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Article

Smooth K-groups for Monoid Algebras and K-regularity

Hvedri Inassaridze

A. Razmadze Mathematical Institute of Tbilisi State University, 6, Tamarashvili Str., Tbilisi 0179, Georgia; E-Mail: inassari@gmail.com; Tel.: +995-599-157-836

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Abstract: The isomorphism of Karoubi-Villamayor K-groups with smooth K-groups for monoid algebras over quasi stable locally convex algebras is established. We prove that the Quillen K-groups are isomorphic to smooth K-groups for monoid algebras over quasi-stable Frechet algebras having a properly uniformly bounded approximate unit and not necessarily m-convex. Based on these results the K-regularity property for quasi-stable Frechet algebras having a properly uniformly bounded approximate unit is established.

Keywords: smooth K-groups; K-regularity; properly uniformly bounded approximate unit; quasi stable locally convex algebra

1. Introduction

Our aim is to find a wide class of locally convex algebras for which the K-regularity property holds [1]. For this purpose first we will investigate the relationship between Karoubi-Villamayor algebraic K-theory and topological K-theory in the category of locally convex complex algebras. Then the existing comparison of Quillen algebraic K-theory and topological K-theory [2–8] will be extended to monoid algebras over locally convex algebras. To establish these results the topological invariants introduced in [4,9] and called smooth K-groups will be used. It should be noted that K-theories related to smooth K-groups have been treated in [2,10].

More recent results on K-regularity of rings were obtained by Gubeladze [11,12] and by Cortinas, Haesemeyer and Weibel [13]. For commutative $C*$-algebras the K-regularity was established by Rosenberg [14]. Further results about K-regularity of operator algebras were obtained for stable C^*-algebras [15] and for stable profinite C^*-algebras [16].

All previous results of Wodzicki, Suslin-Wodzicki, Cortinas-Thom and Cuntz in that direction were obtained exclusively for multiplicatively convex (m-convex) Frechet algebras. The novelty of our investigation consists of considering arbitrary Frechet algebras, and we don't assume the m-convexity property. The confirmation of Karoubi Conjecture for arbitrary Frechet algebras is based on the compatibility property of smooth maps with respect to Grothendieck tensor product of locally convex algebras (Lemma 3.1). The continuous maps don't satisfy this property even for Banach algebras. That is the reason for which topological K-groups and homotopy functors are replaced by smooth K-groups and smooth homotopy functors respectively. That gave us the possibility to apply Higson and Wodzicki-Suslin methods to extend comparison theorems for arbitrary Frechet algebras (Theorems 3.8 and 3.9).

The Frechet algebras considered in this article are not necessarily locally multiplicatively convex (m-convex). Many important examples of functional algebras occur in analysis that are not locally multiplicatively convex and are Frechet algebras with properly uniformly bounded approximate unit (for instance the Frechet algebra of all complex-valued measurable functions on the unit interval and the convolution Schwartz algebras, see Examples 1–4, [4]).

2. Preliminaries

In this section we recall some definitions and propositions given in [4] which will be used later.

As noted above the smooth K-theory was introduced in [4,9] for locally convex algebras. The same definition is valid for arbitrary real or complex topological algebras and is completely similar. Namely, let $A^{\infty(I)}$ be the topological algebra of smooth maps from the unit interval I to the topological algebra A. Any continuous homomorphism of topological algebras $\varphi : A \to A'$ induces a homomorphism of topological algebras $\varphi^{\infty(I)} : A^{\infty(I)} \to A'^{\infty(I)}$. For any topological algebra A consider the evaluation maps at $t = 0$ and $t = 1$

$$A^{\infty(I)} \to A, i = 0, 1, \varepsilon_0(f) = f(0), \varepsilon_1 = f(1).$$

Denote by $\mathfrak{I}(A)$ the kernel of ε_0 and by $\tau_A : \mathfrak{I}(A) \to A$ the restriction of ε_1 on $\mathfrak{I}(A)$. There is a smooth homomorphism $\delta_A : \mathfrak{I}(A) \to \mathfrak{I}^2(A)$ sending $f \in \mathfrak{I}(A)$ to $\delta_A(f)(s,t) = f(st)$. One gets the smooth path cotriple $\mathfrak{I}$ (for locally convex algebras see [4]) which induces the augmented simplicial group

$$GL(\mathfrak{I}^+_*(A)) = GL(\mathfrak{I}_*(A)) \to GL(A).$$

Definition 2.1. *For any topological algebra A the smooth K-functors $K^{sm}_n, n \geq 0$, are defined as follows*

$$K^{sm}_n(A) = \pi_{n-2}GL(\mathfrak{I}_*(A))$$

for $n \geq 3$, $K^{sm}_0(A) = K_0(A)$ and for $n = 1, 2$ are defined by the exact sequence

$$0 \to K^{sm}_2(A) \to \pi_0(GL(\mathfrak{I}_*(A)) \to GL(A) \to K^{sm}_1(A) \to 0.$$

Definition 2.2. *Let $f : B \to A$ be a continuous injective algebra homomorphism for a Banach algebra B. If an approximate unit of the Frechet algebra A is the image of a bounded approximate unit of B, then it is called properly uniformly bounded approximate unit of the Frechet algebra A.*

This is a substantial generalization of the notion of uniformly bounded approximate unit of m-convex Frechet algebras.

Definition 2.3. *It will be said that a ring A possesses the property TF_{right} if for any collection $a_1, ..., a_m \in A$ there exist $b_1, .., b_m, c, d \in A$ such that $a_i = b_i cd$ for $1 \leq i \leq m$ and the left annihilators in A of c and cd are equal.*

This definition was introduced in [6].

Theorem 2.4. *If A is a Frechet algebra with properly uniformly bounded approximate unit and not necessarily m-convex, then it possesses the TF_{right}-property and therefore the excision property in algebraic K-theory and the H-unitality property.*

3. Smooth Karoubi Conjecture and K-regularity

Besides the aforementioned assertions given in [4] for locally convex algebras we need the following important property of smooth maps which will be used to prove Theorems 3.6, 3.8 and 3.9:

Lemma 3.1. *There is an isomorphism*

$$A^{\infty(I)} \widehat{\otimes} B \approx (A \widehat{\otimes} B)^{\infty(I)}$$

for any A and B locally convex algebras.

Remark 3.2. *(1) For m-convex locally convex algebras Lemma 3.1 is proved in [17]. The proof for the general case is due to Larry Schweitzer.*
(2) It should be noted that this property doesn't hold for continuous maps and arbitrary locally convex algebras even for Banach algebras.

Theorem 3.3. *Let $\mathbf{B}$ be a full subcategory of the category $\mathbf{A}$ of topological algebras containing with any topological algebra A the topological algebra $A^{\infty(I)}$.Then*

(1) the functors K_1 and K_1^{sm} are isomorphic on the category $\mathbf{B}$ if and only if K_1 is a smooth homotopy functor on $\mathbf{B}$,

(2) the functors KV_1 and K_1^{sm} are isomorphic on the category $\mathbf{B}$ if and only if KV_1 is a smooth homotopy functor on $\mathbf{B}$,

where KV_* denotes Karoubi-Villamayour's algebraic K-theory.

Let M be a monoid and $A[M]$ a monoid algebra over a locally convex algebra A. Regarding the topology on the monoid algebra $A[M]$ over a locally convex algebra A, it can be considered as the union of the set S of finite products of copies of A indexed by finite subsets of M and partially ordered by inclusion. Then we take on $A[M]$ the union topology induced by the topology of these finite products.

Proposition 3.4. *If a ring A has the $(TF)_{right}$ property, then the monoid algebra $A[M]$ has also the $(TF)_{right}$ property for any monoid M.*

This proposition generalizes Lemma 16 [15]. Therefore the polynomial algebra $A[x_1, x_2, \ldots, x_m]$, $n \geq 1$, and the Laurent polynomial algebra $A[t, t^{-1}]$ over a Frechet algebra A with properly uniformly bounded approximate unit possess the excision property in algebraic K-theory and the H-unitality property.

In what follows the following property of smooth maps for any locally convex algebra algebra A will be also used

$$(A[M])^{\infty(I)} \approx (A^{\infty(I)})[M]. \tag{3.1}$$

Definition 3.5. *A functor T from the category of locally convex algebras to the category of groups is called smooth homotopy functor if for two continuous maps $f, g : A \to B$ of locally convex algebras for which there exists a continuous map $h : A \to B^{\infty(I)}$ such that $\varepsilon_0(h) = \varepsilon_1(h)$ one has $T(f) = T(g)$. These f and g continuous maps are called smoothly homotopic.*

In K-theory the smooth homotopy property was also investigated in [2,18] and [19,20].

Let M be a monoid and denote by $\mathbf{ALC}[M]$ the category of monoid algebras $A[M]$ over locally convex algebras A and by $\mathbb{C}*$ the category of $C*$-algebras. Let T be an arbitrary functor T from the category $\mathbf{ALC}[M]$ to the category $\mathbf{Ab}$ of abelian groups.

Theorem 3.6. *If the functor*

$$T((A\widehat{\otimes}(- \otimes \mathcal{K}))[M]) : \mathbb{C}* \to \mathbf{Ab}$$

is a stable and split exact functor for any locally convex algebra A, then the functor

$$T((-\widehat{\otimes}\mathcal{K})[M]) : \mathbf{ALC} \to \mathbf{Ab}$$

is a smooth homotopy functor.

Definition 3.7. *A locally convex algebra B is called quasi-stable if it has the form $A\widehat{\otimes}\mathcal{K}$ for some locally convex algebra A. where $\mathcal{K}$ is the C^*-algebra of compact operators on the infinite dimensional Hilbert space $\mathcal{H}$.*

Theorem 3.8. *For any locally convex algebra A there is an isomorphism for all $n \geq 1$*

$$KV_n((A\widehat{\otimes}\mathcal{K})[M]) \to K_n^{sm}((A\widehat{\otimes}\mathcal{K})[M])$$

Theorem 3.8 generalizes Higson's result [21] on the isomorphism of Karoubi-Villamayor algebraic K-functors and topological K-functors for stable $C*$-algebras.

Theorem 3.9. *For any Frechet algebra A with properly uniformly bounded approximate unit and not necessarily m-convex there is an isomorphism*

$$K_n((A\widehat{\otimes}\mathcal{K})[M]) \to K_n^{sm}((A\widehat{\otimes}\mathcal{K})[M])$$

for all $n \geq 1$, where $K_n(-)$ denotes Quillen algebraic K-functor.

Theorem 3.10. *For any Frechet algebra A with properly uniformly bounded approximate unit one has isomorphisms*

$$K_n(A\widehat{\otimes}\mathcal{K}) \approx K_n((A\widehat{\otimes}\mathcal{K})[x_1, x_2, \ldots, x_m])$$

for $n, m \geq 1$.

Theorem 3.10 is a consequence of Theorems 3.8 and 3.9.

Corollary 3.11. *Let A be a Frechet algebra with properly uniformly bounded approximate unit. Then one has isomorphisms for all $n \geq 1$*

$$K_n((A\widehat{\otimes}\mathcal{K})[t,t^{-1}]) \approx K_n(A\widehat{\otimes}\mathcal{K}) \oplus K_{n-1}(A\widehat{\otimes}\mathcal{K})$$

and

$$K_n^{sm}((A\widehat{\otimes}\mathcal{K})[t,t^{-1}]) \approx K_n^{sm}(A\widehat{\otimes}\mathcal{K}) \oplus K_{n-1}^{sm}(A\widehat{\otimes}\mathcal{K}).$$

Remark 3.12. *Guillermo Cortinas informed me that in particular cases, namely for countable monoid M, Theorems 3.8 and 3.9 can be obtained respectively by Theorem 6.2.1 [2] and for m-convex Frechet algebra with uniformly bounded approximate unit by applying argument of Theorem 12.1.1 and Remark 12.1.4 [18].*

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Conflicts of Interest

The author declares no conflict of interest.

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