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Abstract

This paper is devoted to study a homology theory on the category of groups supporting a given action of a fixed group.

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1 Introduction

Groups enriched with an action (by automorphisms) of a given group Γ, that is, Γ-groups, provide a suitable setting for the treatment of an extensive list of subjects with recognized mathematical interest. This paper is concerned with a homology theory on the category of Γ-groups and Γ-equivariant homomorphisms between them.

The use of cohomological tools in the study of groups with operators goes back to Whitehead in [18]. There he made an appropriate treatment of Γ-group extensions by working with a vector cohomology theory of the kind considered by Lyndon in [13]. Some recent results have come to justify a renewed interest in this body of research, starting with [2] where a study on a specific cohomology theory for Γ-groups, denoted $H^n_\Gamma(G,A)$, is done (see also [11] for a relative theory, in which applications in algebraic K-theory are established). Indeed, by involving the low dimensional cohomology groups

$H^2_\Gamma(G, A)$ and $H^3_\Gamma(G, A)$, precise theorems on the homotopy classification of graded categorical groups and their homomorphisms are stated in [3]. Also, the classification of several types of equivariant crossed products constructions, such as Rédei extensions of monoids by groups with operators, crossed product Γ-algebras, equivariant graded Clifford systems or strongly graded Hopf Γ-algebras, is given in [3]. Furthermore, the following theorem, which states a suitable counterpart for an equivariant situation of the classic Brauer-Hasse-Noether result, is proved in [5]: if F/K is a Galois finite field extension on which a group Γ is acting by automorphisms, then the equivariant Brauer group $Br_\Gamma(F/K)$ and the cohomology group $H^2_\Gamma\big(\mathrm{Gal}(F/K), F^\times\big)$ are isomorphic.

The present work is embedded in this research program that Whitehead began about cohomological properties of Γ-groups. An early justification for studying homology of groups with operators came from their cohomology theory and however, by now, there is no any systematic study on a specific homology theory for these algebraic structures in the literature. Hence the purpose of this paper is to provide an appropriate and fundamental source of information on that subject. Indeed, we specialize here Barr-Beck cotriple homology [1] to define the homology groups of a Γ-group G with coefficients in an equivariant G-module A, denoted by $H_n^\Gamma(G, A)$. The article is then mainly dedicated to state and prove several desirable properties of this homology theory.

The article is organized in seven sections, with the following headings:

Section 1: Introduction
Section 2: Equivariant derivations and differentials of a Γ-group
Section 3: Homology of Γ-groups
Section 4: Low-dimensional Homology of Γ-group extensions
Section 5: The integral homology of a Γ-group
Section 6: Homology and the lower Γ-central series
Section 7: Universal central equivariant extensions of Γ-groups

2 Equivariant derivations and differentials of a Γ-group

Throughout the paper Γ stands for any fixed group. We denote by Γ-**Gp** the category of Γ-groups, that is, the category whose objects are groups enriched with a left Γ-action by automorphisms and whose morphisms are those homomorphisms $p : G \to H$ that are Γ-equivariant, in the sense that $p({}^\sigma x) = {}^\sigma p(x)$, $\sigma \in \Gamma$, $x \in G$. Such a morphism is usually termed a Γ-homomorphism. The category of abelian Γ-groups, that is, of Γ-modules, is denoted by Γ-**Mod**.

If G is a Γ-group, then a Γ-*equivariant* G-*module* A [2, Definition 2.1] is a Γ-module, also denoted by A, equipped with a G-module structure by a Γ-equivariant action map $G \times A \to A$, which means that the two actions of Γ and G on A are compatible in the following sense:

$$^{\sigma}({}^{x}a) = {}^{({}^{\sigma}x)}({}^{\sigma}a) \qquad (\sigma \in \Gamma, x \in G, a \in A)\,. \tag{1}$$

Homomorphisms between Γ-equivariant G-modules $f : A \to B$ are those abelian group homomorphisms that are of both Γ- and G-modules, that is, such that $f({}^{\sigma}a) = {}^{\sigma}f(a)$ and $f({}^{x}a) = {}^{x}f(a)$.

For any Γ-group G, the category of Γ-equivariant G-modules is isomorphic to the category $(G \rtimes \Gamma)$-**Mod** of modules over the semidirect product group $G \rtimes \Gamma$ (by means of the identification ${}^{(x,\sigma)}a = {}^{x}({}^{\sigma}a)$). Henceforth we will make no distinction between a Γ-equivariant G-module and a $(G \rtimes \Gamma)$-module. Moreover, we should note that the category $(G \rtimes \Gamma)$-**Mod** is equivalent to the category of abelian group objects in the comma category Γ-$\mathbf{Gp}/G$ of Γ-groups over G (by the functor $A \mapsto (A \rtimes G \xrightarrow{pr} G)$, see [2, Theorem 2.1]).

If $p : H \to G$ is a Γ-homomorphism, then any Γ-equivariant G-module A can be given a Γ-equivariant H-module structure "via" p by defining

$$^{h}a = {}^{p(h)}a \qquad (a \in A, h \in H)\,,$$

and keeping the same Γ-action on A. We also denote this Γ-equivariant H-module by A, p being understood.

Let A be a Γ-equivariant G-module. A Γ-*derivation* from G into A is a Γ-equivariant derivation from the group G into the G-module A, that is, a map $d : G \to A$ with the properties

i) $d(xy) = {}^{x}d(y) + d(x) \qquad (x, y \in G)$,
ii) $d({}^{\sigma}x) = {}^{\sigma}d(x), \qquad (\sigma \in \Gamma, x \in G)$.

The abelian group of all Γ-derivations $d : G \to A$, is denoted $\mathrm{Der}_{\Gamma}(G, A)$. If $p : H \to G$ is any Γ-homomorphism and $f : A \to B$ is any morphism of Γ-equivariant G-modules, then there is an induced homomorphism

$$p^{*}f_{*} = f_{*}p^{*} : \mathrm{Der}_{\Gamma}(G, A) \to \mathrm{Der}_{\Gamma}(H, B), \quad d \mapsto f\,d\,p\,.$$

Thus $\mathrm{Der}_{\Gamma}(-,-)$ becomes a functor from the cartesian product category of the comma category of Γ-groups over a given Γ-group G by the category of Γ-equivariant G-modules into the category of abelian groups. For any Γ-homomorphism $p : H \to G$ and any Γ-equivariant G-module A, there is a natural isomorphism [2, Proposition 2.5]

$$\mathrm{Der}_{\Gamma}(H, A) \cong \mathrm{Hom}_{\Gamma\text{-}\mathbf{Gp}/G}\big(H \xrightarrow{p} G,\ A \rtimes G \overset{pr}{\twoheadrightarrow} G\big)\,. \tag{2}$$

For any Γ-group G, the ring-group $\mathbb{Z}(G)$ is a Γ-equivariant G-module with the Γ-action:

$$^{\sigma}\Big(\sum_{x\in G} m_x x\Big) = \sum_{x \in G} m_x {}^{\sigma}x\,.$$

Then, the augmentation $\varepsilon : \mathbb{Z}(G) \to \mathbb{Z}$, $\sum\limits_{x\in G} m_x x \mapsto \sum\limits_{x\in G} m_x$, becomes a homomorphism of Γ-equivariant G-modules, $\mathbb{Z}$ being trivial both as Γ- and G-module. Hence the sequence

$$0 \to \mathbb{I}(G) \to \mathbb{Z}(G) \xrightarrow{\varepsilon} \mathbb{Z} \to 0\,, \tag{3}$$

where $\mathbb{I}(G)$ is the augmentation ideal of the group G, is an exact sequence of Γ-equivariant G-modules.

If A is a Γ-equivariant G-module, we know that $\mathrm{Der}(G,A) \cong \mathrm{Hom}_G(\mathbb{I}(G),A)$ (by the mapping $d \mapsto f_d$,where $f_d(x-1) = d(x)$), and it is immediate that this isomorphism carries the subgroup of Γ-equivariant derivations onto the subgroup of all Γ-equivariant G-module homomorphisms from $\mathbb{I}(G)$ to A. Thus there is a natural isomorphism

$$\mathrm{Der}_\Gamma(G,A) \cong \mathrm{Hom}_{G\rtimes\Gamma}(\mathbb{I}(G),A)\,, \tag{4}$$

and therefore $\mathbb{I}(G)$ is a Γ-*equivariant* G-*module of differential forms* of the Γ-group G. For any Γ-group over G, $H \to G$, we have the natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{G\rtimes\Gamma}\big(\mathbb{Z}(G\rtimes\Gamma)\otimes_{H\rtimes\Gamma}\mathbb{I}(H),A\big) &\cong \mathrm{Hom}_{H\rtimes\Gamma}\big(\mathbb{I}(H),A\big) \cong \mathrm{Der}_\Gamma(H,A) \\ &\cong \mathrm{Hom}_{\Gamma\text{-}\mathbf{Gp}/G}\big(H \xrightarrow{p} G,\ A\rtimes G \overset{pr}{\twoheadrightarrow} G\big)\,, \end{aligned}$$

which means that $\mathrm{Diff}^\Gamma_G(H) = \mathbb{Z}(G\rtimes\Gamma)\otimes_{H\rtimes\Gamma}\mathbb{I}(H)$ is a Γ-equivariant G-module of relative differential forms of H over G, and also that the functor of relative differentials

$$\mathrm{Diff}^\Gamma_G(-) : \Gamma\text{-}\mathbf{Gp}/G \longrightarrow (G\rtimes\Gamma)\text{-}\mathbf{Mod}\,,$$

is a left adjoint functor to the forgetful-embedding functor $(G\rtimes\Gamma)$-$\mathbf{Mod} \to \Gamma$-$\mathbf{Gp}/G$, $A \mapsto (A\rtimes G \xrightarrow{pr} G)$.

3 Homology of Γ-groups

The category of Γ-groups is tripleable over the category of sets [14], since it is a variety of universal algebras, and so it is natural to specialize Barr-Beck cotriple (co)homology [1] to the definition of (co)homology of a Γ-group G with coefficients in a Γ-equivariant G-module A.

Given a Γ-group G, the resulting cotriple $(\mathbb{G},\varepsilon,\delta)$ in the comma category Γ-$\mathbf{Gp}/G$ is as follows. For each Γ-group $H \xrightarrow{\varphi} G$ over G, $\mathbb{G}(H \xrightarrow{\varphi} G) = \mathbb{F}H \xrightarrow{\overline{\varphi}} G$, where $\mathbb{F}H$ is the free Γ-group on the set H (i.e., the free group on the set $H\times\Gamma$ with the Γ-action such that ${}^\sigma(h,\tau) = (h,\sigma\tau)$), and $\overline{\varphi} : \mathbb{F}H \to G$ is the Γ-homomorphism such that $\overline{\varphi}(h,\sigma) = {}^\sigma\varphi(h)$. The counit $\delta : \mathbb{G} \to id$ sends $H \to G$ to the Γ-homomorphism $\mathbb{F}H \to H$ such that

$\delta(h,\sigma) = {}^{\sigma}h$, and the comultiplication $\varepsilon : \mathbb{G} \to \mathbb{G}^2$ sends $H \to G$ to the Γ-homomorphism $\mathbb{F}H \to \mathbb{F}\mathbb{F}H$ such that $\varepsilon(h,\sigma) = ((h,1),\sigma)$, for each $h \in H$ and $\sigma \in \Gamma$. This cotriple produces an augmented simplicial object in the category of endofunctors in Γ-$\mathbf{Gp}/G$, $\mathbb{G}_\bullet \xrightarrow{\delta} id$, which is defined by $\mathbb{G}_n = \mathbb{G}^{n+1}$, with face and degeneracy operators $d_i = \mathbb{G}^{n-i}\delta\mathbb{G}^i : \mathbb{G}_n \to \mathbb{G}_{n-1}$, $0 \le i \le n$, and $s_j = \mathbb{G}^{n-j-1}\varepsilon\mathbb{G}^j : \mathbb{G}_{n-1} \to \mathbb{G}_n$, $0 \le j \le n-1$. Hence, for any Γ-equivariant G-module A, one obtains a cosimplicial object in the category of abelian group valuated functors from Γ-$\mathbf{Gp}/G$

$$\mathrm{Der}_\Gamma(\mathbb{G}_\bullet(-), A)\,,$$

and a simplicial object

$$A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^\Gamma \mathbb{G}_\bullet(-)$$

(as usual, we regard A as a right $(G \rtimes \Gamma)$-module by setting $a^{(x,\sigma)} = {}^{(x,\sigma)^{-1}}a = {}^{\sigma^{-1}}({}^{x^{-1}}a))$. Then one gets the corresponding associated (co)chain complexes (also denoted by $\mathrm{Der}_\Gamma(\mathbb{G}_\bullet(-), A)$ and $A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^\Gamma \mathbb{G}_\bullet(-)$, respectively), obtained by taking alternating sums of the (co)face operators

$$0 \to \mathrm{Der}_\Gamma(\mathbb{G}(-), A) \to \mathrm{Der}_\Gamma(\mathbb{G}^2(-), A) \to \mathrm{Der}_\Gamma(\mathbb{G}^3(-), A) \to \cdots$$

and

$$\cdots \to A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^\Gamma \mathbb{G}^3(-) \to A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^\Gamma \mathbb{G}^2(-) \to A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^\Gamma \mathbb{G}(-) \to 0\,.$$

These (co)chain complexes give the cotriple (co)homology groups of the Γ-group G with values in A:

$$\begin{aligned} H_\Gamma^n(G,A) &= H^{n-1}\big(\mathrm{Der}_\Gamma(\mathbb{G}_\bullet(G), A)\big)\,, \\ H_n^\Gamma(G,A) &= H_{n-1}\big(A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^\Gamma \mathbb{G}_\bullet(G)\big)\,, \quad n \ge 1\,. \end{aligned}$$

A systematic study on the cohomology groups $H_\Gamma^n(G,A)$ was done in [2]. Here, our goal is the homology theory. Some basic properties are immediate consequences of its definition; thus for example:

$$H_1^\Gamma(G,A) = A \otimes_{G\rtimes\Gamma} \mathbb{I}(G)\,, \tag{5}$$

$$H_n^\Gamma(F,A) = 0 \text{ for all } n \ge 2, \text{ whenever } F \text{ is a free } \Gamma\text{-group}, \tag{6}$$

Any short exact sequence $0 \to A \to B \to C \to 0$ of Γ-equivariant G-modules provides a long exact sequence

$$\cdots \to H_{n+1}^\Gamma(G,C) \to H_n^\Gamma(G,A) \to H_n^\Gamma(G,B) \to H_n^\Gamma(G,C) \to \cdots\,. \tag{7}$$

Theorem 3.1 *For any Γ-group G and any Γ-equivariant G-module A, there are natural isomorphisms*

$$H_n^{\Gamma}(G,A) \cong \mathrm{Tor}_{n-1}^{G\rtimes\Gamma}\big(A,\mathbb{I}(G)\big)\,, n \geq 1\,. \tag{8}$$

Proof: By definition $H_n^{\Gamma}(G,A) = H_{n-1}\big(A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^{\Gamma}\mathbb{G}_\bullet(G)\big)$. The theorem follows from the fact that the augmented complex of $(G \rtimes \Gamma)$-modules

$$\mathrm{Diff}_G^{\Gamma}\mathbb{G}_\bullet(G) \to \mathbb{I}(G) \to 0$$

is a projective resolution of $\mathbb{I}(G)$. Indeed, every $\mathrm{Diff}_G^{\Gamma}\mathbb{G}^n(G)$ is actually a free, hence projective, $G\rtimes\Gamma$-module, and for the exactness it suffices to observe that for any injective $(G\rtimes\Gamma)$-module I, the cochain complex $\mathrm{Hom}_{G\rtimes\Gamma}(\mathrm{Diff}_G^{\Gamma}\mathbb{G}_\bullet(G), I)$ is exact at dimensions ≥ 1. But, for all $n \geq 1$,

$$H^n\Big(\mathrm{Hom}_{G\rtimes\Gamma}(\mathrm{Diff}_G^{\Gamma}\mathbb{G}_\bullet(G), I)\Big) \cong H^n\Big(\mathrm{Der}_{\Gamma}(\mathbb{G}_\bullet(G), I)\Big) = H_{\Gamma}^{n+1}(G,I) = 0\,,$$

by [2, Section 3, point c)]. ∎

In the following theorem we show a basic relationship between the cohomology groups $H_n^{\Gamma}(G,A)$ with the ordinary homology groups $H_n(G\rtimes\Gamma, A)$ and $H_n(\Gamma, A)$, by means of a long exact sequence linking these groups. We should stress that the projection $G \rtimes \Gamma \twoheadrightarrow \Gamma$ induces, in general, no homomorphism $H_n(G \rtimes \Gamma, A) \to H_n(\Gamma, A)$.

Moreover, let us remark that from Theorem 3.2 below and the isomorphism (5), it follows that there are natural isomorphisms between the homology groups $H_n^{\Gamma}(G,A)$ and the (ordinary) relative homology groups $H_n(G\rtimes\Gamma, \Gamma, A)$ (cf. [2, (12)] for the corresponding fact in cohomology).

Theorem 3.2 *Let G be a Γ-group and let A be a Γ-equivariant G-module. Then there is a natural long exact sequence*

$$\begin{array}{l} \cdots \longrightarrow H_3^{\Gamma}(G,A) \longrightarrow H_2(\Gamma,A) \longrightarrow H_2(G\rtimes\Gamma,A) \longrightarrow \\ H_2^{\Gamma}(G,A) \longrightarrow H_1(\Gamma,A) \longrightarrow H_1(G\rtimes\Gamma,A) \longrightarrow \\ H_1^{\Gamma}(G,A) \longrightarrow H_0(\Gamma,A) \longrightarrow H_0(G\rtimes\Gamma,A) \longrightarrow 0\,. \end{array} \tag{9}$$

Proof: Let us apply the functor $A \otimes_{G\rtimes\Gamma} -$ to the exact sequence (3). Then we get the long exact sequence

$$\begin{array}{l} \cdots \longrightarrow \mathrm{Tor}_{n+1}^{G\rtimes\Gamma}\big(A,\mathbb{I}(G)\big) \longrightarrow \mathrm{Tor}_{n+1}^{G\rtimes\Gamma}\big(A,\mathbb{Z}(G)\big) \longrightarrow \mathrm{Tor}_{n+1}^{G\rtimes\Gamma}\big(A,\mathbb{Z}\big) \longrightarrow \\ \mathrm{Tor}_{n}^{G\rtimes\Gamma}\big(A,\mathbb{I}(G)\big) \longrightarrow \mathrm{Tor}_{n}^{G\rtimes\Gamma}\big(A,\mathbb{Z}(G)\big) \longrightarrow \cdots\,. \end{array} \tag{10}$$

By Theorem 3.2 $\mathrm{Tor}_n^{G\rtimes\Gamma}\big(A,\mathbb{I}(G)\big) \cong H^{\Gamma}_{n+1}(G,A)$ and the lemma below shows that $\mathrm{Tor}_n^{G\rtimes\Gamma}\big(A,\mathbb{Z}(G)\big) \cong H_n(\Gamma,A)$. Since $\mathrm{Tor}_n^{G\rtimes\Gamma}\big(A,\mathbb{Z}\big) \cong H_n(G\rtimes\Gamma,A)$, the announced long exact sequence (9) follows from the sequence (10). ∎

Lemma 3.3 *For any* Γ*-group* G *and any* Γ*-equivariant* G*-module* A*, there are natural isomorphisms*

$$\mathrm{Tor}_n^{G\rtimes\Gamma}\big(A,\mathbb{Z}(G)\big) \cong H_n(\Gamma,A)\,, \quad n\geq 0\,. \tag{11}$$

$$\mathrm{Tor}_n^{G\rtimes\Gamma}\big(\mathbb{Z}(\Gamma),A\big) \cong H_n(G,A)\,, \quad n\geq 0\,, \tag{12}$$

where $\mathbb{Z}(\Gamma)$ *is considered as a trivial* G*-module.*

Proof: We only prove (11) since the proof of (12) is entirely parallel.
Case $n=0$: First observe that, for any Γ-equivariant G-module B, there are isomorphisms

$$\begin{array}{ccccc} \mathrm{Hom}_{G\rtimes\Gamma}(\mathbb{Z}(G),B) & \cong & B^{\Gamma} & \cong & \mathrm{Hom}_{\Gamma}(\mathbb{Z},B)\,. \\ f & \longmapsto & f(1) & \longleftarrow & f \end{array} \tag{13}$$

Then, for any abelian group X, we have

$$\begin{aligned} \mathrm{Hom}\big(A\otimes_{G\rtimes\Gamma}\mathbb{Z}(G),X\big) &\cong \mathrm{Hom}_{G\rtimes\Gamma}\big(\mathbb{Z}(G),\mathrm{Hom}(A,X)\big)\\ &\overset{(13)}{\cong} \mathrm{Hom}_{\Gamma}\big(\mathbb{Z},\mathrm{Hom}(A,X)\big)\\ &\cong \mathrm{Hom}\big(A\otimes_{\Gamma}\mathbb{Z},X\big)\,, \end{aligned}$$

where $\mathrm{Hom}(A,X)$ is a $G\rtimes\Gamma$-module by the action ${}^{(x,\sigma)}f : a \mapsto f({}^{\sigma^{-1}}({}^{x^{-1}}a))$. Therefore

$$A\otimes_{G\rtimes\Gamma}\mathbb{Z}(G) \cong A\otimes_{\Gamma}\mathbb{Z}\,, \tag{14}$$

which proves (11) for $n=0$.

Arbitrary n: Let $P_\bullet \to A$ be any projective $G\rtimes\Gamma$-module presentation of A. Then it is also a Γ-module projective presentation of A and therefore

$$\mathrm{Tor}_n^{G\rtimes\Gamma}\big(A,\mathbb{Z}(G)\big) = H_n\big(P_\bullet\otimes_{G\rtimes\Gamma}\mathbb{Z}(G)\big) \overset{(14)}{\cong} H_n\big(P_\bullet\otimes_{\Gamma}\mathbb{Z}\big) = H_n(\Gamma,A)\,.$$ ∎

The ordinary integral homology groups, $H_n(G) = H_n(G,\mathbb{Z})$, are actually particular equivariant homology groups of a Γ-group G:

Proposition 3.4 *There are natural isomorphisms*

$$H^{\Gamma}_n(G,\mathbb{Z}(\Gamma)) \cong H_n(G)\,, \quad n\geq 1\,, \tag{15}$$

where $\mathbb{Z}(\Gamma)$ *is considered as a trivial* G*-module.*

Proof: For $n = 1$, we have

$$H_n^{\Gamma}\big(G, \mathbb{Z}(\Gamma)\big) \overset{(5)}{\cong} \mathbb{Z}(\Gamma) \otimes_{G \rtimes \Gamma} \mathbb{I}(G) \overset{(12)}{\cong} H_0\big(G, \mathbb{I}(G)\big) \cong H_1(G, \mathbb{Z}).$$

Let $n \geq 2$ and consider the long exact sequence (9) for $A = \mathbb{Z}(\Gamma)$. Since, for every $m \geq 1$, $H_m\big(\Gamma, \mathbb{Z}(\Gamma)\big) = 0$ we get isomorphisms

$$H_n^{\Gamma}\big(G, \mathbb{Z}(\Gamma)\big) \cong H_n\big(G \rtimes \Gamma, \mathbb{Z}(\Gamma)\big) = \mathrm{Tor}_n^{G \rtimes \Gamma}\big(\mathbb{Z}(\Gamma), \mathbb{Z}\big) \overset{(12)}{\cong} H_n(G, \mathbb{Z}). \qquad \blacksquare$$

4 Low-dimensional Homology of Γ-group extensions

Let $1 \to U \xrightarrow{i} E \xrightarrow{p} G \to 1$ be a short exact sequence of Γ-groups, thus U can be identified with a normal Γ-subgroup of E and $E/U \cong G$ as Γ-groups.

The abelianized group $U^{ab} = U/[U, U]$ becomes both a Γ- and a G-module with actions

$$\begin{aligned} {}^{\sigma}(u[U, U]) &= {}^{\sigma}u[U, U] && (\sigma \in \Gamma, u \in U), \\ {}^{x}(u[U, U]) &= eue^{-1}[U, U] && (x \in G, u \in U, e \in p^{-1}(x)) . \end{aligned}$$

Furthermore, since ${}^{\sigma}(eue^{-1}) = {}^{\sigma}e\,{}^{\sigma}u\,({}^{\sigma}e)^{-1}$ and $p({}^{\sigma}e) = {}^{\sigma}p(e)$, it follows that ${}^{\sigma}({}^{x}(u[U, U]) = {}^{({}^{\sigma}x)}({}^{\sigma}(u[U, U]))$, for all $\sigma \in \Gamma$, $x \in G$ and $u \in U$. Hence U^{ab} is a Γ-equivariant G-module, and, for any Γ-equivariant G-module A, we have

Theorem 4.1 *There is a natural exact sequence*

$$H_2^{\Gamma}(E, A) \to H_2^{\Gamma}(G, A) \to A \otimes_{G \rtimes \Gamma} U^{ab} \to H_1^{\Gamma}(E, A) \to H_1^{\Gamma}(G, A) \to 0 . \quad (16)$$

Proof: The group extension $1 \to U \to E \xrightarrow{p} G \to 1$ induces a short exact sequence of G-modules [10, Theorem VI.6.3],

$$0 \to U^{ab} \xrightarrow{\kappa} \mathbb{Z}(G) \otimes_E \mathbb{I}(E) \xrightarrow{\nu} \mathbb{I}(E) \to 0 , \quad (17)$$

in which $\kappa(u[U, U]) = 1 \otimes (u - 1)$ and $\nu(x \otimes (e - 1)) = x(p(e) - 1) = xp(e) - x$. The G-action on $\mathbb{Z}(G) \otimes_E \mathbb{I}(E)$ being given by ${}^{x}(y \otimes (e - 1)) = xy \otimes (e - 1)$.

Short exact sequence (17) is actually of Γ-equivariant G-modules, where Γ acts on $\mathbb{Z}(G) \otimes_E \mathbb{I}(E)$ by ${}^{\sigma}(x \otimes (e - 1)) = \sigma x \otimes ({}^{\sigma}e - 1)$. By applying the functor $A \otimes_{G \rtimes \Gamma} -$, we obtain the exact sequence

$$\begin{array}{l} \mathrm{Tor}_1^{G \rtimes \Gamma}\big(A, \mathbb{Z}(G) \otimes_E \mathbb{I}(G)\big) \longrightarrow \mathrm{Tor}_1^{G \rtimes \Gamma}\big(A, \mathbb{I}(G)\big) \\ \qquad \longrightarrow A \otimes_{G \rtimes \Gamma} U^{ab} \longrightarrow A \otimes_{G \rtimes \Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E) \longrightarrow A \otimes_{G \rtimes \Gamma} \mathbb{I}(G) \longrightarrow 0 , \end{array} \quad (18)$$

in which we see that $A \otimes_{G\rtimes\Gamma} \mathbb{I}(G) = H_1^{\Gamma}(G,A)$, $\mathrm{Tor}_1^{G\rtimes\Gamma}\big(A, \mathbb{I}(G)\big) \overset{(8)}{\cong} H_2^{\Gamma}(G,A)$ and

$$\begin{aligned} A \otimes_{G\rtimes\Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E) &\overset{(11)}{\cong} A \otimes_{\Gamma} \mathbb{Z} \otimes_E \mathbb{I}(E) \overset{(12)}{\cong} A \otimes_{\Gamma} \mathbb{Z}(\Gamma) \otimes_{E\rtimes\Gamma} \mathbb{I}(E) \\ &\cong A \otimes_{E\rtimes\Gamma} \mathbb{I}(E) \overset{(12)}{\cong} H_1^{\Gamma}(E,A)\,. \end{aligned} \tag{19}$$

The exact sequence (16) follows from (18) since, we claim, there is a natural epimorphism

$$H_2^{\Gamma}(E,A) \longrightarrow \mathrm{Tor}_1^{G\rtimes\Gamma}\big(A, \mathbb{Z}(G) \otimes_E \mathbb{I}(E)\big) \to 0\,. \tag{20}$$

To see this, note that all isomorphisms in (19) are natural in A, so that

$$H_1^{\Gamma}(E,-) \cong - \otimes_{G\rtimes\Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E)\,.$$

If $0 \to K \to P \to A \to 0$ is any Γ-equivariant G-module projective presentation of A, there is an induced exact sequence (7)

$$\to H_2^{\Gamma}(E,P) \to H_2^{\Gamma}(E,A) \to K \otimes_{G\rtimes\Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E) \to P \otimes_{G\rtimes\Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E) \to\ .$$

Since

$$\mathrm{Ker}\Big(K \otimes_{G\rtimes\Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E) \to P \otimes_{G\rtimes\Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E)\Big) = \mathrm{Tor}_1^{G\rtimes\Gamma}\big(A, \mathbb{Z}(G) \otimes_E \mathbb{I}(E)\big)\,,$$

the claimed epimorphism (20) arises as the one induced by the homomorphism $H_2^{\Gamma}(E,A) \to K \otimes_{G\rtimes\Gamma} \mathbb{Z}(G) \otimes_E \mathbb{I}(E)$ in the above exact sequence. ▮

Because of the isomorphisms (15), when one takes $A = \mathbb{Z}(\Gamma)$ in Theorem 4.1, the resulting 5-term exact sequence (16) becomes the known Stallings-Stammbach [16, 17] exact sequence in integral homology

$$H_2(E) \to H_2(G) \to U/[E,U] \to E^{ab} \to G^{ab} \to 0\,,$$

since $\mathbb{Z}(\Gamma) \otimes_{G\rtimes\Gamma} U^{ab} \overset{(12)}{\cong} \mathbb{Z} \otimes_G U^{ab}$ and $\mathbb{Z} \otimes_G U^{ab} \cong U/[E,U]$ by the mapping $1 \otimes u[U,U] \mapsto u[E,U]$. Actually, by naturalness, Stallings-Stammbach 5-term exact sequence is of Γ-modules (note that a Γ-group is the same as a functor $G : \Gamma \to \mathbf{Gp}$, so that, by functoriality, the integral homology groups $H_n(G)$ of a Γ-group are indeed Γ-modules).

5 The integral homology of a Γ-group

In this section we study the homology groups $H_n^{\Gamma}(G, A)$, of a Γ-group G with trivial coefficients, that is, with coefficients in abelian groups A, regarded as Γ-equivariant G-modules on which both groups Γ and G are acting trivially. Particularly, we consider the *equivariant integral homology groups*

$$H_n^{\Gamma}(G) = H_n^{\Gamma}(G, \mathbb{Z}), \quad n \geq 1 . \tag{21}$$

The category of abelian groups is a reflexive subcategory of the category of Γ-groups. To describe the quotient (reflector) functor Γ-**Gp** $\twoheadrightarrow$ **Ab** we introduce the following concept:

Definition 5.1 *The Γ-commutator $[G, U]_{\Gamma}$ for a Γ-subgroup U of a Γ-group G is the subgroup generated by the Γ-commutator elements*

$$x\,{}^{\sigma}u\,x^{-1}u^{-1} = [x, u; \sigma]\,, \quad x \in G,\ u \in U,\ \sigma \in \Gamma\,.$$

Proposition 5.2 *Let $U \subseteq G$ be a Γ-subgroup.*

1. *$[G, U]_{\Gamma} \subseteq G$ is a Γ-subgroup.*
2. *If U is a normal subgroup, then $[G, U]_{\Gamma}$ is a normal subgroup of G and moreover $[G, U]_{\Gamma} \subseteq U$.*
3. *$[G, U]_{\Gamma} = 1$ if and only if U is central in G and Γ acts trivially on U.*

Proof:

1. This follows from the equality ${}^{\tau}[x, u; \sigma] = [{}^{\tau}x, {}^{\tau}u; \tau\sigma\tau^{-1}]$.
2. Let U be normal in G. Then we have $[x, u; \sigma] = (x\,{}^{\sigma}ux^{-1})u^{-1} \in UU = U$, and thus $[G, U]_{\Gamma} \subseteq U$. Further, the equality

   $$y[x, u; \sigma]y^{-1} = [y, [x, u; \sigma]; 1][x, u; \sigma]\,,$$

   shows that $[G, U]_{\Gamma}$ is closed under conjugation in G.
3. Suppose that $[G, U]_{\Gamma} = 1$. Since $[G, U] = [G, U]_1 \subseteq [G, U]_{\Gamma}$, it follows that U is central in G. Furthermore, for any $\sigma \in \Gamma$ and $u \in U$,

   $$1 = [1, u; \sigma] = {}^{\sigma}u\,u^{-1}\,,$$

   and Γ acts trivially on U. The converse is trivial.

∎

Let G be a Γ-group. The quotient $G/[G,G]_\Gamma$ is an abelian group on which both Γ and G act trivially. Moreover, any Γ-homomorphism $G \to A$, from G to an abelian group A, endowed with the trivial Γ- and G-actions, factorizes uniquely through $G/[G,G]_\Gamma$. That is, the canonical projection $G \twoheadrightarrow G/[G,G]_\Gamma$ induces a bijection

$$\mathrm{Hom}\big(G/[G,G]_\Gamma, A\big) \cong \mathrm{Hom}_{\Gamma\text{-}\mathbf{Gp}}(G,A)\,. \tag{22}$$

Hence, the " Γ-abelianization functor "

$$\mathrm{U} : \Gamma\text{-}\mathbf{Gp} \to \mathbf{Ab}, \quad G \mapsto G/[G,G]_\Gamma\,, \tag{23}$$

is left adjoint to the inclusion $\mathbf{Ab} \hookrightarrow \Gamma\text{-}\mathbf{Gp}$. The next theorem shows that this Γ-abelianization functor is the same as the equivariant homology functor $H_1^\Gamma(-)$.

Theorem 5.3 *For any Γ-group G,*

$$H_1^\Gamma(G) \cong G/[G,G]_\Gamma\,. \tag{24}$$

Proof: Let A be any abelian group considered as a trivial Γ- and G-module. Then, we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}\big(H_1^\Gamma(G),A\big) &\overset{(5)}{\cong} \mathrm{Hom}\big(\mathbb{Z}\otimes_{G\rtimes\Gamma}\mathbb{I}(G),A\big) \cong \mathrm{Hom}_{G\rtimes\Gamma}\big(\mathbb{I}(G),\mathrm{Hom}(\mathbb{Z},A)\big)\\ &\cong \mathrm{Hom}_{G\rtimes\Gamma}\big(\mathbb{I}(G),A\big) \overset{(4)}{\cong} \mathrm{Der}_\Gamma(G,A) = \mathrm{Hom}_{\Gamma\text{-}\mathbf{Gp}}(G,A)\\ &\overset{(22)}{\cong} \mathrm{Hom}\big(G/[G,G]_\Gamma,A\big)\,,\end{aligned}$$

from where the theorem follows. An explicit description of the isomorphism (24), $H_1^\Gamma(G) = \mathbb{Z}\otimes_{G\rtimes\Gamma}\mathbb{I}(G) \cong G/[G,G]_\Gamma$, is: $1\otimes(x-1)\mapsto x[G,G]_\Gamma$. ∎

Note also that

$$H_1^\Gamma(G) \cong \mathbb{Z}\otimes_{G\rtimes\Gamma}\mathbb{I}(G) = H_0\big(G\rtimes\Gamma,\mathbb{I}(G)\big) \cong \frac{\mathbb{I}(G)}{\mathbb{I}(G\rtimes\Gamma)\mathbb{I}(G)}\,.$$

Corollary 5.4 *For each $n \geq 1$, the functor $H_n^\Gamma(-) : \Gamma\text{-}\mathbf{Gp} \to \mathbf{Ab}$ is the cotriple $(n-1)$-th derived functor of the Γ-abelianization functor* (23). *Moreover, if A is any abelian group regarded as a trivial Γ-equivariant G-module, then each $H_n^\Gamma(-,A)$ is the cotriple $(n-1)$-th derived functor of the functor $G \mapsto A\otimes G/[G,G]_\Gamma$.*

Proof: Let G be any Γ-group. Then,

$$\begin{aligned} H_n^{\Gamma}(G,A) &= H_{n-1}\big(A \otimes_{G\rtimes\Gamma} \mathrm{Diff}_G^{\Gamma}\mathbb{G}_\bullet(G)\big) \cong H_{n-1}\big(A \otimes_{\mathbb{G}_\bullet(G)\rtimes\Gamma} \mathbb{I}(\mathbb{G}_\bullet(G))\big) \\ &\cong H_{n-1}\big(A \otimes \mathbb{Z} \otimes_{\mathbb{G}_\bullet(G)\rtimes\Gamma} \mathbb{I}(\mathbb{G}_\bullet(G))\big) \overset{(5)}{\cong} H_{n-1}\big(A \otimes H_1^{\Gamma}(\mathbb{G}_\bullet(G))\big) \\ &\overset{(24)}{\cong} H_{n-1}\big(A \otimes \mathbb{G}_\bullet(G)/[\mathbb{G}_\bullet(G),\mathbb{G}_\bullet(G)]_\Gamma\big)\,. \end{aligned}$$

▮

Analogously, as it happens for ordinary group cohomology, there are universal coefficient formulas which allow us to compute the (co)homology of a group with operators with trivial coefficient modules from its integral homology:

Theorem 5.5 *Let G be a Γ-group and let A be an abelian group considered as a trivial Γ-equivariant G-module. There are isomorphisms*

$$H_1^{\Gamma}(G,A) \cong A \otimes H_1^{\Gamma}(G)\,, \quad H^1_{\Gamma}(G,A) \cong \mathrm{Hom}\big(H_1^{\Gamma}(G),A\big)\,, \tag{25}$$

and, for any $n \geq 1$, short exact split sequences

$$0 \to A \otimes H_{n+1}^{\Gamma}(G) \to H_{n+1}^{\Gamma}(G,A) \to \mathrm{Tor}\big(A, H_n^{\Gamma}(G)\big) \to 0\,, \tag{26}$$

$$0 \to \mathrm{Ext}\big(H_n^{\Gamma}(G),A\big) \to H^{n+1}_{\Gamma}(G,A) \to \mathrm{Hom}\big(H_{n+1}^{\Gamma}(G),A\big) \to 0\,. \tag{27}$$

Proof: For the isomorphisms (25), we have

$$H_1^{\Gamma}(G,A) \overset{(5)}{=} A \otimes_{G\rtimes\Gamma} \mathbb{I}(G) \cong A \otimes \mathbb{Z} \otimes_{G\rtimes\Gamma} \mathbb{I}(G) \overset{(5)}{=} A \otimes H_1^{\Gamma}(G)\,,$$

and

$$H^1_{\Gamma}(G,A) = \mathrm{Der}_{\Gamma}(G,A) = \mathrm{Hom}_{\mathbf{\Gamma\text{-}Gp}}(G,A) \overset{(22,24)}{\cong} \mathrm{Hom}\big(H_1^{\Gamma}(G),A\big)\,.$$

To prove the existence of exact sequences (26) and (27), let $F_\bullet \twoheadrightarrow \mathbb{I}(G)$ be any free resolution of $\mathbb{I}(G)$ as a $(G \rtimes \Gamma)$-module. Then $A \otimes_{G\rtimes\Gamma} F_\bullet \cong A \otimes \big(\mathbb{Z} \otimes_{G\rtimes\Gamma} F_\bullet\big)$ and $\mathrm{Hom}_{G\rtimes\Gamma}\big(F_\bullet, A\big) \cong \mathrm{Hom}\big(\mathbb{Z}\otimes_{G\rtimes\Gamma} F_\bullet, A\big)$, where $\mathbb{Z}\otimes_{G\rtimes\Gamma} F_\bullet$ is a complex of free abelian groups. Universal Coefficient Theorem in (co)homology gives the (split) short exact sequences

$$A \otimes H_n\big(\mathbb{Z} \otimes_{G\rtimes\Gamma} F_\bullet\big) \rightarrowtail H_n\big(A \otimes_{G\rtimes\Gamma} F_\bullet\big) \twoheadrightarrow \mathrm{Tor}\big(A, H_{n-1}(\mathbb{Z} \otimes_{G\rtimes\Gamma} F_\bullet)\big)\,,$$

$$\mathrm{Ext}\big(H_{n-1}(\mathbb{Z}\otimes_{G\rtimes\Gamma} F_\bullet),A\big) \rightarrowtail H^n\big(\mathrm{Hom}_{G\rtimes\Gamma}(F_\bullet,A)\big) \twoheadrightarrow \mathrm{Hom}\big(H_n(\mathbb{Z}\otimes_{G\rtimes\Gamma} F_\bullet),A\big)\,,$$

which can be identified with (26) and (27) respectively, by taking into account (8) and that, by [2, Theorem 2.6] and (4), there are natural isomorphisms $H^{n+1}_{\Gamma}(G,A) \cong \mathrm{Ext}^n_{G\rtimes\Gamma}(\mathbb{I}(G),A)$ for all $n \geq 0$. ▮

Next we focus our attention on the homology groups $H_2^{\Gamma}(G)$.

Theorem 5.6 *Any extension of* Γ*-groups* $1 \to U \to E \xrightarrow{p} G \to 1$ *induces an exact sequence*

$$H_2^{\Gamma}(E) \to H_2^{\Gamma}(G) \to U/[E,U]_{\Gamma} \to E/[E,E]_{\Gamma} \to G/[G,G]_{\Gamma} \to 0\,. \tag{28}$$

Proof: Sequence (28) is obtained from the sequences (16) in Theorem 4.1 where coefficients are taken in the trivial Γ-equivariant G-module $A = \mathbb{Z}$. Indeed, by (24), we know that $H_1^{\Gamma}(E) \cong E/[E,E]_{\Gamma}$ and $H_1^{\Gamma}(G) \cong G/[G,G]_{\Gamma}$. Also $\mathbb{Z} \otimes_{G\rtimes\Gamma} U^{ab} = H_1\big(G \rtimes \Gamma, U^{ab}\big) \cong U^{ab}/\mathbb{I}(G \rtimes \Gamma)U^{ab}$, where $\mathbb{I}(G \rtimes \Gamma)U^{ab}$ is the subgroup of U^{ab} generated by the elements

$$\big((p(e),\sigma) - (1,1)\big)(u[U,U]) \;=\; e\,{}^{\sigma}u\, e^{-1}u^{-1}[U,U]\,, \quad e \in E,\, u \in U\,, \sigma \in \Gamma\,.$$

And thus we see that $\mathbb{Z} \otimes_{G\rtimes\Gamma} U^{ab} \cong U/[E,U]_{\Gamma}$. ▮

Let us now consider $1 \to R \to F \to G \to 1$, any exact sequence of Γ-groups with F a free Γ-group. By (6) we have $H_2^{\Gamma}(F) = 0$ and the corresponding associated exact sequence (28) yields a natural isomorphism

$$H_2^{\Gamma}(G) \cong \operatorname{Ker}\big(R/[F,R]_{\Gamma} \longrightarrow F/[F,F]_{\Gamma}\big)\,,$$

whence we obtain the "Hopf formula" for groups with operators

$$H_2^{\Gamma}(G) \cong \frac{R \cap [F,F]_{\Gamma}}{[F,R]_{\Gamma}}\,, \tag{29}$$

which, in particular, implies that its right term does not depend of the chosen free presentation of the Γ-group G. In a natural sense, (29) shows that $H_2^{\Gamma}(G)$ generalizes the idea of the Schur multiplier. Moreover, the abelian group $R \cap [F,F]_{\Gamma}/[F,R]_{\Gamma}$ is the Baer-invariant of G relative to the variety of abelian groups, defined by Fröhlich in [7] and by Furtado-Coelho in [9] (where it is denoted by $D_1\mathrm{U}(G)$, U being the quotient functor (23), $G \mapsto G/[G,G]_{\Gamma}$).

6 Homology and the lower Γ-central series

The homology of a group in low dimensions is related to the lower central series of the group. The key result, from which a large number of interesting applications can be found in the literature, is due to Stallings [16] and Stammbach [17]. This result establishes that if $h: G \to G'$ is a homomorphism inducing an isomorphism $H_1(G) \cong H_1(G')$ and mapping $H_2(G)$ onto $H_2(G')$, then for each integer $n \geq 0$, h induces an isomorphism $G/G_n \cong G'/G'_n$ and an embedding $G/G_\infty \subseteq G'/G'_\infty$. Here G_n denotes the n-th term of the lower central series of G and $G_\infty = \bigcap_n G_n$. We shall show next that Stallings-Stammbach basic theorem is an instance of a more general result concerning Γ-groups (cf. [9] for a very general treatment of the subject).

Let us recall from Proposition 5.2 that, for any normal Γ-subgroup $U \trianglelefteq G$ of a Γ-group G, the Γ-commutator (see Definition 5.1) $[G,U]_{_\Gamma} \subseteq U$ is contained in U, and it is again a normal Γ-subgroup of G. Furthermore, $[G,U]_{_\Gamma} = 1$ means that $U \subseteq Z_{_\Gamma}(G)$, where $Z_{_\Gamma}(G)$ is the "Γ-center" of G, that is,

$$Z_{_\Gamma}(G) = \{x \in G \mid xy = yx,\ {}^\sigma x = x \text{ for all } y \in G,\ \sigma \in \Gamma\}\,.$$

We introduce the lower Γ-central series $\{\Gamma_n G,\ n \geq 0\}$ of a Γ-group G,

$$\cdots \subseteq \Gamma_{n+1}G \subseteq \Gamma_n G \subseteq \cdots \subseteq \Gamma_1 G \subseteq \Gamma_0 G = G\,,$$

by

$$\Gamma_{n+1}G = [G, \Gamma_n G]_{_\Gamma}\,.$$

Also let $\Gamma_\infty G = \bigcap_n \Gamma_n G$.

A Γ-group is called Γ-nilpotent (of class $\leq n$) whenever $\Gamma_n G = 1$.

Theorem 6.1 *Let $f : G \to G'$ be an equivariant homomorphism of Γ-groups inducing an isomorphism $H_1^{^\Gamma}(G) \cong H_1^{^\Gamma}(G')$ and an epimorphism $H_2^{^\Gamma}(G) \twoheadrightarrow H_2^{^\Gamma}(G')$. Then f induces isomorphisms*

$$\Gamma_n G/\Gamma_{n+1}G \cong \Gamma_n G'/\Gamma_{n+1}G'\,, \quad G/\Gamma_n G \cong G'/\Gamma_n G'\,,$$

for all $n \geq 0$, and a monomorphism

$$G/\Gamma_\infty G \rightarrowtail G'/\Gamma_\infty G'\,.$$

If G and G' are both Γ-nilpotent, then $f : G \cong G'$ is an isomorphism.

Proof: This is parallel to Stallings proof when Γ is trivial (also to the proof of Theorem 1 in [6]). We proceed by induction. For $n = 0$ the assertion is obvious and if $n = 1$ it is part of the hypothesis. For $n \geq 2$, consider the exact sequences of Γ-groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_{n-1}G & \longrightarrow & G & \longrightarrow & G/\Gamma_{n-1}G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\
1 & \longrightarrow & \Gamma_{n-1}G' & \longrightarrow & G' & \longrightarrow & G'/\Gamma_{n-1}G' & \longrightarrow & 1
\end{array}$$

and the associated exact sequences (28)

$$\begin{array}{ccccccccccc}
H_2^{^\Gamma}(G) & \longrightarrow & H_2^{^\Gamma}(G/\Gamma_{n-1}G) & \longrightarrow & \dfrac{\Gamma_{n-1}G}{\Gamma_n G} & \longrightarrow & H_1^{^\Gamma}(G) & \longrightarrow & H_1^{^\Gamma}(G/\Gamma_{n-1}G) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} & & \\
H_2^{^\Gamma}(G') & \longrightarrow & H_2^{^\Gamma}(G'/\Gamma_{n-1}G') & \longrightarrow & \dfrac{\Gamma_{n-1}G'}{\Gamma_n G'} & \longrightarrow & H_1^{^\Gamma}(G') & \longrightarrow & H_1^{^\Gamma}(G'/\Gamma_{n-1}G') & \longrightarrow & 0\,.
\end{array}$$

By hypothesis, f_1 is an epimorphism and f_4 is an isomorphism. By induction, f_2 and f_5 are isomorphisms. Hence f_3 is an isomorphism.

We now consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_{n-1}G/\Gamma_n G & \longrightarrow & G/\Gamma_n G & \longrightarrow & G/\Gamma_{n-1}G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_6} & & \downarrow{\scriptstyle f_7} & & \\ 1 & \longrightarrow & \Gamma_{n-1}G'/\Gamma_n G' & \longrightarrow & G'/\Gamma_n G' & \longrightarrow & G'/\Gamma_{n-1}G' & \longrightarrow & 1\,. \end{array}$$

Since f_7 is an isomorphism by induction, we conclude that f_6 is an isomorphism for all n.

If the induced homomorphism $G/\Gamma_\infty G \to G'/\Gamma_\infty G'$ had a nontrivial kernel, then for some n the homomorphism $G/\Gamma_n G \to G'/\Gamma_n G'$ would have a nontrivial kernel.

Finally, if both G and G' are Γ-nilpotent, then the assertion follows from the remark that there exists $n \geq 0$ such that $\Gamma_n G = 1 = \Gamma_n G'$. ∎

7 Universal central equivariant extensions of Γ-groups

In this section we study central equivariant extensions of Γ-groups. It was pointed out by Loday in [12] that these extensions take an interesting role in relative K-theory of rings.

Definition 7.1 *A* central equivariant extension *of a Γ-group G is a short exact sequence E of Γ-groups*

$$(E,p):\ 1 \to A \to E \xrightarrow{p} G \to 1$$

such that $A \subseteq Z_\Gamma(E)$ (or equivalently, such that $[E,A]_\Gamma = 1$ what means that A belongs to the center of E and Γ acts trivially on A).

A central equivariant extension (E,p) of G is called universal *if for any central equivariant extension (E',p') of G there is a unique Γ-homomorphism $E \to E'$ over G.*

For any Γ-group G and abelian group A, equivalence classes of central equivariant extensions of G by A are classified by the equivariant cohomology group $H^2_\Gamma(G,A)$ [2, Theorem 3.3].

Definition 7.2 *[12, §3] A Γ-group G is called* Γ-perfect*, if G coincides with its Γ-commutator subgroup $[G,G]_\Gamma$.*

By the isomorphism (24) in Theorem 5.3, a Γ-group G is Γ-perfect if and only if $H_1^{\Gamma}(G) = 0$.

Many interesting examples of Γ-groups which are Γ-perfect but not perfect arise for instance in algebraic K-theory by considering the elementary group $E(R)$ and the Steinberg group $St(R)$ of a ring R which in general are not perfect but are $St(\mathbb{Z})$-perfect groups (see [11]).

The results in the following lemma are due to Loday (see [12, Lemmas 8 and 10 and the proof of Lemma 9].

Lemma 7.3 *(i) Let $(E,p),(E',p')$ be two central equivariant extensions of a Γ-group G. If E is Γ-perfect, there exists at most one equivariant homomorphism from E to E' over G.*

(ii) Let (E,p) be a a central equivariant extension of a Γ-group G. If E is not Γ-perfect, then there is a suitable central equivariant extension (E',p') of G such that there is more than one Γ-homomorphism from E to E' over G.

(iii) If (E,p) is a central equivariant extension of a Γ-perfect Γ-group G, then the Γ-commutator subgroup $[E,E]_{\Gamma}$ is Γ-perfect, and maps onto G.

Proof: (i) Let $\varphi_1, \varphi_2 : E \to E'$ be two Γ-homomorphisms over G. Then for any $x, y \in E$ and $\sigma \in \Gamma$ we can write

$$\varphi_1(x) = \varphi_2(x)c, \quad \varphi_1(y) = \varphi_2(y)c', \quad \varphi_1({}^{\sigma}y) = \varphi_2({}^{\sigma}y)c'\,,$$

where $c, c' \in \mathrm{Ker}(p') \subseteq Z_{\Gamma}(E')$. Therefore $\varphi_1(x\,{}^{\sigma}yx^{-1}y^{-1}) = \varphi_2(x\,{}^{\sigma}yx^{-1}y^{-1})$. Since $E = [E,E]_{\Gamma}$ is generated by Γ-commutators we conclude that $\varphi_1 = \varphi_2$.

(ii) Let h be the projection of E onto $H_1^{\Gamma}(E) = E/[E,E]_{\Gamma}$. Then h is a nontrivial Γ-equivariant homomorphism, where the group Γ acts trivially on $H_1^{\Gamma}(E)$. Thus for the central equivariant split extension

$$1 \to H_1^{\Gamma}(E) \to H_1^{\Gamma}(E) \times G \to G \to 1$$

of the Γ-group G we have two distinct Γ-homomorphisms φ_1 and φ_2 from E to $H_1^{\Gamma}(E) \times G$ over G given by $\varphi_1(x) = (h(x), p(x))$ and $\varphi_2(x) = (1, p(x))$, respectively.

(iii) Since G is generated by Γ-commutators, it is clear that p maps $[E,E]_{\Gamma}$ onto G. Hence every element x of E can be written as a product $x = x'c$ with $x' \in [E,E]_{\Gamma}$ and $c \in Z_{\Gamma}(E)$. Therefore every generator $[x_1, x_2; \sigma]$ of $[E,E]_{\Gamma}$ is equal to $[x_1'c_1, x_2'c_2; \sigma] = [x_1', x_2'; \sigma]$ for some $x_1', x_2' \in [E,E]_{\Gamma}$. Thus $\big[[E,E]_{\Gamma}, [E,E]_{\Gamma}\big]_{\Gamma} = [E,E]_{\Gamma}$. ∎

The next theorem characterizes universal central equivariant extensions (cf. [15, Theorem 5.3]. We use the following terminology: a central equivariant extension (E,p) of G *splits* whenever it admits an Γ-equivariant section, that is a Γ-homomorphism $s : G \to E$ with $ps = id_G$.

Theorem 7.4 *A central equivariant extension (E, p) of a Γ-group G is universal if and only if E is Γ-perfect and every central equivariant extension (H, q) of E with Γ acting trivially on* $\mathrm{Ker}(pq)$ *splits.*

Proof:Assume that the central equivariant extension (E, p) of G satisfies conditions of the theorem. It will be shown that (E, p) is a universal central equivariant extension of G. Let (E', p') be a central equivariant extension of G. Then one has a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Ker}(p') & \xrightarrow{i_1} & E' \times_G E & \xrightarrow{\pi_2} & E & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle p} & & \\ 1 & \longrightarrow & \mathrm{Ker}(p') & \longrightarrow & E' & \xrightarrow{p'} & G & \longrightarrow & 1\,. \end{array}$$

It is clear that the top row is a central equivariant extension of E and Γ acts trivially on $\mathrm{Ker}(p\pi_2) \cong \mathrm{Ker}(p') \times \mathrm{Ker}(p)$. Therefore there is a Γ-homomorphism $s : E \to E' \times_G E$ such that $ps = id_E$. One gets a Γ-homomorphism $\varphi = \pi_1 s : E \to E'$ over G. The uniqueness of such φ follows from Lemma 7.3. Therefore (E, p) is a universal central equivariant extension of the Γ-group G.

To prove the converse, let (E, p) be a universal central equivariant extension of the Γ-group G. By Lemma 7.3(ii) E is Γ-perfect. Let (H, q) be a central equivariant extension of E such that Γ acts trivially on $\mathrm{Ker}(pq)$. We will show that (H, pq) is a central equivariant extension of G. Take $x \in \mathrm{Ker}(pq)$. Since Γ acts trivially on x, we obtain a Γ-homomorphism $\varphi : H \to H$ over E given by $\varphi(h) = xhx^{-1}$, $h \in H$. Therefore, by Lemma 7.3, the restriction of φ to the Γ-perfect group $[H, H]_\Gamma$ is the identity map. It follows that x commutes with the elements of $[H, H]_\Gamma$. Since q maps $[H, H]_\Gamma$ onto E, it follows that E is generated as a group by $\mathrm{Ker}(q)$ and $[H, H]_\Gamma$, whence that x belongs to the center of H for any $x \in \mathrm{Ker}(pq)$. Thus (H, pq) is a central equivariant extension of G and there is a unique Γ-homomorphism $s : E \to H$ over G, since (E, p) is a universal central equivariant extension of G. Clearly the composite $qks : E \to E$ is a Γ-homomorphism over G, hence equals the identity map. This shows that s is a Γ-equivariant section of (H, q). ∎

Let $1 \to R \to F \xrightarrow{p} G \to 1$ be any exact sequence of Γ-groups with F a free Γ-group. Then the Γ-homomorphism p sends the normal subgroup $[F, R]_\Gamma$ of F to 1 and therefore induces a surjective Γ-homomorphism $p' : [F, F]_\Gamma / [F, R]_\Gamma \to [G, G]_\Gamma$.

Theorem 7.5 *If G is Γ-perfect, then $\big([F, F]_\Gamma / [F, R]_\Gamma, p'\big)$ is a universal central equivariant extension of G and Ker(p') is isomorphic to $H_2^\Gamma(G)$.*

Proof:It is easily checked that $\big(F/[F, R]_\Gamma, p'\big)$ is a central equivariant extension of G. Thus, by Lemma 7.3(iii), the group $[F, F]_\Gamma / [F, R]_\Gamma$ is Γ-perfect and it is mapped onto G. Therefore $\big([F,F]_\Gamma/[F,R]_\Gamma, p'\big)$ is a central equivariant extension of the Γ-group G. Let (E,q)be any other central equivariant extension of G. There is a Γ-homomorphism $\varphi : F \to E$ over G. Since (E,q) is a central equivariant extension of G, it is easily seen that $\varphi\big([F,R]_\Gamma\big) = 1$. Hence the restriction of φ to $[F,F]_\Gamma$ induces a Γ-homomorphism $[F,F]_\Gamma/[F,R]_\Gamma \to E$ over G, which is the unique one by Lemma 7.3(i). Therefore the sequence

$$1 \to \frac{R \cap [F,F]_\Gamma}{[F,R]_\Gamma} \longrightarrow \frac{[F,F]_\Gamma}{[F,R]_\Gamma} \xrightarrow{p'} G \to 1$$

is a universal central extension of G. According to (29), $\mathrm{Ker}(p')$ is isomorphic to $H_2^\Gamma(G)$. ∎

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