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NONLOCAL AND INITIAL PROBLEMS FOR QUASILINEAR, NONSTRICTLY HYPERBOLIC EQUATIONS WITH GENERAL SOLUTIONS REPRESENTED BY SUPERPOSITION OF ARBITRARY FUNCTIONS

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Abstract. We have selected a class of hyperbolic quasilinear equations of second order, admitting parabolic degeneracy by the following criterion: they have a general solution represented by superposition of two arbitrary functions. For equations of this class we consider the initial Cauchy problem and nonlocal characteristic problems for which sufficient conditions are established for the solution solvability and uniquness; the domains of solution definition are described.

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1. Introductory. Solutions Represented by Superposition of Arbitrary Functions

As is known, a general solution or a general integral of partial equations are represented by the presence in them of free parameters – arbitrary functions and constants. The number of these arbitrary parameters does not always coincide with the order of the equation [1]. We will consider below quasilinear, nonstrictly hyperbolic equations on the plane of variables x, t, whose general solutions contain two arbitrary functions as free parameters. In the representation of a general solution of linear equations arbitrary functions are contained linearly and the presence of derivatives and antiderivatives of arbitrary functions is not excluded either. As for the arguments on which arbitrary functions depend, they are defined by differential relations of characteristic directions. In the nonlinear case, the structure of general solutions is essentially more complicated and diverse. We will consider the class of quasilinear equations of special type, whose general solutions are represented by superposition of arbitrary functions f, g as follows:

$$u(x,t) = \alpha(t) f\big\{x + g[\beta(x) - t]\big\}, \tag{1.1}$$

where the given functions $\alpha(t)$, $\beta(x)$ are sufficiently smooth. In particular, if we deal with regular solutions, we should at least require their twice differentiability under the natural condition that arbitrary functions f, g, too, are smooth.

If we follow the linear theory, the differentials of the arguments of arbitrary functions, set equal to zero, should define the characteristic roots of the differential equation corresponding to the solution (1.1). As regards the argument of

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an arbitrary function g, here we do not observe any digression from the linear theory because the characteristic root $d[\beta(x)-t]=0$ is defined immediately from the relation $\lambda_1=\beta'(x)$. We have a different situation with the argument of the function f. In this case, the characteristic root, which by the theory, should give the direction of the characteristics, depends on the derivative g' of an arbitrary function g, $\lambda_2=\beta'+[g']^{-1}$. Therefore the second characteristic root remains undefined and hence the principal part of the equation is not defined either. Such a phenomenon falls out of the framework of the linear theory and is typical only of nonlinear equations.

To explain this phenomenon, we will try here to construct an equation by using its general solution. For this, we will calculate all derivatives of first and second order of the general solution, and remove arbitrary functions f, g and their derivatives of first and second order from the obtained six relations. In our case this scheme is quite realizable and yields a quasilinear equation of second order

$$\Big(\frac{\partial u}{\partial t}+au\Big)\frac{\partial^2 u}{\partial x^2}-\Big(\frac{\partial u}{\partial x}-b\,\frac{\partial u}{\partial t}-abu\Big)\frac{\partial^2 u}{\partial x\partial t}-b\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial t^2}=c, \tag{1.2}$$

where

$$b=\beta'(x),\quad a=-\frac{\alpha'(t)}{\alpha(t)}\,,$$

and the right-hand side has the form

$$c=a\Big(\frac{\partial u}{\partial x}\Big)^2-\Big(\frac{\partial u}{\partial t}+au\Big)^2 b'+b\,\frac{\partial(au)}{\partial t}\,\frac{\partial u}{\partial x}\,. \tag{1.3}$$

It is assumed that the functions a, b are at least twice continuously differentiable for all finite values of their arguments.

2. Construction of a General Integral

For any right-hand side c depending on the sought solution with its derivatives of first order, equation (1.2) has two characteristic real roots, the first of which coincides with $\lambda_1=\beta'(x)$. The second root has the form

$$\lambda_2=-\frac{\partial u}{\partial t}\left[\frac{\partial u}{\partial t}+au\right]^{-1}. \tag{2.1}$$

For some solutions the roots λ_1 and λ_2 can be equal. Solutions for which $\lambda_1=\lambda_2$ everywhere are parabolic. If this equality is fulfilled on some set of points, then the equation is hyperbolic everywhere except for the said set, where the equation parabolically degenerates. The parabolic degeneracy can be both free and characteristic. The condition $\lambda_1\neq\lambda_2$ rewritten in the form

$$\frac{\partial u}{\partial x}+\Big(\frac{\partial u}{\partial t}+au\Big)\neq 0 \tag{2.2}$$

defines the class of hyperbolic solutions of equation (1.2). Hence the equation that we have constructed using solution (1.1) belongs to the class of nonstrictly hyperbolic or, speaking more exactly, hyperbolic-parabolic equations.

It is natural now to pose the question whether expression (1.1) is really a general solution of equation (1.2), (1.3), i.e., whether all solutions are represented by (1.1) or this equation has other solutions that are not representable in this form.

In order to answer this question, we will investigate equation (1,2), (1.3) by the method of characteristics. For convenience, in our further reasoning we will use Monge's notation $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial t}$.

According to the general theory of characteristics ([1], [4]), the family corresponding to the root λ_1 is completely defined by two differential relations $dt = \lambda_1 dx$, $dp + \lambda_2 dq = c[q + au]^{-1} dx$, the first of which is called a differential relation of a characteristic direction, and the second a differential characteristic relation [5]. In detailed form they can be written as

$$\begin{cases} dt - b(x)\, dx = 0, \\ (q + au)\, dp - p\, dq = c\, dx. \end{cases} \tag{2.3}$$

We should complement the pair of these equalities by the compatibility condition $p\, dx + q\, dt = du$ and consider the problem of construction of the first integrals of the obtained system.

System (2.3) is equivalent, for its first integral ξ, to the system of two partial equations of first order

$$\begin{cases} \xi_x + b\xi_t + (p + bq)\xi_u - \Big[ap + abq + a'bu - \dfrac{b'}{p}\,(q + au)^2\Big]\xi_q = 0, \\ p\xi_p + (q + au)\xi_q = 0. \end{cases} \tag{2.4}$$

This system is not complete in the sense of Jacobi and can be extended by using the Poisson brackets:

$$\xi_u - a\xi_q = 0.$$

The extended system is already complete because further use of the Poisson brackets gives trivial relations. The integral of the latter equation is represented as an arbitrary function of the expression

$$r = q + au,$$

which we introduce by a new argument instead of the value u. Introducing the notation $\xi(x, t, u, p, q) \equiv h(x, t, p, q, r)$, we rewrite the system as follows:

$$\begin{cases} h_x + bh_t + \dfrac{b' r^2}{p}\, h_r = 0, \\ ph_p + rh_r = 0, \\ h_q = 0. \end{cases}$$

The last equation shows that the firts integral h of this system does not depend on the value q. The second equation is integrable. Its integral is represented by an arbitrary function depending on the quotient

$$s = \frac{r}{p}\,.$$

Now, instead of the remaining quatruple of arguments x, t, p, r we introduce a new combination of variables x, t, r, s. Denoting the first integral h in terms of new arguments by $H(x,t,r,s)$, we obtain for it the equivalent system of two equations

$$\begin{cases} H_x + bH_t + b's^2 H_s = 0, \\ H_r = 0. \end{cases}$$

The second equation of this system excludes the dependence of the integral H on the argument r. The other equation is equivalent to the system of ordinary equations

$$dx = \frac{dt}{b} = \frac{ds}{b's^2}\,,$$

which upon integration gives the following two combinations:

$$\int b(x)\,dx - t, \quad b + \frac{1}{s}\,.$$

Therefore all first integrals of the latter system are represented by an arbitrary function of these two combinations. Going back to the initial arguments, we conclude that system (2.4) and thereby the characteristic differential relations (2.3) admit two and only two independent first integrals

$$\xi = \int b(x)\,dx - t, \tag{2.5}$$

$$\xi_1 = \frac{p + b(q + au)}{q + au)}\,. \tag{2.6}$$

We have thus constructed explicitly the combinations ξ, ξ_1 taking constant values along any curve of the family of characteristics which is described by relations (2.3) and corresponds to the characteristic root λ_1. Hence between them there exists a dependence

$$\xi_1 = G(\xi) \tag{2.7}$$

with an arbitrary function G whose smoothness will be discussed below. To construct the solution of equation (1.2), (1.3), we have either to integrate the intermediate integral (2.7) or to find one more intermediate integral if such exists. It is necessary to check whether the realization of the second of these approaches is possible. For this, by similar arguments let us consider the characteristic differential relations corresponding to the root λ_2:

$$\begin{cases} dt = -\dfrac{p}{q + au}\,dx, \\ (q + au)(dp + bdq) = cdx. \end{cases} \tag{2.8}$$

For the first integral in (2.8), which we denote by η, we obtain with the condition $du = pdx + qdt$ taken into account the system of two compatible equations of first order

$$\begin{cases} (q + au)\eta_x - p\eta_t + aup\eta_u + c\eta_p = 0, \\ b\eta_p - \eta_q = 0. \end{cases} \tag{2.9}$$

Using the Poisson brackets, system (2.9) can be extended by the equation

$$-\eta_t + au\eta_u + (ap + abq + a'bu)\eta_p = 0.$$

This equation together with (2.9) is a complete system in the sense of Jacobi because all further operations with the Poisson brackets give only a trivial result. The extended system can be written in a simpler form

$$\begin{cases} -\eta_t + au\eta_u + (ap_abq + a'bu)\eta_p = 0, \\ \eta_x - b'(q + au)\eta_p = 0, \\ b\eta_p - \eta_q = 0. \end{cases}$$

The last equation is satisfied by any function depending only on one argument $r = p + bq$. Instead of the value p, let us introduce the combination r in the role of a new argument and denote the sought integral η in terms of the variables x, t, u, q, r by k. Then for the integral k we obtain the system of three homogeneous equations of first order

$$\begin{cases} -k_t + auk_u + (ar + a'bu)k_r = 0, \\ k_x - ab'uk_r = 0, \\ k_q = 0. \end{cases}$$

It follows from the third equation that the first integral k does not depend directly on the argument q. All solutions of the second equation are represented by an arbitrary function of the combination $s = abu + r$. We replace the remaining quadruple of arguments x, t, u, r by another quadruple t, u, r, s and denote the first integral k in terms of the new variables by $\mathcal{K}(t, u, r, s)$. As a result of such a transformation we obtain the system of two homogeneous equations

$$\begin{cases} -\mathcal{K}_t + au\mathcal{K}_u + as\mathcal{K}_s = 0, \\ \mathcal{K}_r = 0. \end{cases}$$

According to the second equation, the direct dependence of the first integral $\mathcal{K}$ on the argument r is excluded. The first equation, equivalent to the system of two ordinary equations

$$-dt = \frac{du}{au} = \frac{ds}{as},$$

gives, upon integrations, two independent first integrals which in terms of the initial arguments are expressed explicitly:

$$\eta = u \exp \int a(t)\, dt, \tag{2.10}$$

$$\eta_1 = (p + bq + abu) \exp \int a(t)\, dt. \tag{2.11}$$

The integral η defined by (2.10) could have been obtained by immediate integration of the first of the differential relations (2.8). However (2.10) allows us to describe only partly the family of characteristics, which corresponds to the root λ_2. By the same reasoning as above we establish that the characteristic system

(2.9) and thereby the differential relations (2.8) admit two and only two first integrals η_1, η_2. This system does not have any singular integrals and therefore the first integrals (2.10), (2.11) completely define the structure of the considered family of characteristics. Moreover, these first integrals make it possible to construct one more intermediate integral of equation (1.2), (1.3):

$$\eta_1 = F(\eta), \tag{2.12}$$

where F is an arbitrary, sufficiently smooth function.

Now, using the characteristic transformation defined by (2.5), (2.10), let us pass over to the plane of variables ξ, η. According to the classical scheme (see, for example, [1]), we are to define all values x, t, u, p, q in terms of the characteristic variables ξ, η, and, after substituting them into the differential relation

$$du = p\,dx + q\,dt \tag{2.13}$$

to obtain the integral of equation (1.2), (1.3) by its next integration. However in our case this scheme can be realized in a much simpler way.

By the differentiated relations (2.5), (2.10)

$$d\xi = b(x)\,dx - dt,$$
$$d\eta = e^{\int a(t)\,dt}\big[p\,dx + (q + au)\,dt\big]$$

we define the differential

$$dx = \frac{1}{p + b(q + au)}\Big[(q + au)\,d\xi + e^{-\int a\,dt}\,d\eta\Big].$$

Next we use the intermediate integrals (2.7) and (2.13). By (2.6), the coefficient at $d\xi$ is equal to $\frac{1}{\xi_1}$. Analogously, the coefficient at the differential $d\eta$ is defined by (2.11) and is equal to $\frac{1}{\eta_1}$. Taking into account (2.7) and (2,13), we have

$$dx = \frac{1}{G(\xi)}\,d\xi + \frac{1}{F(\eta)}\,d\eta.$$

If arbitrary functions F and G in the intermediate integrals are denoted by $(1/F')$ and $(-1/g')$, respectively, then by integrating the differential dx we obtain

$$x = -g(\xi) + F(\eta),$$

which, upon substitution of the characteristic variables ξ, η, gives the representation of the general integral

$$F\big(ue^{\int a(t)\,dt}\big) = x + g\Big(\int b(x)\,dx - t\Big) \tag{2.14}$$

of equation (1.2), (1.3). In representation (2.14), arbitrary functions F, g really depend on the arguments defined by differential relations of the characteristic directions. Bearing in mind that the function F is arbitrary and assuming its invertibility, from (2.14) we come to the representation of the general solution (1.1) of the considered equation.

Thus the method of characteristics allows us to integrate equation (1.2), (1.3) without imposing any additional restrictions on the coefficients of the equation. Furthermore, we have established an equivalent connection between the equation and its general solution (1.1) since this equation does not have other solutions, including singular ones. It should also be said that a direct connection between the equation and its general solution has been established by the above reasoning without using some correctly posed problem as an additional intermediate link.

The availability of the explicit representation (1.1) of a general solution makes it possible to extend D'Alembert method to the nonlinear case and thus to consider quite a number of problems for equation (1.2), (1.3).

3. The Initial Problem

Let us assume temporarily that a pair of functions $\tau \in C^2(J)$, $\nu \in C^1(J)$ is arbitrarily given on the interval $J = \{(x,t):\ 0 \le x \le \ell,\ t = 0\}$. We will consider the problem of constructing the solution $u(x,t)$ of equation (1.2), (1.3) by its initial perturbations

$$u\big|_J = \tau(x), \tag{3.1}$$

$$u_t\big|_J = \nu(x) \tag{3.2}$$

and simultaneously establish the domain of its definition by conditions (3.1), (3.2).

In the first place note that for $\beta(x) \ne \text{const}$ the carrier of the initial conditions has the characteristic direction corresponding to the root λ_1 at none of the points. The situation with the other family is different. If $\tau(x) = \text{const}$, then the interval J turns out to be the characteristic of the family corresponding to the root λ_2. Hence it immediately follows that if

$$\nu(x) \ne a(0)\tau(x), \quad [\nu(x)\cdot b(x)]' + a(0)\tau(x)b'(x) = 0,$$

then the initial problem has an infinite number of solutions. Otherwise, the problem does not have a solution. We exclude the latter case by the condition

$$\tau'(x) \ne 0 \tag{3.3}$$

everywhere in J. This condition guarantees that the carrier of the initial conditions is free, i.e., does not have the characteristic direction anywhere. Then we can expect of the initial problem to be correct under some additional requirements as to the initial perturbations. We will discuss these requirements below. In the first place, let us consider condition (2.2) on the interval J:

$$\tau'(x) + b(x)\big[\nu(x) + a(0)\tau(x)\big] \ne 0 \tag{3.4}$$

the fulfillment of which excludes the parabolic degeneracy on the carrier of data. Otherwise, the carrier of data J turns out to be the set of points of parabolic Tricomian degeneracy by virtue of the fact that the carrier itself is not a characteristic [2]. In what follows it will be assumed that conditions (3.3) and (3.4) are fulfilled.

Before we begin the investigation of the posed problem by means of the general solution (1.1), it should be noted that the function $\alpha(t)$ does not vanish for $t = 0$ due to the relation $\alpha = \exp[-\int a(t)\,dt]$, where the function $a(t)$ is assumed to be bounded. Therefore the initial conditions (3.1), (3.2), their carrier and general solution (1.1) are not contradictory. Hence the general solution (1.1) can be subjected to conditions (3.1), (3.2) to obtain, as a result, two functional relations for defining arbitrary f, g:

$$\begin{gathered}\alpha(0)f\big\{x+g[\beta(x)]\big\}=\tau(x),\\ \alpha'(0)f\big\{x+g[\beta(x)]\big\}-\alpha(0)f'\big\{x+g[\beta(x)]\big\}g'[\beta(x)]=\nu(x)\end{gathered}\tag{3.5}$$

on the interval $0 \le x \le \ell$. Excluding the function f from these equalities, for the derivatives f', g' we have

$$\alpha^2(0)f'\big\{x+g[\beta(x)]\big\}g'[\beta(x)]=\alpha'(0)\tau(x)-\alpha(0)\nu(x).\tag{3.6}$$

Along with relation (3.6), let us consider the equality obtained by differentiation of (3.5) and exclude from this pair the expression $g'[\beta(x)]$. As a result of this operation we come to the equality

$$[\alpha(0)]^2f'\big\{x+g[\beta(x)]\big\}=\alpha(0)\tau'(x)-\beta'(x)\big[\alpha'(0)\tau(x)-\alpha(0)\nu(x)\big].$$

Dividing relation (3.6) by this equality, we define the complete derivative of the superposition $g[\beta(x)]$

$$g'[\beta(x)]\beta'(x)=-1-h'(x),\tag{3.7}$$

where the notation

$$h'(x)=\alpha(0)\tau'(x)\Big\{\big[\alpha'(0)\tau(x)-\alpha(0)\nu(x)\big]\beta'(x)-\alpha(0)\tau'(x)\Big\}^{-1}\tag{3.8}$$

is used.

By integrating equality (3.7) in the interval (x_0, x), where $0 \le x_0 \le \ell$ is taken arbitrarily, we have

$$g[\beta(x)]=x_0-x+g[\beta(x_0)]-h(x)+h(x_0).\tag{3.9}$$

Thus we have expressed the superposition $g[\beta(x)]$ with an arbitrary function g by means of the initial functions τ, ν, the function $\beta(x)$ that appears in the coefficients of the equation, and the value $\alpha(t)$ and its derivative for $t = 0$.

Let us now substitute $g[\beta(x)]$ defined by into equality (3.5):

$$\alpha(0)f\big\{-h(x)+x_0+h(x_0)+g[\beta(x_0)]\big\}=\tau(x).\tag{3.10}$$

As follows from (3.10), the argument of an arbitrary function f depends both on an argument $x \in [0, \ell]$ and on an arbitrarily taken value x_0 from the same interval. However, after differentiating this argument with respect to x_0 and taking into account equality (3.7), we see that this derivative is identically zero. Therefore in (3.10) the argument of the function f does not in fact depend on an arbitrary taken value x_0 and we have

$$\alpha(0)f[-h(x)]=\tau(x).\tag{3.11}$$

Arbitrary functions g, f are defined by (3.9) and (3.11), but they are not yet suitable for substitution into the general solution (1.1) because the right-hand sides of these formulas are expressed in terms of functions of the argument x. To represent the solution of problem (1.2), (1.3), (3.1), (3.2) on the basis of (1.1), the right-hand side $-x-h(x)$ of equality (3.9) should be expressed as a function of the argument

$$z \equiv \beta(x) \tag{3.12}$$

while the right-hand side $\tau(x)$ should be rewritten in terms of values of the argument

$$\zeta \equiv -h(x). \tag{3.13}$$

To make (3.12) invertible, the function $\beta(x)$ must be assumed to be strictly monotone for $x \in R^1$, which is equivalent to the requirement that

$$0 < \beta_0 \leq |\beta'(x)| \leq \beta_1 < \infty \tag{3.14}$$

for all values x, including $x \in [0, \ell]$. In that case the function β is invertible. Denote the solution of equation (3.12) by

$$x = B(z) \tag{3.15}$$

where the inverse function B is defined on the interval $[\beta(0), \beta(\ell)]$. The function g is defined on this interval, too:

$$g(z) = -B(z) - h[B(z)], \quad z \in [\beta(0), \beta(\ell)]. \tag{3.16}$$

When z passes through the values from the above-mentioned interval, the function $B(z)$ takes the values from the interval $[0, \ell]$, where, properly speaking, the function $h(x)$ is defined. In order that equation (3.13) have a solution, we must also require of this function that it be strictly monotone. This condition is fulfilled if the right-hand side of equality (3.8) is bounded and has fixed sign and in that case there exists a unique inverse function

$$x \in H(\zeta) \tag{3.17}$$

whose values are contained within the interval $[0, \ell]$. In some cases such a choice of the boundary of the set of values of the inverse function H plays the role of normalization conditions for choosing an appropriate branch. The strict monotonicity of the function $h(x)$ can be provided by imposing restrictions on its derivative which is defined by the initial functions by the formula (3.8). If we assume that the inequalities

$$0 < \tau_0 \leq |\tau'(x)| \leq T_0 < \infty, \tag{3.18}$$

$$0 < \tau_1 \leq \left| \beta'(x) \Big[\frac{\alpha'(0)}{\alpha(0)} \tau(x) - \nu(x) \Big] - \tau'(x) \right| \leq T_1 < \infty \tag{3.19}$$

with some constants τ_0, τ_1, T_0, $T_1 > 0$ are fulfilled, then there exists a unique inverse function H and, by virtue of (3.11) we have

$$f(\zeta) = \frac{1}{\alpha(0)} \tau[H(\zeta)] \tag{3.20}$$

for

$$\zeta \in [-h(0), -h(\ell)]. \tag{3.21}$$

To obtain the solution $u(x,t)$ of the considered initial problem, we substitute the functions g, f defined by (3.16), (3.20), respectively, into the representation of the general solution (1.1):

$$u(x,t) = \frac{\alpha(t)}{\alpha(0)}\,\tau\Big\{H\Big[x - B(\beta(x)-t) - h\big(B(\beta(x)-t)\big)\Big]\Big\}. \tag{3.22}$$

The question where solution (3.22) is defined is no less important. The structure of the domain of definition of solution (3.22) can be clarified by using the intervals of the arguments z, ζ represented in (3.16) and (3.21). According to these inequalities we have

$$\beta(0) \le \beta(x) - t \le \beta(\ell) \ \text{ for } \ \beta'(x) > 0,$$

or (3.23)

$$\beta(\ell) \le \beta(x) - t \le \beta(0) \ \text{ for } \ \beta'(x) < 0,$$

and, moreover,

$$x - B[\beta(x)-t] - h\big\{B[\beta(x)-t]\big\} \in [-h(0), -h(\ell)]. \tag{3.24}$$

Relations (3.23), (3.24) and underlie the structure of the domain of definition of solution (3.22). Restrictions (3.23) of the characteristic argument $\beta(x) - t$ do not in fact differ from analogous inequalities for characteristic variables in the case of linear equations when the data carrier is finite. Such a family of characteristic curves is completely defined since it does not depend on solution values. Each curve of this family, drawn from an arbitrary point $(x_0, 0)$ of the data carrier J, is given by the equation

$$\beta(x) - t = \beta(x_0), \tag{3.25}$$

where $\beta(x_0)$ can be assumed to be the parameter if we want to represent this one-parameter family analytically. It is constructed by a parallel transfer of anyone of its curves along the ordinate axis. By condition (3.14) all these curves are strictly monotone. Therefore family (3.25) has neither discriminant points nor singular curves, including enveloping ones. For $x_0 \in [0,\ell]$ the set of curves (3.25) entirely covers the curvilinear strip (3.23) and through each point of this strip there passes one and only one characteristic.

The nonlinear effect is observed when we consider the other family of characteristics which in fact is given by the relation $\eta \equiv \text{const}$, where the characteristic argument η is defined by formula (2.10) and is equivalent to the equality $u(x,t)[\alpha(t)]^{-1} = \text{const}$. Taking into account the structure of solution (3.22), we conclude that the latter relations are fulfilled when the argument, on which the superposition of functions τ, H depends, has constant values. Thus the family of characteristics, which corresponds to the root λ_2, is given by the constant values of the combination of functions which is not at all a characteristic argument. Any characteristic of this family drawn, say, from the point $(x_0, 0)$ of the carier J is defined, according to the above-given reasoning, by the equation

$$x - B[\beta(x)-t] - h\big\{B[\beta(x)-t]\big\} = -h(x_0). \tag{3.26}$$

Assuming that x_0 varies in the interval $[0, \ell]$ and the right-hand side $h(x_0)$ plays the role of the parameter, the considered family of curves is completely represented by formula (3.26).

The family of characteristics (3.26) has a more complicated structure than family (3.25) though it is also one-parametric and solvable with respect to the parameter $-h(x_0) \equiv c$. The parameter c continuously passes through all values of the interval $[-h(0), -h(\ell)]$. We want to find out whether family (3.26) has any singularities. To this end, we represent family (3.26) in the equivalent form

$$\beta(x) - \beta\Big\{x - h\big[B(\beta(x) - t)\big] - c\Big\} - t = 0.$$

We differentiate it with respect to the parameter c and consider the result obtained together with equality (3.26). If this family has singularities like discriminant curves or points, then we must have

$$\beta'\big\{B[\beta(x) - t]\big\} = 0.$$

But this equality is inadmissible, since it contradicts condition (3.14) that we have introduced. We have thereby proved

Lemma. *If the functions β, τ, ν are continuously differentiable in the interval* $[0, \ell]$ *and satisfy conditions* (3.14), (3.18), (3.19), *then family* (3.26) *does not have discriminant points and curves.*

Along with this fact, we need to establish whether the curves of family (3.26) intersect. The answer to this question can be obtained in different ways. Here we consider the complete differential on the right-hand side of (3.26), make it equal to zero and define the slope of each characteristic of the family under consideration at any point of the domain, where this left-hand side is defined. As a result, we have the identity

$$\frac{dt}{dx} = \beta'(x) - \Big\{1 + h'\big[B(\beta(x) - t)\big]\Big\}^{-1} \beta'\big\{B[\beta(x) - t]\big\}, \tag{3.27}$$

which is fulfilled for every function $t = t(x, c)$ defined by (3.26).

Let us consider (3.27) as an ordinary differential equation of first order. In the neighborhood of each point in the domain of its definition the right-hand side of this equation is bounded and continuously differentiable with respect to the arguments x, t because it has been assumed above that $\nu(x) \neq a(0)\tau(x)$. According to the well-known local theorem on the existence and uniqueness of a solution of the Cauchy problem (see, for example, [6]), through each point of the domain, where the equation is given, there passes one and only one integral curve.

Lemma. *Under the conditions* (3.14), (3.18), (3.19) *the curves of family* (3.26) *do not intersect.*

The proof follows from the assumption that any two curves of family (3.26) intersect at some point (x_0, t_0), which contradicts the uniqueness theorem mentioned above.

Therefore, through any point of the curvilinear strip (3.24) there passes one and only one curve of the family of characteristics (3.26).

Summarizing the above discussion, we come to a conclusion that the function $u(x,t)$ represented by formula (3.22) is uniquely defined on the intersection D of two curvilinear strips (3.23), (3.24). The intersection D is a simply connected domain containing the initial data carrier J and bounded by characteristics (3.25), (3.26) which are given by the values $x_0 = 0$, $x_0 = \ell$ of a parameter $x_0 \in J$ and thereby drawn from the end points of J.

Thus we have proved

Theorem. *If the functions* α, $\beta \in C^2(R^1)$, $\tau, \nu \in C^2(J)$ *satisfy conditions* (3.14), (3.18), (3.19), *then there exists a unique real regular solution* $u(x,t)$ *of problem* (1.2), (1.3), (3.1), (3.2) *defined by formula* (3.22) *in the characteristic quadrangle* D.

It should be specially noted that we deal with hyperbolic solutions of the problem under consideration, since condition (3.19) excludes the parabolic degeneracy of equation (1.2) not only on the data carrier J, but also throughout the domain D. Indeed, if it is assumed that condition (2.2) for the hypebolicity of equation (1.2) is violated at some point $(\ell_0, 0) \in J$, then we have

$$u_x + \beta'(u_t + au) = 0. \tag{3.28}$$

Taking into account conditions (3.1), (3.2) and the notation $a = -\alpha'\alpha^{-1}$ introduced above, the latter equality takes the form

$$\tau'(\ell_0) + \beta'(\ell_0)\Big\{\nu(\ell_0) - \frac{\alpha'(0)}{\alpha(0)}\,\tau(\ell_0)\Big\} = 0 \tag{3.29}$$

which contradicts condition (3.19), since there does not exists a constant τ_1 that bounds from below the modulus of the expression appearing in this condition, while the upper estimate remains valid. The absence of the parabolic degeneracy of equation (1.2) within the domain D easily follows from the comparison of identity (3.27) and the differential equation of the characteristics of the family of the root λ_1, whose right-hand sides differ from each other at all points of the domain D according to conditions (3.14), (3.19).

It should be said here that the smoothness of the initial functions τ, ν is also essential. The question concerning the influence of discontinuities of the functions τ, ν on the solution of the problem under consideration is interesting beyond any doubt.

4. The Modified Initial Problem with Discontinuous Initial Data

For convenience, let us consider the problem with the same initial data carrier $J = \{(x,t) :\ t = 0,\ 0 \le x \le \ell\}$, on which condition (3.1) is given and condition (3.2) is replaced by the given combination

$$\frac{u_x(x,t)}{u_x(x,t) - \beta'(x)[\frac{\alpha'(0)}{\alpha(0)}\,u(x,t) - u_t(x,t)]}\bigg|_J = h'(x), \tag{4.1}$$

where the function $h(x)$ is twice continuously differentiable in the closed interval $[0,\ell]$. The function $\tau(x)$ also given on this interval is twice continuously differentiable and satisfies inequalities (3.18).

If the function $h(x)$ has a unique real inverse function denoted by H and for the derivative $h'(x)$ there exist such positive constants h_0, H_0 that the inequalities

$$0 < h_0 \le |h'(x)| \le H_0 < \infty, \tag{4.2}$$

are fulfilled, then problem (3.1), (4.1), (1.2) is equivalent to the initial problem we have considered above. Therefore its solution is represented by formula (3.22), and the existence and uniqueness theorem remains valid. The structure of the definition domain of a regular solution is preserved, too.

Retaining the uniqueness and conditions (4.2) for the function $h(x)$, let us consider the case where it has, at some point $x_0(0,\ell)$, a finite jump

$$h(x) = \begin{cases} h_1(x), & 0 \le x \le \ell_0, \\ h_2(x), & \ell_0 < x \le \ell, \end{cases} \tag{4.3}$$

provided that $h_1(\ell_0) \neq h_2(\ell_0+)$. Moreover, it is assumed that the sets of values of functions h_1, h_2 do not intersect. For the smooth function τ the behavior of $h(x)$ defined by (4.3) in terms of conditions (3.1), (3.2) is equivalent to the finite discontinuity of the function $\nu(x)$ at the same point $x = \ell_0$.

In order to establish how the solution of the problem posed behaves, we make a combination of the general solution (1.1), appearing in the left-hand side of equation (4.1), and equate its value taken on the data carrier J to $h'(x)$. As a result, for an arbitrary function g we obtain an analogous relation to (3.9)

$$g[\beta(x)] = \begin{cases} -x + h_1(x), & 0 \le x \le \ell_0, \\ -x + h_2(x), & \ell_0 < x \le \ell. \end{cases}$$

Next we subject the general solution (1.1) to condition (3.1) and substitute into it the superposition defined above by the function $g[\beta(x)]$. This allows us to define the function f as follows;

$$\frac{\tau(x)}{\alpha(0)} = \begin{cases} f[h_1(x)], & x \in [0,\ell_0], \\ f[h_2(x)], & x \in (\ell_0,\ell]. \end{cases}$$

Denote the inverse of the function $h_1(x)$ by $H_1(\zeta)$. It will be defined on the interval $[h_1(0), h_1(\ell_0)]$. We take the branch which vanishes for $\zeta = h_1(0)$. Accordingly, for the inverse function $h_2(\zeta)$ we introduce the notation $H_2(x)$, $H_2[h_2(\ell)] = \ell$. The functions H_1, H_2 are sufficiently smooth and strictly monotone on the intervals of their definition. Note that for the limiting values of H_1, H_2 we have

$$\lim_{\zeta \to h_1(\ell_0-)} H_1(\zeta) = \lim_{\zeta \to h_2(\ell_+)} H_2(\zeta) = \ell_0.$$

An arbitrary function f is defined in terms of our notation as

$$f(\zeta)=\begin{cases}\dfrac{1}{\alpha(0)}\,\tau[H_1(\zeta)], & \zeta\in[h_1(0),h_1(\ell_0)],\\ \dfrac{1}{\alpha(0)}\,\tau[H_2(\zeta)], & \zeta\in[h_2(\ell_0+),h_2(\ell)].\end{cases}$$

Taking into account that the function $\beta(x)$ has a unique real inverse B and arbitrary functions f, g are defined, we substitute them into (1.1) and obtain the following solution of the considered problem

$$u_1(x,t)=\frac{\alpha(t)}{\alpha(0)}\,\tau\Big\{H_1\big[x-B(\beta(x)-t)+h_1[B(\beta(x)-t)]\big]\Big\} \tag{4.4}$$

for

$$D_1=\begin{cases}[\beta(x)-t]\in[\beta(0),\beta(\ell)],\\ x-B[\beta(x)-t]+h_1\big\{B[\beta(x)-t]\big\}\in[h_1(0),h_1(\ell_0)],\end{cases} \tag{4.5}$$

while for

$$D_2=\begin{cases}[\beta(x)-t]\in[\beta(\ell_0),\beta(\ell)],\\ x-B[\beta(x)-t]+h_2\big\{B[\beta(x)-t]\big\}\in(h_2(\ell_0),h_2(\ell)],\end{cases} \tag{4.6}$$

we have

$$u_2(x,t)=\frac{\alpha(t)}{\alpha(0)}\,\tau\Big\{H_2\Big[x-B(\beta(x)-t)+h_2[B(\beta(x)-t)]\Big]\Big\}. \tag{4.7}$$

According to the general linear theory of wave propagation [7], the solution

$$u(x,t)=\begin{cases}u_1(x,t), & (x,t)\in D_1,\\ u_2(x,t), & (x,t)\in D_2\end{cases} \tag{4.8}$$

must have a discontinuity along the characteristics drawn from the point $(\ell_0,0)$. But in our case the behavior of the solution may turn out to be more complicated because of the structure of the solution definition domain. If it is assumed that the functions β', h_1', h_2' are positive,

$$h_1'(\ell_0)>\lim_{x\to\ell_0+}h_2'(x)>0 \tag{4.9}$$

and

$$h_1''(x)<0,\quad h_2''(x)<0 \tag{4.10}$$

then the characteristics of the family of the root λ_2 satisfy a differential identity of form (3.27)

$$\frac{dt}{dx}=\beta'(x)-\frac{\beta'\{B[\beta(x)-t]\}}{1+h_j'\{B[\beta(x)-t]\}}\,,\quad j=1,2,$$

and therefore the family does not have singular points or curves by virtue of conditions (3.14), (4.2) which are satisfied by the functions β' and h' by this time having no modulus any longer.

For the characteristics drawn from the points $[0,\ell]$ of the straight line $t=0$ we obtain the relations

$$\frac{dt}{dx}=\beta'(x)\,\frac{h_j'(x)}{1+h_j'(x)}<\beta'(x),\quad j=1,2,$$

which show that the slope of the characteristics of the root λ_1 on the carrier J always exceeds the slope of the characteristics of family (2.10). According to conditions (4.10) the characteristics of the family of the root λ_2, which are drawn from the points $(x_0,0)$, $x_0\le\ell_0$ of the carrier J, lie over the curve represented by the equation

$$x-B[\beta(x)-t]+h_1\bigl\{B[\beta(x)-t]\bigr\}=h_1(\ell_0),\quad x>\ell_0, \tag{4.11}$$

which itself is a characteristic. All of the characteristics of the same family which pass through the points $(x_0,0)$, $x_0>\ell_0$ lie below the curve given in implicit form

$$x-B[\beta(x)-t]+h_2\bigl\{B[\beta(x)-t]\bigr\}=\lim_{x\to\ell_0+}h_2(x). \tag{4.12}$$

Except for $(\ell_0,0)$ the curves represented by (4.11) and (4.12) do not have other common points and for $t>0$ the first curve is situated above the second one by virtue of inequalities (4.9). Hence we come to a conclusion that when all our assumptions as to the functions α, β, τ, h are fulfilled, the closures of the domains $\overline{D}_1$, $\overline{D}_2$ do not have common points in the half-plane $t\ge0$. But the domains D_1, D_2 given respectively by relations (4.5) and (4.6) are the definition domains of the unique solution (4.8) of the considered problem.

Hence the following statement is true.

Theorem. *If conditions* (3.18), (3.14), (4.2), (4.9), (4.10) *are fulfilled for functions* $\alpha\in C^2(R^1_+)$, $\beta\in C^2(R^1)$, $\beta'>0$, $\tau\in C^2(\overline{J})$ *and* $h(x)$ *defined by* (4.3) *with* $h_1\in C^2[0,\ell_0]$, $h_2\in C^2(\ell_0,\ell]$, *then in the upper half-plane* $t\ge0$ *there exists the unique real solution* (4.8) *of problem* (1.2), (1.3), (3.1), (4.1), *where* u_1 *is defined by* (4.4) *in the domain* D_1 *and* u_2 *is defined by* (4.7) *in the domain* D_2.

Thus we see that the discontinuity of an initial condition of type (4.3) can be the cause for which this discontinuity not only will increase in the respective solution outside the data carrier, but also will lead to the discontinuity of the definition domain of this solution. This is a purely nonlinear effect.

Note that that one should not treat the problem as a combination of two analogous problems with conditions of type (4.1) on two adjacent intervals $[0,\ell_0]$ and $[\ell_0,\ell_1]$ of the axis $t=0$. In that case, the functions u_1, u_2 would be solutions of two independent problems and the function u_1 defined by (4.4) would be constructed not in the domain D_1, but only on its part which is adjacent to the corresponding carrier and obtained by cutting it from the domain D_1 by the characteristics $\beta(x)-t=\beta(\ell_0)$.

5. Nonlocal Problems

Let some given function $\gamma \in C^2[0,\infty)$ satisfy the conditions

$$\gamma(0)=0, \quad \gamma(x)>0, \;\; 0<\gamma_0 \le \gamma'(x)<1, \tag{5.1}$$

and on the interval, where it is given, have a unique real inverse function which we denote by Γ.

Nonlocal Problem. Find a definition domain of the solution $u(x,t)$ of equation (1.2), (1.3) and the solution itself satisfying the conditions (3.1) and

$$u[x,\beta(x)-\beta(0)] = \frac{\alpha[\beta(x)-\beta(0)]}{[\alpha(0)]}\, u[\gamma(x),0]. \tag{5.2}$$

In a certain sense this problem is skin to the well-known Darboux problem. If instead of condition (5.2) only its left-hand side were given, then we would have the formulation of the first linear Darboux problem [3]. Condition (5.2) connects the values of the sought solution at the points of the interval J and the arc of the characteristic $t=\beta(x)-\beta(0)$ drawn from the initial point of the interval J.

Theorem. *If the functions τ and β together with their derivatives of second order are positive and satisfy inequalities* (3.18) *and* (3.14)*, respectively, and the function γ obeys conditions* (5.1)*, then problem* (1.2)*,* (1.3)*,* (3.1)*,* (5.2) *has a unique real solution and it is defined in the triangular domain bounded by the segment J, the arc of the characteristic $t=\beta(x)-\beta(0)$ and the characteristic of the other family defined by the equation*

$$x - B[\beta(x)-t] + \Gamma\big\{B[\beta(x)-t]\big\} = \Gamma(\ell) \tag{5.3}$$

and drawn from the point $(\ell,0)$.

Proof. The family of characteristics, which corresponds to the root λ_2, is completely given by the invariants η, η_1 defined by (2.10), (2.11). Therefore no matter what values of the functions f, g are the relation

$$\eta = \frac{u(x,t)}{\alpha(t)} = f\big\{x+g[\beta(x)-t]\big\},$$

must be constant along each characteristic of this family. Condition (5.2) shows that the values of this relation at the points $(x,\beta(x)-\beta(0))$ and $(\gamma(x),0)$ are the same. Restrictions on the functions τ, β, γ provide the strict monotonicity of the value of the sought solution on the interval J and the arc of the characteristic $t=\beta(x)-\beta(0)$. Hence for the solution the repetition of some value on these arcs is excluded. It turns out that these points lie on the same characteristics in the case of the given fixed x.

The invariant η will have constant values for the fixed f, which may happen when the only argument of this function is constant, $x+g[\beta(x)-t]=\text{const}$. The value of this argument is chosen by the point $(x,\beta(x)-\beta(0))$ and defined to within an arbitrary summand $x_0+g[\beta(0)]$. An arbitrary function g itself is defined by another point $(\gamma(x_0), 0) \in J$, through which the given characteristic also passes:

$$g\big\{\beta[\gamma(x_0)]\big\} = x_0 - \gamma(x_0) + g[\beta(0)], \quad \forall\, x_0 \in [0, \ell].$$

Using the notation $\gamma(x_0) \equiv x^*$, the obtained relation can be represented in an equivalent manner as

$$g\{\beta(x^*)\} = -x^* + \Gamma(x^*) + g[\beta(0)], \quad \forall\, x^* \in [0, \gamma(\ell)],$$

where, as said above, Γ is an inverse function with respect to γ. In view of the fact that the equation $\beta(x^*) = z$ is also uniquely solvable, an arbitrary function g is defined in the form

$$g(z) = -B(z) + \Gamma[B(z)] + g[\beta(0)] \tag{5.4}$$

for

$$z \in [\beta(0), \beta(\gamma(\ell))]. \tag{5.5}$$

Now we can define another arbitrary function, too, by substituting (5.4) into the representation of the general solution (1.1) and, after that, by subjecting the obtained expression to condition (3.1). We have

$$f\big\{\Gamma(x) + g[\beta(0)]\big\} = \frac{\tau(x)}{\alpha(0)}\,.$$

Taking the argument of the function f as the new argument ζ, we transform the relation $\Gamma(x) = \zeta - g[\beta(0)]$ and define x as the function of ζ:

$$x = \gamma\big\{\zeta - g[\beta(0)]\big\}$$

for

$$\zeta \in \big[g[\beta(0)], \ell + g[\beta(0)]\big]. \tag{5.6}$$

We have thus managed to define the function f in interval (5.6) with still arbitrary boundaries:

$$f(\zeta) = \frac{1}{\alpha(0)}\,\tau\big\{\gamma\big[\zeta - g[\beta(0)]\big]\big\}. \tag{5.7}$$

Now, using (5.4) and (5.7), we construct the solution of the problem posed:

$$u(x,t) = \frac{\alpha(t)}{\alpha(0)}\,\tau\Big\{\gamma[x - B(\beta(x) - t) + \Gamma(B(\beta(x) - t))]\Big\}, \tag{5.8}$$

which does not any longer contain any arbitrary values and is defined uniquely. The definition domain of solution (5.8) is bounded in accordance with the definition intervals of arbitrary functions f, g not only by the carrier of data (3.1), (5.2), but also by the family characteristic depending on the solution. Since the family is defined by constant values of the invariant η, we can express their corresponding equations by means of the constant values of the argument of superposition of the functions τ, γ in the right-hand part of (5.8):

$$x - B[\beta(x) - t] + \Gamma[B(\beta(x) - t)] = \text{const}\,. \tag{5.9}$$

According to the propositions as to the functions β, γ the slope of these characteristics

$$\frac{dt}{dx} = \beta'(x) - \Big\{B'[\beta(x)-t]\,\big[1-\Gamma'(B(\beta(x)-t)]\Big\}^{-1}$$

always exceeds the slope of the characteristics of the other family. Therefore solution (5.8) is defined in the domain bounded by the data carrier and the characteristic of the family (5.9) drawn from the end point $(\ell, 0)$ of the interval J. It is this characteristic that is represented by equation (5.3), which was required to prove. □

The problem considered here is one of many variants of nonlocal problems. When formulating any one of them, it is necessary to take into account the construction of a general integral or a general solution of the equation given. The general integral structure can be interpreted in different ways. One of the forms of describing this structure is based on Asgeirsson's principle (see [8], [9]). In the case of equation (1.2) with the right-hand side (1.3) its general solution (1.1) is a nonlinear analogue of Asgeisson's principle that can be formulated as follows: sums of the abscissas of the opposite vertices of any characteristic quadrangle are equal.

Asgeirsson's principle was repeatedly used in posing correct boundary value problems for hyperbolic equations of second order (see, for example, [10]). To exemplify the use of this principle in the case of a nonlinear equation, let us consider the problem akin to the Goursat characteristic linear problem.

Let, on an interval $\ell \le x \le k$, a function $\mu(x)$ be given, which satisfies the following conditions:

$$\begin{gathered}\mu \in C^2[\ell, k], \quad 0 < \beta_0 \le \beta'(x) \le \mu'(x), \\ \mu(\ell) = 0, \quad \mu(k) = \beta(k) - \beta(0),\end{gathered} \tag{5.10}$$

where β_0 is the number from the condition (3.14) which is obeyed by the positive function $\beta(x)$.

As we see, the relation $t = \mu(x)$ describes the monotonically increasing curve connecting the point $(\ell, 0)$ of the axis $t = 0$ with the point $(k, \mu(k))$ lying on the characteristic $t = \beta(x) - \beta(0)$. Take on this arc an arbitrary point $(x^*, \mu(x^*))$ and draw the characteristic

$$\beta(x) - t = \beta(x^*) - \mu(x^*)$$

which intersects the straight line $t = 0$ at the point with the abscissa

$$x_1 = B\big\{\beta(x^*) - \mu(x^*)\big\}, \tag{5.11}$$

where B denotes the inverse function with respect to β. Choose, on the characteristic $\beta(x) - t = \beta(0)$, the point with the abscissa

$$x = B\big\{\beta(x^*) - \mu(x^*)\big\} - x^* + k. \tag{5.12}$$

The quadruple of points

$$(x^*, \mu(x^*)), \quad (k, \mu(k)), \quad (x_1, 0), \quad (x, \beta(x) - \beta(0))$$

may turn out to be the vertices of the characteristic quadrangle under certain conditions; in particular, when in the last pair of points from this quadruple the values of the solution $u(x,t)$ of equation (1.2),(1.3) are interconnected by a relation of type (5.2). To establish such an interconnection, we need to define the relation $x_1 = \gamma(x)$ between the abscissas x_1, x. They are functions of the value x^* which is to be excluded from (5.11), (5.12). Let us assume that relation (5.12) is solvable with respect to x^*. To this end, by conditions (3.14) and (5.10) we obtain the inequalities

$$-1 < \frac{dx^*}{dx} = -\Big\{1 + B'\big[\beta(x^*) - \mu(x^*)\big] \cdot \big(\mu'(x^*) - \beta'(x^*)\big)\Big\}^{-1} < 0.$$

Denote the relation between x^* and x by

$$x^* = X(x),$$

where the function X tends to the value k as $x \to \ell$, and is equal to zero for $x = k$.

By substituting the function $X(x)$ into (5.11) we obtain the relation

$$x_1 = B\big\{\beta[X(x)] - \mu[X(x)]\big\} \equiv \gamma(x), \tag{5.13}$$

where γ satisfies all conditions (5.1).

After defining the function $\gamma(x)$, we can consider condition (5.2) too. Note that this condition can also be given for $x = k$ when $\gamma(k) = \ell$ by virtue of (5.13). Therefore points with these abscissas will lie on the common characteristic of the family of the root λ_2. Thus the problem with conditions (3.1) and (5.2), where γ is defined by relation (5.13), will be solvable, and characteristic (5.3) will coincide with the curve $t = \mu(x)$.

Condition (5.2) with the function $\gamma(x)$ defined by (5.13) can be combined not only with condition (3.1), but also with other data, for example, with the values of the sought solution given on the characteristic $t = \beta(x) - \beta(0)$ when $0 \le x \le k$.

Nonlocal Characteristic Problem. Find a regular solution $u(x,t)$ of equation (1.2), (1.3) in the domain bounded by the curves $t = \beta(x) - \beta(0)$, $0 \le x \le k$; $t = \mu(x)$, $\ell \le x \le k$, a segment $0 \le x \le \ell$ of the axis $t = 0$ if it satisfies conditions (5.2) with the function $\gamma(x)$ defined by equality (5.13) and

$$u\big|_t = \beta(x) - \beta(0) = \varphi(x), \quad 0 \le x \le k. \tag{5.14}$$

Simple calculations show that if, besides conditions (5.10), the difference $\beta(x) - \mu(x)$ is uniquely invertible for $x \in [\ell, k]$, then problem (1.2), (1.3), (5.2), (5.14) is solvable.

Indeed, let us rewrite relation (5.11) in the equivalent form $\beta(x_1) = \beta(x^*) - \mu(x^*)$ and denote the inverse function in the right-hand part of the equality by M. Then we have

$$x^* = M[\beta(x_1)],$$

and, after substituting it into (5.12), obtain

$$x = x_1 - M[\beta(x_1)] + k.$$

We introduce the notation

$$x = \Gamma(x_1) \equiv x_1 - M[\beta(x_1)] + k. \tag{5.15}$$

Now equation (5.2) can be rewritten as follows:

$$u\big\{\Gamma(x), \beta[\Gamma(x) - \beta(0)]\big\} = \frac{\alpha\{\beta[\Gamma(x)] - \beta(0)\}}{\alpha(0)}\, u(x,0).$$

Similarly to the preceding problem, using this condition, we define an arbitrary function g by formula (5.4), where the function Γ is defined by (5.15). Further, by substituting an arbitrary function g we subject the general solution (1.1) to condition (5.14) on the characteristic. This results in

$$\alpha[\beta(x) - \beta(0)] f\big\{x - B(\beta(0)) + \Gamma[B(\beta(0))] + g(\beta(0))\big\} = \varphi(x),$$

from which, using the notation $\zeta = x + g[\beta(0)]$, we define an arbitrary function f

$$f(\zeta) = \varphi\big\{\zeta - g(\beta(0))\big\} \cdot \Big\{\alpha\big[\beta\big(\zeta - g(\beta(0))\big) - \beta(0)\big]\Big\}^{-1} \equiv \varphi_1(\zeta).$$

Having defined arbitrary functions f and g, we construct the solution

$$u(x,t) = \alpha(t)\varphi_1\Big\{x - B[\beta(x) - t] + \Gamma[B(\beta(x) - t)]\Big\}, \tag{5.16}$$

which is defined without an arbitrary number $G(\beta(0))$.

We are to find out how, having the constructed solution, to represent the characteristic drawn from the point $(\ell, 0)$. We certainly mean the characteristic of the family of the root λ_2, which depends on solution values. As said above, this family is represented by relation (5.9), where the function Γ is defined by formula (5.15). The curve we are interested in must be given by equation (5.3). Substituting (5.15) into (5.3), we obtain

$$x - M\{\beta(x) - t\} = \Gamma(\ell) - k = 0.$$

By inverting the relation

$$M\{\beta(x) - t\} = x$$

we have

$$\beta(x) - t = \beta(x) - \mu(x),$$

which implies that the characteristic drawn from the point $(\ell, 0)$ coincides with the curve $t = \mu(x)$.

To conclude, note that having a general integral or a general solution of the quasilinear hyperbolic equation with an admissible parabolic degeneracy, we succeed in establishing quite a number of nonstandard phenomena for the initial problem ([11] –[14]) as well as for characteristic problems [15], [16].

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References

1. E. Goursat, Leçons sur l'integration des equations aux derivées partielles du second ordre, vol. 2. *Hermann, Paris*, 1898.
2. F. Tricomi, Sulle equazioni lineari alle derivate parziali di 2^0 ordine di tipo misto. (Italian) *Acc. Linc. Rend. (5)* **14**(1923), 133–247.
3. A. Bitsadze, Some classes of partial differential equations. (Russian) *Nauka, Moscow*, 1981.
4. G. Monge, Second memoir sur le calcul integral de quelques equations aux differences partielles. *Mem. Turin, années* 1770-73, 5(2), 79–122, (1776).
5. I. M. Gel'fand, Some problems in the theory of quasi-linear equations. (Russian) *Uspehi Mat. Nauk* **14**(1959), No. 2(86), 87–158.
6. E. Goursat, Cours d'analyse mathematique, vol. 2. *Gauthier-Villars, Paris*, 1912.
7. J. Hadamard, Leçons sur la propagation des ondes et les equations de l'hydrodinamique. *Hermann, Paris*, 1903.
8. L. Asgeirsson, Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten. *Math. Ann.* **113**(1936), 321–346.
9. F. John, Plane waves and spherical means applied to partial differential equations. *Interscience Publishers, New York–London*, 1955.
10. A. Nakhushev, A method of formulating correctly posed problems for second-order linear hyperbolic equations in the plane. (Russian) *Differentsial'niye Uravneniya* **6**(1970), No. 1, 192–195.
11. J. Gvazava, Some classes of quasilinear equations of mixed type. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **67**(1981), 1–94.
12. J. K. Gvazava, Problems with initial and characteristic conditions for hyperbolic equations with a nonlinear principal part. (Russian) *Differentsial'nye Uravneniya* **19**(1983), No. 1, 22–27.
13. G. Bagaturia, On perturbationns. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **14**(1999), No. 1, 3–6.
14. M. Menteshashvili, On the Cauchy problem with initial data on the unit circle. (Russian) *Soobshch. Akad. Nauk Gruzii* **148**(1993), No. 2, 190–192.
15. J. Gvazava, Some classes of hyperbolic equations and equations of mixed type. *Proc. A. Razmadze Math. Inst.* **108**(1992), 1–176.
16. J. Gvazava, On second order nonlinear equations with rectilinear characteristics. *Georgian Math. J.* **7**(2000), No. 2, 299–316.
17. M. Klebanskaya, Some nonlinear versions of Darboux and Goursat problems for an hyperbolic equation with parabolic degeneracy. *Abstracts of the Intern. Symp. on Differential Equations and Mathematical Physics, Tbilisi, Georgia, June* 21-25, 1997, p. 79, http://www.rmi.acnet.ge/DEMPh.
18. J. K. Gvazava, General integrals and initial-characteristic problems for second-order nonlinear equations with rectilinear characteristics. (Russian) *Liet. Mat. Rink.* **40**(2000), No. 4, 459–474; English transl.: *Lith. Math. J.* **40**(2000), No. 4, 352–363.

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