5.9}$$

This set is called the set of derivations from P to (M, μ).

We introduce on $Der(P, (M, \mu))$ the following operations:

$$(\gamma, r) + (\gamma', s) = (\gamma + \gamma', r + s),$$
$$\lambda(\gamma, r) = (\lambda\gamma, \lambda r),$$
$$\big[(\gamma, r), (\gamma', s)\big] = (\gamma * \gamma', [r, s]),$$

for all $(\gamma, r), (\gamma', s) \in Der(P, (M, \mu))$ and $\lambda \in \Lambda$, where $\gamma * \gamma'$ is given by $(\gamma * \gamma')(p) = \gamma({}^s p) - \gamma'({}^r p)$, $p \in P$.

Proposition 5.21. *Under the aforementioned operations* $\mathrm{Der}(P, (M, \mu))$ *becomes a Lie algebra.*

Proof. We only show that $(\gamma * \gamma', [r, s]) \in \mathrm{Der}(P, (M, \mu))$. First, we prove that $\gamma * \gamma'$ is a crossed homomorphism. In fact,

$$(\gamma * \gamma')([p, q]) = \gamma({}^s[p, q]) - \gamma'({}^r[p, q]) = \gamma([{}^s p, q]) + \gamma([p, {}^s q]) - \gamma'([{}^r p, q]) - \gamma'([p, {}^r q]) =$$
$$= {}^{({}^s p)}\gamma(q) - {}^q\gamma({}^s p) + {}^p\gamma({}^s q) - {}^{({}^s q)}\gamma(p) - {}^{({}^r p)}\gamma'(q) + {}^q\gamma'({}^r p) - {}^p\gamma'({}^r q) + {}^{({}^r q)}\gamma'(p).$$

On the other hand,

$${}^p(\gamma * \gamma')(q) - {}^q(\gamma * \gamma')(p) = {}^p\gamma({}^s q) - {}^p\gamma'({}^r q) - {}^q\gamma({}^s p) + {}^q\gamma'({}^r p).$$

Moreover, using (iii) of Definition 5.19 and (5.9) we have

$$^{(^s p)}\gamma(q) - {}^{(^s q)}\gamma(p) - {}^{(^r p)}\gamma'(q) + {}^{(^r q)}\gamma'(p) = 0,$$

implying $(\gamma * \gamma')([p, q]) = {}^p(\gamma * \gamma')(q) - {}^q(\gamma * \gamma')(p)$.

Further, by (i) of Definition 5.19 we have

$$\mu(\gamma * \gamma')(p) = \mu\gamma(^s p) - \mu\gamma'(^r p) = -^{(^s p)}r + {}^{(^r p)}s = -[r, {}^p s] + [s, {}^p r] = -^p[r, s].$$

Thus $(\gamma * \gamma', [r, s]) \in \mathrm{Der}(P, (M, \mu))$.

The details are easy to verify and is omitted from the text. $\square$

Remark 5.22. Let (N, ν) and (M, μ) be precrossed and crossed R-modules respectively. Then (M, μ) is an N-crossed R-module induced by ν, and $\mathrm{Der}(N, (M, \mu))$ coincides with the Lie algebra $\mathrm{Der}_R(N, M)$ defined in [55]. In particular, it coincides with the Lie algebra $\mathrm{Der}_R(R, M)$ from [55] when (M, μ) is viewed as an R-crossed R-module.

At the same time, assume that (M, μ) is a P-crossed R-module and a P'-crossed R-module and $f : P' \longrightarrow P$ is a Lie homomorphism such that

$$^{f(p')}m = {}^{p'}m, \quad {}^{f(p')}r = {}^{p'}r, \quad p' \in P', \quad r \in R, \quad m \in M.$$

Then there is a Λ-homomorphism

$$\widetilde{f} : \mathrm{Der}(P, (M, \mu)) \longrightarrow \mathrm{Der}(P', (M, \mu))$$

given by $\widetilde{f}(\gamma, r) = (\gamma f, r)$, $(\gamma, r) \in \mathrm{Der}(P, (M, \mu))$. If, in addition, f satisfy the condition

$$f(^r p') = {}^r f(p'), \quad p' \in P', \quad r \in R,$$

then $\widetilde{f}$ is a Lie homomorphism.

Now assume that P and R act on each other compatibly, i.e., the conditions (5.1) hold. Then there is an action of P on $\mathrm{Der}(P, (M, \mu))$ defined by

$$^p(\gamma, r) = (\gamma', {}^p r), \quad p \in P, \quad (\gamma, r) \in \mathrm{Der}(P, (M, \mu)), \tag{5.10}$$

where $\gamma'(q) = {}^q\gamma(p)$, $q \in P$. There is also an action of R on $\mathrm{Der}(P, (M, \mu))$ given by

$$^s(\gamma, r) = (\gamma'', [s, r]), \quad s \in R, \quad (\gamma, r) \in \mathrm{Der}(P, (M, \mu)), \tag{5.11}$$

where $\gamma''(q) = {}^s\gamma(q) - \gamma(^s q)$, $q \in P$. It is routine to show that the elements $(\gamma', {}^p r)$ and $(\gamma'', [s, r])$ belong to $\mathrm{Der}(P, (M, \mu))$ and that (5.10), (5.11) define Lie actions.

Proposition 5.23. *Let (M, μ) be a P-crossed R-module and the actions of P and R on each other satisfy the compatibility conditions (5.1). Then the Lie homomorphism $\xi : \mathrm{Der}(P, (M, \mu)) \longrightarrow R$ given by $(\gamma, r) \longmapsto r$ with the aforementioned actions of P and R on $\mathrm{Der}(P, (M, \mu))$ is a P-precrossed R-module.*

Proof. We show only the following equality:

$$^{(^p r)}(\gamma, s) = {}^p(^r(\gamma, s)) - {}^r(^p(\gamma, s))$$

for all $r \in R$, $p \in P$ and $(\gamma, s) \in \mathrm{Der}(P, (M, \mu))$.

In fact,

$$^{(^p r)}(\gamma, s) = (\gamma', [^p r, s]),$$

where

$$\gamma'(q) = {}^{(^p r)}\gamma(q) - \gamma(^{(^p r)}q) = {}^{(^p r)}\gamma(q) - \gamma([q, {}^r p]) = {}^{(^p r)}\gamma(q) - {}^q\gamma(^r p) + {}^{(^r p)}\gamma(q) = -^q\gamma(^r p).$$

On the other hand,

$$^p(^r(\gamma, s)) - {}^r(^p(\gamma, s)) = (\gamma_1, {}^p[r, s]) - (\gamma_2, [r, {}^p s]) = (\gamma_1 - \gamma_2, [^p r, s]),$$

where

$$(\gamma_1-\gamma_2)(q) = {}^q({}^r\gamma(p)-\gamma({}^rp)) - {}^r({}^q\gamma(p)) + {}^{({}^rq)}\gamma(p) = -{}^q\gamma({}^rp).$$

The remaining details are omitted as they are routine. $\square$

6. Non-Abelian Cohomology

Before introducing our original definition of the second non-Abelian cohomology of Lie algebras, we recall the definitions of Guin [55] of the first and the zero non-Abelian cohomologies of Lie algebras with coefficients in crossed modules.

Let R be a Lie algebra and (M,μ) a crossed R-module. Define the zero non-Abelian cohomology as an ideal of M of all R-invariant elements,

$$H^0(R,M) = \big\{m\in M \mid {}^rm = 0 \text{ for all } r\in R\big\}.$$

The crossed module relation ${}^{\mu(m)}m' = [m,m']$ implies that $H^0(R,M)$ is contained in the center of M and therefore has only a Λ-module structure.

The first non-Abelian cohomology is a Lie algebra defined by

$$H^1(R,M) = \mathrm{Der}_R(R,M)/\mathfrak{I},$$

where $\mathfrak{I}$ is the following ideal of the Lie algebra $\mathrm{Der}_R(R,M)$:

$$\mathfrak{I} = \big\{(\eta_m, -\mu(m)+c) \mid m\in M, c\in Z(R)\big\},$$

η_m is the principal crossed homomorphism (or derivation) induced by m, namely $\eta_m(x) = {}^xm$, $x\in R$, and $Z(R)$ is the center of the Lie algebra R.

We need the following characterization of the second classical cohomology $H^2(R,M)$ of the Lie algebra R with coefficients in an R-module M.

Let us consider the diagram of Lie algebras

$$P \underset{d_1}{\overset{d_0}{\rightrightarrows}} F \xrightarrow{\ \epsilon\ } R\ , \tag{5.12}$$

where F is a free Lie algebra over some Λ-module, ϵ is a Lie homomorphism having a Λ-linear splitting and (P,d_0,d_1) is a simplicial kernel of ϵ in the category $\mathcal{L}ie$, i.e., $P=\{(x,y)\in F\times F \mid \epsilon(x)=\epsilon(y)\}$, $d_0(x,y)=x$, and $d_1(x,y)=y$. Assume that Δ denotes the Lie subalgebra $\{(x,x)\in F\times F\mid x\in F\}$ of P.

Let us claim M as an R-module; then M is also an F-module and P-module via the Lie homomorphisms ϵ and ϵd_i $(i=0,1)$ respectively. Denote by $\mathrm{Der}(P,M)$ (resp. $\mathrm{Der}(F,M)$) the Λ-module of crossed homomorphisms from P to M (resp. from F to M). Let $\widetilde{\mathrm{Der}}(P,M)$ be a submodule of $\mathrm{Der}(P,M)$ of all crossed homomorphisms γ such that $\gamma(\Delta)=0$. There is a Λ-homomorphism

$$\kappa : \mathrm{Der}(F,M) \longrightarrow \widetilde{\mathrm{Der}}(P,M)\ ,$$

given by $\beta \longmapsto \beta d_0 - \beta d_1$.

Proposition 5.24. *There is a natural isomorphism*

$$H^2(R,M) \cong \mathrm{Coker}\,\kappa.$$

Proof. As in Proposition 5.6, the classical cohomology of the Lie algebra R with coefficients in the R-module M is isomorphic, up to dimension shift, to the non-Abelian right derived functors $\mathcal{R}^k_{\mathbb{P}}\mathrm{Der}(-,M)(R)$, $k\geq 0$, of the contravariant functor $\mathrm{Der}(-,M) : \mathfrak{A}_M \longrightarrow \Lambda - mod$ (here we mean that M is an Abelian Lie algebra acting trivially on each object of $\mathfrak{A}_M$). Hence we only need to construct an isomorphism of Λ-modules

$$\mathcal{R}^1_{\mathbb{P}}\mathrm{Der}(-, M)(R) \cong \mathrm{Coker}\,\kappa.$$

Let us consider a $\mathbb{P}$-projective simplicial resolution of the object R in the category $\mathfrak{A}_M$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & F_2 \arrow[r, shift left=2, "d_0^2"] \arrow[r] \arrow[r, shift right=2, "d_2^2"'] & F_1 \arrow[rr, shift left, "d_0^1"] \arrow[rr, shift right, "d_1^1"'] \arrow[dr, "d"'] & & F_0 \arrow[r, "\epsilon"] & R \\
& & & P \arrow[ur, shift left, "d_0"] \arrow[ur, shift right, "d_1"'] & &
\end{tikzcd}, \tag{5.13}$$

where $F_0 = F$ and d is the unique Lie homomorphism such that $d_i^1 = d_i d$ $(i = 0, 1)$, which by Lemma 5.5 is surjective.

Applying the functor $\mathrm{Der}(-, A)$ to (5.13) yields a cochain complex of Λ-modules

$$\mathrm{Der}(F_0, A) \xrightarrow{\partial_0} \mathrm{Der}(F_1, A) \xrightarrow{\partial_1} \mathrm{Der}(F_2, A) \xrightarrow{\partial_2} \cdots .$$

Now define a Λ-homomorphism $\varphi : \widetilde{\mathrm{Der}}(P, M) \longrightarrow \mathrm{Ker}\,\partial_1$ by $\varphi(\gamma) = \gamma d$, $\gamma \in \widetilde{\mathrm{Der}}(P, M)$. To show that the crossed homomorphism $\gamma d : F_1 \longrightarrow A$ belongs to $\mathrm{Ker}\,\partial_1$, we need only to examine the following lemma.

Lemma 5.25. *For $\gamma \in \widetilde{\mathrm{Der}}(P, M)$ there is an equality*

$$\gamma(x, y) = \gamma(x, z) + \gamma(z, y)$$

for all $x, y, z \in F$ such that $\epsilon(x) = \epsilon(y) = \epsilon(z)$.

Proof. It is straightforward. $\square$

Returning to the main proof, construct a Λ-homomorphism

$$\psi : \mathrm{Ker}\,\partial_1 \longrightarrow \widetilde{\mathrm{Der}}(P, M)$$

by $\psi(\beta) = \gamma$, $\beta \in \mathrm{Ker}\,\partial_1$, where the map $\gamma : P \longrightarrow A$ is given by $\gamma(x, y) = \beta(z)$, $(x, y) \in P$, and $z \in F_1$ such that $d(z) = (x, y)$.

We show that γ is well defined. In fact, let $z' \in F_1$ such that $d(z') = (x, y)$. Using again Lemma 5.5 implies that there exists an element $w \in F_2$ such that $d_0^2(w) = d_1^2(w) = 0$ and $d_2^2(w) = z - z'$. Hence,

$$\gamma(z) - \gamma(z') = \gamma d_2^2(w) = \partial_1(\beta)(w) = 0.$$

It is easy to show that γ is a crossed homomorphism and $\psi\varphi$, $\varphi\psi$ are identity maps. Moreover, it is clear that the above-given isomorphism induces the isomorphism $H^2(R, M) \cong \mathrm{Coker}\,\kappa$. $\square$

Now we are ready to construct our second non-Abelian cohomology of Lie algebras.

Assume that in the diagram (5.12) R acts on F, and ϵ preserves the actions (here we mean that R acts on itself by Lie multiplication), implying the induced action of R on P. Note that all these conditions are satisfied when $F = \mathcal{F}(R)$ is the free Lie algebra on the underlying Λ-module R with canonical Lie homomorphism $\epsilon = \tau_R : \mathcal{F}(R) \longrightarrow R$ and the above-defined action of R on $\mathcal{F}(R)$ (see Sec. 2 of this chapter). Hence, with no lose of generality, we can assume that the center $Z(R)$ of R acts trivially on F (see Theorem 5.29).

Let (M, μ) be a crossed R-module. Then (M, μ) can be viewed as a P-crossed R-module induced by ϵd_i $(i = 0, 1)$ and a F-crossed R-module induced by ϵ. Denote by $\widetilde{\mathrm{Der}}(P, (M, \mu))$ the subset of $\mathrm{Der}(P, (M, \mu))$ consisting of all elements of the form $(\gamma, 0)$ satisfying the condition $\gamma(\Delta) = 0$. Clearly

$\widetilde{\mathrm{Der}}(P,(M,\mu))$ is a Λ-submodule of $\mathrm{Der}(P,(M,\mu))$, since in it the Lie multiplication of $\mathrm{Der}(P,(M,\mu))$ is killed.

Let us consider the Λ-submodule $B(P,(M,\mu))$ of $\widetilde{\mathrm{Der}}(P,(M,\mu))$ consisting of all elements $(\gamma,0)$ for which there exists $(\beta,h) \in \mathrm{Der}(F,(M,\mu))$ such that $\beta d_0 - \beta d_1 = \gamma$.

Proposition 5.26. *The Λ-module $\widetilde{\mathrm{Der}}(P,(M,\mu))/B(P,(M,\mu))$ is unique up to isomorphism of choosing the diagram* (5.12) *for the crossed R-module (M,μ).*

Proof. Consider the commutative diagram of Lie algebras

$$\begin{tikzcd}
P \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, shift right, "\widetilde{\omega}_1"'] \arrow[d, shift left, "\widetilde{\omega}_2"] & F \arrow[r, "\epsilon"] \arrow[d, shift right, "\omega_1"'] \arrow[d, shift left, "\omega_2"] & R \arrow[d, equal] \\
P' \arrow[r, shift left, "d'_0"] \arrow[r, shift right, "d'_1"'] & F' \arrow[r, "\epsilon"] & R
\end{tikzcd}$$

where the bottom row is another diagram of the form (5.12). The existence of such ω_i and $\widetilde{\omega}_i$ $(i=0,1)$, not preserving the actions of R in general, is clear.

As noted in Sec. 5, we have the induced Λ-homomorphisms, which will be denoted by $\overline{\omega}_i$: $\mathrm{Der}(P',(M,\mu)) \longrightarrow \mathrm{Der}(P,(M,\mu))$, $\overline{\omega}_i(\gamma,r) = (\gamma\widetilde{\omega}_i, r)$, $i=0,1$.

It is easy to see that $(\gamma\widetilde{\omega}_i,0) \in \widetilde{\mathrm{Der}}(P,(M,\mu))$ if $(\gamma,0) \in \widetilde{\mathrm{Der}}(P',(M,\mu))$. Let $(\gamma,0) \in B(P',(M,\mu))$, i.e., there exists $(\beta,h) \in \mathrm{Der}(F',(M,\mu))$ such that $\beta d_0' - \beta d_1' = \gamma$; then

$$\gamma\widetilde{\omega}_i = (\beta d_0' - \beta d_1')\widetilde{\omega}_i = \beta\omega_i d_0 - \beta\omega_i d_1.$$

Thus $(\gamma\widetilde{\omega}_i,0) \in B(P,(M,\mu))$. Hence we have the natural homomorphisms of Λ-modules χ_i : $\widetilde{\mathrm{Der}}(P',(M,\mu))/B(P',(M,\mu)) \longrightarrow \widetilde{\mathrm{Der}}(P,(M,\mu))/B(P,(M,\mu))$, $i=0,1$, induced by $\overline{\omega}_i$.

Now we show that $\chi_1 = \chi_2$. Take the Lie homomorphism $s : F \longrightarrow P'$ given by $s(x) = (\omega_1(x),\omega_2(x))$. For $(\gamma,0) \in \widetilde{\mathrm{Der}}(P',(M,\mu))$ we have $(\gamma s,0) \in \mathrm{Der}(F,(M,\mu))$ and the equality

$$\begin{aligned}
(\gamma s d_0 - \gamma s d_1)(x,y) &= \gamma s(x) - \gamma s(y) = \gamma(\omega_1(x),\omega_2(x)) - \gamma(\omega_1(y),\omega_2(y)) = \\
&= \gamma(\omega_1(x)-\omega_1(y),\omega_2(x)-\omega_2(y)) + \gamma\big(\omega_1(y)-\omega_2(x),\omega_1(y)-\omega_2(x)\big) = \\
&= \gamma\big(\omega_1(x)-\omega_2(x),\omega_1(y)-\omega_2(y)\big) = (\gamma\widetilde{\omega}_1 - \gamma\widetilde{\omega}_2)(x,y)
\end{aligned}$$

for $(x,y) \in P$. Therefore $(\gamma\widetilde{\omega}_1,0) - (\gamma\widetilde{\omega}_2,0) \in B(P,(M,\mu))$ and $\chi_1 = \chi_2$.

The rest of the proof is standard. □

Proposition 5.27. *Let R be a Lie algebra and (M,μ) a crossed R-module.*

(i) *There is a canonical epimorphism of Λ-modules*

$$\vartheta : H^2(R,\mathrm{Ker}\,\mu) \longrightarrow \widetilde{\mathrm{Der}}(P,(M,\mu))/B(P,(M,\mu)),$$

given by $\vartheta(|\beta|) = |(\psi(\beta),0)|$, $|\beta| \in H^2(R,\mathrm{Ker}\,\mu)$ (for the definition of ψ see Proposition 5.24).

(ii) *If $r \in Z(R)$ for any element $(\alpha,r) \in \mathrm{Der}(F,(M,\mu))$, then ϑ is an isomorphism.*

Proof. It directly follows from Proposition 5.24. □

Note that the condition of Proposition 5.27 (ii) is fulfilled when either R is an Abelian Lie algebra or M is an R-module thought of as the crossed R-module $(M,0)$. This assertion motivates our definition of the second non-Abelian cohomology of Lie algebras with coefficients in crossed modules.

Definition 5.28. Let R be a Lie algebra and (M,μ) a crossed R-module. Then the Λ-module $\widetilde{\mathrm{Der}}(P,(M,\mu))/B(P,(M,\mu))$ will be called *the second non-Abelian cohomology* of R with coefficients in (M,μ) and will be denoted by $H^2(R,M)$.

It is easy to see that a morphism of crossed R-modules $\theta : (M, \mu) \longrightarrow (N, \nu)$ induces a Λ-homomorphism

$$\theta^2 : H^2(R, M) \longrightarrow H^2(R, N), \qquad \theta^2\big(|(\alpha, 0)|\big) = |(\theta\alpha, 0)|.$$

Finally, using our second non-Abelian cohomology of Lie algebras, we obtain a nine-term exact cohomology sequence that prolongs Guin's seven-term exact cohomology sequence but under one additional necessary condition on the coefficient of the short exact sequence lacking in [55, Theorem 2.8].

Theorem 5.29. *Let R be a Lie algebra and $0 \longrightarrow (L, 0) \xrightarrow{\xi} (M, \mu) \xrightarrow{\theta} (N, \nu) \longrightarrow 0$ an exact sequence of crossed R-modules, having a Λ-linear splitting. Then there is an exact sequence of Λ-modules*

$$\begin{aligned} 0 \longrightarrow H^0(R, L) \xrightarrow{\xi^0} H^0(R, M) \xrightarrow{\theta^0} H^0(R, N) \xrightarrow{\delta^0} H^1(R, L) \xrightarrow{\xi^1} H^1(R, M) \xrightarrow{\theta^1} \\ \xrightarrow{\theta^1} H^1(R, N) \xrightarrow{\delta^1} H^2(R, L) \xrightarrow{\xi^2} H^2(R, M) \xrightarrow{\theta^2} H^2(R, N)\ , \end{aligned}$$

where θ^1 is a Lie homomorphism and δ^1 is a crossed homomorphism with the action of $H^1(R, N)$ on $H^2(R, L)$ induced by the action of R on P.

Proof. According to Theorem 2.8 [55] there is an exact sequence of Λ-modules

$$\begin{aligned} 0 \longrightarrow H^0(R, L) \xrightarrow{\xi^0} H^0(R, M) \xrightarrow{\theta^0} H^0(R, N) \xrightarrow{\delta^0} \\ \xrightarrow{\delta^0} H^1(R, L) \xrightarrow{\xi^1} H^1(R, M) \xrightarrow{\theta^1} H^1(R, N)\ , \end{aligned}$$

where θ^1 is a Lie homomorphism. *Note that the Λ-linear splitting on coefficient sequence is needed to construct the connecting map δ^1.*

We must only define the crossed homomorphism δ^1 and the action of the Lie algebra $H^1(R, N)$ on the Λ-module $H^2(R, L)$ (in our setting), and then show the exactness of the following sequence:

$$H^1(R, M) \xrightarrow{\theta^1} H^1(R, N) \xrightarrow{\delta^1} H^2(R, L) \xrightarrow{\xi^2} H^2(R, M) \xrightarrow{\theta^2} H^2(R, N). \tag{5.14}$$

Let us take an element $|(\alpha, r)| \in H^1(R, N)$ and consider the diagram

$$\begin{tikzcd} P \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & F \arrow[r, "\epsilon"] \arrow[d, "\beta"] & R \arrow[d, "\alpha"] \\ L \arrow[r, "\xi"'] & M \arrow[r, "\theta"'] & N \end{tikzcd}\ , \tag{5.15}$$

where $\beta : F \longrightarrow M$ is a crossed homomorphism such that $\theta\beta = \alpha\epsilon$. The existence of such β follows from the following fact: let F be a free Lie algebra (over some Λ-module X) acting on a Lie algebra M; then any Λ-linear map from X to M could be naturally extended to a crossed homomorphism from F to M.

Then there is a (unique) crossed homomorphism $\gamma : P \longrightarrow L$ such that $\xi\gamma = \beta d_0 - \beta d_1$. It is clear that $\gamma(\Delta) = 0$. Define

$$\delta^1|(\alpha, r)| = |(\gamma, 0)|.$$

We must verify the correctness of δ^1. Let $\beta' : F \longrightarrow M$ be another crossed homomorphism such that $\theta\beta' = \alpha\epsilon$, and hence $\gamma' : P \longrightarrow L$ be the induced crossed homomorphism satisfying $\xi\gamma' = \beta' d_0 - \beta' d_1$.

Then $\theta\beta' = \theta\beta$ and there is a crossed homomorphism $\sigma : F \longrightarrow L$ such that $\beta' = \beta + \xi\sigma$. Thus we have

$$\xi\gamma' = \beta' d_0 - \beta' d_1 = \beta d_0 + \xi\sigma d_0 - \beta d_1 - \xi\sigma d_1 = \xi\gamma + \xi\sigma d_0 - \xi\sigma d_1,$$

implying $|(\gamma, 0)| = |(\gamma', 0)|$.

Now, if (α', r') is another representative of the class $|(\alpha, r)|$, then there exists $n \in N$ such that $\alpha' = \alpha + \eta_n$. Take $\beta' : F \longrightarrow M$ such that $\beta' = \beta + \eta_m$, where $m \in M$ with $\theta(m) = n$ and $\theta\beta = \alpha\epsilon$. It is clear that $\theta\beta' = \alpha'\epsilon$. Moreover,

$$\xi\gamma' = \beta' d_0 - \beta' d_1 = \beta d_0 + \eta_m d_0 - \beta d_1 - \eta_m d_1 = \beta d_0 - \beta d_1 = \xi\gamma.$$

Whence $\gamma' = \gamma$ and the connecting map δ^1 is correctly constructed.

Now define the action of $H^1(R, N)$ on $H^2(R, L)$ by the formula

$$^{|(\alpha,r)|}|(\gamma, 0)| = |(\widetilde{\gamma}, 0)|, \quad |(\alpha, r)| \in H^1(R, N), \quad |(\gamma, 0)| \in H^2(R, L),$$

where $\widetilde{\gamma}(x, y) = \gamma({}^r x, {}^r y)$, $(x, y) \in P$. The following equality in the Lie algebra $\mathrm{Der}(P, (M, \mu))$:

$$(\xi\widetilde{\gamma}, 0) = \big[(\xi\gamma, 0), (\beta d_0, r)\big],$$

where $\beta : F \longrightarrow M$ is the mentioned crossed homomorphism (see diagram (5.15)), implies that $\widetilde{\gamma} : P \longrightarrow L$ is a crossed homomorphism. Furthermore, it is obvious that $\widetilde{\gamma}(\Delta) = 0$. We must verify that this action is correctly defined. Assume that $|(\alpha', r')| = |(\alpha, r)| \in H^1(R, N)$; hence $\alpha' = \alpha - \eta_n$ and $r' = r + \nu(n) - c$ for some $n \in N$ and $c \in Z(R)$. We have

$$\gamma({}^{r'} x, {}^{r'} y) = \gamma({}^r x, {}^r y) + \gamma\big({}^{\nu(n)} x, {}^{\nu(n)} y\big) - \gamma({}^c x, {}^c y).$$

As was mentioned above, we can assume, without loss of generality, that $Z(R)$ acts trivially on F; hence $\gamma({}^c x, {}^c y) = \gamma(0, 0) = 0$. Now we can deduce the correctness of such defined action from the following lemma.

Lemma 5.30. *A map $\beta : F \longrightarrow L$ given by $\beta(x) = \gamma({}^{\nu(n)} x, [u\nu(n), x])$ is a crossed homomorphism, where $u : R \longrightarrow F$ is the required Λ-linear splitting, and there is an equality*

$$\gamma\big({}^{\nu(n)} x, {}^{\nu(n)} y\big) = (\beta d_0 - \beta d_1)(x, y), \quad (x, y) \in P.$$

Proof. To show that β is a crossed homomorphism we make the following calculations:

$$\begin{aligned}
{}^x\beta(y) - {}^y\beta(x) &= {}^x\gamma\big({}^{\nu(n)} y, [u\nu(n), y]\big) - {}^y\gamma\big({}^{\nu(n)} x, [u\nu(n), x]\big) = \\
&= {}^{(x,x)}\gamma\big({}^{\nu(n)} y, [u\nu(n), y]\big) - {}^{(y,y)}\gamma\big({}^{\nu(n)} x, [u\nu(n), x]\big) = \\
&= \gamma\Big[(x, x), \big({}^{\nu(n)} y, [u\nu(n), y]\big)\Big] - \gamma\Big[(y, y), \big({}^{\nu(n)} x, [u\nu(n), x]\big)\Big] =
\end{aligned}$$

$$= \gamma\Big([x, {}^{\nu(n)} y], \big[x, [u\nu(n), y]\big]\Big) - \gamma\Big([y, {}^{\nu(n)} x], \big[y, [u\nu(n), x]\big]\Big) = \gamma\Big({}^{\nu(n)}[x, y], \big[u\nu(n), [x, y]\big]\Big) = \beta[x, y].$$

Let $m \in M$ such that $\theta(m) = n$. Then

$$\begin{aligned}
\gamma\big([u\nu(n), x], [u\nu(n), y]\big) &= \gamma\big[(u\nu(n), u\nu(n)), (x, y)\big] = \\
&= {}^{\nu(n)}\gamma(x, y) - {}^{(x,y)}\gamma(u\nu(n), u\nu(n)) = {}^{\mu(m)}\gamma(x, y) = [m, \gamma(x, y)] = 0,
\end{aligned}$$

since L is contained in the center of M.

Thus by lemma 5.25 we have

$$\begin{aligned}
(\beta d_0 - \beta d_1)(x, y) &= \beta(x) - \beta(y) = \gamma\big({}^{\nu(n)} x, [u\nu(n), x]\big) - \gamma\big({}^{\nu(n)} y, [u\nu(n), y]\big) = \\
&= \gamma\big({}^{\nu(n)} x, [u\nu(n), x]\big) + \gamma\big([u\nu(n), x], [u\nu(n), y]\big) + \gamma\big([u\nu(n), y], {}^{\nu(n)} y\big) = \gamma({}^{\nu(n)} x, {}^{\nu(n)} y).
\end{aligned}$$

□

Now we verify the exactness of the sequence (5.14).

Let $|(\alpha,r)| \in H^1(R,M)$. Then $\delta^1\theta^1|(\alpha,r)| = \delta^1|(\theta\alpha,r)| = |(\gamma,0)|$, where $\xi\gamma = \alpha\epsilon d_0 - \alpha\epsilon d_1 = 0$. Therefore $\operatorname{Im}\theta^1 \subseteq \operatorname{Ker}\delta^1$.

Let $|(\alpha,r)| \in H^1(R,N)$ such that $\delta^1|(\alpha,r)| = |(\gamma,0)| = 0$, where $\xi\gamma = \beta d_0 - \beta d_1$ (see diagram (5.15)). Then there exists a crossed homomorphism $\eta : F \longrightarrow L$ satisfying $\gamma = \eta d_0 - \eta d_1$. Hence we obtain that $(\beta - \xi\eta)d_0 = (\beta - \xi\eta)d_1$ implies the existence of $(\overline{\alpha}, r) \in \operatorname{Der}_R(R,M)$ with $\beta - \xi\eta = \overline{\alpha}\epsilon$. It is obvious that $\theta^1|(\overline{\alpha},r)| = |(\alpha,r)|$. Hence $\operatorname{Ker}\delta^1 \subseteq \operatorname{Im}\theta^1$.

Let $|(\alpha,r)| \in H^1(R,N)$, then $\xi^2\delta^1|(\alpha,r)| = \xi^2|(\gamma,0)| = |(\xi\gamma,0)| = 0$, since there exists $(\beta,r) \in \operatorname{Der}(F,(M,\mu))$ such that $\xi\gamma = \beta d_0 - \beta d_1$. Therefore $\operatorname{Im}\delta^1 \subseteq \operatorname{Ker}\xi^2$.

Let $|(\gamma,0)| \in H^2(R,L)$ such that $|(\xi\gamma,0)| = 0 \in H^2(R,M)$. Then there exists $(\beta,s) \in \operatorname{Der}(F,(M,\mu))$ such that $\xi\gamma = \beta d_0 - \beta d_1$, whence $\theta\beta d_0 = \theta\beta d_1$. It follows that there is a unique crossed homomorphism $\alpha : R \longrightarrow N$ such that $\alpha\epsilon = \theta\beta$. It is easy to verify that the pair (α,s) belongs to $\operatorname{Der}_R(R,N)$ and $\delta^1|(\alpha,s)| = |(\gamma,0)|$. Therefore $\operatorname{Ker}\xi^2 \subseteq \operatorname{Im}\delta^1$.

The rest of the exactness of the sequence (5.14) is similar to the group theoretic case (see [61, Theorem 13]) and will be omitted. □

Chapter 6

MOD q NON-ABELIAN TENSOR PRODUCTS AND (CO)HOMOLOGY OF GROUPS

The aim of this chapter is to study of some mod q theories. In particular, the non-Abelian tensor and exterior product modulo q of Conduché and Rodriguez–Fernández [32] of crossed modules, generalizing definitions of Brown [13] and Ellis and Rodriguez [48] (see also [47] and [112]) and having properties similar to the Brown–Loday non-Abelian tensor product [18] (see Chap. 4, Sec. 1), is investigated in various aspects. Mod q group homology and cohomology theories are introduced and studied as the homologies of the mapping cones of the q multiplication on the standard homological and cohomological complexes, respectively, as in the case of the mod q Hochschild homology [81]. Then both theories are unified into a mod q Tate–Farrell–Vogel group cohomology theory.

In Sec. 1, we give some functorial properties of the non-Abelian tensor product modulo q of crossed modules; in particular, we investigate for the non-Abelian tensor product modulo q of crossed modules the properties of right exactness (Proposition 6.4, 6.5) and compatibility with the direct limit of crossed modules (Proposition 6.6). We show that the 'absolute' tensor product modulo q of two groups G and H with compatible actions is the quotient of non-Abelian tensor product $G \otimes H$ by $q(H_1(G,H) \cap H_1(H,G))$ (Theorem 6.8). D.Guin's isomorphism [12] is generalized for the tensor product modulo q by giving the short exact sequence of groups

$$0 \longrightarrow G \otimes {}^q A \longrightarrow I(G,q) \otimes_G A \longrightarrow q\mathbb{Z} \otimes_G A \longrightarrow 0\,, \quad q \geq 0,$$

where G is a group, A is a G-module, and $I(G,q)$ is the kernel of the morphism $\widetilde{\epsilon} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}_q$ (see Proposition 6.13).

Then we give an application of tensor product modulo q to algebraic K-theory with $\mathbb{Z}_q$ coefficients [9]. In particular, for a (noncommutative) local ring A such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$, we give the relationship between non-Abelian tensor product modulo q and $K_2(A,\mathbb{Z}_q)$ which is an analog in q-modular aspect of D.Guin's six-term exact sequence relating the non-Abelian homology of groups with Milnor's K_2 and the symbol group Sym (Theorem 6.17).

In Sec. 2, given a chain complex, we provide the definition of its mod q homology and Φ-(co)homology (Definition 6.18). We prove the universal coefficient formulas (Proposition 6.19 and Corollary 6.20) and show that mod q (co)homology of chain complexes reduce to the case $q = p^m$ with p a prime (Theorem 6.22 and Corollary 6.23).

The study of non-Abelian left derived functors of the 'absolute' tensor product modulo q of groups inspired our definition of mod q homology, $H_*(G, A; \mathbb{Z}/q)$, of a group G with coefficients in a G-module A, where q is a positive integer, introduced in Sec. 3 (Definition 3.1). We develop certain aspects of this mod q version of the homology theory of discrete groups. According to Proposition 6.29 we can think of mod q homology (in case $q = 0$) as a generalization of classical group homology. We give universal coefficient formulas for the mod q homology of groups (Proposition 6.29). Then we calculate the mod q homology of free groups and finite cyclic groups (Proposition 6.33, Example 6.34, Proposition 6.35).

In Sec. 4, we investigate the derived functors of the non-Abelian 'absolute' tensor product modulo q of groups establishing their relations with classical homology and q-homology of groups (Propositions 6.37, 6.38, 6.41). The main result of Sec. 4 is Theorem 6.40, showing that if A is a q-torsion free G-module and $q > 0$, then there are natural isomorphisms $L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) \cong H_n(G, A; \mathbb{Z}/q)$ for $n \geq 2$.

In Sec. 5, we introduce the mod q cohomology, $H^*(G, A; \mathbb{Z}/q)$, of a group G with coefficients in a G-module A (Definition 6.27). Given a group G we introduce the notion of a (G, q)-torsor over a G-module A (Definition 6.49) and describe the first mod q cohomology group in terms of (G, q)-torsors over A (Theorem 6.50). Using our notions of pointed q-extension and q-extension (Definitions 6.51 and 6.54), we describe the second mod q cohomology of groups (Theorems 6.52 and 6.56).

In Sec. 6, we express the mod q cohomology of groups in terms of cotriple derived functors of the kernels of higher dimensions of the mapping cone of the q multiplication on the standard cohomological complex (Theorem 6.57).

In Sec. 7, we give an account of Vogel cohomology theory [125]. In [52] Goichot gave a detailed exposition of Vogel's homology theory and its relations to Tate and Farrell theories. We shall give here the cohomological approach (see also [128, Sec. 5]). At least in the case of finite groups it is the same, but the point of view is slightly different.

In Sec. 8, the mod q Tate–Farrell–Vogel cohomology of groups is introduced (Definition 6.74). Finally we show how periodicity properties of finite periodic groups extend to mod q Tate cohomology (Theorem 6.81) and give a property of cohomogically trivial G-modules for G a p-group (Theorem 6.85).

In this chapter, q denotes a positive integer, and its product on any module A is represented by qA and $A/q = A/qA$. We denote by IG the augmentation ideal of the group ring $\mathbb{Z}[G]$ over a group G. The groups $\mathbb{Z}$ and $\mathbb{Z}/q$ are trivial G-modules. We consider the group $H^{-1}(G, A)$ trivial. A ring R is always associative and unitary; an R-module A is a left R-module. $\mathcal{D}_R$ is the category of (unbounded) complexes of projective R-modules, and $\mathcal{C}_R$ is the category of complexes of R-modules. Considering a group G, and given two G-modules A and A' we write $A \otimes_G A'$ and $\mathrm{Hom}_G(A, A')$ for $A \otimes_{\mathbb{Z}[G]} A'$ and $\mathrm{Hom}_{\mathbb{Z}[G]}(A, A')$, respectively.

1. The Tensor Product Modulo q of Groups

We begin this section by recalling some definitions of the mod q non-Abelian tensor product of groups [32].

1.1. Various definitions. Let $\mu : M \longrightarrow P$ and $\nu : N \longrightarrow P$ be two crossed P-modules and consider the pullback

$$(pull) \qquad \begin{array}{ccc} M \times_P N & \xrightarrow{\pi_1} & M \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & P \end{array} .$$

Let $K = M \times_P N = \{(m,n) \in M \times N \mid m \in M,\ n \in N,\ \mu(m) = \nu(n)\}$. In this diagram each group acts on any other group via its image in the group P.

Definition 6.1. *The tensor product modulo q, $M \otimes^q N$, of the crossed P-modules μ and ν* is the group generated by the symbols $m \otimes n$ and $\{k\}$, $m \in M$, $n \in N$, $k \in K$ subject to the following relations:

$$mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n), \tag{6.1}$$

$$m \otimes nn' = (m \otimes n)({}^n m \otimes {}^n n'), \tag{6.2}$$

$$\{k\}(m \otimes n)\{k\}^{-1} = {}^{k^q} m \otimes {}^{k^q} n, \tag{6.3}$$

$$\{kk'\} = \{k\} \prod_{i=1}^{q-1} \left(\pi_1 k^{-1} \otimes ({}^{k^{1-q+i}} \pi_2 k')^i\right) \{k'\}, \tag{6.4}$$

$$\left[\{k\}, \{k'\}\right] = \pi_1 k^q \otimes \pi_2 k'^q, \tag{6.5}$$

$$\left\{(m {}^n m^{-1}, {}^m n n^{-1})\right\} = (m \otimes n)^q \tag{6.6}$$

for all $m, m' \in M$, $n, n' \in N$, $k, k' \in K$.

Definition 6.2 ([32]). *The exterior product modulo q, $M \wedge^q N$, of the crossed P-modules μ and ν* is obtained from the tensor product $M \otimes {}^q N$ by imposing the additional relation

$$\pi_1 k \otimes \pi_2 k = 1, \quad k \in K. \tag{6.7}$$

The image of a generic element $m \otimes n$ in $M \wedge^q N$ is written $m \wedge n$.

Note that we can add the case $q = 0$; then under $M \otimes {}^0 N$ we mean the tensor product of Brown and Loday, $M \otimes N$, which is the group generated by elements $m \otimes n$, $m \in M$, $n \in N$ and subject to relations (6.1) and (6.2) (see Chap. 4, Sec. 1 or [16–18]). Furthermore, under $M \wedge^0 N$ we mean Brown–Loday's exterior product, $M \wedge N$, which is the group generated by elements $m \wedge n$, $m \in M$, $n \in N$ and subject to relations (6.1), (6.2), and (6.7) (see [16, 18, 43]).

Assume that (M, μ), (N, ν) are crossed P-modules and (M', μ'), (N', ν') are crossed P'-modules. Assume that $\alpha = (f, \varphi) : (M, \mu) \longrightarrow (M', \mu')$ and $\beta = (g, \psi) : (N, \nu) \longrightarrow (N', \nu')$ are crossed module morphisms such that $\varphi = \psi$. Then there is a unique homomorphism

$$\alpha \otimes {}^q \beta : M \otimes {}^q N \longrightarrow M' \otimes {}^q N' \quad \left(\alpha \wedge^q \beta : M \wedge^q N \longrightarrow M' \wedge^q N'\right),$$

such that

$$(\alpha \otimes {}^q \beta)(m \otimes n) = f(m) \otimes g(n) \quad \left((\alpha \wedge^q \beta)(m \wedge n) = f(m) \wedge g(n)\right),$$

$$(\alpha \otimes {}^q \beta)(\{k\}) = \left\{(f(\pi_1 k), g(\pi_2 k))\right\} \quad \left((\alpha \wedge^q \beta)(\{k\}) = \{(f(\pi_1 k), g(\pi_2 k))\}\right)$$

for all $m \in M$, $n \in N$, and $k \in K$. Further, if α, β are onto, so also is $\alpha \otimes {}^q \beta$ ($\alpha \wedge^q \beta$).

Recall the definition of the function $\beta_t(k, k')$ from [32], where $k, k' \in K$ and t is a positive integer:

$$\beta_t(k, k') = \prod_{i=1}^{t-1} (\pi_1 k^{-1} \otimes {}^{k^{1-t+i}} \pi_2 k'^i). \tag{6.8}$$

Then for any $k, k' \in K$ and any positive integer t we have the following equality:

$$\beta_t(k, k') = \prod_{i=1}^{t-1} ({}^{k'^{i-1}} \pi_1 k^{i-t} \otimes \pi_2 k'). \tag{6.9}$$

We only prove this equality when t is odd, since the case where t is even is similar. In fact, by the definition of β in (6.8), we have

$$\beta_t(k,k') = \prod_{i=1}^{t-1}(\pi_1 k^{-1} \otimes {}^{k^{1-t+i}}\pi_2 k'^i) = (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')(\pi_1 k^{-1} \otimes {}^{k^{3-t}}\pi_2 k'^2)\cdots$$
$$\cdots(\pi_1 k^{-1} \otimes {}^{k^{-j}}\pi_2 k'^j)\cdots(\pi_1 k^{-1} \otimes {}^{k^{-1}}\pi_2 k'^{t-2})(\pi_1 k^{-1} \otimes \pi_2 k'^{t-1}),$$

where $j = \frac{t-1}{2}$.

Thus,

$$\beta_t(k,k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')\cdot(\pi_1 k^{-1} \otimes {}^{k^{3-t}}\pi_2 k'^2)\cdots(\pi_1 k^{-1} \otimes {}^{k^{-j}}\pi_2 k'^j)\cdots$$
$$\cdots(\pi_1 k^{-1} \otimes {}^{k^{-1}}\pi_2 k'^{t-2})\cdot(\pi_1 k^{-1} \otimes \pi_2 k'^{t-2})({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')\cdot(\pi_1 k^{-1} \otimes {}^{k^{3-t}}\pi_2 k'^2)\cdots$$
$$\cdots(\pi_1 k^{-1} \otimes {}^{k^{-j}}\pi_2 k'^j)\cdots(\pi_1 k^{-2} \otimes \pi_2 k'^{t-2})({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')\cdot(\pi_1 k^{-1} \otimes {}^{k^{3-t}}\pi_2 k'^2)\cdots$$
$$\cdots(\pi_1 k^{-1} \otimes {}^{k^{-j}}\pi_2 k'^j)(\pi_1 k^{-j} \otimes \pi_2 k'^{j+1})\cdot({}^{k'^{j+1}}\pi_1 k^{1-j} \otimes \pi_2 k')\cdots({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')\cdot(\pi_1 k^{-1} \otimes {}^{k^{3-t}}\pi_2 k'^2)\cdots$$
$$\cdots(\pi_1 k^{-1} \otimes {}^{k^{-j}}\pi_2 k'^j)\cdot(\pi_1 k^{-j} \otimes \pi_2 k'^j)\cdot({}^{k'^j}\pi_1 k^{-j} \otimes \pi_2 k')\cdot({}^{k'^{j+1}}\pi_1 k^{1-j} \otimes \pi_2 k')\cdots({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k').$$

Now we compute the other part:

$$\prod_{i=1}^{t-1}({}^{k'^{i-1}}\pi_1 k^{i-t} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{1-t} \otimes \pi_2 k')({}^{k'}\pi_1 k^{2-t} \otimes \pi_2 k')\cdots({}^{k'^{t-3}}\pi_1 k^{-2} \otimes \pi_2 k')\cdot({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')(\pi_1 k^{2-t} \otimes \pi_2 k')\cdot({}^{k'}\pi_1 k^{2-t} \otimes \pi_2 k')\cdots$$
$$\cdots({}^{k'^j}\pi_1 k^{-j} \otimes \pi_2 k')\cdots({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')\cdot(\pi_1 k^{2-t} \otimes \pi_2 k'^2)\cdots({}^{k'^j}\pi_1 k^{-j} \otimes \pi_2 k')\cdots({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')\cdot(\pi_1 k^{-1} \otimes {}^{k^{3-t}}\pi_2 k'^2)\cdots$$
$$\cdots(\pi_1 k^{-j-1} \otimes \pi_2 k'^j)({}^{k'^j}\pi_1 k^{-j} \otimes \pi_2 k')\cdots({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}}\pi_2 k')\cdot(\pi_1 k^{-1} \otimes {}^{k^{3-t}}\pi_2 k'^2)\cdots$$
$$\cdots\pi_1 k^{-1} \otimes {}^{k^{-j}}\pi_2 k'^j)(\pi_1 k^{-j} \otimes \pi_2 k'^j)\cdot({}^{k'^j}\pi_1 k^{-j} \otimes \pi_2 k')\cdots({}^{k'^{t-2}}\pi_1 k^{-1} \otimes \pi_2 k').$$

Hence we have proved that

$$be_t(k,k') = \prod_{i=1}^{t-1}({}^{k'^{i-1}}\pi_1 k^{i-t} \otimes \pi_2 k'),$$

when t is odd.

Then there is a unique isomorphism

$$\tau : M \otimes {}^q N \longrightarrow N \otimes {}^q M \quad (\tau : M \wedge^q N \longrightarrow N \wedge^q M),$$

such that $\tau(m \otimes n) = (n \otimes m)^{-1}$ $(\tau(m \wedge n) = (n \wedge m)^{-1})$, $\tau(\{k\}) = \{\overline{k}^{-1}\}^{-1}$, where $\overline{k} = (\pi_2 k, \pi_1 k)$ for all $m \in M$, $n \in N$, and $k \in K$.

In fact, We have only to show that τ commutes with relations (6.1)–(6.7); for instance, by (6.9) we have

$$\begin{aligned}\tau(\{kk'\}) = \{\overline{k}'^{-1}\overline{k}^{-1}\}^{-1} = \{\overline{k}^{-1}\}^{-1}\beta_q(\overline{k}'^{-1},\overline{k}^{-1})^{-1}\{\overline{k}'^{-1}\}^{-1} = \\ = \tau(\{k\})\cdot\left[\prod_{i=1}^{q-1}\left({}^{\overline{k}^{1-i}}\pi_1\overline{k}'^{q-i}\otimes\pi_2\overline{k}^{-1}\right)\right]^{-1}\tau(\{k'\}) = \\ = \tau(\{k\})\prod_{i=1}^{q-1}\left({}^{k^{1-q+i}}\pi_2k'^i\otimes\pi_1k^{-1}\right)^{-1}\cdot\tau(\{k'\}) = \\ = \tau(\{k\})\prod_{i=1}^{q-1}\tau\left(\pi_1k^{-1}\otimes({}^{k^{1-q+i}}\pi_2k')^i\right)\tau(\{k'\}).\end{aligned}$$

Now let G and H be groups that act on themselves by conjugation (${}^xy = xyx^{-1}$) and each of which acts upon the other in such a way that the compatibility conditions (4.1) hold.

Consider the Peiffer product $G \bowtie H$, which was defined by Whitehead in [126] and is the quotient of the free product $G * H$ by the normal subgroup generated by the elements ${}^ghgh^{-1}g^{-1}$ and ${}^hghg^{-1}h^{-1}$ for all $g \in G$, $h \in H$. As a consequence of the compatibility conditions (4.1), the actions of $G * H$ on G and on H factor through $G \bowtie H$ and the canonical maps $G \longrightarrow G \bowtie H$ and $H \longrightarrow G \bowtie H$ are crossed modules; see [51] for more details.

Definition 6.3 ([32]). *An 'absolute' tensor product modulo q of two groups G and H with compatible actions on each other is the tensor product modulo q of crossed modules G and H over the group $G \bowtie H$.*

It is easy to deduce [32] that for groups G and H acting trivially on each other there is a natural isomorphism

$$G \otimes {}^qH \cong (G^{ab}/q) \otimes_{\mathbb{Z}/q} (H^{ab}/q).$$

1.2. Functorial properties. Now some properties of non-Abelian tensor product modulo q will be given.

We begin by giving for the non-Abelian tensor product modulo q of crossed modules the properties of right exactness and compatibility with the direct limit of crossed modules.

Let (L,λ), (M,μ), and (N,ν) be crossed P-modules. A short exact sequence of groups

$$1 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 1$$

is called a short exact sequence of crossed P-modules if α and β are morphisms of crossed P-modules.

Proposition 6.4. *Let $1 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 1$ be a short exact sequence of crossed P-modules. Then there is an exact sequence of groups*

$$P \otimes {}^qL \xrightarrow{\alpha'} P \otimes {}^qM \xrightarrow{\beta'} P \otimes {}^qN \longrightarrow 1, \tag{6.10}$$

where $\alpha' = 1_P \otimes {}^q\alpha$, $\beta' = 1_P \otimes {}^q\beta$.

Proof. Using (6.1), (6.2), and (6.4) we can easily check that $\beta'\alpha'$ is the trivial homomorphism. It is also clear that β' is surjective (see Sec. 1.1).

Now we show that $\operatorname{Im}\alpha'$ is a normal subgroup of $P \otimes {}^qM$. In effect, by [16, Propostion 3]

$$(p\otimes m)(p'\otimes\alpha(l))(p\otimes m)^{-1} = {}^{[p,m]}p' \otimes {}^{[p,m]}\alpha(l) \in \operatorname{Im}\alpha',$$

for all $l \in L$, $m \in M$, $p \in P$. By (6.3)

$$\{(p,m)\}(p' \otimes \alpha(l))\{(p,m)\}^{-1} = {}^{p^q}p' \otimes {}^{p^q}\alpha(l) \in \operatorname{Im}\alpha',$$

for all $l \in L$, $m \in M$, $p, p' \in P$, where $\mu(m) = p$. By [32, Proposition 1.16 and Lemma 1.10]

$$\{(p,m)\}\alpha'\{(1,l)\}\{(p,m)\}^{-1} = (p^q \otimes \alpha(l)^q)\{(1,\alpha(l))\} \in \operatorname{Im}\alpha'$$

and

$$(p' \otimes m')\alpha'\{(1,l)\}(p' \otimes m')^{-1} = (p'\mu(m')p'^{-1}\mu(m')^{-1} \otimes \alpha(l)^q)\{(1,\alpha(l))\} \in \operatorname{Im}\alpha',$$

for all $l \in L$, $m, m' \in M$, $p, p' \in P$, where $\mu(m) = p$, $\lambda(l) = 1$.

Therefore, we have the diagram of groups

$$\begin{tikzcd}
P \otimes {}^q L \arrow[r, "\alpha'"] \arrow[d, equal] & P \otimes {}^q M \arrow[r, "\beta'"] \arrow[d, equal] & P \otimes {}^q N \arrow[r] & 1 \\
P \otimes {}^q L \arrow[r, "\alpha'"'] & P \otimes {}^q M \arrow[r, "\tau"'] & \operatorname{Coker}\alpha' \arrow[r] & 1
\end{tikzcd},$$

where the bottom row is exact. Thus, there exists a natural homomorphism $\gamma : \operatorname{Coker}\alpha' \longrightarrow P \otimes {}^q N$ such that $\gamma\tau = \beta'$.

Let us define a homomorphism $\gamma' : P \otimes {}^q N \longrightarrow \operatorname{Coker}\alpha'$ as follows: $\gamma'(p \otimes n) = [p \otimes m]$, $\gamma'\{(p', n')\} = [\{(p', m')\}]$, where $\beta(m) = n$, $\beta(m') = n'$ and $\mu(m') = \nu(n') = p'$. It is correctly defined. In effect, let $m_1 = m\alpha(l)$ and $m_1' = \alpha(l)m'$; then

$$p \otimes m_1 = p \otimes m\alpha(l) = (p \otimes m)(\mu(m)p\mu(m)^{-1} \otimes m\alpha(l)m^{-1}),$$

and

$$\{(p', m_1')\} = \big\{(p', \alpha(l')m')\big\} = \big\{(1, \alpha(l'))(p', m')\big\} = \big\{(1, \alpha(l'))\big\}\{(p', m')\}.$$

Hence $[p \otimes m_1 = p \otimes m]$ and $[\{(p', m_1')\}] = [\{(p', m')\}]$.

It is easy to verify that γ' is compatible with the relations (6.1)–(6.6) and $\gamma\gamma'$, $\gamma'\gamma$ are identity maps. □

Note that in general the non-Abelian tensor product modulo q of crossed P-modules is not a right exact functor, i.e., the sequence (6.10) is not exact when the group P is replaced by any crossed P-module A.

Let M and N be crossed P-modules. By [32, Lemma 1.3] we have two homomorphisms $\xi : M \otimes {}^q N \longrightarrow M$ and $\xi' : M \otimes {}^q N \longrightarrow N$ defined by

$$\xi(m \otimes n) = m^n m^{-1}, \qquad \xi(\{k\}) = \pi_1 k^q, \tag{6.11}$$

$$\xi'(m \otimes n) = {}^m n n^{-1}, \qquad \xi'(\{k\}) = \pi_2 k^q. \tag{6.12}$$

Furthermore, these homomorphisms factor through $M \wedge^q N$.

Proposition 6.5. *Let* $1 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 1$ *be a short exact sequence of crossed P-modules. Then there is an exact sequence of groups*

$$\operatorname{Ker}\xi_L' \longrightarrow \operatorname{Ker}\xi_M' \longrightarrow \operatorname{Ker}\xi_N' \longrightarrow \operatorname{Coker}\xi_L' \longrightarrow \operatorname{Coker}\xi_M' \longrightarrow \operatorname{Coker}\xi_N' \longrightarrow 1,$$

where the homomorphisms $\xi_L' : P \otimes {}^q L \longrightarrow L$, $\xi_M' : P \otimes {}^q M \longrightarrow M$, *and* $\xi_N' : P \otimes {}^q N \longrightarrow N$ *are defined according to* (6.12).

Proof. It follows from the commutative diagram of groups with exact rows and columns

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
 & \operatorname{Ker}\xi'_L \arrow[r] \arrow[d] & \operatorname{Ker}\xi'_M \arrow[r] \arrow[d] & \operatorname{Ker}\xi'_N \arrow[r] \arrow[d] & {} \\
 & P \otimes {}^q L \arrow[r] \arrow[d, "\xi'_L"'] & P \otimes {}^q M \arrow[r] \arrow[d, "\xi'_M"'] & P \otimes {}^q N \arrow[r] \arrow[d, "\xi'_N"'] & 1 \\
1 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 1 \\
 & \operatorname{Coker}\xi'_L \arrow[r] \arrow[d] & \operatorname{Coker}\xi'_M \arrow[r] \arrow[d] & \operatorname{Coker}\xi'_N \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
.
$$

∎

Proposition 6.6. *Let $\{M_\alpha, \Phi_\alpha^\beta, \alpha \le \beta\}$ and $\{P_\alpha, \Psi_\alpha^\beta, \alpha \le \beta\}$ be two directed systems of groups. Let $\mu_\alpha : M_\alpha \longrightarrow P_\alpha$ be a crossed P_α-module for every α such that $(\Phi_\alpha^\beta, \Psi_\alpha^\beta) : (M_\alpha, P_\alpha) \longrightarrow (M_\beta, P_\beta)$, $\alpha \le \beta$, is a crossed module morphism. Let $\nu_\alpha : N \longrightarrow P_\alpha$ be a crossed P_α-module for every α such that $(1, \Psi_\alpha^\beta) : (N, \nu_\alpha) \longrightarrow (N, \nu_\beta)$, $\alpha \le \beta$, is a crossed module morphism. Then there is a natural isomorphism*

$$\big(\varinjlim_\alpha \{M_\alpha\}\big) \otimes {}^q N \cong \varinjlim_\alpha \{M_\alpha \otimes {}^q N\}, \quad \big(\varinjlim_\alpha \{M_\alpha\}\big) \wedge^q N \cong \varinjlim_\alpha \{M_\alpha \wedge^q N\}.$$

Proof. First, note that the tensor product modulo q of $\varinjlim_\alpha \{M_\alpha\}$ and N are considered as crossed modules over the group $\varinjlim_\alpha \{P_\alpha\}$.

It is clear that a homomorphism $\nu : N \longrightarrow \varinjlim_\alpha \{P_\alpha\}$ defined as $\nu(n) = [\nu_\alpha(n)]$, $n \in N$, with action ${}^{[p_\alpha]}n = {}^{p_\alpha}n$, is a crossed module.

Now we show that a homomorphism $\mu : \varinjlim_\alpha \{M_\alpha\} \longrightarrow \varinjlim_\alpha \{P_\alpha\}$ defined by $\mu([m_\alpha]) = [\mu_\alpha(m_\alpha)]$ with action ${}^{[p_\alpha]}[m_\beta] = [{}^{\Psi_\alpha^\gamma(p_\alpha)}\Phi_\beta^\gamma(m_\beta)]$, where $\gamma \ge \alpha, \beta$ (the existence of such γ follows from the directness of the system), is a crossed module. Proof of correctness here is easy and is omitted. Next

$$\begin{aligned}
\mu({}^{[p_\alpha]}[m_\beta]) &= \mu\big({}^{\Psi_\alpha^\gamma}\Phi_\beta^\gamma(m_\beta)\big) = \big[\mu_\gamma\big({}^{\Psi_\alpha^\gamma(p_\alpha)}\Phi_\beta^\gamma(m_\beta)\big)\big] = \big[\Psi_\alpha^\gamma(p_\alpha)\mu_\gamma\Phi_\beta^\gamma(m_\beta)\Psi_\alpha^\gamma(p_\alpha^{-1})\big] = \\
&= \big[\Psi_\alpha^\gamma(p_\alpha)\Psi_\beta^\gamma\mu_\beta(m_\beta)\Psi_\alpha^\gamma(p_\alpha^{-1})\big] = [p_\alpha]\mu[m_\beta][p_\alpha^{-1}], \quad \text{where} \quad \gamma \ge \alpha, \beta; \\
{}^{\mu([m_\alpha])}[m_\beta] &= {}^{[\mu_\alpha(m_\alpha)]}[m_\beta] = \big[{}^{\Psi_\alpha^\gamma\mu_\alpha(m_\alpha)}\Phi_\beta^\gamma(m_\beta)\big] = \big[{}^{\mu_g m\Phi_\alpha^\gamma(m_\alpha)}\Phi_\beta^\gamma(m_\beta)\big] = \\
&= \big[\Phi_\alpha^\gamma(m_\alpha)\Phi_\beta^\gamma(m_\beta)\Phi_\alpha^\gamma(m_\alpha^{-1})\big] = [m_\alpha][m_\beta][m_\alpha^{-1}], \quad \text{where} \quad \gamma \ge \alpha, \beta.
\end{aligned}$$

Let us define a homomorphism

$$\kappa : \big(\varinjlim_\alpha \{M_\alpha\}\big) \otimes {}^q N \longrightarrow \varinjlim_\alpha \{M_\alpha \otimes {}^q N\} \quad \big(\kappa : (\varinjlim_\alpha \{M_\alpha\}) \wedge^q N \longrightarrow \varinjlim_\alpha \{M_\alpha \wedge^q N\}\big)$$

as follows: $\kappa([m_\alpha]\otimes n)=[m_\alpha\otimes n]$ ($\kappa([m_\alpha]\wedge n)=[m_\alpha\wedge n]$) and $\kappa(\{([m_\alpha],n)\})=[\{(\Phi_\alpha^\beta(m_\alpha),n)\}]$, since $[\mu_\alpha(m_\alpha)]=[\nu_\alpha(n)]$ and therefore there exists $\beta\geq\alpha$ such that $\Psi_\alpha^\beta\mu_\alpha(m_\alpha)=\Psi_\alpha^\beta\nu_\alpha(n)$; $\mu_\beta\Phi_\alpha^\beta(m_\alpha)=\nu_\beta(n)$.

Let $(\Phi_\alpha^\gamma)'=\Phi_\alpha^\gamma\otimes{}^q1_N$ and $(\Phi_\beta^\gamma)'=\Phi_\beta^\gamma\otimes{}^q1_N$. If $[m_\alpha]=[m_\beta]$, then there exists $\gamma\geq\alpha,\beta$ such that $\Phi_\alpha^\gamma(m_\alpha)=\Phi_\beta^\gamma(m_\beta)$. Thus,

$$(\Phi_\alpha^\gamma)'(m_\alpha\otimes n)=\Phi_\alpha^\gamma(m_\alpha)\otimes n=\Phi_\alpha^\gamma(m_\beta)\otimes n=(\Phi_\beta^\gamma)'(m_\beta\otimes n).$$

If there exists $\beta'\geq\alpha$ such that $\Psi_\alpha^{\beta'}\mu_\alpha(m_\alpha)=\Psi_\alpha^{\beta'}\nu_\alpha(n)$ and therefore $\mu_{\beta'}\Phi_\alpha^{\beta'}(m_\alpha)=\nu_{\beta'}(n)$, then there exists $\gamma\geq\beta,\beta'$. Thus

$$(\Phi_\beta^\gamma)'\big(\{(\Phi_\alpha^\beta(m_\alpha),n)\}\big)=\big\{(\Phi_\alpha^\gamma(m_\alpha),n)\big\}=(\Phi_{\beta'}^\gamma)'\big(\{(\Phi_\alpha^{\beta'}(m_\alpha),n)\}\big).$$

If $[m_\alpha]=[m_{\alpha'}]$, then there exists $\gamma\geq\alpha,\alpha'$ such that $\Phi_\alpha^\gamma(m_\alpha)=\Phi_{\alpha'}^\gamma(m_{\alpha'})$. Hence we have

$$(\Phi_\beta^{\gamma'})'\big(\{(\Phi_\alpha^\beta(m_\alpha),n)\}\big)=\big\{\big(\Phi_\alpha^{\gamma'}(m_\alpha),n\big)\big\}=(\Phi_{\beta'}^{\gamma'})'\big(\{(\Phi_{\alpha'}^{\beta'}(m_{\alpha'}),n)\}\big),$$

where $\gamma'\geq\gamma,\beta,\beta'$. Therefore κ is correctly defined. Commutativity of κ with relations (6.1)–(6.7) is easy to verify and is omitted from the text.

On the other hand, the canonical homomorphisms $\Phi_\alpha : M_\alpha\otimes{}^qN\longrightarrow(\varinjlim_\alpha\{M_\alpha\})\otimes{}^qN$ ($\Phi_\alpha : M_\alpha\wedge^q N\longrightarrow(\varinjlim_\alpha\{M_\alpha\})\wedge^q N$), $\Phi_\alpha(m_\alpha\otimes n)=[m_\alpha]\otimes n$ ($\Phi_\alpha(m_\alpha\wedge n)=[m_\alpha]\wedge n$), and $\Phi_\alpha(\{(m_\alpha,n)\})=\{([m_\alpha],n)\}$, induce a homomorphism $\kappa' : \varinjlim_\alpha\{M_\alpha\otimes{}^qN\}\longrightarrow\varinjlim_\alpha\{M_\alpha\}\otimes{}^qN$ ($\kappa' : \varinjlim_\alpha\{M_\alpha\wedge^q N\}\longrightarrow\varinjlim_\alpha\{M_\alpha\}\wedge^q N$), and it is easy to see that $\kappa\kappa'$, $\kappa'\kappa$ are identity maps. $\square$

Let r be a nonnegative integer and $\mu : M\longrightarrow P$ be a crossed P-module. (M,μ) is called a r-crossed P-module if $a^r=1$ for all $a\in\operatorname{Ker}\mu$.

We have the following proposition.

Proposition 6.7. *Let $\mu : M\longrightarrow P$ and $\nu : N\longrightarrow P$ be r-crossed and l-crossed P-modules respectively. Let s be the least common multiple of r and l. Then $\alpha : M\wedge^q N\longrightarrow P$, given by $\alpha(m\wedge n)=[\mu(m),\nu(n)]$, $\alpha(\{k\})=\mu(\pi_1k)^q=\nu(\pi_2k)^q$, is a qs-crossed P-module.*

Proof. By [32, Corollary 1.17], $(M\wedge^q N,\alpha)$ is a crossed P-module. Let $x\in\operatorname{Ker}\alpha$. Then by [32, Corollary 1.21] and [32, Lemma 1.20], we have

$$x^{qs}=\{\partial'x\}^s=\{\partial'x^s\},$$

where $\partial' : M\wedge^q N\longrightarrow K$ is defined by the following way:

$$\partial'x=(\xi x,\xi'x)$$

and the homomorphisms ξ and ξ' are defined according to (6.11) and (6.12).

Since $\alpha=\mu\xi=\nu\xi'$, we have

$$x^{qs}=\{\partial'x^s\}=\big\{(\xi x,\xi'x)^s\big\}=\big\{(\xi x^s,\xi'x^s)\big\}=\{(1,1)\}=1.$$

$\square$

From here till the end of this subsection we investigate the 'absolute' tensor product modulo q. In effect, we have the following theorem.

Theorem 6.8. *Let G and H be groups acting compatibly on each other. Then we have the following exact sequence of groups*

$$0\longrightarrow q(H_1(G,H)\cap H_1(H,G))\longrightarrow G\otimes H\xrightarrow{\varphi}G\otimes{}^qH\longrightarrow 1,$$

where φ is given by $\varphi(g \otimes h) = g \otimes h$.

Proof. By [32, Proposition 1.6] for any crossed modules $\mu : G \longrightarrow P$ and $\nu : H \longrightarrow P$ over any group P we have the exact sequence of groups

$$G \otimes H \xrightarrow{\varphi} G \otimes {}^q H \longrightarrow K/[G,H] \longrightarrow 1, \tag{6.13}$$

where $[G,H]$ is the subgroup of $K = G \times_P H$ generated by the elements $(g^h g^{-1}, {}^g h h^{-1})$, $g \in G$, $h \in H$.

In our case, when P is the Peiffer product of G and H, according to [12], $P = G \bowtie H = (G \rtimes H)/L$, where L is the subgroup of $G \rtimes H$ (semidirect product of G and H) generated by the elements $(g^h g^{-1}, h^g h^{-1})$, $g \in G$, $h \in H$ and we show that $K = [G,H]$. In effect, let $k = (g,h) \in K$. Then $(g, h^{-1}) \in L \subseteq G \rtimes H$ and therefore we have

$$(g,h^{-1}) = \big({g_1}^{h_1} g_1^{-1}, {h_1}^{g_1} h_1^{-1}\big) \cdots \big({g_k}^{h_k} g_k^{-1}, {h_k}^{g_k} h_k^{-1}\big) = \big({g_k}^{h_k} g_k^{-1} \cdots {g_1}^{h_1} g_1^{-1}, {h_1}^{g_1} h_1^{-1} \cdots {h_k}^{g_k} h_k^{-1}\big).$$

Hence,

$$g = {g_k}^{h_k} g_k^{-1} \cdots {g_1}^{h_1} g_1^{-1} \quad \text{and} \quad h = {}^{g_k} h_k h_k^{-1} \cdots {}^{g_1} h_1 h_1^{-1}.$$

Thus $k = ({g_k}^{h_k} g_k^{-1}, {}^{g_k} h_k h_k^{-1}) \cdots ({g_1}^{h_1} g_1^{-1}, {}^{g_1} h_1 h_1^{-1})$, i.e., $K = [G,H]$ and by (6.13) the homomorphism φ is surjective.

By Definition 4.13, $H_1(G,H)$ and $H_1(H,G)$ are the kernels of the homomorphisms $G \otimes H \xrightarrow{\lambda'} H$, $\lambda'(g \otimes h) = {}^g h h^{-1}$ and $G \otimes H \xrightarrow{\lambda} G$, $\lambda(g \otimes h) = g^h g^{-1}$ respectively, which are crossed modules (see Proposition 4.4), and therefore

$$q(H_1(G,H) \cap H_1(H,G)) \subseteq (H_1(G,H) \cap H_1(H,G)) \subseteq Z(G \otimes H).$$

First, we show that $\varphi(q(H_1(G,H) \cap H_1(H,G))) = 1$.

Let $(g_1 \otimes h_1) \cdots (g_m \otimes h_m) \in H_1(G,H) \cap H_1(H,G) \subseteq G \otimes H$; then by the formulas (6.4), (6.6), and [32, Lemma 1.13], we have

$$\begin{aligned}\varphi\big((g_1 \otimes h_1) \cdots (g_m \otimes h_m)\big)^q &= \big((g_1 \otimes h_1) \cdots (g_m \otimes h_m)\big)^q = \\ &= \Big\{({g_1}^{h_1} {g_1}^{-1}, {}^{g_1} h_1 h_1^{-1}) \cdots ({g_m}^{h_m} {g_m}^{-1}, {}^{g_m} h_m h_m^{-1})\Big\} = \big\{(1,1)\big\} = 1.\end{aligned}$$

Hence φ induces a natural homomorphism

$$\Phi : G \otimes H / q(H_1(G,H) \cap H_1(H,G)) \longrightarrow G \otimes {}^q H.$$

Now define a homomorphism $\Psi : G \otimes {}^q H \longrightarrow G \otimes H / q(H_1(G,H) \cap H_1(H,G))$ as follows:

$$\Psi(g \otimes h) = [g \otimes h], \qquad \Psi(\{k\}) = \big[\big((g_1 \otimes h_1) \cdots (g_n \otimes h_n)\big)^q\big],$$

since $k = ({g_1}^{h_1} {g_1}^{-1} \cdots {g_n}^{h_n} {g_n}^{-1}, {}^{g_1} h_1 {h_1}^{-1} \cdots {}^{g_n} h_n {h_n}^{-1})$ (see above). If

$$k = ({g'_1}^{h'_1} {g'_1}^{-1} \cdots {g'_m}^{h'_m} {g'_m}^{-1}, {}^{g'_1} h'_1 {h'_1}^{-1} \cdots {}^{g'_m} h'_m {h'_m}^{-1}),$$

then

$$(g_1 \otimes h_1) \cdots (g_n \otimes h_n)\big((g'_1 \otimes h'_1) \cdots (g'_m \otimes h'_m)\big)^{-1} \in H_1(G,H) \cap H_1(H,G) \subseteq Z(G \otimes H).$$

Hence

$$\begin{aligned}&\big((g_1 \otimes h_1) \cdots (g_n \otimes h_n)\big)^q \big((g'_1 \otimes h'_1) \cdots (g'_m \otimes h'_m)\big)^{-q} = \\ &\quad = \Big((g_1 \otimes h_1) \cdots (g_n \otimes h_n) \cdot \big((g'_1 \otimes h'_1) \cdots (g'_m \otimes h'_m)\big)^{-1}\Big)^q \in q\big(H_1(G,H) \cap H_1(H,G)\big).\end{aligned}$$

Thus Ψ is correctly defined. It is easy to see that Ψ commutes with relations (6.1)–(6.6) and $\Phi\Psi$ and $\Psi\Phi$ are identity maps. $\square$

Remark 6.9. Let G and H be two normal subgroups of a group P, and let them act on each other by conjugation in P. Then the tensor product modulo q, $G \otimes {}^q H$, of G and H may be considered as crossed $G \bowtie H$-modules ('absolute') or as crossed P-modules with canonical inclusions. In general these two definitions are different, but when $[G, H] = G \cap H$ (in the first case $K = [G, H]$ and in the second case $K = G \cap H$), they coincide, e.g., when $G = H = P$ is a perfect group.

Corollary 6.10. *If G is a perfect group then we have the following short exact sequence of groups*:

$$0 \longrightarrow qH_2(G) \longrightarrow G \otimes G \longrightarrow G \otimes {}^q G \longrightarrow 1.$$

Proof. It follows from Theorem 6.8 and the fact that if G is perfect, then by Proposition 4.5 and [97], $H_1(G, G) = H_2(G)$. $\square$

Now let us consider two infinite cyclic groups X and Y generated by x and y respectively, acting on each other by the following 'funny' actions:

$${}^x y = y^{-1}, \quad {}^y x = x^{-1},$$

which cannot arise as conjugation in a big group containing X and Y as normal subgroups.

In [51] it is proved that these actions are compatible and $X \otimes Y \cong \mathbb{Z}^2$ with basis $x \otimes y$ and $x^2 \otimes y$.

Proposition 6.11. *Let X and Y be infinite cyclic groups generated by x and y respectively, acting on each other by ${}^x y = y^{-1}$, ${}^y x = x^{-1}$. Then*

$$X \otimes {}^q Y \cong X \otimes Y \cong \mathbb{Z}^2.$$

Proof. Using Theorem 6.8 we have

$$X \otimes {}^q Y \cong X \otimes Y / q(\operatorname{Ker} \lambda \cap \operatorname{Ker} \lambda'),$$

where $\lambda : X \otimes Y \longrightarrow X$, $\lambda' : X \otimes Y \longrightarrow Y$ are canonical homomorphisms.

Now compute $\operatorname{Ker} \lambda \cap \operatorname{Ker} \lambda'$ using the fact that $X \otimes Y \cong \mathbb{Z} \times \mathbb{Z}$ and the basis of $X \otimes Y$ is $x \otimes y$, $x^2 \otimes y$.

We have

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{Z} \arrow[r, "\overline{\lambda}"] \arrow[d, "\cong"'] & \mathbb{Z} \arrow[d, "i\cong"] \\ X \otimes Y \arrow[r, "\lambda"'] & X \end{tikzcd}, \qquad \begin{tikzcd} \mathbb{Z} \times \mathbb{Z} \arrow[r, "\overline{\lambda}'"] \arrow[d, "\cong"'] & \mathbb{Z} \arrow[d, "j\cong"] \\ X \otimes Y \arrow[r, "\lambda'"'] & Y \end{tikzcd},$$

where $i(x) = 1$, $j(y) = 1$, and $\overline{\lambda}$, $\overline{\lambda}'$ are defined as follows:

$$\overline{\lambda}(m, n) = i\lambda\big((x \otimes y)^m (x^2 \otimes y)^n\big) = 2m + 4n,$$
$$\overline{\lambda}'(m, n) = j\overline{\lambda}\big((x \otimes y)^m (x^2 \otimes y)^n\big) = -2m.$$

It is easy to see that $\operatorname{Ker} \overline{\lambda} \cap \operatorname{Ker} \overline{\lambda}' = 0$. $\square$

Let $I(G, q) = \operatorname{Ker} \widetilde{\epsilon}$ with $\widetilde{\epsilon} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}/q$, $\widetilde{\epsilon}\Big(\sum\limits_i n_i g_i\Big) = \Big[\sum\limits_i n_i\Big]$. It is easy to see that an element $\sum\limits_i n_i g_i$ of $\mathbb{Z}[G]$ belongs to $I(G, q)$ if and only if q divides $\sum\limits_i n_i$.

Proposition 6.12. *There is a short exact sequence of G-modules:*

$$0 \longrightarrow IG \xrightarrow{\alpha} I(G, q) \xrightarrow{\varphi} \mathbb{Z} \longrightarrow 0,$$

where α is the natural inclusion.

Proof. Let define the homomorphism φ by the following formula:

$$\varphi(n_1g_1+\cdots+n_kg_k)=\frac{n_1+\cdots+n_k}{q},$$

since q divides $n_1+\cdots+n_k$.

It is easy to see that $\varphi\alpha=0$ and if $\varphi(n_1g_1+\cdots+n_kg_k)=0$ for any $n_1g_1+\cdots+n_kg_k\in I(G,q)$, then $n_1+\cdots+n_k=0$ and thus $n_1g_1+\cdots+n_kg_k\in IG$. $\square$

Assume that G is a group and A is a G-module. Let us define a homomorphism $\Phi : G\otimes{}^qA\cong G\otimes A/qH_1(G,A)\longrightarrow I(G,q)\otimes_G A$ as follows: $\Phi[g\otimes a]=(g-e)\otimes a$. We must show correctness. In effect, let $g_1\otimes a_1\cdots g_n\otimes a_n\in H_1(G,A)\subseteq G\otimes A$ i.e., $g_1a_1-a_1+\cdots+g_na_n-a_n=0$; then $(g_1\otimes a_1\cdots g_n\otimes a_n)^q\longmapsto q(g_1-e)\otimes a_1+\cdots+q(g_n-e)\otimes a_n=qe\otimes(g_1a_1-a_1+\cdots+g_na_n-a_n)=qe\otimes 0=0.$

Proposition 6.13. *Let G be a group and A be a G-module. There is a short exact sequence of groups*

$$0\longrightarrow G\otimes{}^qA\xrightarrow{\Phi}I(G,q)\otimes_G A\xrightarrow{\varphi\otimes 1_A}q\mathbb{Z}\otimes_G A\longrightarrow 0,$$

where φ is defined in Proposition 6.12.

Proof. By Proposition 6.12 and if we replace $\mathbb{Z}$ by its isomorphic G-module $q\mathbb{Z}$, we obtain the commutative diagram of groups with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & IG \arrow[r] \arrow[d, "\times q"] & \mathbb{Z}[G] \arrow[r, "\epsilon"] \arrow[d, "\times q"] & q\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & IG \arrow[r, "\alpha"'] & I(G,q) \arrow[r, "\varphi"'] & q\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

This diagram induces the following commutative diagram of left derived functors of the functor $-\otimes_G A$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & H_1(G,A) \arrow[r] \arrow[d, equal] & IG\otimes_G A \arrow[r] \arrow[d, "\times q"] & \mathbb{Z}[G]\otimes_G A \arrow[r] \arrow[d, "\times q"] & q\mathbb{Z}\otimes_G A \arrow[r] & 0 \\
\cdots \arrow[r] & \bullet \arrow[r] & H_1(G,A) \arrow[r] & IG\otimes_G A \arrow[r] & I(G,q)\otimes_G A \arrow[r] & q\mathbb{Z}\otimes_G A \arrow[r] & 0
\end{tikzcd}, \tag{6.14}$$

where the rows are long exact sequences of groups. From [53], Theorem 6.8, and (6.14) follows the assertion. $\square$

Remark 6.14. We can consider this theorem as a generalization of D.Guin's isomorphism

$$G\otimes A\cong IG\otimes_G A,\qquad g\otimes a\longmapsto (g-e)\otimes a,$$

when $q=0$ (see [53, Proposition 3.2]).

1.3. Applications to algebraic K-theory with $\mathbb{Z}/q$ coefficients of local rings. This subsection is devoted to an application of the non-Abelian tensor product modulo q to the algebraic K-theory with $\mathbb{Z}/q$ coefficients of Browder [9].

Recall the definition of algebraic K-functors with $\mathbb{Z}/q$ coefficients [9].

The algebraic K-functors of a discrete ring A has been defined by Quillen [110] by

$$K_i(A)=\pi_i(BGL(A)^+),\quad i\geq 1,$$

where $GL(A)=\varinjlim_n GL(n,A)$ and $BGL(A)$ is its classifying space, and $BGL(A)^+$ is the plus construction on $BGL(A)$ obtained by attaching 2-cells and 3-cells to kill the subgroup of elementary matrices $E(A)\subset GL(A)=\pi_1(BGL(A))$ in such a way that $H_*(BGL(A))\cong H_*(BGL(A)^+)$. Then define

$$K_i(A;\mathbb{Z}/q)=\pi_i(BGL(A)^+;\mathbb{Z}/q)=\text{the homotopy group with } \mathbb{Z}/q \text{ coefficients},$$

where $\pi_i(X) = [S^i, X]$ =homotopy classes of maps of the sphere S^i to X, and $\pi_i(X; \mathbb{Z}/q) = [Y^i, X]$, where $Y^i = S^{i-1} \cup_q e^i$ (attaching an i-cell by a map of degree q).

We can obtain the universal coefficient formula for algebraic K-functors with $\mathbb{Z}/q$ coefficients (see [9, 102]).

Proposition 6.15. *Let A be a discrete ring. Then for $q > 1$, there is a short exact sequence of groups*

$$0 \longrightarrow K_n(A) \otimes \mathbb{Z}/q \longrightarrow K_n(A, \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(K_{n-1}(A), \mathbb{Z}/q) \longrightarrow 0.$$

Let A be a ring with unit. Then $\mathrm{Sym}(A)$ is the group generated by the symbols $\{u, v\}$, where $u, v \in A^*$ (A^* is the group of units of the ring A), subject to the relations

(S1) $\{u, 1-u\} = 1,\ u \neq 1,\ u,\ 1-u \in A^*$;

(S2) $\{uu', v\} = \{u, v\}\{u', v\}$;

(S3) $\{u, vv'\} = \{u, v\}\{u, v'\}$.

Note that $\mathrm{Sym}(A)$ is an Abelian group. It is well known, by Matsumoto's theorem, that for any field A there exists the isomorphism $K_2(A) \cong \mathrm{Sym}(A)$ (see [97, Sec. 11, Theorem 11.1]).

Assume A is a (noncommutative) local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$. From [53, 84] it is known that there exists a group $D_0(A)$ generated by elements $\{u, v\}$, where $u, v \in A^*$, subject to the relations

(U0) $\{u, 1-u\} = 1,\ u \neq 1,\ u,\ 1-u \in A^*$;

(U1) $\{uu', v\} = {}^u\{u', v\}\{u, v\}$;

(U2) $\{u, vw\}\{v, wu\}\{w, uv\} = 1$,

where ${}^u\{v, w\} = \{{}^u v, {}^u w\}$, such that there is a short exact sequence of crossed A^*-modules

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(A) & \longrightarrow & D_0(A) & \longrightarrow & [A^*, A^*] & \longrightarrow & 1\ , \\ & & & & \{u, v\} & \longmapsto & [u, v]\ . & & \end{array} \tag{6.15}$$

From this exact sequence D.Guin obtained his six-term exact non-Abelian homology sequence [53, Theorem 4.2]

$$\begin{aligned} &(A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A) \longrightarrow H_1(A^*, D_0(A)) \longrightarrow H_1(A^*, [A^*, A^*]) \longrightarrow \\ &\qquad \longrightarrow K_2(A) \longrightarrow \mathrm{Sym}(A) \longrightarrow [A^*, A^*]/[A^*, [A^*, A^*]] \longrightarrow 1\ , \end{aligned}$$

where the first non-Abelian homology of group A^* is defined as $H_1(A^*, D_0(A)) = \mathrm{Ker}(\lambda' : A^* \otimes D_0(A) \longrightarrow D_0(A))$ and $H_1(A^*, [A^*, A^*]) = \mathrm{Ker}(\lambda' : A^* \otimes [A^*, A^*] \longrightarrow [A^*, A^*])$ (for the definition of λ', see the proof of Theorem 6.8).

Let A be a local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$. Let us denote by $\mathrm{Sym}(A; \mathbb{Z}/q)$ the pushout

$$(push) \qquad \begin{array}{ccc} K_2(A)/q & \longrightarrow & K_2(A; \mathbb{Z}/q) \\ \downarrow & & \downarrow \\ \mathrm{Sym}(A)/q & \longrightarrow & \mathrm{Sym}(A; \mathbb{Z}/q) \end{array} .$$

Proposition 6.16. *Let A be a field and $q > 1$. Then there exists an isomorphism*

$$K_2(A; \mathbb{Z}/q) \cong \mathrm{Sym}(A; \mathbb{Z}/q).$$

Proof. From the definition of $\mathrm{Sym}(A;\mathbb{Z}/q)$, Proposition 6.15, and the Matsumoto theorem, we have the following commutative diagram of groups with exact rows:

$$\begin{tikzcd}
K_2(A) \arrow[r, "q"] \arrow[d, "\cong"'] & K_2(A) \arrow[r] \arrow[d, "\cong"'] & K_2(A;\mathbb{Z}/q) \arrow[r] \arrow[d] & K_1(A) \arrow[r, "q"] \arrow[d, equal] & K_1(A) \arrow[d, equal] \\
\mathrm{Sym}(A) \arrow[r, "q"'] & \mathrm{Sym}(A) \arrow[r] & \mathrm{Sym}(A;\mathbb{Z}/q) \arrow[r] & K_1(A) \arrow[r, "q"'] & K_1(A)
\end{tikzcd}.$$

Now the assertion follows from the five lemma. □

This is an analog of Matsumoto's Theorem in the q-modular aspect.

Let A be a (noncommutative) local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$ and $q > 1$. By Proposition 6.5 the short exact sequence (6.15) of crossed A^*-modules induces the exact sequence of groups

$$\operatorname{Ker}\xi'_{K_2(A)} \longrightarrow \operatorname{Ker}\xi'_{D_0(A)} \longrightarrow \operatorname{Ker}\xi'_{[A^*,A^*]} \longrightarrow$$
$$\longrightarrow \operatorname{Coker}\xi'_{K_2(A)} \longrightarrow \operatorname{Coker}\xi'_{D_0(A)} \longrightarrow \operatorname{Coker}\xi'_{[A^*,A^*]} \longrightarrow 0,$$

where $\xi'_{K_2(A)} : A^* \otimes {}^q K_2(A) \longrightarrow K_2(A)$, $\xi'_{D_0(A)} : A^* \otimes {}^q D_0(A) \longrightarrow D_0(A)$ and $\xi'_{[A^*,A^*]} : A^* \otimes {}^q[A^*,A^*] \longrightarrow [A^*,A^*]$ are defined according to (6.12). It is easy to see that $\operatorname{Coker}\xi' = (\operatorname{Coker}\lambda')/q$; therefore, the calculations of D. Guin [53] imply that

$$\operatorname{Coker}\xi'_{K_2(A)} = (\operatorname{Coker}\lambda'_{K_2(A)})/q = K_2(A)/q,$$
$$\operatorname{Coker}\xi'_{D_0(A)} = (\operatorname{Coker}\lambda'_{D_0(A)})/q = \mathrm{Sym}(A)/q$$

and

$$\operatorname{Coker}\xi'_{[A^*,A^*]} = (\operatorname{Coker}\lambda'_{[A^*,A^*]})/q = \big([A^*,A^*]/\big[A^*,[A^*,A^*]\big]\big)/q.$$

By [53, Lemma 4.1], A^* acts trivially on $K_2(A)$. Then using Proposition 1.6 [32], the same arguments as in Theorem 6.8, and the splitting $\psi : A^* \otimes {}^q K_2(A) \longrightarrow (A^* \otimes K_2(A))/q$ defined by $\psi(a \otimes x) = [a \otimes x]$, $\psi(\{(1,y)\}) = 1$ for $a \in A$, $x, y \in K_2$, we see that there is an exact sequence of Abelian groups

$$1 \longrightarrow q(A^* \otimes K_2(A)) \longrightarrow A^* \otimes K_2(A) \longrightarrow A^* \otimes {}^q K_2(A) \longrightarrow K_2(A) \longrightarrow 1\ .$$

Moreover, $A^* \otimes {}^q K_2(A) \cong (A^* \otimes K_2(A))/q \oplus K_2(A)$. The triviality of actions induces $A^* \otimes K_2(A) \cong (A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A)$. Now it is easy to see that $\operatorname{Ker}\xi'_{K_2(A)} \cong ((A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A))/q \oplus \operatorname{Tor}(K_2(A),\mathbb{Z}/q)$.

We can construct the connecting homomorphism

$$\operatorname{Ker}\xi'_{[A^*,A^*]} \longrightarrow \operatorname{Coker}\xi'_{K_2(A)} = K_2(A)/q \longrightarrow K_2(A;\mathbb{Z}/q).$$

Then the following commutative diagram of groups with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & K_2(A)/q \arrow[r] \arrow[d] & K_2(A;\mathbb{Z}/q) \arrow[r] \arrow[d] & \operatorname{Tor}(K_1(A),\mathbb{Z}/q) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Sym}(A)/q \arrow[r] \arrow[d] & \mathrm{Sym}(A;\mathbb{Z}/q) \arrow[r] \arrow[d] & \operatorname{Tor}(K_1(A),\mathbb{Z}/q) \arrow[r] & 0 \\
 & \big([A^*,A^*]/\big[A^*,[A^*,A^*]\big]\big)/q \arrow[d] & \big([A^*,A^*]/\big[A^*,[A^*,A^*]\big]\big)/q \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

induces

Theorem 6.17. *Let A be a (noncommutative) local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$ and $q > 1$. Then there is an exact sequence of groups*

$$\begin{aligned}\big((A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A)\big)/q \oplus \operatorname{Tor}(K_2(A), \mathbb{Z}/q) \longrightarrow \operatorname{Ker} \xi'_{D_0(A)} \longrightarrow \operatorname{Ker} \xi'_{[A^*,A^*]} \longrightarrow \\ \longrightarrow K_2(A; \mathbb{Z}/q) \longrightarrow \operatorname{Sym}(A; \mathbb{Z}/q) \longrightarrow ([A^*, A^*]/[A^*, [A^*, A^*]])/q \longrightarrow 0.\end{aligned}$$

2. Mod q Homology and Cohomology of Chain Complexes

Given a covariant (contravariant) functor $\Phi : \mathcal{D}_R \longrightarrow \mathcal{C}_{\mathbb{Z}}$ and an object in the category $\mathcal{D}_R$

$$C_* \equiv \ \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots , \quad n \in \mathbb{Z},$$

the product by q defines a morphism ${}_{\times}q : C_* \longrightarrow C_*$ of chain complexes and the mapping cone of this morphism

$$\cdots \longrightarrow C_{n+1} \oplus C_n \xrightarrow{\widetilde{\partial}_{n+1}} C_n \oplus C_{n-1} \xrightarrow{\widetilde{\partial}_n} C_{n-1} \oplus C_{n-2} \xrightarrow{\widetilde{\partial}_{n-1}} \cdots ,$$

$\widetilde{\partial}_n(x_n, x_{n-1}) = (\partial_n(x_n) + qx_{n-1}, -\partial_{n-1}(x_{n-1}))$, denoted by $\operatorname{Mc}(C_*, q)_*$.

Definition 6.18. For $n \in \mathbb{Z}$

(i) the mod q homology of the complex C_* is given by

$$H_n(C_*; \mathbb{Z}/q) := H_n(\operatorname{Mc}(C_*, q)_*);$$

(ii) the Φ-homology (Φ-cohomology) of C_* is given by

$$H_n^{\Phi}(C_*) := H_n(\Phi(C_*)) \qquad (H^n_{\Phi}(C_*) := H_{-n}(\Phi(C_*)))$$

and the mod q Φ-homology (Φ-cohomology) of C_* is given by

$$\begin{gathered} H_n^{\Phi}(C_*; \mathbb{Z}/q) := H_n(\Phi(C_*); \mathbb{Z}/q) = H_n(\operatorname{Mc}(\Phi(C_*), q)_*) \\ \big(H^n_{\Phi}(C_*; \mathbb{Z}/q) := H_{-n+1}(\Phi(C_*); \mathbb{Z}/q) = H_{-n+1}(\operatorname{Mc}(\Phi(C_*), q)_*) \big). \end{gathered}$$

Proposition 6.19 (universal coefficient formula for mod q homology)**.** *Given $C_* \in \mathcal{D}_R$, we have an exact sequence*

$$0 \longrightarrow H_n(C_*) \otimes \mathbb{Z}/q \longrightarrow H_n(C_*; \mathbb{Z}/q) \longrightarrow \operatorname{Tor}(H_{n-1}(C_*), \mathbb{Z}/q) \longrightarrow 0 , \quad n \in \mathbb{Z}.$$

Proof. The mapping cone gives rise to an exact homology sequence

$$\cdots \longrightarrow H_n(C_*) \xrightarrow{\times q} H_n(C_*) \longrightarrow H_n(C_*; \mathbb{Z}/q) \longrightarrow H_{n-1}(C_*) \xrightarrow{\times q} H_{n-1}(C_*) \longrightarrow \cdots .$$

Now the product by q in a module A has cokernel $A/qA \cong A \otimes \mathbb{Z}/q$ and kernel $\operatorname{Tor}(A, \mathbb{Z}/q)$. The exactness of the homology sequence gives the result. □

Corollary 6.20 (universal coefficient formula for mod q Φ-cohomology)**.** *Given $C_* \in \mathcal{D}_R$ and a covariant (contravariant) functor $\Phi : \mathcal{D}_R \longrightarrow \mathcal{C}_{\mathbb{Z}}$ we have an exact sequences*

$$0 \longrightarrow H_n^{\Phi}(C_*) \otimes \mathbb{Z}/q \longrightarrow H_n^{\Phi}(C_*; \mathbb{Z}/q) \longrightarrow \operatorname{Tor}(H_{n-1}^{\Phi}(C_*), \mathbb{Z}/q) \longrightarrow 0$$

$$\big(\ 0 \longrightarrow H^{n-1}_{\Phi}(C_*) \otimes \mathbb{Z}/q \longrightarrow H^n_{\Phi}(C_*; \mathbb{Z}/q) \longrightarrow \operatorname{Tor}(H^n_{\Phi}(C_*), \mathbb{Z}/q) \longrightarrow 0 \ \big),$$

$n \in \mathbb{Z}$.

Remark 6.21. In the examples we shall consider that the morphism $\Phi({}_{\times}q)$ is the product by q and, up to isomorphism, $\mathrm{Mc}(\Phi(C_*), q)_n = \Phi(\mathrm{Mc}(C_*, q))_n$ for a covariant functor Φ and $\mathrm{Mc}(\Phi(C_*), q)_{n+1} = \Phi(\mathrm{Mc}(C_*, q))_n$ for a contravariant functor Φ. This motivates the index shift in the definition of mod q Φ-cohomology.

Let $q = rs$; then there are canonical morphisms of chain complexes

$$\alpha_{r,*} : \mathrm{Mc}(C_*, q)_* \longrightarrow \mathrm{Mc}(C_*, r)_* \quad \text{and} \quad \alpha_{s,*} : \mathrm{Mc}(C_*, q)_* \longrightarrow \mathrm{Mc}(C_*, s)_*$$

given by $\alpha_{r,n}(x_n, x_{n-1}) = (x_n, sx_{n-1})$ and $\alpha_{s,n}(x_n, x_{n-1}) = (x_n, rx_{n-1})$ for all $n \in \mathbb{Z}$, respectively. It follows that we obtain a canonical homomorphism

$$\alpha_n : H_n(C_*; \mathbb{Z}/q) \longrightarrow H_n(C_*; \mathbb{Z}/r) \times H_n(C_*; \mathbb{Z}/s), \quad n \in \mathbb{Z}.$$

Theorem 6.22. *If $q = rs$ and the integers r, s are relatively prime, we have a canonical isomorphism*

$$H_n(C_*; \mathbb{Z}/q) \cong H_n(C_*; \mathbb{Z}/r) \times H_n(C_*; \mathbb{Z}/s)$$

for all $n \in \mathbb{Z}$.

Proof. The inverse homomorphism to α_n, $n \in \mathbb{Z}$, will be constructed. Since r and s are relatively prime, there exist $k, l \in \mathbb{Z}$ such that

$$kr + ls = 1. \tag{6.16}$$

Define two morphisms of chain complexes

$$\beta_{r,*} : \mathrm{Mc}(C_*, r)_* \longrightarrow \mathrm{Mc}(C_*, q)_*$$

and

$$\beta_{s,*} : \mathrm{Mc}(C_*, s)_* \longrightarrow \mathrm{Mc}(C_*, q)_*$$

by

$$\beta_{r,n}(x_n, x_{n-1}) = (lsx_n, lx_{n-1}) \quad \text{and} \quad \beta_{s,n}(x_n, x_{n-1}) = (krx_n, kx_{n-1})$$

for $n \in \mathbb{Z}$. These maps are compatible with boundary operators. We verify it for $\beta_{r,*}$. In fact,

$$\begin{aligned}\widetilde{\partial}_n \beta_{r,n}(x_n, x_{n-1}) = \widetilde{\partial}_n(lsx_n, lx_{n-1}) = (ls\partial_n(x_n) + lqx_{n-1}, -l\partial_{n-1}(x_{n-1})) = \\ = \beta_{r,n-1}\big(\partial_n(x_n) + rx_{n-1}, -\partial_{n-1}(x_{n-1})\big) = \beta_{r,n-1}\widetilde{\partial}_n(x_n, x_{n-1}).\end{aligned}$$

Therefore, we obtain a homomorphism

$$\beta_n : H_n(C_*; \mathbb{Z}/r) \times H_n(C_*; \mathbb{Z}/s) \longrightarrow H_n(C_*; \mathbb{Z}/q), \quad n \in \mathbb{Z},$$

induced by $\beta_{r,n}$ and $\beta_{s,n}$. It remains to prove that $\alpha_*\beta_*$ and $\beta_*\alpha_*$ are identity maps.

Let (x_n, x_{n-1}) be an nth chain of $\mathrm{Mc}(C_*, q)_*$. Then, using (6.16), we have

$$\beta_n\alpha_n(x_n, x_{n-1}) = \beta_n\big((x_n, sx_{n-1}), (x_n, rx_{n-1})\big) = (lsx_n, lsx_{n-1}) + (krx_n, krx_{n-1}) = (x_n, x_{n-1}),$$

thus $\beta_*\alpha_* = 1$.

Let (x_n, x_{n-1}) be an nth cycle of $\mathrm{Mc}(C_*, r)_*$, i.e.,

$$\partial_n(x_n) + rx_{n-1} = 0, \qquad \partial_{n-1}(x_{n-1}) = 0. \tag{6.17}$$

We have

$$\alpha_n\beta_n(x_n, x_{n-1}) = \alpha_n(lsx_n, lx_{n-1}) = (lsx_n, lsx_{n-1}) + (lsx_n, lrx_{n-1}).$$

Whence the equality

$$(x_n, x_{n-1}) - \alpha_n\beta_n(x_n, x_{n-1}) = (krx_n, krx_{n-1}) + (-lsx_n, -lrx_{n-1})$$

in the R-module $\mathrm{Mc}(C_*, r)_n \times \mathrm{Mc}(C_*, s)_n$.

By (6.17) we get

$$\widetilde{\partial}_{n+1}(0, kx_n) = (krx_n, -k\partial_n(x_n)) = (krx_n, krx_{n-1})$$

and

$$\widetilde{\partial}_{n+1}(0, -lx_n) = (-lsx_n, l\partial_n(x_n)) = (-lsx_n, -lrx_{n-1}).$$

Therefore

$$(x_n, x_{n-1}) - \alpha_n\beta_n(x_n, x_{n-1}) = \widetilde{\partial}_{n+1}\big((0, kx_n), (0, -lx_n)\big).$$

Obviously, the same is true for an nth cycle of $\mathrm{Mc}(C_*, s)_*$. Thus $\alpha_*\beta_* = 1$. □

Corollary 6.23. *Let $\Phi : \mathcal{D}_R \longrightarrow \mathcal{C}_{\mathbb{Z}}$ be a covariant (contravariant) functor, $C_* \in \mathcal{D}_R$ and $q = rs$ with r and s relatively prime integers. Then there is a canonical isomorphism*

$$H_n^{\Phi}(C_*; \mathbb{Z}/q) \cong H_n^{\Phi}(C_*; \mathbb{Z}/r) \times H_n^{\Phi}(C_*; \mathbb{Z}/s)$$
$$\big(\, H^n_{\Phi}(C_*; \mathbb{Z}/q) \cong H^n_{\Phi}(C_*; \mathbb{Z}/r) \times H^n_{\Phi}(C_*; \mathbb{Z}/s)\,\big)$$

for all $n \in \mathbb{Z}$.

As the product by q is obviously functorial, the homotopy properties of Φ, if any, induce homotopy properties on the mod q Φ-homology (Φ-cohomology).

Lemma 6.24.

(i) *Let $C_* \in \mathcal{D}_R$, $\Phi' : \mathcal{D}_R \longrightarrow \mathcal{C}_{\mathbb{Z}}$ be a second covariant (contravariant) functor and $\theta : \Phi \longrightarrow \Phi'$ a natural transformation, such that $\theta(C_*)$ is a weak equivalence between $\Phi(C_*)$ and $\Phi'(C_*)$. Then θ induces isomorphisms*

$$H_n^{\Phi}(C_*; \mathbb{Z}/q) \cong H_n^{\Phi'}(C_*; \mathbb{Z}/q) \quad \big(\, H^n_{\Phi}(C_*; \mathbb{Z}/q) \cong H^n_{\Phi'}(C_*; \mathbb{Z}/q)\,\big)$$

for all $n \in \mathbb{Z}$.

(ii) *Assume that Φ is a homotopy functor, i.e., homotopic complexes are sent to homotopic complexes. Let $C_*, C'_* \in \mathcal{D}_R$ be homotopic. Then we have isomorphisms*

$$H_n^{\Phi}(C_*; \mathbb{Z}/q) \cong H_n^{\Phi}(C'_*; \mathbb{Z}/q) \quad \big(\, H^n_{\Phi}(C_*; \mathbb{Z}/q) \cong H^n_{\Phi}(C'_*; \mathbb{Z}/q)\,\big)$$

for all $n \in \mathbb{Z}$.

Proof. The proof will be only for the mod q Φ-cohomology.

(i) As the mapping cone construction is functorial we have a commutative diagram with exact rows

$$\begin{tikzcd}
H^{n-1}_{\Phi}(C_*) \arrow[r] \arrow[d] & H^{n-1}_{\Phi}(C_*) \arrow[r] \arrow[d] & H^n_{\Phi}(C_*; \mathbb{Z}/q) \arrow[r] \arrow[d] & H^n_{\Phi}(C_*) \arrow[r] \arrow[d] & H^n_{\Phi}(C_*) \arrow[d] \\
H^{n-1}_{\Phi'}(C_*) \arrow[r] & H^{n-1}_{\Phi'}(C_*) \arrow[r] & H^n_{\Phi'}(C_*; \mathbb{Z}/q) \arrow[r] & H^n_{\Phi'}(C_*) \arrow[r] & H^n_{\Phi'}(C_*)
\end{tikzcd}.$$

By hypothesis the two vertical maps on the left and the two on the right are isomorphisms. The five-lemma gives the result.

(ii) It works the same with the diagram

$$\begin{tikzcd}
H^{n-1}_{\Phi}(C'_*) \arrow[r] \arrow[d] & H^{n-1}_{\Phi}(C'_*) \arrow[r] \arrow[d] & H^n_{\Phi}(C'_*; \mathbb{Z}/q) \arrow[r] \arrow[d] & H^n_{\Phi}(C'_*) \arrow[r] \arrow[d] & H^n_{\Phi}(C'_*) \arrow[d] \\
H^{n-1}_{\Phi}(C_*) \arrow[r] & H^{n-1}_{\Phi}(C_*) \arrow[r] & H^n_{\Phi}(C_*; \mathbb{Z}/q) \arrow[r] & H^n_{\Phi}(C_*) \arrow[r] & H^n_{\Phi}(C_*)
\end{tikzcd}.$$

□

Example 6.25. Let K_* be an object of the category $\mathcal{C}_R$.

(i) Let $\Phi : \mathcal{D}_R \longrightarrow \mathcal{C}_{\mathbb{Z}}$ be the covariant functor defined by the tensor product complex, i.e., $\Phi(C_*) = (C_* \otimes K_*)_*$, where

$$(C_* \otimes K_*)_n = \bigoplus_{i\in\mathbb{Z}} C_i \otimes_R K_{n-i}$$

with the differential Δ given by

$$\Delta(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy, \quad x \in C_i,\ y \in K_{n-i}.$$

(See any book on algebraic homology.) Then we write $H_n(C_*, K_*) = H_n^{\Phi}(C_*)$ and $H_n(C_*, K_*; \mathbb{Z}/q) = H_n^{\Phi}(C_*; \mathbb{Z}/q)$.

(ii) Let $\Phi : \mathcal{D}_R \longrightarrow \mathcal{C}_{\mathbb{Z}}$ be the contravariant functor defined by $\Phi(C_*) = \mathcal{H}om(C_*, K_*)_*$, where

$$\mathcal{H}om(C_*, K_*)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, K_{i+n})$$

with the differential Δ given by

$$(\Delta f)_i(x) = d(f_i(x)) + (-1)^{n+1} f_{i-1}(d(x))$$

for $f = (f_i) \in \mathcal{H}om(C_*, K_*)_n$ and $x \in C_i$. Then we write $H^n(C_*, K_*) = H^n_{\Phi}(C_*)$ and $H^n(C_*, K_*; \mathbb{Z}/q) = H^n_{\Phi}(C_*; \mathbb{Z}/q)$.

If the complex K_* is concentrated in degree 0, we get with $H_n(C_*, K_*)$ and $H^n(C_*, K_*)$ the usual homology and cohomology, respectively, with coefficients in K_0. If A is an R-module and K_* a resolution of A, the morphism $K_* \longrightarrow A$ defined by the map $K_0 \longrightarrow A$ induces isomorphisms

$$H_n(C_*, K_*; \mathbb{Z}/q) \longrightarrow H_n(C_*, A; \mathbb{Z}/q), \quad H^n(C_*, K_*; \mathbb{Z}/q) \longrightarrow H^n(C_*, A; \mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$ by Lemma 6.24 (i).

The "internal Hom functor" in the category of chain complexes of R-modules was first studied by R. Brown [11].

Lemma 6.26. *Let $C_* \in \mathcal{D}_R$ and $K_* \in \mathcal{C}_R$. We have, for all $n \in \mathbb{Z}$, a canonical isomorphism*

$$\mathcal{H}om\big(\mathrm{Mc}(C_*, q)_*, K_*\big)_n \cong \mathrm{Mc}\left(\mathcal{H}om(C_*, K_*), q\right)_{n+1}.$$

Proof. We have

$$\mathrm{Hom}_R(\mathrm{Mc}(C_*, q)_i, K_{n+i}) = \mathrm{Hom}_R(C_i \oplus C_{i-1}, K_{n+i}) \cong \mathrm{Hom}_R(C_i, K_{n+i}) \oplus \mathrm{Hom}_R(C_{i-1}, K_{n+i})$$

which gives, taking the product over $\mathbb{Z}$ and exchanging the factors on the right-hand side of the equality,

$$\mathcal{H}om(\mathrm{Mc}(C_*, q)_*, K_*)_n \cong \mathcal{H}om(C_*, K_*)_n \oplus \mathcal{H}om(C_*, K_*)_{n+1} = \mathrm{Mc}(\mathcal{H}om(C_*, K_*), q)_{n+1}.$$

$\square$

Another example of the functor Φ will be considered in Sec. 7. Note that all results of this section are true when $\mathcal{D}_R$ is an additive subcategory of $\mathcal{C}_R$.

3. Mod q Homology of Groups

We begin this section by introducing a mod q homology of groups by using Definition 6.18 and then expressing it as the Tor_* functors. Then we will establish some properties of mod q homology groups and do some calculations. The relation of mod q homology of groups to the non-Abelian tensor product modulo q of groups, particularly with its non-Abelian left derived functors, will be studied in the next section.

Let G be a group, A a G-module, and $P_* \longrightarrow \mathbb{Z}$ a projective G-resolution of $\mathbb{Z}$. According to Example 6.25 $-\otimes A$ is a covariant functor from $\mathcal{D}_{\mathbb{Z}[G]}$ to $\mathcal{C}_{\mathbb{Z}}$.

Definition 6.27. The nth mod q homology, $H_n(G, A; \mathbb{Z}/q)$, of the group G with coefficients in the G-module A is

$$H_n(G, A; \mathbb{Z}/q) := H_n^{-\otimes A}(P_*; \mathbb{Z}/q), \quad n \geq 0.$$

Note that by Lemma 6.24(ii) these homology groups are well defined and do not depend on the choice of the projective G-resolution of $\mathbb{Z}$.

The following lemma is useful for expressing mod q homology of groups as the Tor_* functors.

Lemma 6.28. *The morphism* $\mathrm{Mc}(P_*, q)_* \longrightarrow \mathbb{Z}/q$ *defined by the composed map* $\mathrm{Mc}(P_*, q)_0 = P_0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/q$ *is a projective G-resolution of* $\mathbb{Z}/q$.

Proof. It is straightforward by the exact homology sequence of mapping cone [93] and the fact that $\mathbb{Z}$ is torsion free. □

Proposition 6.29. $H_n(G, A; \mathbb{Z}/q) \cong \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}/q, A), \quad n \geq 0.$

Proof. It follows from Lemma 6.28 and the fact that there is an isomorphism $(\mathrm{Mc}(P_*, q)_* \otimes A)_n \cong \mathrm{Mc}((P_* \otimes A)_*, q)_n$, $n \geq 0$. □

It is easy to see that $H_0(G, A; \mathbb{Z}/q) \cong H_0(G, A)/q$ and $H_n(G, \mathbb{Z}; \mathbb{Z}/q) \cong H_n(G, \mathbb{Z}/q)$, $n \geq 0$.

Proposition 6.30 (universal coefficient formulas)**.** *Let G be any group, A be a G-module, and $n \geq 0$. Then*

(a) *there is a short exact sequence of groups*

$$0 \longrightarrow H_n(G, A) \otimes \mathbb{Z}/q \longrightarrow H_n(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(H_{n-1}(G, A), \mathbb{Z}/q) \longrightarrow 0;$$

(b) *for a trivial G-module A there is a short exact sequence* (*splits nonnaturally*) *of groups*

$$0 \longrightarrow H_n(G, \mathbb{Z}/q) \otimes A \longrightarrow H_n(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(H_{n-1}(G, \mathbb{Z}/q), A) \longrightarrow 0.$$

Proof.

Assertion (a) follows directly from Corollary 6.20; assertion (b) can be proved classically. □

Proposition 6.31. *Let G be a finite group of order k ($|G| = k$) and $(k, q) = 1$; then*

$$H_n(G, A; \mathbb{Z}/q) = 0, \quad n \geq 2.$$

Proof. It is well known (see [93]) that $H_n(G, A)$, $n > 0$ is a group of exponent k. Then $H_n(G, A) \otimes \mathbb{Z}/q$, $n > 0$ is a group of exponent q and k and by assumption $H_n(G, A) \otimes \mathbb{Z}/q = 0$, $n > 0$. By the same reasoning $\mathrm{Tor}(H_n(G, A), \mathbb{Z}/q) = 0$, $n > 0$. It remains to apply Proposition 6.30 (a). □

Proposition 6.32. *Let G be a finite group and A be a divisible, torsion free G-module. Then*

$$\begin{aligned} &H_n(G, A; \mathbb{Z}/q) = 0, \quad n \geq 2, \\ &H_1(G, A; \mathbb{Z}/q) = \mathrm{Tor}(H_0(G, A), \mathbb{Z}/q), \\ &H_0(G, A; \mathbb{Z}/q) = 0. \end{aligned}$$

Proof. By Proposition 6.30(a) and [93], $H_n(G,A;\mathbb{Z}/q) = 0$, $n \geq 2$ and $H_1(G,A;\mathbb{Z}/q) = \mathrm{Tor}(H_0(G,A),\mathbb{Z}/q)$. As we know $H_0(G,A;\mathbb{Z}/q) \cong H_0(G,A)/q$. But $H_0(G,A) = A/I_G A$ and since A is divisible, then $H_0(G,A)$ is also a divisible group. Therefore $H_0(G,A;\mathbb{Z}/q) = 0$. □

Proposition 6.33. *Let F be a free group and A be a F-module, then $H_n(F,A;\mathbb{Z}/q) = 0$, $n \geq 3$.*

Proof. It follows from Proposition 6.30(a) and $H_n(F,A) = 0$, $n \geq 2$. □

Example 6.34. Let $\mathbb{Z}$ be the additive group of integers and $\mathbb{Q}/\mathbb{Z}$ be the quotient of the additive group of rational numbers by $\mathbb{Z}$. Assume that $\mathbb{Z}$ acts trivially on $\mathbb{Q}/\mathbb{Z}$. By Proposition 6.33 we obtain $H_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) = 0$, $n \geq 3$. Using Proposition 6.30(a) we have the following short exact sequences of groups

$$0 \longrightarrow H_2(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}/q \longrightarrow H_2(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) \longrightarrow \mathrm{Tor}(H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z}),\mathbb{Z}/q) \longrightarrow 0\ ,$$

$$0 \longrightarrow H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}/q \longrightarrow H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) \longrightarrow \mathrm{Tor}(H_0(\mathbb{Z},\mathbb{Q}/\mathbb{Z}),\mathbb{Z}/q) \longrightarrow 0\ .$$

It is easy to see that

$$H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) = H_0(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}.$$

Since

$$H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}/q = H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z})/q = (\mathbb{Q}/\mathbb{Z})/q = 0$$

(because $\mathbb{Q}$ is divisible) and $H_2(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) = 0$, we have

$$H_2(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) = H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) = \mathrm{Tor}(\mathbb{Q}/\mathbb{Z},\mathbb{Z}/q).$$

$\mathrm{Tor}(\mathbb{Q}/\mathbb{Z},\mathbb{Z}/q)$ is the kernel of the homomorphism $\mathbb{Q}/\mathbb{Z} \xrightarrow{\times q} \mathbb{Q}/\mathbb{Z}$ and by [22, Chap. VII, Proposition 2.2] we have $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z},\mathbb{Z}/q) = \mathbb{Z}/q$. Therefore

$$H_2(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) = H_1(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) = \mathbb{Z}/q.$$

Finally,

$$H_0(\mathbb{Z},\mathbb{Q}/\mathbb{Z};\mathbb{Z}/q) = H_0(\mathbb{Z},\mathbb{Q}/\mathbb{Z})/q = (\mathbb{Q}/\mathbb{Z})/q = 0.$$

A good example is to show that $H_*(G,A;\mathbb{Z}/q)$ is not isomorphic to $H_*(G,A)/q$ or $H_*(G,A/q)$. In this case $H_n(\mathbb{Z},(\mathbb{Q}/\mathbb{Z})/q) = 0$, $n \geq 0$ (since $(\mathbb{Q}/\mathbb{Z})/q = 0$) and $H_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z})/q = 0$, $n \geq 0$.

Now we compute the mod q homology of finite cyclic groups. Let $G = C_m(t)$ be a multiplicative cyclic group of order m and generated by t. It is well known that the elements

$$N = 1 + t + \cdots + t^{m-1}, \quad D = t - 1$$

induce G-module homomorphisms

$$N_* : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G], \quad N_* u = Nu,$$
$$D_* : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G], \quad D_* u = Du, \quad u \in \mathbb{Z}[G],$$

respectively, and the following long exact sequence of G-modules:

$$\cdots \xrightarrow{N_*} \mathbb{Z}[G] \xrightarrow{D_*} \mathbb{Z}[G] \xrightarrow{N_*} \mathbb{Z}[G] \xrightarrow{D_*} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a free G-resolution of $\mathbb{Z}$, which gives us the possibility to compute Eilenberg–Maclane homology $H_*(G,A)$ of the finite cyclic group $G = C_m(t)$ with coefficients in any G-module A (see [93]).

By Lemma 6.28, we obtain the following free G-resolution of $\mathbb{Z}/q$:

$$\cdots \xrightarrow{\widetilde{N}_*} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{\widetilde{D}_*} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{\widetilde{N}_*} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{\widetilde{\widetilde{D}}_*} \mathbb{Z}[G] \xrightarrow{\widetilde{\epsilon}} \mathbb{Z}/q \longrightarrow 0\ ,$$

where $\widetilde{\widetilde{D}}_*(u,u') = Du + qu'$, $\widetilde{N}_*(u,u') = (Nu + qu', -Du')$, and $\widetilde{D}_*(u,u') = (Du + qu', -Nu')$ for all $u, u' \in \mathbb{Z}[G]$. The mod q homology groups of the finite cyclic group $G = C_m(t)$ with coefficients in G-module A are the homology groups of the following chain complex of G-modules:

$$\cdots \xrightarrow{\widetilde{N}^*} A \oplus A \xrightarrow{\widetilde{D}^*} A \oplus A \xrightarrow{\widetilde{N}^*} A \oplus A \xrightarrow{\widetilde{\widetilde{D}}^*} A,$$

where

$$\widetilde{\widetilde{D}}^*(a,a') = Da + qa', \quad \widetilde{N}^*(a,a') = (Na + qa', -Da'), \quad \widetilde{D}^*(a,a') = (Da + qa', -Na')$$

for all $a, a' \in A$.

It is easy to see that the mod q homology of finite cyclic groups is periodic from $n \geq 2$ with period 2. We obtain the following proposition.

Proposition 6.35. *For a finite cyclic group $C_m(t)$ of order m and with generator t and for any $C_m(t)$-module A*

$$\begin{aligned}
H_0(C_m, A; \mathbb{Z}/q) &= A/\widetilde{\widetilde{D}}^* = H_0(C_m, A)/q = \operatorname{Coker} D^*, \quad \textit{where } D^* : A/q \longrightarrow A/q,\ D^*[a] = [Da],\\
H_1(C_m, A; \mathbb{Z}/q) &= \big[(a,a') | Da + qa' = 0\big]/\widetilde{N}^*(A \oplus A),\\
H_{2n}(C_m, A; \mathbb{Z}/q) &= \big[(a,a') | Na + qa' = 0, ta' = a'\big]/\widetilde{D}^*(A \oplus A), \quad n \geq 1,\\
H_{2n+1}(C_m, A; \mathbb{Z}/q) &= \big[(a,a') | Da + qa' = 0, Na = 0\big]/\widetilde{N}^*(A \oplus A), \quad n \geq 1.
\end{aligned}$$

Proposition 6.36. *Let G be a group and $0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0$ be a short exact sequence of G-modules. Then there is a long exact sequence of mod q homology groups*

$$\begin{aligned}
\cdots \longrightarrow H_n(G, A_1; \mathbb{Z}/q) \longrightarrow H_n(G, A; \mathbb{Z}/q) \longrightarrow H_n(G, A_2; \mathbb{Z}/q) \longrightarrow \cdots \longrightarrow\\
\longrightarrow H_0(G, A_1; \mathbb{Z}/q) \longrightarrow H_0(G, A; \mathbb{Z}/q) \longrightarrow H_0(G, A_2; \mathbb{Z}/q) \longrightarrow 0.
\end{aligned}$$

Proof. It is straightforward by Proposition 6.29. □

4. Derived Functors of the Non-Abelian Tensor Product Modulo q of Groups

Let A denote a fixed Abelian group and consider the category $\mathcal{A}'_A$ a subcategory to that of $\mathcal{A}_A$ examined in Chap. 4, Sec. 2, which denotes the category whose objects are all groups G together with an action of G on A (and a trivial action of A on G). Morphisms in the category $\mathcal{A}'_A$ are all group homomorphisms $\alpha : G \longrightarrow H$ that preserve the actions, namely ${}^g a = {}^{\alpha(g)}a$, for all $a \in A$ and $g \in G$.

Let $F' : \mathcal{A}'_A \longrightarrow \mathcal{A}'_A$ be the restriction of the endofunctor $F : \mathcal{A}_A \longrightarrow \mathcal{A}_A$ while $\tau' : F' \longrightarrow 1_{\mathcal{A}'_A}$ and $\delta' : F' \longrightarrow F'^2$ be the restrictions of the natural transformations $\tau : F \longrightarrow 1_{\mathcal{A}_A}$ and $\delta : F \longrightarrow F^2$, respectively, given in Chap. 4, Sec. 2. We obtain a cotriple $\mathcal{F}' = (F', \tau', \delta')$, and we denote by $\mathcal{P}'$ the projective class induced by this cotriple $\mathcal{F}'$.

Since the actions of the groups G and A on each other satisfy the compatibility conditions (4.1) for any object $G \in \mathcal{A}'_A$, the non-Abelian 'absolute' tensor product modulo q of groups defines a covariant functor $- \otimes^q A : \mathcal{A}'_A \longrightarrow \mathcal{A}b\mathfrak{G}r$. Consider the non-Abelian left derived functors $L_n^{\mathcal{P}'}(- \otimes^q A)$, $n \geq 0$, of the functor $- \otimes^q A$ relative to the projective class $\mathcal{P}'$ (see Chap. 1).

Proposition 6.37. *Assume that G is a group and the groups A and A/q are trivial G-modules. Then there are isomorphisms*

$$H_n(G, A/q) \cong L_{n-1}^{\mathcal{P}'}(G \otimes^q A), \quad n \geq 2,$$
$$H_1(G, A/q) \cong \operatorname{Ker} \xi', \qquad H_0(G, A/q) \cong (\operatorname{Coker} \xi')/q,$$

where $\xi' : G \otimes {}^q A \longrightarrow A$, $\xi'(g \otimes a) = {}^g a \cdot a^{-1}$, $\xi'(\{k\}) = \pi_2 k^q$.

Proof. It is obvious that $H_0(G, A/q) \cong A/q$, $H_1(G, A/q) \cong G \otimes A/q \cong G^{ab} \otimes_{\mathbb{Z}} A/q \cong (G^{ab} \otimes_{\mathbb{Z}} A)/q$ and $H_n(G, A/q) \cong \pi_{n-1} C_*$, $n \geq 2$, where C_* is the following simplicial group:

$$\begin{array}{ccccccccc}
\cdots \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & F^n(G) \otimes A/q & \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots & \equiv\!\!\!> & F^2(G) \otimes A/q & \rightrightarrows & F^1(G) \otimes A/q & \\
& \downarrow \cong & & & \downarrow \cong & & \downarrow \cong & , \\
\cdots \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & (F^n(G)^{ab} \otimes_{\mathbb{Z}} A)/q & \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots & \equiv\!\!\!> & (F^2(G)^{ab} \otimes_{\mathbb{Z}} A)/q & \rightrightarrows & (F^1(G)^{ab} \otimes_{\mathbb{Z}} A)/q &
\end{array}$$

and

$$\cdots \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} F^n(G) \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots \equiv\!\!\!> F^2(G) \rightrightarrows F^1(G) \longrightarrow G$$

is the cotriple resolution of G.

By Theorem 6.8 $(\operatorname{Coker} \xi')/q = (\operatorname{Coker} \lambda')/q = A/q$, $\operatorname{Ker} \xi' = \operatorname{Ker} \lambda' / q \operatorname{Ker} \lambda' = (G^{ab} \otimes_{\mathbb{Z}} A)/q$, and $L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) = \pi_{n-1} C'_*$, $n \geq 2$, where C'_* is the following simplicial group:

$$\begin{array}{ccccccccc}
\cdots \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & F^n(G) \otimes {}^q A & \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots & \equiv\!\!\!> & F^2(G) \otimes {}^q A & \rightrightarrows & F^1(G) \otimes {}^q A & \\
& \downarrow \cong & & & \downarrow \cong & & \downarrow \cong & . \\
\cdots \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & (F^n(G)^{ab} \otimes_{\mathbb{Z}} A)/q & \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots & \equiv\!\!\!> & (F^2(G)^{ab} \otimes_{\mathbb{Z}} A)/q & \rightrightarrows & (F^1(G)^{ab} \otimes_{\mathbb{Z}} A)/q &
\end{array}$$

□

Now the relation between classical homology of groups and non-Abelian left derived functors of the tensor product modulo q will be established.

Proposition 6.38. *Let G be a group and A be a G-module. Then there is a long exact sequence of groups*

$$\begin{aligned}
\cdots \longrightarrow L_{n-1}^{\mathcal{P}'}(qH_1(G,A)) \longrightarrow H_n(G,A) \longrightarrow L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) \longrightarrow \cdots \longrightarrow \\
\longrightarrow L_2^{\mathcal{P}'}(G \otimes {}^q A) \longrightarrow L_1^{\mathcal{P}'}(qH_1(G,A)) \longrightarrow H_2(G,A) \longrightarrow L_1^{\mathcal{P}'}(G \otimes {}^q A) \longrightarrow \\
\longrightarrow L_0^{\mathcal{P}'}(qH_1(G,A)) \longrightarrow H_1(G,A) \longrightarrow \operatorname{Ker} \xi' \longrightarrow 0 \,. \qquad (6.18)
\end{aligned}$$

Proof. Let consider the cotriple resolution of G in the category $\mathcal{A}'_A$

$$\cdots \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} F^n(G) \stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots \equiv\!\!\!> F^2(G) \rightrightarrows F^1(G) \longrightarrow G.$$

By Theorem 6.8 we have a commutative diagram of groups

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] & qH_1(F^n(G),A) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] \arrow[d] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & qH_1(F^2(G),A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & qH_1(F^1(G),A) \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] & F^n(G)\otimes A \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] \arrow[d] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(G)\otimes A \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F^1(G)\otimes A \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] & F^n(G)\otimes {}^qA \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] \arrow[d] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F^2(G)\otimes {}^qA \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F^1(G)\otimes {}^qA \arrow[d] \\
 & 0 & & & 0 & 0
\end{tikzcd}
\quad ,
$$

where the columns are short exact sequences of groups.

From this diagram we have exactness till $L_0^{\mathcal{P}'}(qH_1(G,A))$. Exactness in $H_1(G,A)$ and in $\operatorname{Ker}\xi'$ follows from Theorem 6.8 and from the fact that $H_1(F^1(G),A) \longrightarrow H_1(G,A)$ and hence $qH_1(F^1(G),A) \longrightarrow qH_1(G,A)$ are epimorphisms and therefore $L_0^{\mathcal{P}'}(qH_1(G,A)) \longrightarrow qH_1(G,A)$ is also an epimorphism. $\square$

We establish sufficient conditions for the isomorphism between non-Abelian left derived functors of the tensor product modulo q of groups and mod q homology of groups.

Lemma 6.39. *Let G be a group and A be a q-torsion free G-module. There are natural isomorphisms*

$$H_n(G,A/q) \cong H_n(G,A;\mathbb{Z}/q), \quad n \geq 0.$$

Proof. Consider two sequences of functors

1) $H_0(G,-/q), H_1(G,-/q), \ldots, H_n(G,-/q), \ldots;$
2) $H_0(G,-;\mathbb{Z}/q), H_1(G,-;\mathbb{Z}/q), \ldots, H_n(G,-;\mathbb{Z}/q), \ldots.$

These both sequences satisfy the following axioms for a connected sequence of additive functors $\{T_n,\ \delta_n,\ n \geq 0\}$ from the category of q-torsion free G-modules to the category of Abelian groups:

(i) $T_0(-) = H_0(G,-/q) = H_0(G,-;\mathbb{Z}/q)$;

(ii) for any short exact sequence of q-torsion free G-modules

$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0$$

there exists a long exact sequence of Abelian groups

$$\cdots \longrightarrow T_{n+1}(A_2) \xrightarrow{\delta n+1} T_n(A_1) \longrightarrow T_n(A) \longrightarrow$$
$$\longrightarrow T_n(A_2) \xrightarrow{\delta_n} \cdots T_1(A_2) \xrightarrow{\delta_1} T_0(A_1) \longrightarrow T_0(A) \longrightarrow T_0(A_2) \longrightarrow 0\ ;$$

(iii) if A is an induced q-torsion free G-module (it is easy to see that if M is q-torsion free G-module, then $\mathbb{Z}[G]\otimes_{\mathbb{Z}} M$ is also a q-torsion free G-module), then $T_n(A) = 0$, $n \geq 1$.

This proves the existence of the required natural isomorphisms. $\square$

Theorem 6.40. *Let G be a group and A be a q-torsion free G-module. There are natural isomorphisms*

$$L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) \cong H_n(G, A; \mathbb{Z}/q), \quad n \geq 2.$$

Proof. Let us consider the non-Abelian left derived functors, $L_n^{\mathcal{P}'}((- \otimes A)/q)$, $n \geq 0$, of the functor $(- \otimes A)/q$. By [53] we have the isomorphisms

$$G \otimes A/q \cong IG \otimes_G A/q \cong (IG \otimes_G A)/q \cong (G \otimes A)/q.$$

Therefore by [68, Theorem 6]

$$L_{n-1}^{\mathcal{P}'}((G \otimes A)/q) \cong H_n(G, A/q), \quad n \geq 2. \tag{6.19}$$

If F is a free group, then the well-known fact that IF is a free $\mathbb{Z}F$-module, i.e., $IF \cong \sum \mathbb{Z}F$, implies the following isomorphism:

$$F \otimes A \cong IF \otimes_F A \cong \sum A.$$

Assume that A is a q-torsion free F-module; then, according to this isomorphism $H_1(F, A) \subseteq F \otimes A$ is also q-torsion free. Thus $H_1(F, A) \overset{\times q}{\cong} qH_1(F, A)$. From this fact and Theorem 4.20 we can conclude that $L_n^{\mathcal{P}'}(qH_1(G, A)) \cong L_n^{\mathcal{P}}(H_1(G, A)) \cong H_{n+1}(G, A)$, $n \geq 0$. Therefore by Proposition 6.38 it is easy to get the following long exact sequence of groups:

$$\begin{aligned} \cdots \longrightarrow H_n(G, A) \xrightarrow{q} H_n(G, A) \longrightarrow L_{n-1}^{\mathcal{P}}(G \otimes {}^q A) \longrightarrow \cdots \longrightarrow L_2^{\mathcal{P}}(G \otimes {}^q A) \longrightarrow \\ \longrightarrow H_2(G, A) \xrightarrow{q} H_2(G, A) \longrightarrow L_1^{\mathcal{P}}(G \otimes {}^q A) \longrightarrow \\ \longrightarrow H_1(G, A) \xrightarrow{q} H_1(G, A) \longrightarrow (H_1(G, A))_q \longrightarrow 0. \end{aligned} \tag{6.20}$$

(6.20) implies the short exact sequences of groups

$$0 \longrightarrow H_n(G, A) \otimes \mathbb{Z}/q \longrightarrow L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) \longrightarrow \mathrm{Tor}(H_{n-1}(G, A), \mathbb{Z}/q) \longrightarrow 0\,, \quad n \geq 2.$$

To prove the assertion, we must only construct homomorphisms $\kappa : L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) \longrightarrow H_n(G, A; \mathbb{Z}/q)$, $n \geq 2$, in our case (when A is q-torsion free) such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & H_n(G, A) \otimes \mathbb{Z}/q \arrow[r] \arrow[d, equal] & L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) \arrow[r] \arrow[d, "\kappa"] & \mathrm{Tor}(H_{n-1}(G, A), \mathbb{Z}/q) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H_n(G, A) \otimes \mathbb{Z}/q \arrow[r] & H_n(G, A; \mathbb{Z}/q) \arrow[r] & \mathrm{Tor}(H_{n-1}(G, A), \mathbb{Z}/q) \arrow[r] & 0 \end{tikzcd}\,, \tag{6.21}$$

where the bottom row is by Proposition 6.30 (a) in the case where A is a q-torsion free G-module.

By Theorem 6.8 there is a natural homomorphism

$$G \otimes {}^q A \cong (G \otimes A)/qH_1(G, A) \longrightarrow (G \otimes A)/q$$

and by (6.19) and Lemma 6.39 it induces homomorphisms $\kappa : L_{n-1}^{\mathcal{P}'}(G \otimes {}^q A) \longrightarrow L_{n-1}^{\mathcal{P}'}((G \otimes A)/q) \cong H_n(G, A/q) \cong H_n(G, A; \mathbb{Z}/q)$, $n \geq 2$. It is easy to see that (6.21) is commutative. □

It is interesting to consider derived functors of the additive functor $G \otimes {}^q -$ from the category of G-modules to the category of Abelian groups. Let us denote these derived functors by $\mathfrak{L}_n(G \otimes {}^q -)$, $n \geq 0$; then we have the following assertion.

Proposition 6.41. *Let A be a G-module; then there are isomorphisms*

$$\mathfrak{L}_n(G \otimes {}^q A) \cong H_{n+1}(G, A), \quad n \geq 1.$$

Proof. Consider a projective G-resolution $P_* \longrightarrow A$ of A.

Since $H_1(G, P_n) = 0$, by Theorem 6.8 and [53] we have

$$G \otimes {}^q P_n \cong (G \otimes P_n)/qH_1(G, P_n) = G \otimes P_n \cong IG \otimes_G P_n.$$

The well-known short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow IG \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

gives a short exact sequence of chain complexes of Abelian groups

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & IG \otimes_G P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & IG \otimes_G P_1 \arrow[r] \arrow[d] & IG \otimes_G P_0 \arrow[d] \\
\cdots \arrow[r] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[d] \\
\cdots \arrow[r] & \mathbb{Z} \otimes_G P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{Z} \otimes_G P_1 \arrow[r] \arrow[d] & \mathbb{Z} \otimes_G P_0 \arrow[d] \\
 & 0 & & 0 & 0
\end{tikzcd} .$$

The induced long homology exact sequence proves the assertion. □

5. Mod q Cohomology of Groups

In this section we shall define a mod q cohomology of groups by using Definition 6.18 and then express it as the Ext^* functors in the same way as the mod q homology of groups is expressed as the Tor_* functors. The first and the second mod q cohomology of groups will be described in terms of q-torsors and q-extensions of groups respectively.

Let G be a group, A be a G-module, and $P_* \longrightarrow \mathbb{Z}$ be a projective G-resolution of $\mathbb{Z}$. According to Example 6.25 $\mathcal{H}om(-, A)$ is a contravariant functor from $\mathcal{D}_{\mathbb{Z}[G]}$ to $\mathcal{C}_{\mathbb{Z}}$.

Definition 6.42. The nth mod q cohomology, $H^n(G, A; \mathbb{Z}/q)$, of the group G with coefficients in the G-module A is

$$H^n(G, A; \mathbb{Z}/q) := H^n_{\mathcal{H}om(-,A)}(P_*; \mathbb{Z}/q), \quad n \geq 0.$$

Note, Lemma 6.24 (ii) implies that these cohomology groups are well defined and do not depend on the choice of the projective G-resolution of $\mathbb{Z}$.

The next proposition immediately follows from Corollary 6.20.

Proposition 6.43 (universal coefficient formula)**.** *Let G be a group and A a G-module. Then there is a short exact sequence of Abelian groups*

$$0 \longrightarrow H^{n-1}(G, A) \otimes \mathbb{Z}/q \longrightarrow H^n(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(H^n(G, A), \mathbb{Z}/q) \longrightarrow 0 \qquad (6.22)$$

for $n \geq 0$.

Now applying Lemmas 6.26 and 6.28, we have the following

Proposition 6.44. *$H^n(G, A; \mathbb{Z}/q) \cong \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}/q, A)$, $n \geq 0$.*

Let us consider the standard bar G-resolution of $\mathbb{Z}$ (see [93])

$$C_*(G):\ \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where C_n is the free G-module generated by all symbols $[x_1, \dots, x_n]$, $n \geq 1$, $x_i \in G$, and C_0 is a free G-module generated by only one symbol $[\quad]$. The differential is defined by the formula

$$\partial[x_1, \dots, x_n] = x_1[x_2, \dots, x_n] + \sum_{i=1}^{n-1}(-1)^i\big[x_1, \dots, x_i x_{i+1}, \dots, x_n\big] + (-1)^n[x_1, \dots, x_{n-1}],$$

and $\epsilon[\quad] = 1$.

According to Theorem 6.44, using also the classical convention converting chain complexes into cochain complexes, we call $\mathcal{H}om(\mathrm{Mc}(C_*(G)_*, q), A)_*$ the standard cochain complex for the mod q cohomology of G with coefficients in A and denote it by $D^*(G, A; \mathbb{Z}/q)$. We denote its cocycles by $Z^*(G, A; \mathbb{Z}/q)$ and its coboundaries by $B^*(G, A; \mathbb{Z}/q)$, while $Z^*(G, A)$ and $B^*(G, A)$ denote the cocycles and coboundaries of the standard cochain complex, respectively.

As usual, we identify $\mathrm{Hom}_G(C_n, A)$ with the G-module $\mathrm{Set}(G^n, A)$ of maps from G^n to A for $n \geq 1$ and with A for $n = 0$. In the complex $D^*(G, A; \mathbb{Z}/q)$ we get, for $(f, g) \in \mathrm{Set}(G^n, A) \times \mathrm{Set}(G^{n-1}, A)$

$$\widetilde{\delta}(f, g) = (\delta(f), qf - \delta(g)), \tag{6.23}$$

where δ is the classical differential given by

$$\begin{aligned}\delta(f)(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) + {}&\\ + \sum_{i=1}^{n}(-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) &+ (-1)^{n+1} f(x_1, \dots, x_n).\end{aligned}$$

In the following example $H^*(G, A; \mathbb{Z}/q)$ is neither isomorphic to $H^*(G, A)/q$ nor to $H^*(G, A/q)$.

Example 6.45. Let $\mathbb{Z}$ be the group of integers, and $\mathbb{Q}/\mathbb{Z}$ the quotient of the group of rational numbers by $\mathbb{Z}$. Assume that $\mathbb{Z}$ acts trivially on $\mathbb{Q}/\mathbb{Z}$. We have, for $n \geq 2$, $H^n(\mathbb{Z}, A) = 0$ for any G-module A, especially $\mathbb{Q}/\mathbb{Z}$, and $H^0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = H^1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. Since the group $\mathbb{Q}/\mathbb{Z}$ is divisible, $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}/q = 0$. Whence the exact sequence (6.22) gives $H^n(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}; \mathbb{Z}/q) = 0$ for $n \geq 2$, and we have

$$H^0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}; \mathbb{Z}/q) = H^1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}; \mathbb{Z}/q) = \mathbb{Z}/q.$$

While, for $n \geq 0$, $H^n(\mathbb{Z}, (\mathbb{Q}/\mathbb{Z})/q) = 0$ and $(H^n(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}))/q = 0$.

Proposition 6.46. *Let G be a group and A a G-module.*

(a) *If A has exponent q, then*

$$H^n(G, A; \mathbb{Z}/q) \cong H^n(G, A) \oplus H^{n-1}(G, A), \quad n \geq 0.$$

(b) *If A is q-torsion-free, then*

$$H^0(G, A; \mathbb{Z}/q) = 0 \quad \textit{and} \quad H^n(G, A; \mathbb{Z}/q) \cong H^{n-1}(G, A/q), \quad n \geq 1.$$

Proof. (a) Follows from the triviality of the homomorphism ${}_\times q$ in the equality (6.23).

(b) Obviously $H^0(G, A; \mathbb{Z}/q) = \mathrm{Tor}(H^0(G, A), \mathbb{Z}/q) = 0$. The short exact sequence

$$0 \longrightarrow A \xrightarrow{\times q} A \longrightarrow A/q \longrightarrow 0 \tag{6.24}$$

induces a long exact cohomology sequence, and we have only to construct the homomorphism $H^{n-1}(G, A/q) \longrightarrow H^n(G, A; \mathbb{Z}/q)$, $n \geq 1$, compatible with the exact cohomology sequences and then apply the five lemma at each level. Using the short exact sequence of standard cochain complexes

$$0 \longrightarrow \mathcal{H}om(C_*, A)_* \xrightarrow{\times q} \mathcal{H}om(C_*, A)_* \longrightarrow \mathcal{H}om(C_*, A/q)_* \longrightarrow 0$$

induced by the exact sequence (6.24), for any $(n-1)$-cocycle of $\mathcal{H}om(C_*, A/q)_*$ we find in a natural way an n-cocycle of $\mathcal{H}om(\mathrm{Mc}(C_*, q)_*, A)_*$. This map of cocyles induces the required homomorphism $H^{n-1}(G, A/q) \longrightarrow H^n(G, A; \mathbb{Z}/q)$, $n \geq 1$. □

Proposition 6.46 provides a general reason why the mod q cohomology and homology of groups play a distinguished role especially for G-modules having torsion elements.

A q-derivation from G to A is a pair (f, a) consisting of a derivation $f : G \longrightarrow A$ and an element $a \in A$ such that $qf(x) = xa - a$ for all $x \in G$.

Let $\mathrm{Der}(G, A; \mathbb{Z}/q)$ denote the Abelian group of q-derivations from G to A. We write $\mathrm{Der}(G, A)$ for the Abelian group of derivation from G to A and $\mathrm{PDer}(G, A)$ for the subset of principal derivations.

Plainly any pair of the form (f_a, qa), with f_a the principal derivation from G to A induced by $a \in A$, is a q-derivation. We call it a *principal q-derivation.* The set $\mathrm{PDer}(G, A; \mathbb{Z}/q)$ of principal q-derivations is a subgroup of $\mathrm{Der}(G, A; \mathbb{Z}/q)$.

Clearly, using the identification of $\mathrm{Hom}_G(C_1, A)$ with $\mathrm{Set}(G, A)$ and of $\mathrm{Hom}_G(C_0, A)$ with A, a pair $(f, a) \in D^1(G, A; \mathbb{Z}/q)$ is a cocycle if and only if it is a q-derivation. Furthermore it is a coboundary if and only if it is a principal q-derivation. Hence the identification induces a natural isomorphism

$$H^1(G, A; \mathbb{Z}/q) \cong \mathrm{Der}(G, A; \mathbb{Z}/q)/\,\mathrm{PDer}(G, A; \mathbb{Z}/q).$$

Note that the map $\mathrm{PDer}(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{PDer}(G, A)$ given by $(f_a, qa) \longmapsto f_a$, $a \in A$, is an isomorphism if and only if $H^0(G, A)$ is a group of exponent q.

Proposition 6.47. *The group $\mathrm{Der}(G, A; \mathbb{Z}/q)$ is isomorphic to the group of pairs (α, a), where α is an automorphism of the semidirect product $A \rtimes G$ inducing identity maps on A and G, and a is an element of A such that α^q is equal to the inner automorphism β_a of $A \rtimes G$ induced by a. Moreover $\mathrm{PDer}(G, A; \mathbb{Z}/q)$ is isomorphic to the group of pairs (β_a, qa).*

Proof. It is similar to the classical case. □

It is well known [93] that any derivation f can be extended to the Abelian group homomorphism $\gamma : \mathbb{Z}[G] \longrightarrow A$ given by $\gamma\Big(\sum\limits_i n_i g_i\Big) = \sum\limits_i n_i f(g_i)$ satisfying the condition $\gamma(rs) = r\gamma(s) + \epsilon(s)\gamma(r)$ for all $r, s \in \mathbb{Z}[G]$. The restriction of γ to IG induces a G-module homomorphism $\beta : \mathrm{IG} \longrightarrow A$, and we obtain the well-known isomorphism $\mathrm{Der}(G, A) \xrightarrow{\vartheta} \mathrm{Hom}_G(\mathrm{IG}, A)$ with $\vartheta(f) = \beta$.

The set K of elements $(f, a) \in \mathrm{Der}(G, A; \mathbb{Z}/q)$ for which there exists a G-module homomorphism $\alpha : I(G, q) \longrightarrow A$ such that $\alpha(x) = \vartheta(f)(x)$ for $x \in \mathrm{IG}$ and $\alpha(q1) = a$, is a subgroup of $\mathrm{Der}(G, A; \mathbb{Z}/q)$. Let $\alpha_a : I(G, q) \longrightarrow A$ be the G-module homomorphism given by $\alpha_a(u) = ua$, $u \in I(G, q)$. Since, for any principal derivation f_a and for $x \in \mathrm{IG}$, $\vartheta(f_a)(x) = xa$, we obtain $\alpha_a(x) = \vartheta(f_a)(x)$, $x \in \mathrm{IG}$ and $\alpha_a(q1) = q1a = qa$. Therefore $K \supseteq \mathrm{PDer}(G, A; \mathbb{Z}/q)$.

Proposition 6.48. *There is a short exact sequence of Abelian groups*

$$0 \longrightarrow \mathrm{Hom}_G(I(G, q), A) \xrightarrow{\varphi} \mathrm{Der}(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Der}(G, A; \mathbb{Z}/q)/K \longrightarrow 0.$$

Proof. Define the homomorphism φ by $\varphi(\alpha) = (f, a)$ for $\alpha \in \mathrm{Hom}_G(I(G, q), A)$, where $\vartheta(f) = \alpha|_{\mathrm{IG}}$ and $a = \alpha(q1)$. The pair (f, a) is a q-derivation. Indeed we have $q\alpha(x) = \alpha(xq1) = xa$ for $x \in \mathrm{IG}$.

Since $\{q1\} \cup \{g-1 | g \in G\}$ is a generating set of $I(G,q)$ as a G-module, $\varphi(\alpha) = \varphi(\alpha')$ implies $\alpha = \alpha'$. Clearly the image of φ is the subgroup of K. $\square$

Now the group $H^1(G, A; \mathbb{Z}/q)$ will be expressed by torsors. Recall [114] that a principal homogeneous space over A is a nonempty G-set P with right action $(p, a) \longmapsto pa$ of A compatible with G-action such that, given $p, p' \in P$, there exists a unique $a \in A$ such that $p' = pa$. We introduce the following notion.

Definition 6.49. A (G,q)-torsor over a G-module A is a pair (P, f), where P is a principal homogeneous space over A and f is a map from P to A subject to the following conditions:

(i) $f(xb) = f(x) + qb$ for $x \in P$, $b \in A$;

(ii) $qa_s = sf(x) - f(x)$ with a_s defined by $sx = xa_s$, $s \in G$, $x \in P$.

Two (G,q)-torsors (P, f) and (P', f') over a G-module A are said to be equivalent if there is a bijection $\vartheta : P \longrightarrow P'$ such that ϑ is compatible with the actions of G and A, and $f = f'\vartheta$.

Denote by $P(G, A; \mathbb{Z}/q)$ the set of equivalence classes of (G,q)-torsors over A. We can construct a natural sum on $P(G, A; \mathbb{Z}/q)$ given by $(P, f) + (P', f') = (P'', f'')$, where P'' is a quotient of $P \times P'$ by the relation $(x, x') = (xa, x'(-a))$ for $x \in P$, $x' \in P'$, $a \in A$, and $f'' = f + f'$. Under this sum $P(G, A; \mathbb{Z}/q)$ is an Abelian group with zero element $(A, {}_\times q)$.

Theorem 6.50. *For any G-module A there is a canonical isomorphism*

$$P(G, A; \mathbb{Z}/q) \cong H^1(G, A; \mathbb{Z}/q).$$

Proof. We have a natural homomorphism

$$\alpha : P(G, A; \mathbb{Z}/q) \longrightarrow H^1(G, A; \mathbb{Z}/q)$$

defined as follows: Given a (G,q)-torsor (P, f), take an element $x \in P$. Then the equality $sx = xa_s$ defines a derivation $\varphi_x : G \longrightarrow A$ given by $\varphi_x(s) = a_s$. It is easily checked that the pair $(\varphi_x, f(x))$ is a q-derivation (use the equality (ii) of Definition 6.49). The element $[(\varphi_x, f(x))]$ does not depend on the element $x \in P$, and therefore the map α given by $\alpha[(P, f)] = [(\varphi_x, f(x))]$ is well defined (use the equality (i) of Definition 6.49).

Conversely, if (φ, a) is a q-derivation, define a (G,q)-torsor (P, f) over A as follows: Take $P = A$; A acts on P by $xa = x + a$ for $x \in P$, $a \in A$. The group G acts on P by ${}^s x = \varphi(s) + sx$. The map $f : P \longrightarrow A$ is given by $f(x) = a + qx$.

It is easily checked that the pair (P, f) is a (G,q)-torsor over A, that we obtain a well-defined homomorphism

$$\beta : H^1(G, A; \mathbb{Z}/q) \longrightarrow P(G, A; \mathbb{Z}/q)$$

given by $\beta([(\varphi, a)]) = [(P, f)]$ and that $\alpha\beta$ and $\beta\alpha$ are identity maps. $\square$

To describe the group $H^2(G, A; \mathbb{Z}/q)$ in terms of extensions, some definitions will be introduced.

Definition 6.51. Let G be a group and A a G-module. A pointed q-extension of G by A is a triple (E, u, g) consisting of an extension $E: \ 0 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 1$ of G by A, a section map $u : G \longrightarrow B$, and a map $g : G \longrightarrow A$, such that

$$qv(x, y) = (\delta g)(x, y) = xg(y) - g(xy) + g(x)$$

for all $x, y \in G$, where $v : G \times G \longrightarrow A$ is the factorization system induced by the section u.

The pointed q-extension (E, u, g) is said to be equivalent to the pointed q-extension (E', u', g') if there exists a morphism $(1_A, \sigma, 1_G) : E \longrightarrow E'$ and an element $a \in A$ such that

$$g'(x) - g(x) = q(u'(x) - \sigma u(x)) - xa + a$$

for all $x \in G$.

This binary relation $\sim$ is an equivalence. The proof is routine and is omitted.

Let us denote by $\mathrm{E}^1(G, A; \mathbb{Z}/q)$ the set of equivalence classes of pointed q-extensions of the group G by the G-module A.

Theorem 6.52. *Let G be a group and A a G-module. There is a natural bijection*

$$\mathrm{E}^1(G, A; \mathbb{Z}/q) \xrightarrow{\omega} H^2(G, A; \mathbb{Z}/q)\ .$$

Proof. Define a map ω by $\omega[(E, u, g)] = [(v, g)]$ for $[(E, u, g)] \in \mathrm{E}^1(G, A; \mathbb{Z}/q)$, where $v : G \times G \longrightarrow A$ is the factorization system induced by the section u.

Correctness: we must show that if $(E, u, g) \sim (E', u', g')$, then $[(v, g)] = [(v', g')]$. It is well known [93] that

$$v'(x, y) - v(x, y) = xh(y) - h(xy) + h(x)$$

for all $x \in G$, where $h(x) = u'(x) - \sigma u(x)$. But there exists an element $a \in A$ such that $g'(x) - g(x) = qh(x) - xa + a$ for all $x \in G$. This means that we have $(v', g') - (v, g) \in B^2(G, A; \mathbb{Z}/q)$.

Injectivity of ω: Let $[(E, u, g)], [(E', u', g')] \in \mathrm{E}^1(G, A; \mathbb{Z}/q)$ and $[(v, g)] = [(v', g')]$, i.e., there exists $h \in \mathrm{Set}(G, A)$ and $a \in A$ such that $v'(x, y) - v(x, y) = (\delta h)(x, y)$ and $g'(x) - g(x) = qh(x) - xa + a$, for all $x, y \in G$. We can choose in the second extension a section u'' and a map g'' in such a way that $v'' = v$ and $g'' = g$. In effect, let us define the section $u''(x) = u'(x) - h(x)$, for $x \in G$, and the map $g'' : G \longrightarrow A$ by $g''(x) = g(x) - qh(x) + xa - a$. It is easy to show that

$$(E', u', g') \sim (E', u'', g'') \sim (E, u, g),$$

implying $[(E, u, g)] = [(E', u', g')]$.

Surjectivity of ω: Let $(v, g) \in Z^2(G, A; \mathbb{Z}/q)$. We take the extension

$$E:\ 0 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 0$$

induced by the 2-cocycle v and the section $u_0(x) = (0, x)$. Then we get the equality $\omega([E, u_0, g]) = [(v, g)]$. $\square$

Remark 6.53. If G is a group and A a G-module, and they satisfy the following condition:

$$\text{for any map} \quad h : G \longrightarrow A, \quad \delta(qh) = 0 \Longrightarrow qh \quad \text{is cohomologically trivial,} \tag{6.25}$$

then the group $H^2(G, A; \mathbb{Z}/q)$ can be described in terms of pairs (E, g) consisting of a map $g : G \longrightarrow A$ and an extension E of G by A having a factorization system v such that $qv = \delta g$. The relation $\sim$ between such pairs will be similar, requiring that the sections u and u' inducing the 2-cocycles v and v', respectively, such that $qv = \delta g$ and $qv' = \delta g'$, satisfy the equality of the equivalence relation. Clearly, the condition $H^1(G, A) = 0$ implies condition (6.25) and both conditions are equivalent to each other if A is a q-divisible group.

Moreover, for any G-module A, it is possible to introduce a "Baer sum" on the set $\mathrm{E}^1(G, A; \mathbb{Z}/q)$, making the map ω an isomorphism.

Before defining q-extensions of groups, we recall some properties on extensions of groups induced by derivations [57].

Let G be a group and

$$E:\ 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

an exact sequence of G-modules. Given a derivation $f : G \longrightarrow A''$, we obtain an induced extension $f^*(E)$. If f is a principal derivation, the induced extension splits. If two derivations $f_1, f_2 : G \longrightarrow A$ are equivalent, the induced extensions $f_1^*(E)$ and $f_2^*(E)$ are equivalent. Given two derivations $f_1, f_2 : G \longrightarrow A$, the extension $(f_1 + f_2)^*(E)$ is the "Baer sum" of the extensions $f_1^*(E)$ and $f_2^*(E)$.

Let G be a group, A a G-module, and $f : G \longrightarrow A/q$ a derivation. We consider the exact sequence $0 \longrightarrow qA \longrightarrow A \xrightarrow{c} A/q \longrightarrow 0$ and call $f^*(q)$ the induced extension of G by qA.

Given an extension

$$E:\ 0 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 1,$$

we call qE the extension induced by the q-multiplication from A to qA.

Definition 6.54. Let G be a group and A a G-module. A q-extension of G by A is a pair (E, f), where E is an extension of G by A and $f : G \longrightarrow A/q$ a derivation, such that the induced extensions qE and $f^*(q)$ are equivalent. Two q-extensions (E, f) and (E', f') are equivalent if the extensions E and E' are equivalent and the derivations f and f' are equivalent.

Let $\mathrm{Ext}(G, A; \mathbb{Z}/q)$ be the set of equivalence classes of q-extensions of G by A. If (E, u, g) is a pointed q-extension, then $(E, c \circ g)$ is a q-extension, since $qv = \delta g$, whence the extensions qE and $(c \circ g)^*(q)$ are equivalent. Furthermore the map $(E, u, g) \longmapsto (E, c \circ g)$ sends two equivalent pointed q-extensions onto two equivalent q-extensions. So it induces a map $\Phi : \mathrm{E}^1(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Ext}(G, A; \mathbb{Z}/q)$.

Lemma 6.55. *Let G be a group, and A a G-module. The map Φ is surjective. Furthermore for any q-extension (E, f) of G by A with $E:\ 0 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 1$ there exists a pair $(u, g) \in \mathrm{Set}(G, B) \times \mathrm{Set}(G, A)$ such that $c \circ g = f$ and $qv = \delta g$, where $v \in Z^2(G, A)$ is induced by u.*

Proof. The first sentence is a consequence of the second. Let $u \in \mathrm{Set}(G, B)$ be a section map of the morphism $B \longrightarrow G$ and $v_1 \in Z^2(G, A)$ be given by

$$v_1(x, y) = u_1(x)u_1(y)u_1(xy)^{-1}.$$

Let $g : G \longrightarrow A/q$ such that $f = c \circ g$; then, as the extensions $f^*(q)$ and qE are equivalent, there is a map $h_0 : G \longrightarrow qA$ such that $\delta(g) = v_1 + \delta(h_0)$. Let $h : G \longrightarrow A$ be such that $h_0 = qh$. Let $u = u_1 h$. There is a section map of $B \longrightarrow G$, and the associated 2-cocycle is $v = v_1 + \delta h$. $\square$

The morphism Φ is not, generally, injective: consider a q-extension (E, f) and two pairs (u_1, g_1) and (u_2, g_2) as in Lemma 6.55; then we have $u_1 = u_2 + h$, $g_1 = g_2 + h'$ with $h, h' : G \longrightarrow A$. Now we must have $q\delta h = q\delta h'$, but this does not imply $h = h'$; we have only the condition $q\delta(h - h') = \delta(q(h - h')) = 0$.

Note that a map $k : G \longrightarrow A$ with $q\delta k = 0$ corresponds to the inclusion

$$Z^1(G, qA) \hookrightarrow \mathrm{Hom}_G(C_1, qA) \hookrightarrow \mathrm{Hom}_G(C_1, A) \hookrightarrow \mathrm{Hom}_G(C_2 \oplus C_1, A).$$

Theorem 6.56. *Let G be a group and A a G-module. The group $\mathrm{Ext}(G, A; \mathbb{Z}/q)$ of equivalence classes of q-extensions of G by A is isomorphic to the quotient $H^2(G, A; \mathbb{Z}/q)/L$, where L is the image of $H^1(G, qA)$ induced by the composed map*

$$\mathrm{Set}(G, qA) \hookrightarrow \mathrm{Set}(G, A) \hookrightarrow \mathrm{Set}(G^2, A) \times \mathrm{Set}(G, A)\ ,$$

where the first map is induced by the inclusion $qA \hookrightarrow A$.

Proof. The remark we made for two pointed q-extensions inducing the same extension works as well for pointed q-extensions inducing equivalent q-extensions. Let (E, f) be a q-extension of G by A. Let (E, u, g) be a pointed q-extension such that $\Phi([(E, u, g)]) = [(E, f)]$. Let $\Psi([(E, f)]) = (c' \circ \omega)([(E, u, f)])$, where $c' : H^2(G, A; \mathbb{Z}/q) \longrightarrow H^2(G, A; \mathbb{Z}/q)/L$ is the canonical map. By Remark 6.53 if (E', f') is equivalent to (E, f) and (E', u', g') such that $\Phi([(E', u', g')]) = [(E', g')]$, we have

$$(c' \circ \omega)\big([(E', u', g')]\big) = (c' \circ \omega)\big([(E, u, g)]\big).$$

Then $\Psi([(E, f)]) = \Psi([(E', f')])$, and the map Ψ is well defined. Furthermore it is surjective, since $(c' \circ \omega)$ is surjective.

Now we assume that $\Psi([(E,f)])=0$. There is a q-extension (E,u,g) such that $\omega([(E,u,g)])\in L$, that is, $\omega([(E,u,g)])=[(v,g')]$ with $v=0$ and $q\delta g'=0$. So the q-extension (E,f) is equivalent to $(E',0)$, where E' is the trivial extension. $\square$

6. Mod q Cohomology of Groups as Cotriple Cohomology

In this section the mod q cohomology of groups will be described as cotriple cohomology.

Let us consider again the category $\mathcal{A}'_A$ and the cotriple $\mathcal{F}'=(F',\tau',\delta')$ in $\mathcal{A}'_A$. Let $F_*(G)\xrightarrow{\tau_G} G$ be the cotriple resolution of an object G in the category $\mathfrak{G}r_A$ (see Chap. 1, Sec. 1.2).

Let $T:\mathfrak{G}r_A\longrightarrow Ab\mathfrak{G}r$ be a contravariant functor to the category of Abelian groups. Applying T dimension-wise to the simplicial group $F_*(G)$ yields an Abelian cosimplicial group $TF_*(G)$. Then the nth cohomology group of the Abelian cosimplicial group $TF_*(G)$ is called the nth right derived functor $R^n_{\mathcal{F}}T$ of the functor T with respect to the cotriple $\mathcal{F}$.

It is well known that the right derived functors of the contravariant functor of derivations $\operatorname{Der}(-,A)=Z^1(-,A):\mathfrak{G}r_A\longrightarrow Ab\mathfrak{G}r$ with respect to the cotriple $\mathcal{F}$ are isomorphic, up to dimension shift, to the group cohomology functors $H^*(-,A)$ [4]. A similar assertion is not true for mod q cohomology of groups, i.e., the cotriple derived functor $R^n_{\mathcal{F}}Z^1(-,A;\mathbb{Z}/q)$ of the contravariant functor of q-derivations $\operatorname{Der}(-,A;\mathbb{Z}/q)=Z^1(-,A;\mathbb{Z}/q):\mathfrak{G}r_A\longrightarrow Ab\mathfrak{G}r$ is not isomorphic to the mod q group cohomology functor $H^{n+1}(-,A;\mathbb{Z}/q)$ for some $n\geq 1$. In effect, if G is a free group acting on A, then $R^1_{\mathcal{F}}Z^1(G,A;\mathbb{Z}/q)=0$, while, using Proposition 6.19, we see that $H^2(G,A;\mathbb{Z}/q)$ is isomorphic to $H^1(G,A)/q$.

Theorem 6.57. *Let G be a group and A a G-module. Then there are natural isomorphisms*

$$R^0_{\mathcal{F}}Z^k(G,A;\mathbb{Z}/q)\cong Z^k(G,A;\mathbb{Z}/q),$$
$$R^n_{\mathcal{F}}Z^k(G,A;\mathbb{Z}/q)\cong H^{n+k}(G,A;\mathbb{Z}/q),$$

for $k>1$ and $n>0$.

Proof. The augmented simplicial group $\tau_G:F_*(G)\longrightarrow G$ is simplicially exact and therefore is left (right) contractible as an augmented simplicial set. Since

$$D^k(L,A;\mathbb{Z}/q)=\operatorname{Set}(L^k,A)\oplus\operatorname{Set}(L^{k-1},A)$$

for any group L acting on A, the Abelian cochain complex

$$\begin{aligned}0\longrightarrow D^k(G,A;\mathbb{Z}/q)\longrightarrow D^k(F_0(G),A;\mathbb{Z}/q)\longrightarrow D^k(F_1(G),A;\mathbb{Z}/q)\longrightarrow\\ \longrightarrow D^k(F_2(G),A;\mathbb{Z}/q)\longrightarrow\cdots\longrightarrow D^k(F_n(G),A;\mathbb{Z}/q)\longrightarrow\cdots\end{aligned}\tag{6.26}$$

becomes exact for $k\geq 0$, implying

$$R^0_{\mathcal{F}}D^k(G,A;\mathbb{Z}/q)\cong D^k(G,A;\mathbb{Z}/q)\quad\text{and}\quad R^n_{\mathcal{F}}D^k(G,A;\mathbb{Z}/q)=0,\quad n>0.$$

For any $k\geq 0$ the short exact sequence of Abelian cochain complexes

$$0\longrightarrow Z^k(F_*(G),A;\mathbb{Z}/q)\longrightarrow D^k(F_*(G),A;\mathbb{Z}/q)\longrightarrow B^{k+1}(F_*(G),A;\mathbb{Z}/q)\longrightarrow 0\tag{6.27}$$

induces a long exact sequence of cotriple derived functors

$$\begin{aligned}0\longrightarrow R^0_{\mathcal{F}}Z^k(G,A;\mathbb{Z}/q)\longrightarrow R^0_{\mathcal{F}}D^k(G,A;\mathbb{Z}/q)\longrightarrow R^0_{\mathcal{F}}B^{k+1}(G,A;\mathbb{Z}/q)\longrightarrow\\ \longrightarrow R^1_{\mathcal{F}}Z^k(G,A;\mathbb{Z}/q)\longrightarrow R^1_{\mathcal{F}}D^k(G,A;\mathbb{Z}/q)\longrightarrow\cdots.\end{aligned}$$

The injection

$$R^0_{\mathcal{F}} B^{k+1}(G,A;\mathbb{Z}/q) \hookrightarrow R^0_{\mathcal{F}} D^{k+1}(G,A;\mathbb{Z}/q) \cong D^{k+1}(G,A;\mathbb{Z}/q)$$

yields the exact sequence

$$0 \longrightarrow R^0_{\mathrm{F}} Z^k(G,A;\mathbb{Z}/q) \longrightarrow D^k(G,A;\mathbb{Z}/q) \longrightarrow D^{k+1}(G,A;\mathbb{Z}/q)\ ,$$

showing that $R^0_{\mathrm{F}} Z^k(G,A;\mathbb{Z}/q) \cong Z^k(G,A;\mathbb{Z}/q)$.

It is easily checked that any short exact sequence of G-modules

$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0$$

induces a long exact cohomology sequence

$$\begin{aligned}0 \longrightarrow Z^k(G,A_1;\mathbb{Z}/q) \longrightarrow Z^k(G,A;\mathbb{Z}/q) \longrightarrow Z^k(G,A_2;\mathbb{Z}/q) \longrightarrow H^{k+1}(G,A_1;\mathbb{Z}/q) \longrightarrow \\ \longrightarrow H^{k+1}(G,A;\mathbb{Z}/q) \longrightarrow H^{k+1}(G,A_2;\mathbb{Z}/q) \longrightarrow H^{k+2}(G,A_1;\mathbb{Z}/q) \longrightarrow \cdots .\end{aligned}$$

It follows that for a free group G the sequence

$$0 \longrightarrow Z^k(G,A_1;\mathbb{Z}/q) \longrightarrow Z^k(G,A;\mathbb{Z}/q) \longrightarrow Z^k(G,A_2;\mathbb{Z}/q) \longrightarrow 0$$

is exact for $k > 1$, since in this case $H^{k+1}(G,A;\mathbb{Z}/q) = 0$. Hence for $k > 1$ there is a long exact sequence of cotriple right derived functors

$$\begin{aligned}0 \longrightarrow Z^k(G,A_1;\mathbb{Z}/q) \longrightarrow Z^k(G,A;\mathbb{Z}/q) \longrightarrow Z^k(G,A_2;\mathbb{Z}/q) \longrightarrow R^1_{\mathcal{F}} Z^k(G,A_1;\mathbb{Z}/q) \longrightarrow \\ \longrightarrow R^1_{\mathcal{F}} Z^k(G,A;\mathbb{Z}/q) \longrightarrow R^1_{\mathcal{F}} Z^k(G,A_2;\mathbb{Z}/q) \longrightarrow R^2_{\mathcal{F}} Z^k(G,A_1;\mathbb{Z}/q) \longrightarrow \cdots .\end{aligned}$$

Now it will be shown that $R^n_{\mathcal{F}} Z^k(G,A;\mathbb{Z}/q) = 0$ for $k \geq 1$ and $n > 0$, if A is an injective G-module.

The following complex of Abelian cosimplicial groups:

$$\begin{aligned}0 \longrightarrow D^0(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta}^0_*} D^1(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta}^1_*} D^2(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta}^2_*} \\ \xrightarrow{\widetilde{\delta}^2_*} D^3(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta}^3_*} \cdots \xrightarrow{\widetilde{\delta}^{k-1}_*} D^k(F_*(G),A;\mathbb{Z}/q) \xrightarrow{\widetilde{\delta}^k_*} \cdots ,\end{aligned} \tag{6.28}$$

is exact at the terms $D^k(F_*(G),A;\mathbb{Z}/q)$, $k \geq 3$, since $H^k(F_*(G),A;\mathbb{Z}/q) = 0$, $k \geq 3$, by the universal coefficient formula (Proposition 6.43).

It is easy to show that any injective G-module is a q-divisible group, and the proof is similar to the case of injective Abelian groups.

Since $F_n(G), n \geq 0$, is a free group, the group $Z^1(F_n(G),A)$ of 1-cocycles is isomorphic to a direct product $\prod\limits_{i \in J} A_i$ of copies $A_i = A$, where the set J is a basis of $F_n(G)$. Hence, if A is injective, then $Z^1(F_n(G),A)$, $n \geq 0$, is q-divisible; thus, $H^1(F_n(G),A)$, $n \geq 0$, is also q-divisible. Therefore for an injective G-module A the short exact sequence of Abelian cosimplicial groups

$$0 \longrightarrow \mathrm{Tor}(H^1(F_*(G),A),\mathbb{Z}/q) \longrightarrow H^1(F_*(G),A) \xrightarrow{\times q} H^1(F_*(G),A) \longrightarrow 0$$

together with the well-known isomorphism $R^n_{\mathcal{F}} H^1(G,A) \cong H^{n+1}(G,A)$, $n \geq 0$, imply the equality $R^n_{\mathcal{F}} \mathrm{Tor}(H^1(G,A),\mathbb{Z}/q) = 0$, $n \geq 0$.

The universal coefficient formula yields a short exact sequence of Abelian cosimplicial groups

$$0 \longrightarrow H^0(F_*(G),A) \otimes \mathbb{Z}/q \longrightarrow H^1(F_*(G),A;\mathbb{Z}/q) \longrightarrow \mathrm{Tor}(H^1(F_*(G),A),\mathbb{Z}/q) \longrightarrow 0\ ,$$

implying the isomorphism $R^n_{\mathcal{F}} H^1(G, A; \mathbb{Z}/q) \cong R^n_{\mathcal{F}} \operatorname{Tor}(H^1(G, A), \mathbb{Z}/q)$, $n > 0$. By (6.27) and the following short exact sequence of Abelian cosimplicial groups

$$0 \longrightarrow B^1(F_*(G), A; \mathbb{Z}/q) \longrightarrow Z^1(F_*(G), A; \mathbb{Z}/q) \longrightarrow H^1(F_*(G), A; \mathbb{Z}/q) \longrightarrow 0$$

it is easily seen that $R^n_{\mathcal{F}} Z^1(G, A; \mathbb{Z}/q) \cong R^n_{\mathcal{F}} H^1(G, A; \mathbb{Z}/q)$, $n > 0$. Hence, for an injective G-module A we deduce that $R^n_{\mathcal{F}} Z^1(G, A; \mathbb{Z}/q) = 0$, $n > 0$, and using again (6.27) we obtain $R^n_{\mathcal{F}} B^2(G, A; \mathbb{Z}/q) = 0$, $n > 0$.

Let us consider the short exact sequence of Abelian cosimplicial groups

$$0 \longrightarrow H^1(F_*(G), A) \otimes \mathbb{Z}/q \longrightarrow H^2(F_*(G), A; \mathbb{Z}/q) \longrightarrow \operatorname{Tor}(H^2(F_*(G), A), \mathbb{Z}/q) \longrightarrow 0$$

induced by the universal coefficient formula, which for an injective G-module A, implies that $H^2(F_n(G), A; \mathbb{Z}/q) \cong \operatorname{Tor}(H^2(F_n(G), A), \mathbb{Z}/q) = 0$ for all $n \geq 0$.

We also have the following short exact sequence of Abelian cosimplicial groups

$$0 \longrightarrow B^2(F_*(G), A; \mathbb{Z}/q) \longrightarrow Z^2(F_*(G), A; \mathbb{Z}/q) \longrightarrow H^2(F_*(G), A; \mathbb{Z}/q) \longrightarrow 0.$$

Finally this implies that $R^n_{\mathcal{F}} Z^2(G, A; \mathbb{Z}/q) = 0$, $n > 0$, if A is an injective G-module.

Now by induction on k, using (6.26) and (6.28), we easily obtain that $R^n_{\mathcal{F}} Z^k(G, A; \mathbb{Z}/q) = 0$, $n > 0$, for an injective G-module A and $k \geq 3$.

Clearly, by the universal coefficient formula, we have $H^n(G, A; \mathbb{Z}/q) = 0$, $n \geq 2$, if A is an injective G-module.

Thus we have shown that two sequences of functors

1) $Z^k(G, -; \mathbb{Z}/q), H^{k+1}(G, -; \mathbb{Z}/q), H^{k+2}(G, -; \mathbb{Z}/q), \ldots;$
2) $Z^k(G, -; \mathbb{Z}/q), R^1_{\mathcal{F}} Z^k(G, -; \mathbb{Z}/q), R^2_{\mathcal{F}} Z^k(G, -; \mathbb{Z}/q), \ldots$

satisfy the following axioms for a connected sequence of additive functors $\{T_n, \theta^n, n \geq 0\}$ from the category of G-modules to the category of Abelian groups:

(i) $T_0(-) = Z^k(G, -; \mathbb{Z}/q)$;

(ii) for any short exact sequence of G-modules $0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0$ there is a long exact sequence of Abelian groups

$$0 \longrightarrow T_0(A_1) \longrightarrow T_0(A) \longrightarrow T_0(A_2) \xrightarrow{\theta^0} T_1(A_1) \longrightarrow$$
$$\longrightarrow \cdots \xrightarrow{\theta^{n-1}} T_n(A_1) \longrightarrow T_n(A) \longrightarrow T_n(A_2) \xrightarrow{\theta^n} T_{n+1}(A_1) \longrightarrow \cdots;$$

(iii) if A is an injective G-module, then $T_n(A) = 0$ for all $n \geq 1$.

□

In particular, Theorem 6.57 allows us to describe the mod q cohomology groups $H^n(G, A; \mathbb{Z}/q)$, $n \geq 3$, in terms of the non-Abelian derived functors of the functor $Z^2(-, A; \mathbb{Z}/q)$.

Remark 6.58. An assertion similar to Theorem 6.57 has been proved in [64] for the classical (co)homology of groups and associative algebras. Moreover, we can obtain a similar result for mod q homology of groups.

7. Vogel Cohomology of Groups

In this section we recall the definition of Vogel cohomology and give the proof, due to Vogel [125], that it is a generalization of Tate–Farrell cohomology.

Recall from Example 6.25 the Hom complex $\mathcal{H}om(C_*, K_*)_*$ in the category $\mathcal{D}_R$. Given C_* and K_* in $\mathcal{D}_R$, the bounded Hom complex $\mathcal{H}om_b(C_*, K_*)_*$ is the subcomplex of $\mathcal{H}om(C_*, K_*)_*$ given by

$$\mathcal{H}om_b(C_*, K_*)_n = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, K_{i+n}).$$

Proposition 6.59. *Let C_i, $i \in \mathbb{Z}$ be a finitely generated R-module. Then there is an isomorphism of complexes*

$$\mathcal{H}om_b(C_*, K_*)_* \cong \mathcal{H}om(C_*, R)_* \otimes K_*.$$

Proof. It is easy to verify that for a finitely generated projective R-module C_i there is an isomorphism

$$\mathrm{Hom}_R(C_i, K_{i+n}) \cong \mathrm{Hom}_R(C_i, R) \otimes_R K_{i+n}.$$

Then we have

$$\begin{aligned}\mathcal{H}om_b(C_*, K_*)_n &= \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, K_{i+n}) \cong \bigoplus_{i\in\mathbb{Z}} (\mathrm{Hom}_R(C_i, R) \otimes_R K_{i+n}) = \\ &= \bigoplus_{i\in\mathbb{Z}} (\mathrm{Hom}_R(C_{-i}, R) \otimes_R K_{n-i}) = (\mathcal{H}om_b(C_*, R)_* \otimes K_*)_n.\end{aligned}$$

□

Let $K_* \in \mathcal{D}_R$. Our second example of a functor Φ (see Sec. 2 of this chapter) associates to $C_* \in \mathcal{D}_R$ the quotient complex

$$\widehat{\mathcal{H}om}(C_*, K_*)_* = \mathcal{H}om(C_*, K_*)_* / \mathcal{H}om_b(C_*, K_*)_*.$$

Then the Φ-cohomology of this complex is written

$$\widehat{H}^n(C_*, K_*) := H^n_\Phi(C_*) = H_{-n}(\widehat{\mathcal{H}om}(C_*, K_*)_*).$$

These cohomology groups have the expected property:

Lemma 6.60. *Let $C_*, K_* \in \mathcal{D}_R$. Then the cohomology groups $\widehat{H}^n(C_*, K_*)$ depend only on the homotopy classes of C_* and K_*.*

This lemma allows the following:

Definition 6.61. Let A and A' be two R-modules. Let L_* be a projective resolution of A and L'_* a projective resolution of A'. Then we set

$$\widehat{\mathrm{Ext}}^n_R(A, A') := \widehat{H}^n(L_*, L'_*).$$

Proposition 6.62. *Let $K_* \in \mathcal{D}_R$ and $0 \longrightarrow C'_* \longrightarrow C_* \longrightarrow C''_* \longrightarrow 0$ be an exact sequence in $\mathcal{D}_R$. Then we have two long exact sequences*

$$\cdots \longrightarrow \widehat{H}^{n-1}(C'_*, K_*) \longrightarrow \widehat{H}^n(C''_*, K_*) \longrightarrow \widehat{H}^n(C_*, K_*) \longrightarrow$$
$$\longrightarrow \widehat{H}^n(C'_*, K_*) \longrightarrow \widehat{H}^{n+1}(C''_*, K_*) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \widehat{H}^{n-1}(K_*, C''_*) \longrightarrow \widehat{H}^n(K_*, C'_*) \longrightarrow \widehat{H}^n(K_*, C_*) \longrightarrow$$
$$\longrightarrow \widehat{H}^n(K_*, C''_*) \longrightarrow \widehat{H}^{n+1}(K_*, C'_*) \longrightarrow \cdots .$$

Proof. The short exact sequence of complexes

$$0 \longrightarrow \mathcal{H}om(C''_*, K_*)_* \longrightarrow \mathcal{H}om(C_*, K_*)_* \longrightarrow \mathcal{H}om(C'_*, K_*)_* \longrightarrow 0$$

is restricted to an exact sequence

$$0 \longrightarrow \mathcal{H}om_b(C''_*, K_*)_* \longrightarrow \mathcal{H}om_b(C_*, K_*)_* \longrightarrow \mathcal{H}om_b(C'_*, K_*)_* \longrightarrow 0\ .$$

By diagram chasing these two exact sequences induce a third one

$$0 \longrightarrow \widehat{\mathcal{H}om}(C''_*, K_*)_* \longrightarrow \widehat{\mathcal{H}om}(C_*, K_*)_* \longrightarrow \widehat{\mathcal{H}om}(C'_*, K_*)_* \longrightarrow 0$$

implying the first long exact sequence. The second long exact sequence is obtained in the same way. $\square$

Corollary 6.63. *Let M be an R-module and $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ an exact sequence of R-modules. Then we have two long exact sequences*

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^{n-1}(A', M) \longrightarrow \widehat{\mathrm{Ext}}^{n}(A'', M) \longrightarrow \widehat{\mathrm{Ext}}^{n}(A, M) \longrightarrow \widehat{\mathrm{Ext}}^{n}(A', M) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^{n-1}(M, A'') \longrightarrow \widehat{\mathrm{Ext}}^{n}(M, A') \longrightarrow \widehat{\mathrm{Ext}}^{n}(M, A) \longrightarrow \widehat{\mathrm{Ext}}^{n}(M, A'') \longrightarrow \cdots\ .$$

Vogel's Ext functors have applications outside group theory [95, 128], but, to keep to our subject, we just relate them, when R is a group ring, to Farrell cohomology theory (see [50] or, e.g., [10]). From now on, the ring R is $\mathbb{Z}[G]$ with G a group, and we give the definition of Vogel cohomology of groups.

Definition 6.64. Let G be a group and A a G-module. Then Vogel cohomology groups are given, for $n \in \mathbb{Z}$, by

$$\widehat{H}^n(G, A) := \widehat{\mathrm{Ext}}^{n}_G(\mathbb{Z}, A).$$

Before giving the proof, due to Vogel, that his cohomology theory is a generalization of Farrell cohomology we recall the definition of Farrell cohomology [50].

Definition 6.65. A complete resolution for a group G is a pair (F_*, F''_*) of complexes of G-modules such that

(i) F_* is acyclic;

(ii) F''_* is a resolution of the G-module $\mathbb{Z}$;

(iii) F_* and F''_* coincide in higher dimensions.

In the sequel, we will always assume that a complete resolution is projective, i.e., F_* and F''_* are complexes of projective G-modules. We shall say that a group G satisfies condition (CR) if there exists a complete resolution (F_*, F''_*) for G, if such a complete resolution is unique up to homotopy, and if there exists a surjective morphism $F_* \longrightarrow F''_*$ which is the identity in higher dimensions. We shall say that G satisfies condition (CR_f) if, furthermore, there exists a complete resolution with each F_i and F''_i finitely generated, $i \in \mathbb{Z}$.

Remark 6.66. The existence of the morphism $F_* \longrightarrow F''_*$ is a consequence of the construction of the complete resolution [10, Proposition X 2.3]. Furthermore this morphism can be made surjective by a change of F_*.

Definition 6.67. Let G be a group satisfying condition (CR), A a G-module and (F_*, F''_*) a complete resolution for G. Then Farrell cohomology groups with coefficients in A are the groups $\widehat{H}^n_{Fa}(G, A) = H^n(F_*, A)$.

Tate cohomology is Farrell cohomology for finite groups. Tate first built complete resolutions in this case by splicing a resolution and a coresolution [10]. Then Farrell checked the condition (CR) for groups with finite virtual cohomological dimension (vcd) [50]. Finally Ikenaga, introducing a generalized cohomological dimension, proved that condition (CR) is valid for a wider class of groups [58].

Theorem 6.68 (see [125])**.** *Let G be a group satisfying condition (CR_f). We assume that, given an acyclic projective complex F_*, the complex $\mathcal{H}om(F_*, \mathbb{Z}[G])_*$ is acyclic. Then the Farrell cohomology of G and the Vogel cohomology of G coincide.*

Proof. Let A be a G-module and L_* a projective G-resolution of A. By condition (CR_f) there exists a complete projective resolution (F_*, F''_*) for G with each F_i and F''_i finitely generated, $i \in \mathbb{Z}$.

Let F'_* be the kernel of the canonical epimorphism $F_* \longrightarrow F''_*$. We have an exact sequence of complexes $0 \longrightarrow F'_* \longrightarrow F_* \longrightarrow oF''_* \longrightarrow 0$, and thus an exact sequence

$$0 \longrightarrow \widehat{\mathcal{H}om}(F''_*, L_*)_* \longrightarrow \widehat{\mathcal{H}om}(F_*, L_*)_* \longrightarrow \widehat{\mathcal{H}om}(F'_*, L_*)_* \longrightarrow 0. \tag{6.29}$$

As L_* is bounded beneath and F'_* is bounded overhead, we have

$$\mathcal{H}om(F'_*, L_*)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_G(F'_i, L_{n+i}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_G(F'_i, L_{n+i}) = \mathcal{H}om_b(F'_*, L_*)_n.$$

Thus $\mathcal{H}om(F'_*, L_*)_* = \mathcal{H}om_b(F'_*, L_*)_*$ and $\widehat{\mathcal{H}om}(F'_*, L_*)_* = 0$. Sequence (6.29) gives an isomorphism $\widehat{\mathcal{H}om}(F''_*, L_*)_* \cong \widehat{\mathcal{H}om}(F_*, L_*)_*$.

As the complex $\mathcal{H}om(F_*, \mathbb{Z}[G])_*$ is acyclic, the complex $\mathcal{H}om(F_*, \mathbb{Z}[G])_* \otimes L_*$ is acyclic and, by Proposition 6.59, $\mathcal{H}om_b(F_*, L_*)_*$ is also acyclic. Thus, the canonical morphism $\mathcal{H}om(F_*, L_*)_* \longrightarrow \widehat{\mathcal{H}om}(F''_*, L_*)_*$ is a homology equivalence. The complexes $\mathcal{H}om(F_*, L_*)_*$ and $\mathcal{H}om(F_*, A[0])_*$, where A is in degree 0, are homotopy equivalent, since L_* is a resolution of A (see Example 6.25). Finally the complexes $\widehat{\mathcal{H}om}(F''_*, L_*)_*$ and $\mathcal{H}om(F_*, A[0])_*$ have the same homology, that is, Vogel and Farrell cohomology coincide. □

Remark 6.69. Let (K_*, K''_*) be another complete projective resolution for G. Plainly, as, by condition (CR), the complexes F_* and K_* are homotopy equivalent the complexes $\mathcal{H}om_b(F_*, L_*)_*$ and $\mathcal{H}om_b(K_*, L_*)_*$ are equivalent. Whence $\mathcal{H}om_b(K_*, L_*)_*$ is acyclic even if the groups K_i are not finitely generated.

Finite groups G satisfy condition (CR_f) and, given an acyclic projective complex F_*, the complex $\mathcal{H}om(F_*, \mathbb{Z}[G])_*$ is acyclic [10]. Thus we have

Corollary 6.70. *For a finite group G, Tate and Vogel cohomology of G coincide.*

Condition (CR) is true for a group G with vcd(G) finite, but condition (CR_f) is not always true. Nevertheless Remark 6.69 allows to extend the corollary in this case.

Corollary 6.71. *Let G be a group satisfying condition (CR). We assume that for any G-module A, if $\widehat{H}^n_{Fa}(H, A) = 0$ for any finite subgroup H of G, then $\widehat{H}^n_{Fa}(G, A) = 0$. Then Farrell and Vogel cohomology of G coincide.*

Proof. Let (F_*, F''_*) be a complete resolution for G. It is a complete resolution as well for any (finite) subgroup H, and a complex L_* of projective G-modules is a complex of projective H-modules. By definition $\widehat{H}^n_{Fa}(G, A) = 0$ (resp $\widehat{H}^n_{Fa}(H, A) = 0$) means that the complex $\mathcal{H}om_G(C_*, A)_*$ (resp $\mathcal{H}om_H(C_*, A)_*$) is acyclic. Either by hand calculation or by use of a spectral sequence associated to the bicomplex $\mathrm{Hom}_G(C_i, L'_j)$, we see that, for any bounded G-complex L'_*, the complex $\mathcal{H}om(C_*, L'_*)_*$

is acyclic if and only if, for each $i \in \mathbb{Z}$, the complex $\mathcal{H}om(C_*, L'_i)_*$ is acyclic. Thus the hypothesis is equivalent to the same hypothesis where the G-module A is replaced by a bounded complex.

Let A be a G-module and L_* a projective G-resolution of A. Then, for any subgroup H of G and bounded subcomplex $L'_* \in \mathcal{D}_R$ of L_*, the complex $\mathcal{H}om_H(F_*, L'_*)$ is acyclic by Proposition 6.59, the proof of Theorem 6.68 and Remark 6.69. Thus, the complex $\mathcal{H}om_G(F_*, L'_*)$ is acyclic. Whence, as a colimit, the complex $\mathcal{H}om(F_*, L_*)$ is acyclic, and we apply the end of the proof of Theorem 6.68. $\square$

The hypothesis in Corollary 6.71 is true for groups with finite vcd [10, Lemma X 5.1]. It does not work for all groups considered by Ikenaga, but he exhibited among them a large class of groups, called C_∞, containing a class of groups with finite vcd but larger and for which this hypothesis is true; see [58] for details. Thus we have

Corollary 6.72. *If the group G belongs to the class C_∞ of Ikenaga, in particular if G has a finite* vcd*, then Farrell and Vogel cohomology of G coincide.*

Remark 6.73.

(i) Vogel introduced also a cohomology theory in which he replaces the complex $\mathcal{H}om_b(C_*, K_*)_*$ by the subcomplex $\mathcal{H}om_f(C_*, K_*)_*$ of morphisms which factor through bounded complexes of finitely generated projective R-modules. This gives the same theory for a finite group but not generally.

(ii) Proposition 6.59 is still valid if instead of C_i finitely generated we assume that K_i, $i \in \mathbb{Z}$, is finitely generated. Hence, if the G-module A admits a projective resolution by finitely generated projective G-modules, Vogel and Ikenaga cohomology with coefficients in A coincide if G satisfies condition (CR).

8. Mod q Vogel Cohomology of Groups

In this section we extend Vogel's definition to get mod q cohomology. Then we investigate its properties, among which we generalize some classical properties of Tate–Farrell cohomology.

Lemmas 6.24 and 6.60 allow the following definition.

Definition 6.74. Let G be a group and A a G-module. Let L_* (respectively, K_*) be a projective G-resolution of $\mathbb{Z}$ (respectively, A). Then mod q Vogel cohomology groups are given by

$$\widehat{H}^n(G, A; \mathbb{Z}/q) := H_{-n+1}(\widehat{\mathcal{H}om}(L_*, K_*)_*; \mathbb{Z}/q).$$

Now, as an immediate consequence of Lemma 6.26, the mapping cones of L_* and of $\widehat{\mathcal{H}om}(L_*, K_*)_*$ are related:

Lemma 6.75. *Let $C_*, K_* \in \mathcal{D}_R$. For all $n \in \mathbb{Z}$ we have a canonical isomorphism*

$$\widehat{\mathcal{H}om}(\mathrm{Mc}(C_*, q)_*, K_*)_n \cong \mathrm{Mc}(\widehat{\mathcal{H}om}(C_*, K_*)_*, q)_{n+1}.$$

Hence we have the following proposition.

Proposition 6.76. *Let G be a group and A a G-module. Then, for all $n \in \mathbb{Z}$, we have an isomorphism*

$$\widehat{H}^n(G, A; \mathbb{Z}/q) = \widehat{\mathrm{Ext}}^n_G(\mathbb{Z}/q, A).$$

Proof. If L_* is a projective G-resolution of $\mathbb{Z}$, we claim that $\mathrm{Mc}(L_*, q)_*$ is a projective G-resolution of $\mathbb{Z}/q$. This is always true, but for our purpose it is enough to consider the standard bar resolution; see Sec. 5 of this chapter. Whence Lemma 6.75 gives the result. $\square$

Remark 6.77. The Farrell mod q cohomology of a group of finite virtual cohomological dimension can be defined in the same way Farrell defined his cohomology by taking a complete resolution of $\mathbb{Z}/q$ instead of $\mathbb{Z}$. For instance, the pair of mapping cones $(\mathrm{Mc}(F_*, q)_*, \mathrm{Mc}(F''_*, q)_*)$, where (F_*, F''_*) is a complete resolution of $\mathbb{Z}$, is a complete resolution of $\mathbb{Z}/q$. Then the proofs of Theorem 6.68 and Corollary 6.71 work to show that Farrell mod q cohomology coincides with Vogel mod q cohomology.

In this context Corollary 6.20 becomes

Proposition 6.78 (universal coefficient formula)**.** *Let G be a group and A a G-module. Then, for all $n \in \mathbb{Z}$, there is a short exact sequence of Abelian groups*

$$0 \longrightarrow \widehat{H}^{n-1}(G, A) \otimes \mathbb{Z}/q \longrightarrow \widehat{H}^n(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(\widehat{H}^n(G, A), \mathbb{Z}/q) \longrightarrow 0\ .$$

Corollary 6.79. *Let G be a group with $\mathrm{vcd}(G) < \infty$ and A a G-module. Then the canonical map $H^n(G, A; \mathbb{Z}/q) \longrightarrow \widehat{H}^n(G, A; \mathbb{Z}/q)$ induces an isomorphism for $n \geq \mathrm{vcd}(G) + 2$ and a surjection for $n = \mathrm{vcd}(G) + 1$.*

Proof. We have the following commutative diagram of groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{n-1}(G, A) \otimes \mathbb{Z}/q & \longrightarrow & H^n(G, A; \mathbb{Z}/q) & \longrightarrow & \mathrm{Tor}(H^n(G, A), \mathbb{Z}/q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{H}^{n-1}(G, A) \otimes \mathbb{Z}/q & \longrightarrow & \widehat{H}^n(G, A; \mathbb{Z}/q) & \longrightarrow & \mathrm{Tor}(\widehat{H}^n(G, A), \mathbb{Z}/q) & \longrightarrow & 0
\end{array}$$

with exact rows; the vertical homomorphisms are the canonical maps. By [10] the first vertical map is surjective, and the third vertical map is an isomorphism for $n - 1 \geq \mathrm{vcd}(G)$; furthermore the first vertical map is an isomorphism still for $n-1 > \mathrm{vcd}(G)$, whence the result given by the five-lemma. $\square$

Corollary 6.80. *Let G be a finite group and A a G-module. Then*

$$\widehat{H}^n(G, A; \mathbb{Z}/q) = H^n(G, A; \mathbb{Z}/q), \quad n \geq 2;$$
$$\widehat{H}^{-n}(G, A; \mathbb{Z}/q) = H_n(G, A; \mathbb{Z}/q), \quad n \geq 2;$$

furthermore, the groups $\widehat{H}^{-1}(G, A; \mathbb{Z}/q)$, $\widehat{H}^0(G, A; \mathbb{Z}/q)$, and $\widehat{H}^1(G, A; \mathbb{Z}/q)$ are new and enter into short exact sequences

$$0 \longrightarrow H_1(G, A) \otimes \mathbb{Z}/q \longrightarrow \widehat{H}^{-1}(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(\widehat{H}^{-1}(G, A), \mathbb{Z}/q) \longrightarrow 0,$$
$$0 \longrightarrow \widehat{H}^{-1}(G, A) \otimes \mathbb{Z}/q \longrightarrow \widehat{H}^0(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(\widehat{H}^0(G, A), \mathbb{Z}/q) \longrightarrow 0,$$
$$0 \longrightarrow \widehat{H}^0(G, A) \otimes \mathbb{Z}/q \longrightarrow \widehat{H}^1(G, A; \mathbb{Z}/q) \longrightarrow \mathrm{Tor}(H^1(G, A), \mathbb{Z}/q) \longrightarrow 0.$$

Using again the universal coefficient formula for a group G of order k, we see that, for $n \in \mathbb{Z}$ and $x \in \widehat{H}^n(G, A; \mathbb{Z}/q)$, we have $k^2 x = 0$. Whence the groups $\widehat{H}^n(G, A; \mathbb{Z}/q)$ are finite when G is finite and A is a finitely generated G-module.

It is easy to verify that Shapiro's lemma holds for mod q Vogel cohomology of groups which states that, if H is a subgroup of finite index in a group G and A is an H-module, we have an isomorphism

$$\widehat{H}^*(H, A; \mathbb{Z}/q) \cong \widehat{H}^*(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A; \mathbb{Z}/q).$$

The proof is similar to the case of Vogel homology [52, Lemma 4.4].

We have a cup product, actually a composition product [10], on Vogel cohomology $\widehat{H}^*(G, -)$ [52]. For $\mathrm{vcd}(G) < \infty$, a fortiori for G finite, we recover the usual cup product. We shall extend this cup product to mod q cohomology for groups with finite virtual cohomology dimensions.

A group G is said to have periodic cohomology if there exists an integer $d \neq 0$ such that, for any $n \in \mathbb{Z}$, the functors $\widehat{H}^n(G,-)$ and $\widehat{H}^{n+d}(G,-)$ are isomorphic. In the case $\operatorname{vcd}(G) \leq \infty$, it is equivalent to the existence of an element $u \in \widehat{H}^d(G,\mathbb{Z})$ that is invertible in the ring $\widehat{H}^*(G,\mathbb{Z})$. Then [10] the cup product with u gives, for any $n \in \mathbb{Z}$ and any G-module A, a periodicity isomorphism

$$u \cup - : \widehat{H}^n(G,A) \cong \widehat{H}^{n+d}(G,A).$$

Note that, at least for $\operatorname{vcd}(G) < \infty$, if G has periodic cohomology, the period d is even.

Theorem 6.81. *Let G be a finite group, L_* a complete resolution of $\mathbb{Z}$ for G, and A and B two G-modules. Then*

(i) *the cochain product $\cup$ of Tate cohomology induces a cup product*

$$\widehat{H}^p(G,A) \otimes \widehat{H}^n(G,B;\mathbb{Z}/q) \xrightarrow{\cup} \widehat{H}^{p+n}(G,A\otimes B;\mathbb{Z}/q)$$

given by

$$f\cdot(g,h) = \big(f\cdot g, (-1)^p f\cdot h\big),$$

where $f \in \operatorname{Hom}_G(L_*,A)_p$ and $(g,h) \in \operatorname{Hom}_G(\operatorname{Mc}(L_*,q)_*,B)_n$;

(ii) *for G with periodic cohomology of period d, the cup product with $u \in \widehat{H}^d(G,\mathbb{Z})$ induces an isomorphism*

$$\widehat{H}^n(G,B;\mathbb{Z}/q) \cong \widehat{H}^{n+d}(G,B;\mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$ and any G-module B.

Proof.

(i) It is easily checked that we have the equality

$$\delta(f(g,h)) = \delta f\cdot(g,h) + (-1)^p f\cdot\delta(g,h)$$

implying the correctness of the cup product.

(ii) To prove the periodicity, the defining properties of the Tate cohomology cup product are used [2, Theorem 7.1]. We obtain the following commutative diagram of groups:

$$\begin{tikzcd}
\cdots \arrow[r, "\times q"] & \widehat{H}^{n-1}(G,B) \arrow[r] \arrow[d] & \widehat{H}^n(G,B;\mathbb{Z}/q) \arrow[r] \arrow[d] & \widehat{H}^n(G,B) \arrow[r, "\times q"] \arrow[d] & \cdots \\
\cdots \arrow[r, "\times q"'] & \widehat{H}^{n-1+d}(G,B) \arrow[r] & \widehat{H}^{n+d}(G,B;\mathbb{Z}/q) \arrow[r] & \widehat{H}^{n+d}(G,B) \arrow[r, "\times q"'] & \cdots
\end{tikzcd}$$

with exact rows, and the vertical homomorphisms are induced by the cup product given in (i). Since the periodicity holds for the Tate cohomology [2, 10], it remains to apply the five lemma. □

Remark 6.82. By the same way the periodicity theorem can be proved for groups with $\operatorname{vcd} G < \infty$ having periodic cohomology.

Notice too that there is a cup product action of Tate cohomology on the right:

$$\widehat{H}^n(G,B;\mathbb{Z}/q) \otimes \widehat{H}^p(G,A) \xrightarrow{\cup} \widehat{H}^{n+p}(G,B\otimes A;\mathbb{Z}/q)$$

given by $(g,h)\cdot f = (g\cdot f, h\cdot f)$. In this case

$$\delta((g,h)\cdot f) = \delta(g,h)\cdot f + (-1)^n\ (g,h)\cdot\delta f,$$

where $(g,h) \in \operatorname{Hom}_G(\operatorname{Mc}(L_*,q)_*,B)_n$, $f \in \operatorname{Hom}_G(L_*,B)_p$, and the mod q Tate cohomology $\widehat{H}^*(G,B;\mathbb{Z}/q)$ becomes an $\widehat{H}^*(G,\mathbb{Z})$-bimodule for any G-module B.

From Theorem 6.81 we deduce that we have periodicity of the mod q Tate cohomology for finite cyclic groups having periodic cohomology of period 2 and for finite subgroups of the multiplicative group of the quaternion algebra having periodic cohomology of period 4. Moreover we have

Corollary 6.83. *Let C_m be a cyclic group of order m and t be a generator of C_m. Then for any C_m-module A, we obtain*

$$\widehat{H}^{2n}(C_m, A; \mathbb{Z}/q) = \{(a, a') \mid Na + qa' = 0,\ ta' = a'\}/\widetilde{D}(A \oplus A), \quad n \in \mathbb{Z},$$
$$\widehat{H}^{2n+1}(C_m, A; \mathbb{Z}/q) = \{(a, a') \mid Da + qa' = 0,\ Na' = 0\}/\widetilde{N}(A \oplus A), \quad n \in \mathbb{Z},$$

with $N = 1 + t + \cdots + t^{m-1}$, $D = t - 1 \in \mathbb{Z}[G]$ and where the homomorphisms $\widetilde{D} : A \oplus A \longrightarrow A \oplus A$ and $\widetilde{N} : A \oplus A \longrightarrow A \oplus A$ are defined by $\widetilde{D}(a, a') = (Da + qa', -Na')$ and $\widetilde{N}(a, a') = (Na + qa', -Da')$.

Proof. It follows from Theorem 6.81 (ii) and Proposition 6.35. $\square$

Remark 6.84. The question of periodic cohomology for a wider class of groups has been considered in classical cohomology in the context of "periodicity after k steps" [121, 122].

Theorem 6.85. *Let G be a p-group whose order $|G| = p^m$ divides q, and A a G-module. Then the following conditions are equivalent:*

(i) $\widehat{H}^n(G, A; \mathbb{Z}/q) = 0$ *for some $n \in \mathbb{Z}$;*

(ii) *A is cohomologically trivial.*

If in addition A is p-torsion-free, then (i) *and* (ii) *are equivalent to*

(iii) *A/pA is free over $(\mathbb{Z}/p)[G]$.*

Proof. First, assume that A is p-torsion-free. According to [2, Theorem 9.2], it suffices to show the equivalence of the following two conditions:

(i) $\widehat{H}^n(G, A; \mathbb{Z}/q) = 0$ for some $n \in \mathbb{Z}$;

(iv) $\widehat{H}^n(G, A) = 0$ for two consecutive integers n.

(iv) $\Longrightarrow$ (i): if $\widehat{H}^n(G, A) = \widehat{H}^{n+1}(G, A) = 0$, then, by Theorem 6.78, $\widehat{H}^{n+1}(G, A; \mathbb{Z}/q) = 0$.

(i) $\Longrightarrow$ (iv): if $\widehat{H}^n(G, A; \mathbb{Z}/q) = 0$, the homomorphism $\widehat{H}^{n-1}(G, A) \xrightarrow{\times q} \widehat{H}^{n-1}(G, A)$ is surjective, and the homomorphism $\widehat{H}^n(G, A) \xrightarrow{\times q} \widehat{H}^n(G, A)$ is injective. Thus, for $x \in \widehat{H}^{n-1}(G, A)$, there is an element $y \in \widehat{H}^{n-1}(G, A)$ with $qy = x$. On the other hand, we have $p^m y = 0$, whence $qy = 0$. If $x \in \widehat{H}^n(G, A)$, the equality $p^m x = 0$ implies $qx = 0$, and therefore $x = 0$.

The equivalence of (i) and (ii) for any G-module A is reduced to the previous case by use of dimension-shifting. Take a short exact sequence of G-modules

$$0 \longrightarrow A' \longrightarrow F \longrightarrow A \longrightarrow 0$$

with F free over $\mathbb{Z}[G]$. Then we have the isomorphisms

$$\widehat{H}^n(G, A) \cong \widehat{H}^{n+1}(G, A') \quad \text{and} \quad \widehat{H}^n(G, A; \mathbb{Z}/q) \cong \widehat{H}^{n+1}(G, A'; \mathbb{Z}/q)$$

for all $n \in \mathbb{Z}$ with A' torsion-free. $\square$

We end with a final example of extension of a classical property to Vogel cohomology:

Proposition 6.86. *Let G be a group and A a projective G-module. Then*

$$\widehat{H}^*(G, A; \mathbb{Z}/q) = 0.$$

Proof. We take $L_0 = A$ and $L_n = 0$ for $n \neq 0$ as a projective resolution of A. $\square$

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